On initial segments of computable linear orders

Richard J. Coles, Rod Downey and Bakhadyr Khoussainov*

Dept. of Mathematics
Victoria University of Wellington
Wellington
New Zealand

Dept. of Computer Science
University of Auckland
New Zealand

Abstract

We show there is a computable linear order with a Π^0_2 initial segment that is not isomorphic to any computable linear order.

1 Introduction

In effective (or computable) mathematics one seeks to understand the interactions of various computability considerations on structures. There is a long and interesting history of such structures, having roots in the work of Max Dehn and David Hilbert. One example of such a structure is that of a linear ordering. A linear ordering is computable if its domain is a computable set and its ordering relation is a computable relation.

*This work is partially supported by Victoria University IGC and the Marsden Fund for Basic Science under grant VIC-509 for all three authors. The third author acknowledges the support of the Computer Science Department at the University of Auckland.

Computable linear orders have been extensively studied by many authors such as Ash [2], Chen [3], Chisholm and Moses [4], Downey [6, 7], Downey and Knight [8], Downey and Moses [9], Feiner [10], Gandy [11], Gregorieff [12], Harrison [13], Jockusch and Soare [14], Kierstead [16, 17], Lerman and Rosenstein [18], Moses [19, 20, 21, 22, 23] and Remmel [25, 26].

In classical mathematics, the principal classification tool is that of isomorphism type. However, when one studies computational aspects of mathematics it becomes very important in how one presents a structure in terms of "computability" considerations. For instance if one considers a linear ordering of type ω the one that springs to mind is the natural numbers with their usual ordering. However one can take another copy $\langle \mathcal{A}, \leq_{\mathcal{A}} \rangle$ for which the ordering is computable (in the sense that the domain is $\mathbb{N}$ and there is an algorithm P such that given $x, y \in \mathbb{N}$, P decides which of $x < y$, $x = y$ or $y < x$ holds) yet for instance one cannot decide in general the question $S_{x,y}$ which asks if x is the successor of y. We refer to the book of Rosenstein [27], and the surveys of Kierstead [16] and Downey [6].

In this article we are concerned with understanding the tension between *classical* and *effective* order types. In particular, we seek to understand what parts of classical types have computable (and hence polynomial time (Gregorieff [12])) presentations within their type.

Such investigation began with the work of Feiner [10]. Our interest here is the longstanding question of understanding what types occur as initial segements of computable linear orderings. Such segments can be very complex. It is possible for $\mathcal{A} + \mathcal{B}$ to be a computable liner order and yet $\mathcal{A}$ to be of order type ω_1^{CK} and hence not even hyperarithmetical.

On the other hand Raw [24] proved that if $\mathcal{A}$ is a Π_1^0 initial segment[1] with computable adjacency relation then in fact $\mathcal{A}$ has a computable presentation.

The limiting result was that there is a Π_3^0 initial segment $\mathcal{A}$ with $\mathcal{A} + \mathcal{B}$ computable which has no computable presentation.

The question to answer then is whether each Π_2^0 initial segment of a computable linear order is isomorphic to a computable linear order, and it has resisted all attempts at solution for several years.

Recently, Ambos-Spies, Cooper and Lempp [1] improved the positive direction of Raw's result to prove that every Σ_2^0 initial segment of a computable

[1]Recall the set A is called Σ_1^0 if there is a computable relation R such that $x \in A$ iff $\exists y R(x, y)$ holds. R is called Π_1^0 if $\mathbb{N} - A$ is Σ_1^0. Finally we call A Σ_{n+1}^0 iff there is a Π_n^0 relation R such that $x \in A$ iff $\exists y R(x, y)$ holds (and similarly for Π_{n+1}). The Σ_n^0, Π_n ($n \in \mathbb{N}$) sets form a proper hierarchy called the arithmetical hierarchy.

linear order is isomorphic to a computably presentable linear order. We close the circle by proving that there is a computable linear order with a Π^0_2 initial segment not isomorphic to a computably presentable linear order. The proof of this result (theorem 2.14) uses a priority/coding argument which we believe to be of independent interest, and may well prove to have applications in other contexts.

We now give the main results proved in this paper together with the definiton of η-like. Any other necessary definitions are given in section 2 below.

Definition 1.1 (η-like order type) Let $\mathcal{L} = \langle L, <_{\mathcal{L}} \rangle$ be a linear order of type η and let h be a bijection $h : \omega \mapsto L$. A linear order $\mathcal{A} = \langle A, <_{\mathcal{A}} \rangle$ is *η-like* if there is an injection $g : \omega \mapsto \omega$ such that $g(i) \geq 1$ for all $i \in \omega$ and

(i) $A = \bigcup_{i<\omega} \{x_{i,1}, \ldots , x_{i,g(i)}\}$,

(ii) $x_{i,j} <_{\mathcal{A}} x_{k,l}$ if and only if $h(i) <_{\mathcal{L}} h(k) \vee (h(i) =_{\mathcal{L}} h(k) \,\&\, j < l)$.

Any η-like linear order $\mathcal{A}$ has infinite domain and has no leftmost or rightmost point. It is also dense with respect to blocks.

Theorem 2.10 *If $\mathcal{A}$ is an η-like computable linear order then its block set S is the range of a $\mathbf{0}'$-limitwise monotonic function.*

Theorem 2.13 *The linear order $\mathcal{A} + \omega^*$ is computably presentable if and only if the block set S of $\mathcal{A} + \omega^*$ is Σ^0_3.*

Theorem 2.14 *There is a computable linear order $\mathcal{L}$ of the form $\mathcal{A} + \omega^*$ such that $\mathcal{A}$ is an η-like Π^0_2 initial segment of $\mathcal{L}$ and $\mathcal{A}$ is not isomorphic to a computable linear order.*

We use three notations for the relation denoting order: $<$ denotes the standard ordering of the set of natural numbers ω, $\preceq$ denotes the ordering of strings on the priority tree used in the proof of the main theorem, and $<_{\mathcal{L}}$ denotes the ordering of elements of the linear order $\mathcal{L}$. We also write $\mu_{\mathcal{L}}$ to denote an unbounded search for the leftmost element of a specified subset of $\mathcal{L}$. If F and G are two sets of natural numbers such that $F \cap G = \emptyset$ then we write $F \sqcup G$ to denote the disjoint union of F and G.

2 Computable linear orders

In this section we give some useful definitions and prove some lemmas and thereoms needed for the main result.

Definition 2.1 Let $\mathcal{L} = \langle L, < \rangle$ be a linear order. An *n-block* of $\mathcal{L}$ is a sequence of $n > 0$ elements $x_1, x_2, \ldots, x_n \in \mathcal{L}$ such that

(i) $x_1 < x_2 < \ldots < x_n$,

(ii) $x_1 \leq y \leq x_n \implies y = x_i$ for some $1 \leq i \leq n$,

(iii) $(\forall y)(\exists z)[y < x_1 \implies y < z < x_1]$,

(iv) $(\forall y)(\exists z)[x_n < y \implies x_n < z < y]$.

We say x_1 and x_n are right and left endpoints (or limit points) respectively.

The notion of n-block can be used to code sets of natural numbers into linear orders. We will refer to this coding technique as the *standard coding* as it is frequently used when working with linear orders.

Let $\mathcal{L} = \langle L, < \rangle$ be a linear order and consider the set

$$S = \{n \mid \mathcal{L} \text{ contains an } n\text{-block}\}.$$

By considering what it means to be an n-block it is easy to see that S has arithmetical complexity $\Sigma_3^{\mathcal{L}}$. So when $\mathcal{L}$ is a computable linear order S is a Σ_3^0 set. Whenever we refer to a set of numbers S below we are thinking of S as the set of numbers n that occur as n-blocks in a given linear ordering.

It is possible to code any Σ_3^0 set in the standard way in a computable linear order, see Raw [24] for example

We say that a linear order has order type η if and only if it is isomorphic to $\mathbb{Q}$ with the usual ordering. We need the following:

Definition 2.2 (η-like order type) Let $\mathcal{L} = \langle L, <_{\mathcal{L}} \rangle$ be a linear order of type η and let h be a bijection $h : \omega \mapsto L$. A linear order $\mathcal{A} = \langle A, <_{\mathcal{A}} \rangle$ is *η-like* if there is an injection $g : \omega \mapsto \omega$ such that $g(i) \geq 1$ for all $i \in \omega$ and

(i) $A = \bigcup_{i<\omega} \{x_{i,1}, \ldots, x_{i,g(i)}\}$,

(ii) $x_{i,j} <_{\mathcal{A}} x_{k,l}$ if and only if $h(i) <_{\mathcal{L}} h(k) \vee (h(i) =_{\mathcal{L}} h(k) \,\&\, j < l)$.

