

SUBSETS OF HYPERSIMPLE SETS

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Structural properties of r.e. sets often have degree theoretic consequences, particularly concerning degrees of supersets. It is our intention to show that such properties can have interesting ramifications on the degrees of subsets, by showing that no hypersimple r.e. set has the universal splitting property (USP). We also show that there are, however, simple sets (indeed, low and promptly simple sets) with USP and thus USP is not invariant under automorphisms of the lattice of r.e. sets.

1. Introduction. An r.e. sequence of disjoint nonempty canonical finite sets is called a *strong array*. Post [Po] defined an r.e. set A to be hypersimple if it was coinfinite and given any infinite strong array $\{D_x\}_{x \in \mathbb{N}}$, then for some x , $D_x \subset A$. Hypersimple sets were first constructed by Dekker [De] who showed that they exist in each r.e. degree, and have since been extensively studied.

A pair of r.e. sets A_1, A_2 is said to *split* an r.e. set A (written $A_1 \sqcup A_2 = A$) if $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$. Following Lerman and Remmel [LR1, 2] we shall say an r.e. set A has the *universal splitting property* (USP) if given any r.e. set $B \leq_T A$ there is an r.e. splitting of $A = A_1 \sqcup A_2$ with $A_1 \equiv_T B$. Sets with and without USP have recently been analysed in several papers, for example [AS2, AF, Do1,2, DW, LR1,2]. Lerman and Remmel [LR2] showed that the degrees containing non-USP r.e. sets were dense in the r.e. degrees. In [Do1], the author improved this to show that in fact every nonzero r.e. degree contained a non-USP r.e. set. The main result of this paper is an extension of this:

THEOREM. (i) *No hypersimple r.e. set has USP.*

(ii) *Indeed, if A is hypersimple and B is an r.e. set with $\emptyset <_T B \leq_T A$ then there exists an r.e. set Q with $Q \leq_T B$ and such that if $A_1 \sqcup A_2 = A$ is an r.e. splitting of A then $A_1 \not\equiv_T Q$.*

(iii) *In particular if $A_1 \sqcup A_2 = A$ is an r.e. splitting of a hypersimple set then A_1 is non-USP.*

We feel that this result is quite interesting since it connects two apparently unrelated notions and means that properties of supersets of r.e. sets can affect degrees of subsets.

As a companion result, we shall prove that no obvious lattice theoretic extension seems possible. To obtain the sharpest formulation, say A has the *SUSP* (*strong USP*) if given any pair of r.e. sets with $B_1 \oplus B_2 \equiv_T A$ there exist r.e. sets A_1, A_2 with $A = A_1 \sqcup A_2$ and $A_i \equiv_T B_i$ for $i = 1, 2$. Recall from Maass [Ma], that A is *promptly simple* if $\text{card}(\omega - A) = \infty$ and there exists a recursive function f such that W_e infinite implies that for some $s, \exists x(x \in (W_{e,s+1} - W_{e,s}) \cap A_{f(x)})$.

Our result here is:

THEOREM. *There exist low promptly simple SUSP r.e. sets.*

As a corollary, following some work of Maass, we see that (S)USP is not invariant under automorphisms of the lattice of r.e. sets.

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2. Notation and conventions. Our notation and terminology is for the most part standard and follows Soare [So1, 2, 3], subject to several conventions we now describe. We denote r.e. sets by upper case roman letters (e.g. U, V, W, \dots) and T -functionals by upper case greek letters ($\Phi, \Gamma, \Delta, \dots$). We denote the use function of a computation by $u(-)$. This is the maximum element used in a computation $(-)$. We suppose that if $\Phi(A; x) \downarrow$ then $u(\Phi(A; x)) > x$.

If $\{\Phi_s\}$ and $\{A_s\}$ are approximations we shall suppose that if $\Phi_s(A_s; x) \downarrow$ and $\Phi_t(A_t; x) \downarrow$ then $u(\Phi_s(A_s; x)) \leq u(\Phi_t(A_t; x))$. We remark that all sets and degrees etc. will be r.e. unless specifically stated otherwise. We let " \sqcup " denote union for disjoint sets so that $A \sqcup B = C$ means $A \cup B = C$ and $A \cap B = \emptyset$. Also \langle, \rangle will denote a standard pairing. For an (r.e.) set A , $A^{(e)}$ denotes $\{\langle e, x \rangle : x \in \omega\}$. We let $A[x] = \{z : z \leq x \text{ \& } z \in A\}$. Finally we adopt the convention that all computations etc. are bounded by s at stage $s + 1$.

3. Results. In [Do1], the author showed that every nonzero r.e. degree contained a non-USP r.e. set. At the time, we felt that the proof was slightly unusual since what it really showed was the following:

Let $A = f(\omega)$ be a 1-1 enumeration of an r.e. nonrecursive set. Define the "filled in" set B associated with A as follows:

Stage 0. Set $B_0 = \emptyset$ and $b_{i,0} = i$ for all $i \in \omega$.

Stage $s + 1$. Set $B_{s+1} = B_s \cup \{b_{i,s} : f(s) \leq i \leq f(s) + s\}$, $b_{i,s+1} = b_{i,s}$ for $i < f(s)$ and $b_{i,s+1} = b_{i+s+1,s}$ otherwise.

Set $B = \bigcup_s B_s$.

Then in [Do1] we proved:

THEOREM 3.1 ([Do1]). *If B is constructed as above, then B is non-USP.*

Our reason for feeling that this proof was slightly unusual was that construction of a *nonsplitting witness*—an r.e. set C such that if E and F are r.e. sets with $E \sqcup F = B$ then $C \not\equiv_T E$ —was virtually independent of our construction of B .