Any η-like linear order $\mathcal{A}$ has infinite domain and has no leftmost or rightmost point. It is also dense with respect to blocks.

Intuitively, an η-like linear order is derived from a linear order of type η by substituting each point in the latter with some block of size 1 or more. This process is sometimes referred to as a *block shuffle.*

We need two lemmas for the proof of the main theorem, the first of which is a relativisation of a result of Khoussainov, Nies and Shore [15]:

Definition 2.3 A function f is *limitwise monotonic* if there exists a recursive function $\phi(x, s)$ such that

(i) $\phi(x, s) \leq \phi(x, s+1)$ for all $x, s \in \omega$,

(ii) $\lim_s \phi(x, s)$ exists for all $x \in \omega$,

(iii) $f(x) = \lim_s \phi(x, s)$.

Relativising this definition to $\mathbf{0}'$ we obtain the following definition:

Definition 2.4 A function f is $\mathbf{0}'$-*limitwise monotonic* if there exists a function $\phi(x, s)$ recursive in $\mathbf{0}'$ such that

(i) $\phi(x, s) \leq \phi(x, s+1)$ for all $x, s \in \omega$,

(ii) $\lim_t \phi(x, s)$ exists for all $x \in \omega$,

(iii) $f(x) = \lim_s \phi(x, s)$.

Lemma 2.5 (Khoussainov, Nies and Shore) *There exists a Δ^0_2 set S which is not the range of any limitwise monotonic function.*

PROOF:

Let ϕ_e, $e \in \omega$, be a uniform enumeration of all partial computable functions ϕ such that for all $t' \geq t$, if $\phi(x, t')$ is defined then $\phi(x, t)$ is defined and $\phi(x, t) \leq \phi(x, t')$. At stage s of the construction we define a finite set A_s such that $A(y) = \lim_s A_s(y)$ exists for all y. We aim to satisfy for all $e \in \omega$, the requirements:

$$\mathcal{R}_e : f(e) \ \lim_t \phi_e(x, t) < \omega \text{ for all } x \implies range(f_e) \neq A.$$

The strategy for a single requirement $\mathcal{R}_e$ as follows: at stage s pick a witness, $m_e say$, and enumerate m_e into A, that is let $A_s(m_e) = 1$. Now $\mathcal{R}_e$ is satisfied unless there is some later stage t_0 such that there is an x with $\phi_e(x, t_0) = m_e$. If there is such a stage t_0 and such an x then $\mathcal{R}_e$ ensures that $A(\phi_e(x, t)) = 0$ for all $t \geq t_0$. Thus either $f_e(x) \uparrow$ or $f_e(x) \downarrow$ and $f_e(x) \notin A$.

However keeping $\phi_e(x, t)$ out of A for all $t \geq t_0$ can conflict with a lower priority ($i > e$) requirement $\mathcal{R}_i$ since the witness m_i chosen for $\mathcal{R}_i$ may equal $\phi_e(x, t')$ for some $t' > t_0$. If $f_e(x) \downarrow$, then this holds permanently for just one number, and if $f_e(x) \uparrow$ then this restriction on choice of witness for $\mathcal{R}_i$ is transitory for each number. Therefore we will be able to argue in the verification that each lower priority $\mathcal{R}_i$ requirement will be able to choose a stable witness at some stage.

Construction.

At stage s we try to determine the values of the parameters m_e, x_e and $n_e = \phi_e(x_e, s)$ for $\mathcal{R}_e$. Each parameter may remain undefined. Furthermore we define the approximation A_s to A at stage s.

Stage 0: Let $A_0 = \emptyset$, and declare all parameters to be undefined.

Stage s : For each $e = 0, \ldots, s-1$ in turn go through substage e by performing the following actions:

(1) If m_e is undefined, let m_e be the least number in $\omega^{[e]}$ greater than all m_i ($i < e$) which is not equal to any n_i. Let $A_s(m_e) = 1$ and proceed to the next substage, or to stage $s + 1$ if $e = s - 1$.

(2) If x_e is undefined and $\phi_e(x, s) = m_e$ for some x then let $x_e = x, n_e = m_e$ and $A_s(n_e) = 0$, and proceed to the next stage $s + 1$ if $e = s - 1$.

(3) Let $n_e = \phi_e(x_e, s)$ and $A_s(n_e) = 0$. If $n_e = m_i$ for some $i > e$, declare all the parameters of requirements $\mathcal{R}_j$ for $j \geq i$ to be undefined.

For each y, if $A_s(y)$ is not determined by the end of stage s, then assign to $A_s(y)$ its previous value $A_{s-1}(y)$. The stage is now completed.

Verification.

We prove three claims to show that the construction succeeds.

Claim 2.6 *Each m_e is defined and is constant from some stage on.*

PROOF:

Suppose inductively that the claim holds for each $i < e$. Let s_0 be a stage such that each m_i has reached its limit for $i < e$ and if x_i ever becomes defined after s_0 and $\lim_s n_{i,s} < \infty$ then the limit has been reached at s_0. Moreover, let $k \geq e$ be the least number which does not equal any of these limits and is greater than all m_i for $i < e$. Also suppose that $n_{i,s_0} > k$ if $\lim_s n_{j,s} = \infty$, $(j < e)$. If m_e is cancelled after stage s_0, then $m_e = k$ is permanent from the next stage on. This proves the claim. $\square$

Claim 2.7 *For each y, $\lim_s A_s(y)$ exists. Therefore the set $A = \lim_s A_s$ is a Δ^0_2 set.*

PROOF:

Suppose that $y \in \omega^{[e]}$, and let s_0 be a stage at which m_e has reached its limit. Since y can be enumerated into A if and only if $y = m_e$, then after stage s_0 $A(y)$ can change at most once. This proves the claim. $\square$

Claim 2.8 *If $f_e(x) = \lim_t \phi_e(x,t)$ exists for each x, then there $A \neq range(f_e)$.*

PROOF:

Suppose for a contradiction that $A = range(f_e)$. Let s_0 be the stage at which m_e reaches its limit. Then at some stage $s > s_0$ we must reach part (2) of the construction, otherwise $A(m_e) = 1$ but $m_e \notin range(f_e)$. Suppose that $\phi_e(x,s) = m_e$ for the minimal $s \geq s_0$ at which we reach part (2). It follows that for $t \geq s$, $n_e = \phi_e(x,t)$ and $A_t(n_e) = 0$. So $A(f_e(x)) = 0$. This contradiction proves the claim. $\square$

This concludes the proof of lemma 2.5. $\square$

Relativising this lemma we have:

Lemma 2.9 *There exists a Δ^0_3 (and hence for our purposes a Σ^0_3) set S which is not the range of any $\mathbf{0}'$-limitwise monotonic function.*

PROOF:

Relativise the proof of lemma 2.5 (see lemma 2.1 in [15]).

Theorem 2.10 *If $\mathcal{A}$ is an η-like computable linear ordering then its block set S is the range of a $\mathbf{0}'$-limitwise monotonic function.*

PROOF:

Let $\mathcal{A} = \langle A, < \rangle$ be an η-like computable linear ordering. Therefore A is an infinite computable set.

Let $N(x,y)$ be the relation x is adjacent to y in $\mathcal{A}$, i.e. $N(x,y)$ holds if and only if

$$(x < y \vee y < x) \,\&\, (\forall z)[(x \leq z \leq y \vee y \leq z \leq x) \implies (x = z \vee y = z)].$$

Then we can observe that N is a Π^0_1 relation since $\mathcal{A}$ is computable. Hence with a $\mathbf{0}'$ oracle we can determine whether two elements in $\mathcal{A}$ are adjacent or not.