It thus seems reasonable to conjecture that perhaps the proof of [Do1] turned on a *property* of B rather than the particular construction of B . Now it is easy to see that B is hypersimple. As our first result we shall show that hypersimplicity is enough to ensure a set is non-USP. We use a modification of the techniques of [Do1]. We show:

THEOREM 3.2. (i) *No hypersimple r.e. set has USP.*

(ii) *Indeed, if A is hypersimple and B is an r.e. set with $\emptyset <_T B \leq_T A$ then there exists an r.e. set $Q \leq_T B$ such that if $A_1 \sqcup A_2 = A$ is an r.e. splitting of A , then $A_1 \not\equiv_T Q$.*

(iii) *In particular, if $A_1 \sqcup A_2$ is an r.e. splitting of a hypersimple set, then A_1 is non-USP.*

Proof. Let A be a given hypersimple r.e. set and B an r.e. set with $\emptyset <_T B \leq_T A$. Let $g(\omega) = B$ be a 1-1 enumeration of B . We shall build a nonsplitting witness Q for A with $Q \leq_T B$ by simple permitting as in [Do1]. Let $(V_e, W_e, \Gamma_e, \Phi_e)$ list all 4-tuples consisting of disjoint pairs of r.e. sets and pairs of functionals. Please note here that we are using *disjoint pairs* of r.e. sets (V_e, W_e) and we shall thus know that at no stage of the construction can $V_{e,s} \cap W_{e,s} \neq \emptyset$. We must ensure Q satisfies the requirements

R_e : One of the following fails:

- (i) $W_e \sqcup V_e = A$.
- (ii) $\Phi_e(W_e) = Q$.
- (iii) $\Gamma_e(Q) = W_e$.

We shall now briefly discuss the strategy for the satisfaction of a single R_e . We shall satisfy the R_e by *followers*. Each follower x is assigned a *target region* $T(x)$. $T(x)$ will be a (finite) subset of ω . Define the “ Q

controllable" length of agreement via

$$1(e, s) = \max\{x : \forall y < x (\Phi_{e,s}(W_{e,s}; y) = Q_s(y) \& \\ \forall z (z \leq u(\Phi_{e,s}(W_{e,s}; y)) \rightarrow \Gamma_{e,s}(Q_s; z) = W_{e,s}(z) \& \\ (W_{e,s} \sqcup V_{e,s})(z) = A_s(z))\}.$$

Associated with the R_e will be a *marker* $M(e, s)$. Roughly speaking, the position of $M(e, s)$ will denote the part of Q currently devoted to attempts to satisfy R_j for $j \leq e$. Our argument will be finite injury, and so we shall suppose that beyond some stage s_0 we need only change $M(e, s)$ for the sake of attacks on R_e (and the R_j for $j < e$ have ceased activity). Also $M(e, s)$ is monotone in both variables.

Now we attack R_e as follows. First we wait for a number x to occur in $\omega^{(e)}$ such that

- (i) $1(e, s) > x$, and
- (ii) $x > \max\{u(\Gamma_{e,s}(Q_s; z)) : z \leq M(e, s)\}$.

At this stage, we *appoint* x as a *follower* of R_e and reset $M(k, s+1)$ to be $M(k, s+1) = s+1$ for $k \geq e$. We then set $T(x) = \{z : M(e, s) < z \leq M(e, s+1)\}$ and cancel all lower priority followers. There are now two key observations.

Observation 1. For all stages $t \geq s$ if we do not add any number $\leq x$ to $Q_t - Q_s$ and if $1(e, t) > x$, we know $W_{e,t}[G] = W_{e,s}[G]$ where $G = u(\Phi_{e,s}(W_{e,s}; x))$. This will follow since we have reset $M(k, s+1)$ to $s+1$ for all $k \geq e$, and our conventions regarding $u(-)$ ensure that all followers of R_j for $j \geq e$ appointed after stage s must exceed $s+1$, and all the R_j for $j < e$ have ceased activity.

Observation 2. For all stages $t \geq s$ if no number $< x$ is added to $Q_t - Q_s$, and if $\Gamma_e(Q) = W_e$ then $W_{e,t}[M(e, s)] = W_{e,s}[M(e, s)]$. This follows by the same reasoning as Observation 1 since we specifically chose $x > \max\{u(\Gamma_{e,s}(Q_s; z)) : z \leq M(e, s)\}$. In particular, provided $Q_s[x-1] = Q[x-1]$ we know $W_e[M(e, s)] = W_{e,s}[M(e, s)]$ (provided that it is truly the case that $\Gamma_e(Q) = W_e$).

Our eventual aim is to enumerate x into Q . To do so immediately would be useless since although it might create a temporary disagreement between $\Phi_{e,s}(W_{e,s}; x)$ and $Q_{e,s+1}(x)$, the fact that we have changed $Q_{e,t}[s]$ allows $W_{e,t}[G] \neq W_{e,s}[G]$ (where $G = u(\Phi_{e,s}(W_{e,s}; x))$) for some $t \geq s$.

However, it is important to note that if we added x to $Q_t - Q_s$ but kept $Q_t[x-1]$ unchanged then Observation 2 tells us that although $W_{e,t}[G] \neq W_{e,s}[G]$ we do know that $W_{e,t}[M(e, s)] = W_{e,s}[M(e, s)]$. That

is, W_e may change, but only above $M(e, s)$. Thus as in [Do1], our idea is not to enumerate x into $Q_t - Q_s$ until a stage where it is impossible for $W_{e,t}[G] \neq W_{e,s}[G]$ and yet $W_{e,t}[M(e, s)] = W_{e,s}[M(e, s)]$.