Observe however that the relation $\mathrm{Suc}(x, y)$ stating that y is the successor of x is a Π^0_2 relation:

$$\mathrm{Suc}(x, y) \iff x < y \,\&\, (\forall z)[x < z \implies (\overline{N}(x, z) \vee y = z)].$$

Construction. Fix a computable enumeration $\{A_s\}_{s\in\omega}$ of A such that $|A_{s+1} - A_s| = 1$ and $A_0 = \emptyset$. Let $\mathcal{A}_s = \langle A_s, <\rangle$.

We build a $\mathbf{0}'$-recursive function $\phi(x, s)$ such that

(i) $\phi(x, s) \leq \phi(x, s + 1)$ for all $x, s \in \omega$,

(ii) $\lim_s \phi(x, s)$ exists for all $x \in \omega$,

(iii) $S = range(\lim_s \phi(x, s))$.

Stage $s = 0 : \phi(i, 0) = 0$ for all $i \in \omega$.

Stage $s + 1$: At the end of stage s we have enumerated s elements of A, $A_s = \{a_0, a_1, \ldots , a_{s-1}\}$ say. We have also defined $\phi(i, s)$ for $0 \leq i < s$. Let a_s be the unique element in $A_{s+1} - A_s$.

For each $0 \leq i < s$ do the following using $\mathbf{0}'$ as an oracle to answer questions about N :

(a) If $N(a_{s+1}, a_i)$ holds then let $\phi(i, s + 1) = \phi(i, s) + 1$.
Also let $\phi(s, s + 1) = \phi(i, s) + 1$.

(b) If $\overline{N}(a_{s+1}, a_i)$ holds then let $\phi(i, s + 1) = \phi(i, s)$.

Finally if after the above process $\phi(s, s + 1) = 0$ then let $\phi(s, s + 1) = 1$.

This ends the description of the construction.

Verification.

We prove two claims to obtain the lemma:

Claim 2.11 *$f(i) = \lim_s \phi(i, s)$ is a $\mathbf{0}'$-limitwise monotonic function.*

PROOF:

From the construction it is clear that $\phi(i, s + 1) \geq \phi(i, s)$ for all $i, s \in \omega$.

Since $\mathcal{A}$ is η-like each element a_i of A is part of an n-block for some $n > 0$. Therefore there is a stage t_0 such that all members in the block containing

a_i have been enumerated in A_{t_0}. Then for all $s \geq t_0$, $\phi(i,s) = \phi(i,t_0) = n$ as $\phi(i,s)$ can only increase when a new element is enumerated which is adjacent to one of the elements in the same block as a_i. Hence $\lim_s \phi(i,s)$ exists for all $i \in \omega$.

Finally observe that $\phi(i,s)$ is computable from a $\mathbf{0}'$ oracle. Hence all the conditions in definition 2.4 are met and so f is a $\mathbf{0}'$-limitwise monotonic function.

Claim 2.12 $S = range(f)$.

PROOF:

Suppose that $n \in S$. Then $\mathcal{A}$ contains a block of size n. There is a stage t_1 when all n elements of an n-block have been enumerated in A_{t_1}. Let a_i be one element of the n-block. Then for all $s > t_1$, $\phi(i,s) = \phi(i,t_1) = n$ and hence $n \in range(f)$.

Conversely, suppose that $n \in range(f)$. Then $n > 0$ and there is some i for which $\lim_s \phi(i,s) = n$. Therefore there must be n elements i$x_1, x_2, \ldots, x_n$ in $\mathcal{A}$ such that x_i is adjacent to x_{i+1} for $1 \leq i < n$. These n elements must form a block because of the density of blocks property of η-like linear orders, and by the fact that $\lim_s \phi(i,s) = n$. Therefore $n \in S$.

This concludes the proof of the lemma.

□

Theorem 2.13 *The linear order $\mathcal{A} + \omega^*$ is computably presentable if and only if the block set S of $\mathcal{A} + \omega^*$ is Σ_3^0.*

PROOF:

This is presented in section 3.

□

Theorem 2.14 *There is a computable linear order $\mathcal{L}$ of the form $\mathcal{A} + \omega^*$ such that $\mathcal{A}$ is an η-like Π_2^0 initial segment of $\mathcal{L}$ and $\mathcal{A}$ is not isomorphic to a computable linear order.*

PROOF:

By lemma 2.9 there is a Σ_3^0 set S which is not the range of a $\mathbf{0}'$-limitwise monotonic function.

Then by theorem 2.13 there is a computable linear order, $\mathcal{L}$ say, which codes S in the standard way such that $\mathcal{L}$ is of the form $\mathcal{A} + \omega^*$ and $\mathcal{A}$ is

η-like. Furthermore $n \in S$ if and only if $\mathcal{A}$ contains an n-block. Therefore by theorem 2.10 $\mathcal{A}$ cannot be isomorphic to a computable linear order.

□

3 Proof of theorem 2.13

($\Rightarrow$) Suppose $\mathcal{L}$ is a linear order of the form $\mathcal{A} + \omega^*$. Then $\mathcal{L}$ contains an n-block if and only if

$$\begin{aligned}&(\exists x_1, x_2, \ldots, x_n)(\forall y)(\exists z_1, z_n)[x_1 < x_2 < \ldots < x_n \,\&\\ &\qquad (x_1 \le y \le x_n \implies (\exists i \le n)[y = x_i] \,\&\\ &\qquad\quad (y < x_1 \implies y < z_1 < x_1) \,\& \,(x_n < y \implies x_n < z_n < y)].\end{aligned}$$

Hence S is $\Sigma_3^{\mathcal{L}}$ and therefore Σ_3^0.
($\Leftarrow$) Let $S \neq \emptyset$ be a given Σ_3^0 set. We construct a computable linear ordering $\mathcal{L}$ of the form $\mathcal{A} + \omega^*$ such that $\mathcal{L}$ contains an n-block if and only if $n \in S$. Further, $\mathcal{A} = \langle A, < \rangle$ will be a Π_2^0 initial segment of $\mathcal{L}$ since
$x \notin A \iff (\exists t)(\forall s \ge t)[x \notin A]$.

Since S is Σ_3^0, $n \in S \iff \exists x \widehat{R}(n, x)$ holds for some Π_2^0 binary relation $\widehat{R}$. Then we can approximate S via $\{S^s\}_{s \in \omega}$ as follows:
$n \in S^s \iff \exists x R(n, x, s)$ for some computable approximation R to $\widehat{R}$. Then $S = \{n \mid (\exists^\infty s)[n \in S^s]\}$.

Fix such an R and hence an approximation to S. We construct $\mathcal{L}$ to satisfy the following set of requirements for $i \ge 1, i \in \omega$:
$\mathcal{P}_i : i \in S \implies \mathcal{L}$ contains an i-block.

Rewriting this in terms of the relation $\widehat{R}$ which approximates S we aim to satisfy the following requirements for all $i, j \in \omega$:
$\mathcal{Q}_{i,j} : \widehat{R}(i, j)$ holds $\implies \mathcal{L}$ contains an i-block.

First we outline the strategy for a single requirement $\mathcal{Q}_{1,0}$. Initially we begin with just one point, x_1 say. Whenever the approximation $R(1, 0, s)$ to $\widehat{R}(1, 0)$ holds at stage s we estimate that $\widehat{R}(1, 0)$ holds and that $1 \in S$. Therefore at stage s we add two new points to $\mathcal{L}$, one immediately to the left of x_1, z_1 say, and another immediately to the right of x_1, z_2 say. Further between any two points to the left of x_1 and between any two points to the right of x_1 we also add a new point (which we will refer to as *density points*).

However we also need to add a point to the left of the leftmost point and to the right of the rightmost point to successfully build η, which we will call *extension points.* (In fact we can add extension points density points and so do not need to deal with them seperately as extension points.)

In the full construction points will be chosen from some computable partition of ω in increasing Gödel number to ensure that $\mathcal{L}$ is a computable linear order.

Whenever the approximation $R(1,0,s)$ to $\widehat{R}(1,0)$ does not hold at a stage s we leave $\mathcal{L}^s$ unchanged.

Then suppose $1 \in S$ with witness 0 and so $\widehat{R}(1,0)$ holds. Therefore there are infinitely many stages s where $R(1,0,s)$ holds and so the strategy outlined above builds a computable linear order of order type $\eta + \mathbf{1} + \eta$. So $\mathcal{L}$ contains a 1-block as required. Otherwise if $1 \notin S$ then $\widehat{R}(1,0)$ does not hold. Therefore there is a stage t such that for all stages $s \geq t$, $R(1,0,s)$ does not hold. Hence after stage t no further points are added to $\mathcal{L}$ and $\mathcal{L}$ has order type k for some $k \in \omega$. (When more requirements are considered this outcome will produce the tail of $\mathcal{L}$ of order type ω^*.)

The sub-linear order of $\mathcal{L}$ constructed because of requirement $\mathcal{Q}_{i,j}$ will be referred to as the (i,j)-*section.* In the full construction each (i,j)-section will be of order type $\eta + \mathbf{i} + \eta$ if j witnesses that $R(i,j)$ holds and so $i \in S$ and $\mathcal{L}$ contains an i-block, or otherwise will contribute finitely many points to the ω^* tail of $\mathcal{L}$.

Now consider all requirements $\mathcal{Q}_{1,j}$ for all $j \in \omega$. $\mathcal{Q}_{1,j_1}$ will have higher priority than $\mathcal{Q}_{1,j_2}$ if and only if $j_1 < j_2$. Also the $(1,j_1)$-section will be built to the right of the $(1,j_2)$ section if and only if $j_1 < j_2$. We can assume without loss of generality that if z is the least number for which $\widehat{R}(1,z)$ holds then $\widehat{R}(1,y)$ also holds for all $y \geq z$. Further we may assume that there are infinitely many stages s such that $R(1,z,s) \implies R(1,y,s)$ for all $y \geq z$.

Let z_s be the least witness to $1 \in S^s$ at stage s. Also assume we have started building the $(1,j)$-section for $j < s$, and that whenever we start building a $(2,j_0)$-section for the first time we add a new point, x_{1,j_0} say, targeted for being a 1-block in case $1 \in S$ via witness j_0. This new point is added in the correct position in $\mathcal{L}$, namely to the left of the $(1,j_0-1)$-section as previously mentioned.

For $(1,j)$-sections with $j < z_s$ we leave $\mathcal{L}$ unchanged at stage s. For $(1,j)$-sections with $z_s \leq j < s$ we add density points between every pair not targeted for a 1-block and extend each $(1,j)$-section left and right with extension points. If no such least z_s exists then leave $\mathcal{L}$ unchanged at this stage.

Now suppose $1 \notin S$. Therefore there is no witness z for which $\widehat{R}(1,z)$ holds. So for each j there is a stage t_j such that for all stages $s \geq t_j$, $R(1,j,s)$ does not hold. In this outcome the strategy builds finitely many points in each $(1,j)$-section so the overall result is to build ω^*. Then $\mathcal{L}$ has no 1-blocks as required.

Otherwise there is some least z such that $R(1,z,s)$ holds for infinitely many stages s. The above strategy builds a computable linear order of order type $(\eta + \mathbf{1}).\eta + \eta + k$ for some $k \in \omega$. Therefore 1 is coded in $\mathcal{L}$ as required.

We consider one further combination of requirements before presenting the full construction. Consider the two requirements $\mathcal{Q}_{1,0}$ and $\mathcal{Q}_{2,0}$. The higher priority $(1,0)$-section is initially constructed to the right of the lower priority $(2,0)$-section. The relative location of the two sections within $\mathcal{L}$ will change depending on the estimated outcome of the strategy at each stage.

Suppose initially we have a single point x_1 targeted for a 1-block in the case that $1 \in S$ and two points y_1 and y_2 targeted for a 2-block in the case that $2 \in S$. There are four cases to distinguish for a stage s depending on which of $R(1,0,s)$ and $R(2,0,s)$ hold. Assume that during the previous stages we have constructed finitely many points in the $(1,0)$ and $(2,0)$ sections.

(i) Suppose neither $R(1,0,s)$ nor $R(2,0,s)$ holds. Then we add no further points to $\mathcal{L}$ at this stage. The $(1,0)$-section is estimated to lie to the right of the $(2,0)$-section in this case.

(ii) Suppose $R(1,0,s)$ does not hold but $R(2,0,s)$ does hold. Then we estimate that $2 \in S$ and that $R(2,0,s')$ will hold for infinitely many stages $s' > s$. We do not add any points to the $(1,0)$-section but locate y_1 and y_2 in the original $(2,0)$-section to the left of the $(1,0)$-section and add density points and extension points as described in earlier discussions of the strategy.

(iii) Suppose $R(1,0,s)$ does hold but $R(2,0,s)$ does not. Then we estimate that $1 \in S$ and that $R(1,0,s')$ will hold for infinitely many stages $s' > s$. Therefore for the $(1,0)$-section we add density and extension points. This outcome however *instigates* a reordering of the sections, and we now estimate that the $(2,0)$-section is to the immediate right of the $(1,0)$-section. Since we estimate that $R(2,0)$ does not hold we do not add any new points to the new location of the $(2,0)$-section. However we add new points to the old $(2,0)$-section which is to the left of the $(1,0)$-section to convert it into $(1,j)$-sections. We can think of this as *initialising* those points in sections to the left of the $(1, 0)$-section. We leave the detail of how to form the other $(1, j)$-sections until the full construction below. When we change at some later stage to estimating that $R(1, 0, s')$ does not hold for some $s' > s$ we revert to the original ordering of the sections (namely with the $(1, 0)$-section to the right of the $(2, 0)$-section) and locate the points y_1 and y_2 to once again target that section for a 2-block. Again full details are delayed until the the construction.

(iv) Suppose both $R(1, 0, s)$ and $R(2, 0, s)$ hold. Then as in (iii) we estimate that the $(2, 0)$-section is to the right of the $(1, 0)$-section. We initialise all points to the left of the $(1, 0)$-section and convert them into $(1, j)$-sections. We add density and extension points to the $(1, 0)$-section, and similarly add density points and extend the new location of the $(2, 0)$-section to be found to the right of the $(1, 0)$-section.

If case (i) is the true outcome then we only add finitely many points for both $\mathcal{Q}_{1,0}$ and $\mathcal{Q}_{2,0}$ thus constructing $\mathcal{L}$ with order type k for some $k \in \omega$. Hence $\mathcal{L}$ has no 1- or 2-blocks and $1, 2 \notin S$ as required.

Suppose case (ii) is the true outcome. Then there is a stage t such that for all $s \geq t$, $R(1, 0, s)$ does not hold and we never again add new points to the $(1, 0)$-section. Further, for infinitely many stages $s \geq t$, $R(2, 0, s)$ holds and so we have 2 coded in $\mathcal{L}$ via the standard coding of a 2-block. In this outcome $\mathcal{L}$ has order type $(\eta + \mathbf{2}).\eta + \eta + k$ for some $k \in \omega$.

If case (iii) is the true outcome then there is a stage t such that for all $s \geq t$, $R(2, 0, s)$ does not hold but for infinitely many stages $s \geq t$, $R(1, 0, s)$ holds. Therefore after stage t we build $\mathcal{L}$ with order type $(\eta + \mathbf{1}).\eta + \eta + k$ for some $k \in \omega$.

When the true outcome is (iv) we construct $\mathcal{L}$ with order type $(\eta + \mathbf{1}).\eta + (\eta + \mathbf{2}) + \eta$. This linear order codes 1 and 2 as required. Notice that the outcome for coding 2 into $\mathcal{L}$ does not affect the construction of points for the coding of 1 into $\mathcal{L}$. That is, there is no interaction between requirements other than to reorder the sections.

Now we will present the full construction to deal with all requirements. This involves the use of a priority tree to assist with the complications arising from nesting the different (i, j)-sections in the correct way.

Definition 3.1 Given a Σ^0_3 set $S \neq \emptyset$ let $\widehat{R}$ be a binary Π^0_2 relation and $R(n, x, s)$ an approximation to it such that:

(i) $n \in S \iff \exists x \widehat{R}(n,x)$,

(ii) $n \in S^s \iff \exists x R(n,x,s)$,

(iii) $S = \{n \mid (\exists^\infty s)[n \in S^s]\}$.

(iv) $(\forall n)(\exists x)[\widehat{R}(n,x)$ does not hold $\&\, (\forall s \leq 2^{\langle n,x\rangle})[R(n,x,s)$ holds $]]$,

(v) $\widehat{R}(n,x)$ holds $\implies \widehat{R}(n,y)$ holds for all $y \geq x$.