For our purposes it would surely suffice to find a stage $t > s$ such that

- (i) $T(x) \subset A_t$, and
- (ii) $1(e, t) > x$, and
- (iii) $Q_t[x] = Q_s[x]$.

By Observation 1, if such a stage occurs, then if W_e and V_e truly split A we know that $W_{e,t}[G] = W_{e,s}[G]$ and for all z if $M(e, s) < z \leq G$, then $z \in W_e$ iff $x \in W_{e,s}$. (Any such z not in $W_{e,s}$ must have entered V_e .) In our construction, when such a stage t occurs we shall declare x as *confirmed*. We shall argue that once x is confirmed if we ever get a chance to enumerate x into Q we shall ensure that

- (a) either $\Phi_e(W_e; x) = 0 \neq 1 = Q(x)$, or
- (b) $\Gamma_e(Q_e; z) \neq W_e(z)$ for some $z \leq M(e, s)$.

The remainder of the proof is an appeal to the hypersimplicity of A to ensure that if R_e fails to be met, infinitely many followers get confirmed. (The $\{T(x)\}$ form a strong array.) We then argue that if we do not get to win a confirmed follower, then B is recursive contrary to hypothesis.

We shall now give the formal details of the argument although the reader—especially the one familiar with [Do1]—may prefer to supply these details for himself.

We shall say R_e *requires attention* at stage $s + 1$ if e is least with one of the following options holding.

- (3.3) R_e has a follower x which is confirmed and $g(s) \leq x$.
- (3.4) Not (3.3), and there is a number $x \in \omega^{(e)}$ such that
 - (a) $1(e, s) > x$, and
 - (b) $x > \max\{u(\Gamma_{e,s}(Q_s; z)); z \leq M(e, s)\}$.

CONSTRUCTION

Stage 0. All $T(x)$ for $x \in \omega$ are undefined. We set $M(e, 0) = 0$ for all $e \in \omega$.

Stage $s + 1$. *Step 1.* For each pair x and j , if x is a follower of R_j and x is not already confirmed, then if

- (i) $T(x) \subset A_s$, and
- (ii) $1(j, s) > x$,

declare x as confirmed.

Step 2. Find the least e (if any) such that R_e requires attention. Cancel all target regions and followers of R_j for $j > e$. Adopt the appropriate case below.

Case 1 (3.3) holds. Find the least such x . Set $Q_{s+1} = Q_s \cup \{x\}$. Cancel all followers y of R_e with $y \neq x$.

Case 2 (3.4) holds. Appoint x as a follower of R_e . Set $M(k, s+1) = s+1$ for $k \geq e$. Now set $T(x) = \{z : M(e, s) < z \leq M(e, s+1)\}$. Note that $T(x) \neq \emptyset$.

End of Construction.

Verification. Evidently $Q \leq_T B$ since numbers enter $Q_{s+1} - Q_s$ only when smaller ones enter $B_{s+1} - B_s$: to decide if $x \in Q$ find the least stage s where $\forall t > s (g(t) > x)$ then $x \in Q$ iff $x \in Q_{s+1}$.

As in [Do1] the argument is finite injury. Let s_0 be the least stage such that $\forall j < e \forall s > s_0 (R_j \text{ does not receive attention at stage } s)$. Since $M(j, s) \neq M(j, s+1)$ only when some R_i for $i \leq j$ receives attention, this means that $\forall j < e \forall s > s_0 (M(j, s) = M(j, s_0))$. Let $M = M(e, s_0) = \max\{M(j, s_0) : j \leq e\}$. By our cancellation procedure—when the R_j receive attention—we may suppose R_e has no follower at stage s_0 .

LEMMA. *If (3.3) holds for some follower x of R_e at some stage $s_1 > s_0$ then*

- (i) R_e is met at stage $s_1 + 1$,
- (ii) $\forall t \geq s_1 (R_e \text{ does not require attention at stage } t)$,
- (iii) $\forall t > s_1 (M(e, t) = M(e, s_1))$.

Proof. Let s_1 be the least stage $> s_0$ where (3.3) pertains to any x following R_e . Now x was appointed at some stage $s_2 + 1$ with $s_0 + 1 < s_2 + 1 < s_1$ (by (3.4)). By definition, we know that

- (a) $1(e, s_2) > x$, and
- (b) $x > \max\{u(\Gamma_{e, s_2}(Q_{s_2}; z)) : z \leq M(e, s_2)\}$.

When x was appointed, we reset $M(k, s_2 + 1) = s_2 + 1$ for all $k \geq e$, and we cancelled all followers of R_j for $j > e$. By choice of s_0 and the fact that the $M(k, s)$ are clearly monotone, it follows that

$$(3.5) \quad Q_{s_1}[s_2 + 1] = Q_{s_2}[s_2 + 1].$$

The important point here is that after stage $s_2 + 1$ followers that are appointed must exceed $s_2 + 1$. Also since s_1 is the *least* stage at which (3.3) pertains we also know that no numbers $\leq s_2 + 1$ have entered $Q_{s_1} - Q_{s_2}$ for the sake of R_e .

Now since (3.3) pertains to x it follows that x must be confirmed, and so there is a stage t with $s_2 + 1 < t \leq s_1$ with

- (i) $1(e, t) > x$, and
- (ii) $T(x) \subset A_t$.