Clause (iv) ensures that the final segment of $\mathcal{L}$ is of order type ω^* (see lemma 3.13) and not of order type k for some finite $k \in \omega$.

We construct stage by stage a computable linear ordering $\mathcal{L} = \langle L, <_{\mathcal{L}}\rangle$ to meet the following requirements for $i \geq 1$, $i, j \in \omega$:

$$\mathcal{Q}_{i,j} : \widehat{R}(i,j) \implies \mathcal{L} \text{ contains an } i\text{-block.}$$

We order the requirements with the following priority: $\mathcal{Q}_{i_0,j_0}$ is of higher priority than $\mathcal{Q}_{i_1,j_1}$ if and only if

$$(i_0 + j_0 < i_1 + j_1) \vee (i_0 + j_0 = i_1 + j_1 \,\&\, i_0 < i_1) \ \ (*)$$

That is, in decreasing order of priority,

$$(1,0),(1,1),(2,0),(1,2),(2,1),(3,0),\ldots$$

Fix a bijection $\langle i,j\rangle : \omega \times \omega \to \omega$ to code ordered pairs via the ordering $(*)$ above. $\mathcal{Q}_{i,j}$ uses numbers exclusively from $\omega^{[\langle i,j\rangle]}$ whenever it needs to add points to $\mathcal{L}$ during the construction.

Definition 3.2 Let the tree of outcomes T be $T = \omega^{<2}$. Requirement $\mathcal{Q}_{i,j}$ is assigned to nodes σ of T with $lh(\sigma) = \langle i,j\rangle$. At a stage s of the construction we build a path through T of length s with $\sigma_s(\langle i,j\rangle) = 1$ denoting that $R(i,j,s)$ holds, and $\sigma_s(\langle i,j\rangle) = 0$ denoting that $R(i,j,s)$ does not hold. We desire the outcome 1 to be to left of outcome 0 on the tree and so order nodes on the tree as follows:

$$\sigma \preceq \tau \iff \sigma \subseteq \tau \vee \sigma(j(\sigma,\tau)) > \tau(j(\sigma,\tau)),$$

where

$$j(\sigma,\tau) = \mu k[\sigma(k)\downarrow \,\&\, \tau(k)\downarrow \,\&\, \sigma(k) \neq \tau(k)].$$

We say that σ is to the *left* of τ (τ is to the *right* of σ) if and only if $\sigma \preceq \tau$ and $\sigma \not\subseteq \tau$. Also we say that σ is *above* τ (τ is *below* σ) if and only if $\sigma \subseteq \tau$.

Definition 3.3 We say τ is *visited* at stage s of the construction if and only if $\tau \subseteq \sigma_s$. A stage s is a *σ-stage* if and only if σ is visited at stage s of the construction. We write σ^- to denote $\sigma \upharpoonright (lh(\sigma - 1))$.

Definition 3.4 $R_\sigma(i,j,s)$ holds if and only if s is a σ-stage and there exists a σ-stage $t < s$ such that

$$(\forall v)[t < v < s](v \text{ is not a } \sigma\text{-stage}) \,\&\, (\exists w)[t < w \leq s](R(i,j,w) \text{ holds}).$$

We can now prove an easy lemma about R_σ :

Lemma 3.5

If σ is visited infinitely often and $\widehat{R}(i,j)$ holds then there are infinitely many σ-stages such that $R_\sigma(i,j,s)$ holds.

PROOF:

$\widehat{R}(i,j)$ holds $\implies (\exists^\infty s)[R(i,j,s)$ holds $]$.

Let t be a stage such that $R(i,j,t)$ holds. Let s be the least stage greater than t such that σ is visited at stage s, there is a stage w such that $t < w \leq s$ and $R(i,j,w)$ holds, and for all v, $t < v < s$, v is not a σ-stage. Such a stage s exists as σ is visited infinitely often.

Then $R_\sigma(i,j,s)$ holds by definition.

□

In the construction which follows we use P_σ to keep account of which points in $\mathcal{L}$ are under the control of which strategy, we use B_σ and D_σ to refer to sets of block points and density points respectively for the strategy σ and α_σ and β_σ are used to record the current leftmost and rightmost points of the section that σ controls.

Construction.

Stage $s = 0$. $P^0_\sigma = \emptyset$ for all $\sigma \in T$. $\sigma_0 = \lambda$, the empty string.

Stages $s+1$. We build a string σ_{s+1} of length $s+1$ through T as follows.

For each $\langle i,j\rangle$ in turn beginning with $\langle i,j\rangle = \langle 1,0\rangle$, let $\sigma = \sigma_{s+1} \upharpoonright \langle i,j\rangle$ and take action according to which case below applies:

(1) $R_\sigma(i,j,s+1)$ does not hold.

ACTION:Let $\sigma_{s+1}(\langle i,j\rangle) = 0$.

Let $P^{s+1}_\sigma = P^s_\sigma$.

Let $B_\sigma^{s+1} = B_\sigma^s$.
Let $D_\sigma^{s+1} = D_\sigma^s$.
Let $\alpha_\sigma^{s+1} = \mu_{\mathcal{L}} p \in P_\sigma^{s+1}[q \in P_\sigma^{s+1} \implies p <_{\mathcal{L}} q]$.
Let $\beta_\sigma^{s+1} = \mu_{\mathcal{L}} p \in P_\sigma^{s+1}[q \in P_\sigma^{s+1} \implies q <_{\mathcal{L}} p]$.

(2) $R_\sigma(i, j, s+1)$ holds and $\sigma(\langle i, k\rangle) = 0$ for all $k < j$.
ACTION:Let $\sigma_{s+1}(\langle i, j\rangle) = 1$.
Let $\widehat{P}_\sigma^{s+1} = \bigcup_{\tau \supset \sigma_s \restriction \langle i,j\rangle ^\frown \langle 0\rangle} P_\tau^s \cup P_\sigma^s$.
For each $x \in \widehat{P}_\sigma^{s+1}$:

(a) if $x \in \omega^{[\langle i,j\rangle]} \cap B_\sigma^s$ then let $x \in \widehat{B}_\sigma^{s+1}$,
(b) if $x \in \omega^{[\langle i,j\rangle]} \cap D_\sigma^s$ then let $x \in \widehat{D}_\sigma^{s+1}$,
(c) if $x \notin \omega^{[\langle i,j\rangle]}$ then let $x \in \widehat{D}_\sigma^{s+1}$.

Add new points to the linear order by choosing unused points x from $\omega^{[\langle i,j\rangle]}$ greater than any point added to $\mathcal{L}$ during the construction so far such that $\beta_\tau^{s+1} <_{\mathcal{L}} x$ for $\tau \preceq \sigma$ with $\sigma(lh(\tau)) = 1$, and $x <_{\mathcal{L}} \alpha_\tau^{s+1}$ for $\tau \preceq \sigma$ with $\sigma(lh(\tau)) = 0$ as follows:

(i) add i new points $x_1, \dots, x_i$ that are adjacently ordered
$x_1 <_{\mathcal{L}} x_2 <_{\mathcal{L}} \dots <_{\mathcal{L}} x_i$.
Let $x_i <_{\mathcal{L}} p$, where $p = (\mu_{\mathcal{L}} p' \in \widehat{P}_\sigma^{s+1})[q \in \widehat{P}_\sigma^{s+1} \implies p' \leq_{\mathcal{L}} q]$, if such a p exists. If no such p exists then let
$p = \mu_{\mathcal{L}} p' \in \{\alpha_{\sigma'}^{s+1} \mid \sigma' \subset \sigma \,\&\, \sigma(lh(\sigma')) = 0\}$. Otherwise x_i becomes the rightmost elemet of $\mathcal{L}$ at this step in the construction.
Let $B_\sigma^{s+1} = \widehat{B}_\sigma^{s+1} \cup \{x_1, x_2, \dots, x_i\}$.

(ii) for all $y_0, y_1 \in \widehat{P}_\sigma^{s+1}$ such that $y_0 \in \widehat{D}_\sigma^{s+1}$ and $y_1 \in \widehat{B}_\sigma^{s+1}$ and y_0 is adjacent to y_1 in $\mathcal{L}_s$ add a new point z such that $y_e <_{\mathcal{L}} z <_{\mathcal{L}} y_{1-e}$ if $y_e <_{\mathcal{L}} y_{1-e}$, for $e = 0, 1$.

(iii) for all $y_0, y_1 \in \widehat{P}_\sigma^{s+1}$ such that $y_0, y_1 \in \widehat{D}_\sigma^{s+1}$ and y_0 is adjacent to y_1 add a new point z such that $y_e <_{\mathcal{L}} z <_{\mathcal{L}} y_{1-e}$ if $y_e <_{\mathcal{L}} y_{1-e}$, or $e = 0, 1$.

(iv) add a new point z adjacent to x_1 such that $p <_{\mathcal{L}} z <_{\mathcal{L}} x_1$ where $p = \mu_{\mathcal{L}} p' \in \{\beta_{\sigma'}^{s+1} \mid \sigma' \subset \sigma \,\&\, \sigma(lh(\sigma)) = 1\}$, if such a p exists. Otherwise let $z <_{\mathcal{L}} x_i$. Also add a new point z adjacent to q such that $q <_{\mathcal{L}} z <_{\mathcal{L}} p$ where $p = \mu_{\mathcal{L}} p' \in \{\alpha_{\sigma'}^{s+1} \mid \sigma' \subset \sigma \,\&\, \sigma(lh(\sigma')) = 0\}$. and $q \geq_{\mathcal{L}} q'$ for all $q' \in \widehat{P}_{\sigma}^{s+1}$.

Let Z_{σ}^{s+1} = set of all points z added through $(ii), (iii)$ or (iv) above.

Let $D_{\sigma}^{s+1} = \widehat{D}_{\sigma}^{s+1} \cup Z_{\sigma}^{s+1}$.

Let $P_{\sigma}^{s+1} = \widehat{P}_{\sigma}^{s+1} \cup B_{\sigma}^{s+1} \cup D_{\sigma}^{s+1}$.

Let $P_{\tau}^{s+1} = D_{\tau}^{s+1} = B_{\tau}^{s+1} = \emptyset$ for all $\tau \supset \sigma\hat{\ }\langle 0\rangle$.

Let $\alpha_{\sigma}^{s+1} = \mu_{\mathcal{L}} p \in P_{\sigma}^{s+1}[q \in P_{\sigma}^{s+1} \implies p <_{\mathcal{L}} q]$.

Let $\beta_{\sigma}^{s+1} = \mu_{\mathcal{L}} p \in P_{\sigma}^{s+1}[q \in P_{\sigma}^{s+1} \implies q <_{\mathcal{L}} p]$.

(3) $R_{\sigma}(i, j, s+1)$ holds and $\sigma(\langle i, k\rangle) = 1$ for some least $k < j$.

Action:Let $\sigma_{s+1}(\langle i, j\rangle) = 1$.

Let $P_{\sigma}^{s+1} = B_{\sigma}^{s+1} = D_{\sigma}^{s+1} = \emptyset$.

Let $\alpha_{\sigma}^{s+1} = \alpha_{\sigma \upharpoonright \langle i,k\rangle}^{s+1}$.

Let $\beta_{\sigma}^{s+1} = \beta_{\sigma \upharpoonright \langle i,k\rangle}^{s+1}$.

Any parameter $\mathfrak{P}$ not defined at stage $s+1$ of the construction is assumed to keep the same value at stage $s+1$ as at stage s, if defined at stage s, otherwise it remains undefined.

Let $\mathcal{L}_{s+1} = \langle L^{s+1}, <_{\mathcal{L}}\rangle$ where $L^{s+1} = \bigcup_{\tau \preceq \sigma_{s+1}} P_{\tau}^{s+1}$ and $<_{\mathcal{L}}$ is as defined through the construction.

To complete the construction let $P_{\sigma} = \{x \mid (\exists t)(\forall s > t)[x \in P_{\sigma}^{s}]\}$. Define $\mathcal{L} = \langle L, <_{\mathcal{L}}\rangle$ where $L = \bigcup_{s \in \omega} L^{s}$ and $<_{\mathcal{L}}$ is as defined through the construction.

Verification.

Note that whenever a point is enumerated into $\mathcal{L}$ at stage s say, it is greater in value than any previous point, hence L is a computable set. Further, its position within $\mathcal{L}_s$ with respect to all other points in $\mathcal{L}_s$ is defined at stage s and never changes at any later stage. Hence $<_{\mathcal{L}}$ is a computable relation. Thus $\mathcal{L}$ is a computably presentable linear ordering.

Definition 3.6 Let $\sigma^* = \liminf_s \sigma_s$. We call σ^* the *true path* through T.

Lemma 3.7 (True path lemma) *The true path exists.*

PROOF:

We proceed by induction. Fix $\langle i,j\rangle \geq 0$.

Suppose $\sigma^* \restriction \langle i,j\rangle = \liminf \sigma_s \restriction \langle i,j\rangle \downarrow$. Let σ denote $\sigma^* \restriction \langle i,j\rangle$. We show that $\sigma^*(\langle i,j\rangle) \downarrow$.

Since σ is visited infinitely often there are infinitely many σ-stages and finitely many τ-stages for $\tau \preceq \sigma$ with $\tau \not\subseteq \sigma$. We consider two cases:

(a) $R(i,j,s)$ holds for infinitely many s.

Then there exist infinitely many σ-stages s for which $R_\sigma(i,j,s)$ holds by lemma 3.5. Hence there are infinitely many σ-stages s such that $\sigma_s(\langle i,j\rangle = 1$, and thus $\sigma^*(\langle i,j\rangle) = 1$.

(b) $R(i,j,s)$ holds for finitely many s.

Then there is a stage t_0 such that for all stages $s > t_0$, $R(i,j,s)$ does not hold. Hence there is a stage $t_1 > t_0$ such that for all $s > t_1$, $R_\sigma(i,j,s)$ does not hold by definition. So for all $s > t_1, \sigma_s(\langle i,j\rangle) = 0$ and thus $\sigma^*(\langle i,j\rangle) = 0$.

$\square$

Lemmas 3.8, 3.9 ,3.10 and 3.11 which follow deal with the properties certain points of $\mathcal{L}$ have depending on the outcome of the strategy for a particular node.

Lemma 3.8 (Truth of outcome lemma)

Let $\sigma \subset \sigma^*$ with $lh(\sigma) = \langle i,j\rangle$. Let $B_\sigma = \{x \mid (\exists t)(\forall s > t)[x \in B^s_\sigma]\}$.

Let $D_\sigma = \{x \mid (\exists t)(\forall s > t)[x \in D^s_\sigma]\}$.

Then

(i) $\sigma(\langle i,j\rangle) = 0 \implies |P_\sigma| < \omega \,\&\, P_\sigma = D_\sigma \sqcup B_\sigma$.

(ii) $\sigma(\langle i,j\rangle) = 1 \,\&\, (\forall k < j)[\sigma(\langle i,k\rangle) = 0]$

$\implies |P_\sigma| = |B_\sigma| = |D_\sigma| = \omega \,\&\; P_\sigma = B_\sigma \sqcup D_\sigma$.

(iii) $\sigma(\langle i,j\rangle) = 1 \,\&\, (\exists k < j)[\sigma(\langle i,k\rangle) = 1]$

$\implies |P_\sigma| = |B_\sigma| = |D_\sigma| = \emptyset$.

(iv) Let $\tau \prec \sigma^* \,\&\, \tau \not\subseteq s^*$ such that $lh(\tau) = \langle i,j\rangle$. Then P_τ is finite.

(v) $(\exists\sigma, t)[x \in P^t_\sigma] \implies (\forall s > t)[x \in L_s]$.

PROOF:

In all the cases below we let t_0 be the least stage such that for all $s > t_0$

$$\sigma^* \restriction \langle i,j\rangle \preceq \sigma_s \restriction \langle i,j\rangle.$$

(i) $\sigma(\langle i,j\rangle) = 0$.

Let $t_1 > t_0$ such that for all $s > t_1$, $R_\sigma(i,j,s) = 0$.