Now by (3.5) and the facts that $1(e, s_2) > x$ and $1(e, t) > x$, we see

$$(3.6) \quad W_{e,t} [u(\Phi_{e,s_2}(W_{e,s_2}; x))] = W_{e,s_2} [u(\Phi_{e,s_2}(W_{e,s_2}; x))].$$

However, we specifically chose $T(x)$ as

$$T(x) = \{z : M(e, s_2) < z \leq s_2 + 1\}.$$

Thus as W_e and V_e are disjoint and $T(x) \subset A$, it must be that

$$(3.7) \quad \text{For all } z \in W_e \text{ if } z \leq u(\Phi_{e,t}(W_{e,t}; x)) \text{ and } z \notin W_{e,t} \\ \text{then } z \leq M(e, s_2).$$

The points to note here are that since $1(e, t) > x$ we must have that

$$A_t[u] = W_{e,t}[u] \sqcup V_{e,t}[u] \quad \text{where } u = u(\Phi_{e,t}(W_{e,t}; x)),$$

and furthermore $T(x) \subset A_t$.

Now by choice of x in (3.4) we know that

$$(3.8) \quad x > u(\Gamma_{e,s_2}(Q_{s_2}; y)) \quad \text{for all } y \leq M(e, s_2).$$

By (3.5) we know that this means that

$$(3.9) \quad x > u(\Gamma_{e,s_1}(Q_{s_1}; y)) \quad \text{for all } y \leq M(e, s_2).$$

When (3.3) pertains and so R_e receives attention via x at stage $s_1 + 1$ we not only add x to Q but also cancel all other followers of R_e . By choice of s_0 and s_1 this means that $Q_{s_1}[x - 1] = Q[x - 1]$. By (3.5) and (3.9) this ensures that

$$(3.10) \quad \text{For all } y \leq M(e, s_2), \Gamma_{e,s_2}(Q_{s_2}; y) = \Gamma_{e,s_1}(Q_{s_1}; y) = \Gamma_e(Q; y).$$

Now (3.3) pertains to x at stage s_1 . There are two cases.

Case 1. $\exists q(t \leq q \leq s_1 \ \& \ 1(e, q) \leq x)$. In this case the only possibility is that some number $z \leq M(e, s_2)$ has entered $W_{e,s_1} - W_{e,t}$. (Again we use the disjointness of W_e and V_e here.) But now (3.10) ensures that

$$\Gamma_e(Q; z) = 0 \neq W_e(z).$$

Case 2. $\forall q(t \leq q \leq s_1 \rightarrow 1(e, q) > x)$. By (3.6) and (3.7) we know that

$$0 = W_{e,s_1} [u(\Phi_{e,s_1}(W_{e,s_1}; x))] = W_{e,s_2} [u(\Phi_{e,s_2}(W_{e,s_2}; x))];$$

hence we create a disagreement

$$\Phi_{e,s_1}(W_{e,s_1}; x) = 0 \neq 1 = Q_{s_1+1}(x).$$

This disagreement will remain unless

$$W_{e,s_1} [u(\Phi_{e,s_2}(W_{e,s_2}; x))] \neq W_{e,s_3} [u(\Phi_{e,s_2}(W_{e,s_2}; x))]$$

for some (least) $s_3 > s_1$. But if this occurs, by (3.7) we know $W_{e,s_3}[M(e, s_2)] \neq W_{e,s_1}[M(e, s_2)]$ and hence for some $z \leq M(e, s_2)$ we know

$$\Gamma_e(Q; z) \neq W_e(z).$$

It follows that R_e is met. Also since $1(e, s) < M(e, s_2 + 1)$ for all $s > s_1 + 1$ it follows that R_e will never again receive attention. This ensures that $M(e, s_1) = M(e, s)$ for all $s > s_1$ as well. Our result now follows.

Conclusion of the proof of (3.2). Thus it suffices to show that (3.3) pertains. Suppose (3.3) does not pertain after stage s_0 , and R_e receives attention infinitely often or fails to be met. Then as $1(e, s) \rightarrow \infty$ (3.4) must pertain at some stage $s_1 > s_0$. At this stage we appoint a pair $(T(x_1), x_1)$ consisting of a follower and its target region. In like fashion, (3.4) must again pertain to eventually get an infinite increasing list

$$(T(x_1), x_1), (T(x_2), x_2), \dots$$

with $x_1 < x_2 < \dots$.

Evidently $\{T(x_i)\}_{i \in \omega}$ is a strong array. Hence as A is hypersimple for infinitely many x_i we have $T(x_i) \subset A$. As $1(e, s) \rightarrow \infty$, each such x_i must eventually get confirmed. We show now that $g(\omega) = B$ is a recursive set, contrary to hypothesis. Let $z \in \omega$. Let $s = s(z)$ be the least stage with $s > s_0$ and such that there is a confirmed follower $x > z$ of R_e at stage s . Then $z \in B$ iff $z \in B_s$ since (3.3) does not pertain. The result now follows. \square

It therefore follows that possessing a fairly sparse lattice of supersets (e.g. a maximal set) imposes strong degree-theoretic restrictions on the possible degrees of subsets. Further results along these lines were recently obtained by Mike Stob and the author in [DS2]. These results concerned splittings of maximal sets. In [DS2] it is shown—for example—that if \mathbf{a} is any nonzero r.e. degree there exist r.e. degrees \mathbf{b} and \mathbf{c} with $\mathbf{0} < \mathbf{b} < \mathbf{a}$ and $\mathbf{0} < \mathbf{c} < \mathbf{a}$ such that

(i) there exist r.e. disjoint sets C and D with $\text{deg}(C) = \mathbf{c}$ and $C \sqcup D$ maximal, and

(ii) there do not exist disjoint r.e. sets B and E with $\text{deg}(B) = \mathbf{b}$ and $B \sqcup E$ maximal (or r -maximal or hh -simple).