Therefore by choice of t_1, case (2) of the construction never applies for σ at any σ-stage $s > t_1$. Furthermore for all $s > t_1$ there is no $\tau \subset \sigma$ for which $\tau^\frown\langle 0\rangle \subset \sigma$ and $\tau^\frown\langle 1\rangle$ is visited at stage $s > t_1$. Now through (1) of the construction it follows that $(\forall s > t_1)[P^s_\sigma = P^{t_1}_\sigma]$ and hence $|P_\sigma| = \omega$. Also we have $(\forall s > t_1)[B^s_\sigma = B^{t_1}_\sigma \,\&\, D^s_\sigma = D^{t_1}_\sigma]$.

Now let $x \in P_\sigma$ and so $x \in P^{t_1}_\sigma$. Suppose $x \in \omega^{[\langle i,j\rangle]}$, then either

$x \in D^{t_1}_\sigma$: therefore $(\forall s > t_1)[x \in D^s_\sigma \,\&\, x \notin B^s_\sigma]$ and $[x \in D_\sigma \,\&\, x \notin B_\sigma]$, or

$x \in B^{t_1}_\sigma$: therefore $(\forall s > t_1)[x \in B^s_\sigma \,\&\, x \notin D^s_\sigma]$ and $[x \in B_\sigma \,\&\, x \notin D_\sigma]$.

Suppose $x \notin \omega^{[\langle i,j\rangle]}$ then $(\forall s > t_1)[x \notin B^s_\sigma]$ and hence $x \notin B_\sigma$.

However $x \in P^{t_1}_\sigma$, therefore $x \in \widehat{P}'_\sigma \cup \widehat{D}'_\sigma$ for some $s' \leq t_1$ through case (2) of the construction. It follows that $x \in D^s_\sigma$ for all $s > t_1$.

Therefore $x \in P_\sigma \implies x \in B_\sigma \sqcup D_\sigma$.

Conversely, it is clear from the construction that $B_\sigma \sqcup D_\sigma \subseteq P_\sigma$.

(ii) $\sigma(\langle i,j\rangle) = 1 \,\&\, (\forall k < j)[\sigma(\langle i,k\rangle) = 0]$.

Hence $\sigma_s \restriction \langle i,k\rangle = 0$ for all $s > t_0$ and $k < j$. Also there are infinitely many stages $s > t_0$ such that $\sigma_s(\langle i,j\rangle) = 1$. Hence case (2) of the construction applies infinitely often.

By choice of a σ-stage t_1 with $t_1 > t_0$, we have the following:

$x \in P^{t_1}_\sigma \implies x \in P_\sigma$,

$(\forall s > t_1)[x \in B^{t_1}_\sigma \implies x \in B^s_\sigma]$,

$(\forall s > t_1)[x \in D^{t_1}_\sigma \implies x \in D^s_\sigma]$.

Let t_2 be any σ-stage greater than t_1. For each $x \in P^{t_2}_\sigma$ a similar argument to (i) proves that

$x \in P_\sigma \implies x \in B_\sigma \sqcup D_\sigma$.

Again the converse is clear from the construction. Hence $P_\sigma = B_\sigma \sqcup D_\sigma$.

Furthermore, if s_1 and s_2 are σ-stages such that $s_1 < s_2$ then it is clear from (i) of case (2) of the construction that $|B_\sigma^{s_2}| > |B_\sigma^{s_1}|$, and so $|B_\sigma| = \omega$.

Similarly $|D_\sigma^{s_2}| > |D_\sigma^{s_1}|$ through (iv) of case (2) of the construction and so $|D_\sigma| = \omega$. Therefore $|P_\sigma| = \omega$.

(iii) $\sigma(\langle i, j\rangle) = 1 \,\&\, (\exists k < j)[\sigma(\langle i, k\rangle) = 0]$.

Let $\langle i, k\rangle$ be the least pair such that $\sigma^*(\langle i, k\rangle) = 1$. Then at all infinitely many σ-stages s we have that $P_\sigma^s = B_\sigma^s = D_\sigma^s = \emptyset$ through the action of case (2) of the construction. Hence $P_\sigma = B_\sigma = D_\sigma = \emptyset$.

(iv) Let $lh(\tau) = \langle i, j\rangle$ with $\tau \prec \sigma \,\&\, \tau \not\subseteq \sigma$.

Then τ is never visited after stage t_0 and hence P_τ is finite.

(v) Suppose $x \in P_\sigma^t$. Then $x \in L_t$.

From inspecting the construction it can be seen that once an element $x \in P_\sigma^t$ for some σ at some stage t, then either $x \in P_\sigma^s$ for all stages $s > t$, or otherwise there is some $\tau \subset \sigma^*$ for which $\tau^\frown\langle 0\rangle \subset \sigma$, $\tau^\frown\langle 1\rangle \subset \sigma^*$ and $x \in P_\tau^s$ for all s greater than t_1, where t_1 is the least stage such that $\sigma^* \restriction lh(\tau) \preceq \sigma_s \restriction lh(\tau)$.

$\square$

Lemma 3.9 (ω^* tail lemma)
Suppose case (i) of lemma 3.8 applies then for any $x \in P_\sigma$ we have that $|\{y \in L \mid x <_{\mathcal{L}} y\}| < \omega$.

PROOF:
Let $\sigma(\langle i, j\rangle) = 0$ and $x \in P_\sigma$. Then for $\tau \subsetneq \sigma$ such that $\sigma(lh(\tau)) = 0$ we have that P_τ is finite by lemma 3.8 and so τ can only contribute finitely many points to the right of x.

For $\tau \subset \sigma$ such that $\sigma(lh(\tau)) = 1$, we have that $x \notin P_\tau$ because $x \in P_\sigma$. Then when x was first enumerated into $\mathcal{L}$ at stage t say, we defined $\beta_\tau^t <_{\mathcal{L}} x$ through case (2) of the construction. Then when τ adds more points to $\mathcal{L}$ at stages s it does so adjacent to β_τ^s and hence to the left of x.

For $\tau \supset \sigma$, τ only adds points to the left of α_σ^s and $\alpha_\sigma^s \leq_{\mathcal{L}} \beta_\sigma^s \leq_{\mathcal{L}} x$ for all $x \in P_\sigma$ and $s \in \omega$.

Therefore there are only finitely many points to the right of x.

$\square$

Lemma 3.10 (η-like initial segment lemma)
Suppose case (ii) of lemma 3.8 applies. Then

(a) $y_0, y_1 \in D_\sigma \implies (\exists z)[z \in D_\sigma \,\&\, (y_e <_{\mathcal{L}} y_{1-e} \implies y_e <_{\mathcal{L}} z <_{\mathcal{L}} y_{1-e})]$, *for* $e = 0, 1$.

(b) $(\forall y \in D_\sigma)(\exists x \in B_\sigma)[x <_{\mathcal{L}} y]$.

(c) $(\forall x \in B_\sigma)(\exists y \in D_\sigma)[y <_{\mathcal{L}} x]$.

(d) $y \in D_\sigma \implies$
$(\exists z_0, z_1 \in D_\sigma)[z_0 <_{\mathcal{L}} y <_{\mathcal{L}} z_1 \,\&\, z_0 <_{\mathcal{L}} y' <_{\mathcal{L}} z_1 \implies y' \in D_\sigma]$.

PROOF:
Let t_0 be the least stage such that for all $s > t_0$

$$\sigma^* \upharpoonright \langle i, j\rangle \preceq \sigma_s \upharpoonright \langle i, j\rangle.$$

Then for all $s > t_0$ we have that $x \in P^s_\sigma \implies x \in P^{s+1}_\sigma$.

(a) There are infinitely many σ-stages greater than t_0 and at all such stages case (2) of the construction applies. Then through part (*ii*) of case (2) whenever $y_0, y_1 \in D^s_\sigma$, y_0 is adjacent to y_1 and $y_0 <_{\mathcal{L}} y_1$ then a new point z is added to D^{s+1}_σ such that $y_0 <_{\mathcal{L}} z <_{\mathcal{L}} y_1$.

If $y_0, y_1 \in D^s_\sigma$ are not adjacent, $y_0 <_{\mathcal{L}} y_1$ but for all y' such that $y_0 <_{\mathcal{L}} y' <_{\mathcal{L}} y_1$, $y' \notin D^s_\sigma$, then there exists some $y' \in B^s_\sigma$ such that we have $y_0 <_{\mathcal{L}} y' <_{\mathcal{L}} y_1$. Then through part (*ii*) of case (2) of the construction a new point z is added to D^{s+1}_σ such that $y_0 <_{\mathcal{L}} z <_{\mathcal{L}} y' <_{\mathcal{L}} y_1$ as required.