We remark that (ii) and several extensions are established by techniques along the lines of those of (3.2). We also remark that the degrees of halves of splittings of maximal sets are particularly interesting, since in [DS2] it is also shown that the r.e. nonrecursive sets A such that there is an r.e. B with $B \neq \emptyset$ with $A \cap B = \emptyset$ and $A \cup B$ maximal, form an orbit in the automorphism group of the lattice of r.e. sets. This is the first known elementarily definable orbit (containing members in all high degrees) satisfying degree theoretic properties like those in (i) and (ii) above.

Before we continue we would like to give one (final!) construction of a non-USP set that we feel provides an interesting counterpoint to (3.2) and the [Do1] construction, both of which seem to work because large "blocks" of elements are put into A . Let B be any nonrecursive r.e. set. Define A as follows:

Stage 0. Let $f(\omega) = B$ be a 1-1 enumeration of B . Let $A_0 = \emptyset$ and $b_{i,0} = i$ for all $i \in \omega$.

Stage $s + 1$. Set $A_{s+1} = A_s \cup \{b_{f(s),s}\}$ and set

$$b_{i,s+1} = \begin{cases} b_{i,s} & \text{for } i < f(s) \\ b_{i+f(s)+s+1}, & \text{otherwise.} \end{cases}$$

Here the reader should think of the $b_{i,s}$ as markers placed on some, but not all, members of \bar{A}_s . Now define $A = \bigcup_s A_s$. Then via exactly the same argument as [Do1], A is non-USP.

It is interesting to speculate on other properties of A constructed as above. Ambos-Spies [AS2] has shown that for a particular choice of B it is possible to make A complete but to also have the property that if $A_1 \sqcup A_2 = A$ is an r.e. splitting of A then one of A_1 or A_2 is low. In particular the author would like to know if it is possible to choose B so that A is *speedable* (cf. [So3]) but have the Ambos-Spies property that if $A_1 \sqcup A_2 = A$ is an r.e. splitting of A then one of A_1 or A_2 is low. (It is not known if all speedable r.e. sets can be split into a pair of speedable subsets. This is an apparently difficult open question due to Jeff Remmel.)

Returning now to the possible lattice properties of non-USP r.e. sets, we might be tempted to conjecture that perhaps simple sets might be non-USP or alternatively suggest that USP might be invariant under automorphisms of the lattice of r.e. sets. To dispose of these possibilities, we prove Theorem 3.11 below.

THEOREM 3.11. *There exist low promptly simple SUSP r.e. sets.*

Proof. The proof of this result is rather more difficult than (3.2) and it relies on a tree of strategies argument. We shall build $A = \bigcup_s A_s$ in stages to satisfy $\text{card}(\omega - A) = \infty$ and

$$P_e : \text{card}(W_e) = \infty \text{ implies } \exists y, s (y \in W_{e,s+1} - W_{e,s} \ \& \ y \in A_s)$$

$$H_e : \exists^\infty s (\Phi_{e,s}(A_s; e) \downarrow) \text{ implies } \Phi_e(A; e) \downarrow$$

$$N_e : \text{If } \Phi_e(A) = W_e \oplus V_e \text{ and } \Gamma_e(W_e \oplus V_e) = A$$

then for some r.e. sets Q_e, R_e we have

$$Q_e \sqcup R_e = A, Q_e \equiv_T W_e \text{ and } R_e \equiv_T V_e.$$

(Here we work over 4-tuples $(\Phi_e, \Gamma_e, W_e, V_e)$ as in (3.2).) Thus P_e ensures that A is promptly simple via the identity function, and H_e is the standard lowness requirement as in [So2]. Thus for the H_e we define a restraint

$$r(e, s) = \begin{cases} 1 + u(\Phi_{e,s}(A_s; e)) & \text{if } \Phi_{e,s}(A_s; e) \downarrow \\ 0 & \text{otherwise} \end{cases}$$

and let $R(e, s) = \max\{r(j, s) : j \leq e\}$. Now, given that we argue the positive restraints are finitary, and if we only add y to $A_{s+1} - A_s$ for the sake of P_j for $j \geq e$ if $y \geq R(e, s)$, the standard argument will show A is low as in Soare [So2]. Hence we will not discuss these further, but refer the reader to [So2] if he wishes further amplification. We also will not consciously make $\text{card}(\omega - A) = \infty$ but simply remark that this is a simple enough matter to arrange (and will actually come out of the construction).

The crucial dynamics of the construction involve the interaction of the N_e requirements and the satisfaction of the P_e . We first briefly attempt to describe our method of satisfaction of the N_e , at least for a single N_e .

Define the “ A controllable” length of agreement as

$$1(e, s) = \max\{x : \forall y < x [\Gamma_{e,s}(W_{e,s} \oplus V_{e,s}; y) = A_s(y) \\ \& \forall z (z \leq u(\Gamma_{e,s}(W_{e,s} \oplus V_{e,s}; y)) \\ \rightarrow \Phi_{e,s}(A_s; z) = (W_{e,s} \oplus V_{e,s})(z))]\}$$

Our strategy was arrived at by a series of approximations, and it would perhaps be useful to use this approach to describe our strategy. The crudest attempt to satisfy the N_e is to define a recursive series of markers $x(e, 0), x(e, 1), \dots$ and for $x = x(e, i)$ to simply wait until the *first stage* s occurs where $1(e, s) > x$. We then set $s(x) = s$ and try to ensure that

$$(3.12) \quad \forall t > s(1(e, t) > 1(e, s)) \text{ implies}$$

$$(W_{e,t}[s(x)] = W_e[s(x)] \text{ iff } Q_{e,t}[x] = Q_e[x]).$$

Of course we must treat R_e and V_e similarly. Assuming that $x(e, i) \rightarrow \infty$, (3.12) will clearly ensure that $W_e \equiv_T Q_e$.

We attempt to implement this strategy by monitoring the progress of $W_{e,t}[s(x)]$ and $V_{e,t}[s(x)]$ at *e-expansionary stages*; that is, stages where $1(e, s) > m1(e, s)$ where $m1(e, s) = \max\{1(e, t) : t < s\}$. The idea is then that if $W_{e,t}[s(x)] \neq W_{e,q}[s(x)]$ —where $q = \max\{t' < t \ \& \ 1(e, t') > m1(e, t')\}$ —hopefully we have arranged matters so that $A_t[x] \neq A_q[x]$. Then we can ensure $Q_{e,t}[x] \neq Q_{e,q}[x]$ by enumerating into Q_e some element $y \leq x$ with $y \in A_t - A_q$.

The reader should note that even in this crude approximation, the fact that $A_t[x] \neq A_q[x]$ is really not enough. Perhaps $y \leq x$ is the unique element to enter $A_t - A_q$ and $y = x(e, j)$ for some $x(e, j) < x$. Now since (3.12) is an “iff” condition for it to be legal for us to add y into Q_e , it must also be the case that $W_{e,t}[s(y)] \neq W_{e,q}[s(y)]$.

It is also important to note here that the dynamics of the construction force us to proceed by first adding elements to A , then waiting until the next *e-expansionary stage* to decide which sets Q_e or R_e to put these new elements into. (In modern terminology this is essentially “closing a gap”.) The reason for this order is that we must wait to see which of W_e or V_e changes to decide which set to place the appropriate elements into. (It might seem feasible to alternatively wait till the end of a gap and see which of W_e or V_e changes, and then perhaps add to A some further elements as traces to the V_e and W_e change. This strategy seems to fail since (for example) $R(e, s)$ might change in intervening stages.)

Now even at the local level, if our strategy has any hope of working, we must try to ensure that whenever $W_e[s(x)]$ has a chance of changing, we need to also ensure that $A[x]$ has a chance of changing also. This is primarily achieved by two devices. First we use a sort of “waiting and cancellation” procedure in some ways along the lines of a “contiguous degree” construction (cf. Stob [St]). As in the confirmation procedure of (3.2), this waiting and cancellation procedure attempts to clear A of potentially injurious numbers z with $z < u(\Phi_{e,s}(A_s; g))$ for all $g \leq s(x)$ and thus tries to force $W_e[s(x)]$ to be able to change only if $Q_e[x]$ changes. This type of device has been used in several other settings and its origins may be traced back to Ladner [Ld2]. (In our construction this is the reason for the $b(\sigma, s)$ function.) The second device we use is to replace the single $x(e, i)$ marker by a finite *block* $\{x(e, i, 1), \dots, x(e, i, n)\}$ of markers. Roughly speaking this is forced upon us because of the interactions of W_e or V_e changes. The point is that the waiting-cancellation procedure ensures us to *some* W_e or V_e change, but if both sets change, perhaps W_e changes through $s(x)$ and V_e changes through $s(y)$, but y is

very much smaller than x . This will be good from R_e 's point of view (because it will be that $A[y]$ changes) but is bad from Q_e 's point of view since we cannot use y (or $y + 1$, etc.) to add to Q_e if W_e does not change through $s(y)$. The use of a block of markers essentially ensures that there will be some appropriate element in $A_i - A_q$ —"one from block x "—that we shall be able to add to Q_e . It is essentially possible to preset this block in advance because we will know precisely the maximum number of "injuries" any block can suffer in the same way as Post's simple set construction.

The final problems we must overcome are due to our lack of knowledge of which $l(e, s) \rightarrow \infty$ and thus how to combine strategies for the various N_e . This problem is much the same as, say, a minimal pair construction. We choose to overcome this problem using a tree of strategies argument. We use $T = 2^{<\omega}$ to encode certain "guesses" as to the behaviour of the various N_j for $j \leq e$ and $e \in \omega$. Hence in particular in the place of Q_e and R_e and $x(e, i, n)$ we shall use certain "guessed" versions which we denote by Q_σ , R_σ and $x(\sigma, i, n(\sigma))$ where σ has length $e + 1$. It suffices to argue for some σ the " σ -strategy" is successful. For more on tree arguments we refer the reader to [So3] and [So4, Chapter XIV]. The conscious use of trees was introduced by Lachlan [La3].

We now give the formal details of the argument. We refer to $\sigma \in 2^{<\omega}$ as *guesses*. Let $\text{lh}(\sigma)$ denote the length of σ . Let $\sigma \subset \tau$ denote σ is an initial segment of τ . We define the usual lexicographic ordering \leq_L via $\sigma \leq_L \tau$ iff $\sigma \subset \tau$ or $\exists \gamma(\gamma \wedge 0 \subset \sigma \ \& \ \gamma \wedge 1 \subset \tau)$. Here we are denoting the concatenation of σ and τ by $\sigma \wedge \tau$.

As we remarked earlier, for the waiting-cancellation strategy, it will be convenient in the construction to make use of a marker $b(\sigma, s)$. We refer to this as the σ -*boundary*. Roughly speaking, the σ -boundary (for $\text{lh}(\sigma) = e + 1$) indicates the amount of A that N_e believes is "covered" as far as guess σ is concerned. Finally in the construction we shall use the word "large fresh number". This is taken to mean a number to exceed all numbers previously considered in the construction; and saves on notation.

We define a σ -*stage* by induction on $\text{lh}(\sigma)$:

- (i) Every stage s is an \emptyset -stage.
- (ii) If s is a τ -stage and $\text{lh}(\tau) = e$ then if $l(e, s) > \max\{l(e, t) : t \text{ is a } \tau\text{-stage and } t < s\}$ and $l(e, s) > b(\tau \wedge 0, s)$ then s is a $\tau \wedge 0$ -stage.