Parts (b), (c) and (d) follow in a similar way to (a) through case (2) of the construction.

$\square$

Lemma 3.11 (ω^* tail left of the true path lemma)
Suppose case (iv) of lemma 3.8 applies. Then for each $x \in P_\tau$, *we have* $|\{y \in \mathcal{L} \mid x <_{\mathcal{L}} y\}| < \omega$.

PROOF:

Let t_0 be a stage such that for all $s > t_0$, τ is not visited at stage s and $\sigma^* \upharpoonright lh(\tau) \preceq \sigma_s \upharpoonright lh(\tau)$. For each $x \in P_\tau$, since τ is to the left of the true path x is added at stage s to the right of $\beta^s_{\tau'}$ for some τ' such that $\tau'^\frown\langle 1\rangle \subset \tau$ and $\tau'^\frown\langle 0\rangle \subset \sigma$. Then for all $s > t_0$, $x \in P^s_\tau$ and $\beta^s_\sigma <_{\mathcal{L}} x$ hence nodes below σ only ever add points to the left of x. Also for nodes σ' of higher priority than σ with $\sigma(lh(\sigma')) = 1$ we have $\beta^s_{\sigma'} <_{\mathcal{L}} x$ and hence σ' never add points to the right of x after stage t_0.

$\square$

Lemma 3.12

Requirement $\mathcal{Q}_{i,j}$ is satisfied for all $i \geq 1$, $i, j \in \omega$.

PROOF:

Let $\sigma = \sigma^* \upharpoonright \langle i, j\rangle$.

If $\widehat{R}(i,j)$ does not hold then there is nothing to prove.

Suppose $\widehat{R}(i,j)$ holds. Then $\sigma(\langle i,j\rangle) = 1$ at infinitely many σ-stages s. At each such stage a set of i elements are added to $\mathcal{L}$ through case (2) of the construction and $x_1, \ldots, x_i \in P^s_\sigma$ B for all $s > t_0$. Then by lemma 3.10, x_1 and x_i are the endpoints of an i-block. Hence $\mathcal{Q}_{i,j}$ is satisfied.

$\square$

Lemma 3.13

$\mathcal{L}$ is a linear order of the form $\mathcal{A} + \omega$ where $\mathcal{A}$ is an η-like Π^0_2 initial segment of $\mathcal{L}$ which codes S in the standard way.

PROOF:

Let $\mathcal{A} = \langle A, <_L\rangle$ be a sub-linear order of $\mathcal{L}$ such that:

$x \in A \iff \forall y \exists z[x \leq_{\mathcal{L}} y \implies y <_{\mathcal{L}} z]$ and

$x \notin A \iff \exists y \forall z[x \leq_{\mathcal{L}} y \implies z \leq_{\mathcal{L}} y]$.

Clearly $\mathcal{A}$ is a Π^0_2 initial segment of $\mathcal{L}$.

$\overline{A}$ is infinite by the choice of R (see definition 3.1) and the fact that $\langle i,j\rangle < 2^{\langle i,j\rangle}$ since each $\sigma \subset \sigma^*$ with $lh(\sigma) = \langle i,j\rangle$ and $\sigma(\langle i,j\rangle) = 0$ contributes at least one point to $\overline{\mathcal{A}}$ though case (2) of the construction.

It follows from lemmas 3.8, 3.9 and 3.11 that $\overline{\mathcal{A}} = \langle A, <_{\mathcal{L}}\rangle$ has order type ω^*.

Since $S \neq \emptyset$ and every requirement $\mathcal{Q}_{i,j}$ is satisfied for $i \geq 1$, $i, j \in \omega$, it follows that A is infinite and η-like and also that $\mathcal{L}$ codes S via the standard coding.

Since $\overline{\mathcal{A}}$ is of order type ω^* it does not contain any n-blocks for $n > 0$. Therefore since $\mathcal{A}$ is η-like S must be coded in $\mathcal{A}$ via the standard coding.

$\square$

This completes the proof of the main theorem.

References

[1] K. Ambos-Spies, S. B. Cooper, S. Lempp. Initial segments of linear orders. *to appear*

[2] C. J. Ash. Categoricity in the hyperarithmetical degrees. *Annals of pure and applied logic*, Vol. 34 (1987), 1–34.

[3] K. H. Chen. Recursive well founded linear orderings. *Annals of mathematical logic*, Vol 13 (1978), 117-147.

[4] J. Chisholm and M. Moses. Undecidable linear orderings that are n-recursive for each n. *to appear.*

[5] M. Dehn. Über unendlische diskontinuierliche gruppen *Annals of Math*, Vol. 71 (1911), 116–144.

[6] R. Downey. Recursion theory and linear orderings. *Handbook of computable algebra* (Yu. L. Ershov, S. S. Goncharov, A. Nerode and J. B. Remmel, eds.)

[7] R. G. Downey. On presentations of algebraic structures. To appear in the proceedings of the EC COLORET network, Marcel Dekker.

[8] R. G. Downey and J. F. Knight. Orderings with α-th jump degree $\mathbf{0}^{(\alpha)}$. *Proc. Amer. Math. Soc.*, Vol 14 (1992), 545–552.

[9] R. G. Downey and M. F. Moses. Recursive linear orderings with incomplete successivities. *Trans. Amer. Math. Soc.*, Vol. 326 (1991) 653–668.

[10] L. J. Feiner. Orderings and boolean algebras not isomorphic to recursive ones. PhD. Thesis, MIT (1967).

[11] R. O. Gandy. General recursive functionals of finite type and hierarchies of functions, *Annals Fac. Sci. Univ. Clermont-Ferrand*, Vol. 35 (1967), 5–24.

[12] S. Gregorieff. Every recursive linear ordering has a copy in DTIME-SPACE $(n, \log(n))$. *Journal of Symbolic Logic*, Vol. 55 (1990), 260–276.

[13] J. Harrison. Recursive pseudo-well-orderings. *Trans. Amer. Math. Soc.*, Vol. 131 (1968), 526–543.

[14] C. G. Jockusch, Jr. and R. I. Soare. Degrees of orderings not isomorphic to recursive linear orderings. *Annals of pure and applied logic*, Vol. 52 (1991), 139–64.

[15] B. Khoussainov, A. Nies and R. A. Shore. Recursive models of theories with few models. *to appear.*

[16] H. Kierstead. Recursive ordered sets. In Combinatorics and Ordered sets, (Ed. I. Rival) Contemporaary Math., Vol. 57 Amer. Math. Soc. (1986).

[17] H. Kierstead. On Π_1-automorphisms of recursive linear orderings. *Journal of Symbolic Logic*, Vol. 52 (1987), 681–688.

[18] M. Lerman and J. Rosenstein. Recursive linear orderings. In Metakides (1982), 123–136.

[19] M. F. Moses. Recursive properties of isomorphism types. PhD thesis, Monash University, Melbourne, Australia (1983).

[20] M. F. Moses. Recursive linear orders with recursive successivities. *Ann. Pure and Appl. Logic*, Vol. 27 (1984), 253–264.

[21] M. F. Moses. Relations intrinsically recursive in linear orderings. *Z. Math. Logik Grundlagen. Math.*, Vol. 32 (5), 467–472.

[22] M. F. Moses. Decidable discrete linear orderings. *Journal of Symbolic Logic*, Vol. 53 (1988), 531–539.

[23] M. F. Moses. n-recursive linear orderings without $n+1$-recursive copies. In Logical methods, (Ed. J. Crossley, J. Remmel, R. A. Shore and M. Sweedler), Birkhäuser (1993), 572–592.

[24] M. J. S. Raw. Complexity of automorphisms of recursive linear orders, PhD. Thesis, University of Wisconsin-Madison (1995).

[25] J. B. Remmel. Recursion theory on orderings II. *Journal of Symbolic Logic*, Vol. 45 (1980), 317–333.

[26] J. B. Remmel. Recursively categorical linear orderings. *Proc. Amer. Math. Soc.*, Vol. 83 (1981), 379–386.

[27] J. Rosenstein. Linear orderings. Academic Press, New York (1982).

[28] R. I. Soare Recursively enumerable sets and degrees. Springer-Verlag, New York (1987).