Otherwise s is a $\tau \wedge 1$ -stage.

Let σ_s denote the unique path of length s such that s is a σ_s -stage. We shall say P_e *requires attention* at stage $s + 1$ if one of the following options holds.

(3.13) P_e has a follower $x > R(e, s)$ such that
 $\exists y(y \in W_{e,s+1} - W_{e,s} \ \& \ y > x)$.

(3.14) P_e has no follower x with $x > R(e, s)$ such that
 x has guess σ_s , and furthermore P_e has not been
declared satisfied.

The reader—especially the one already familiar with tree arguments—should note that (3.13) means that P_e can receive attention via a follower x with guess σ at *other than* σ -stages. This seems crucial if we are needing to achieve *prompt* simplicity rather than just simplicity.

CONSTRUCTION.

Stage 0. Set $A_0 = Q_{\sigma,0} = R_{\sigma,0} = \emptyset$ for all $\sigma \in 2^{<\omega}$. For all such σ , set $b(\sigma, 0) = 0$ and declare $x(\sigma, i, j)$ and $s(\sigma, i)$ as undefined for all i and j .

Stage $s + 1$.

Step 1 (Cancellation). Cancel all followers with guess τ , all $x(\tau, i, j)$ and all $s(\tau, i)$ for all $\tau \not\leq_L \sigma_s$. For all such τ set $R_{\tau,s+1} = \emptyset$ and $Q_{\tau,s+1} = \emptyset$.

Step 2 (Trace assignment). For each $\tau \wedge 0 \subset \sigma_s$, find the least i (if any) with $x(\tau \wedge 0, i)$ defined but $s(\tau \wedge 0, i)$ not yet defined. Define $s(\tau \wedge 0, i) = z$ where $z = \max\{u(\Gamma_{e,s}(W_{e,s} \oplus V_{e,s}; p)) : p \leq 1(e, s)\}$ for $e = \text{lh}(\tau)$.

Step 3 (Marker appointment). For each $\tau \wedge 0 \subset \sigma_s$ define a new block of markers (in order of $\tau \wedge 0$) as follows.

Find the least i with block $x(\tau \wedge 0, i)$ currently undefined. Find $n = 3s + 4$ large fresh numbers y_1, \dots, y_n and appoint the current marker block as

$$x(\tau \wedge 0, i) = \{x(\tau \wedge 0, i, 1), \dots, x(\tau \wedge 0, i, n)\}$$

with $x(\tau \wedge 0, i, j) = y_j$. (The number $3s + 4$ is not critical here, it is simply one we know will be large enough to ensure the block is not used up.) We remark that we may in the next step add two further numbers (followers of some P_j) to each such $x(\tau \wedge 0, i)$ we have defined in this step. This will be done for technical convenience only.

REMARK. We remark that we should possibly use a notation $x(\tau \wedge 0, i)(s)$ to indicate $x(\tau \wedge 0, i)$ at stage s , but we feel that this will be clear from context, since once $x(\tau \wedge 0, i)$ is defined it remains fixed or else becomes undefined later. Furthermore, after some finite stage we will soon see that $x(\tau \wedge 0, i)$ remains fixed for “correct” $\tau \wedge 0$.

Step 4. Find the least e such that P_e requires attention. Adopt the appropriate case below.

Case 1 (Follower appointment). (3.14) holds; find a large fresh number y and appoint y as a follower of P_e with guess σ where $\sigma \subset \sigma_s$ and $\text{lh}(\sigma) = e + 1$. Cancel all followers with guesses $\gamma \supset \sigma$ and $\gamma \neq \sigma$. Now for each $\tau \wedge 0 \subset \sigma_s$ find the set $x(\tau \wedge 0, i)$ defined in Step 4, and enumerate y and $y + 1$ into $x(\tau \wedge 0, i)$ (so that now $x(\tau \wedge 0, i)$ has $3s + 6$ elements). Go to Step 5.

Case 2 (Satisfaction). (3.13) holds. Enumerate x , $x + 1$ and y into $A_{s+1} - A_s$. Let x_1, \dots, x_m list those marker blocks currently defined and satisfying $\forall z \in x_i (z > x + 1)$. (Remark: this will consist of those blocks currently defined and appointed after the stage x was appointed.) For each such block find the least two elements g_i, h_i with

$$g_i, h_i \in x_i - A_s \quad (\text{These will not be followers.})$$

Also enumerate $\{g_i, h_i : 1 \leq i \leq m\}$ into $A_{s+1} - A_s$. Cancel all (remaining) followers of P_j for $j > e$. Declare P_e as satisfied (forever).

REMARK. The reader should note that when some follower x of P_e is enumerated into A we perhaps cancel some follower $z > x$. Now perhaps $z \in x(\gamma \wedge 0, j)$ for several j . It really will not matter whether or not we regard z as still a member of $x(\gamma \wedge 0, j)$ after it is cancelled. The crucial fact will be that z is no longer a follower once it is cancelled. For technical reasons if $x(\gamma \wedge 0, j)$ is defined it is easiest to regard z as still a member of $x(\gamma \wedge 0, j)$ but ask that z never be added to A unless z is added to A for the sake of some P_k for $k < e$. A counting argument ensures that this is possible.

Step 5 (Defining boundaries). Find a large fresh number z and define $b(\tau \wedge 0, s) = z$ for each $\tau \wedge 0 \subset \sigma_s$.

Step 6 (Recovery). In this step we attempt to build our splitting of A . Thus for each $\sigma = \tau \wedge 0 \subset \sigma_s$ perform the following recovery step: let $\text{ls}(\sigma, s)$ denote last σ -stage $< s$, namely

$$\text{ls}(\sigma, s) = \max\{0, t : t \text{ is a } \sigma\text{-stage and } t < s\}.$$

Let $t = \text{ls}(\sigma, s)$ and let $e = \text{lh}(\tau)$. Now if $x(\tau \wedge 0, i)$ is currently defined let $\text{mx}(\tau \wedge 0, i)$ denote the largest member of $x(\tau \wedge 0, i)$. Now find the least i and j (if any) such that $s(\tau \wedge 0, i)$ and $s(\tau \wedge 0, j)$ were defined at

stage t and

- (i) $A_s[mx(\tau \wedge 0, i)] \neq A_t[mx(\tau \wedge 0, i)]$ and
 $W_{e,s}[s(\tau \wedge 0, i)] \neq W_{e,t}[s(\tau \wedge 0, i)]$, and
- (ii) $A_s[mx(\tau \wedge 0, j)] \neq A_t[mx(\tau \wedge 0, j)]$ and
 $V_{e,s}[s(\tau \wedge 0, j)] \neq V_{e,t}[s(\tau \wedge 0, j)]$,

There are several cases to consider:

Case 1. Neither i nor j exists. In this case set $Q_{\sigma,s+1} = Q_{\sigma,s} \cup (A_s - (Q_{\sigma,s} \cup R_{\sigma,s}))$ and $R_{\sigma,s+1} = R_{\sigma,s}$.

Case 2. i exists but j does not. As in Case 1.

Case 3. j exists but i does not. In this case set $R_{\sigma,s+1} = R_{\sigma,s} \cup (A_s - (Q_{\sigma,s} \cup R_{\sigma,s}))$ and $Q_{\sigma,s+1} = Q_{\sigma,s}$. For the remaining cases we have that i and j both exist.

Case 4. $i = j$. In this case let y_1 and y_2 be the least two numbers to enter $A_s - A_t$.

We claim two such numbers exist, and in fact $y_2 = y_1 + 1$ and y_1 is a follower, this being checked in the verification. We similarly claim that not only are both y_1 and $y_2 < mx(\tau \wedge 0, i)$, but in fact y_1 and y_2 are members of $x(\tau \wedge 0, i)$.

Assuming this claim, we shall set $R_{\sigma,s+1} = R_{\sigma,s} \cup \{y_1\}$ and $Q_{\sigma,s+1} = Q_{\sigma,s} \cup (A_s - (Q_{\sigma,s} \cup R_{\sigma,s+1}))$.

Case 5. $i < j$. In this case we claim that

some number in block $x(\tau \wedge 0, j)$ entered $A_s - A_t$.

Let z be the least such number. We set $R_{\sigma,s+1} = R_{\sigma,s} \cup \{z\}$ and $Q_{\sigma,s+1} = Q_{\sigma,s} \cup (A_s - (Q_{\sigma,s} \cup R_{\sigma,s+1}))$.

Case 6. $j < i$. As in Case 5 with the roles of Q and R reversed, noting that some z in block $x(\tau \wedge 0, i)$ entered $A_s - A_t$.

END OF CONSTRUCTION.

Verification. We first verify the P_e and H_e as these are fairly standard arguments.

Let β denote the leftmost path. That is, $\emptyset \subset \beta$ and $\gamma \subset \beta$ implies $\gamma \wedge 0 \subset \beta$ if $\exists^\infty s$ (s is a $\gamma \wedge 0$ -stage) otherwise $\gamma \wedge 1 \subset \beta$. Now by construction (3.13) pertains to P_e at most once. Hence H_e will be met. By [So2] we now argue that P_e is met. Let $\sigma \subset \beta$ with $\text{lh}(\sigma) = e + 1$ and let s_0 be a stage good for σ , namely

- (i) $\forall s > s_0$ ($\sigma \leq_L \sigma_s$),
- (ii) $\forall j < e$ $\forall s > s_0$ (s is a σ -stage implies P_j does not receive attention at stage s),

- (iii) $\forall j < e \forall s > s_0$ ((3.13) does not pertain to P_j at stage s),
 (iv) $\forall s > s_0 \forall j \leq e$ ($r(j, s) = r(j)$).

Without loss, we might as well suppose P_e has no followers. Now the next follower x to be appointed to P_e is clearly uncancellable. Thus at the least stage $s + 1 > s_0$ when some $y > x$ occurs in $W_{e,s+1} - W_{e,s}$, we will meet P_e .

The remainder of the verification is devoted to checking that our apparatus manages to meet the N_e . Thus again let $\sigma = \tau \wedge 0 \subset \beta$ with $\text{lh}(\sigma) = e + 1$ and let s_0 be good for σ as above. First observe that by our cancellation procedure we can suppose that $x(\sigma, 0)$ is undefined at stage s_0 . Now by choice of s_0 and σ it is clear that $\forall i \forall t > s_0 \forall s > t$ ($x(\sigma, i)$ defined at stage t implies $x(\sigma, i)$ defined and unchanged at stage s). Furthermore the same holds for $s(\sigma, i)$. Moreover as $1(e, s) \rightarrow \infty$ we see that for all $i \in \omega$, $x(\sigma, i)$ and $s(\sigma, i)$ eventually become (permanently) defined.

Suppose $x(\sigma, i)$ is defined at stage $t > s_0$. Then $x(\sigma, i)$ is now fixed and has at least $3t + 4$ elements. These elements can only be added to A because of the action of a P_j requirement, and by construction we only add these elements to A (≤ 3 at a time) when some follower $\leq mx(\sigma, i)$ enters A . By the way we appoint followers, all followers appointed after stage t must exceed $mx(\sigma, i)$. A counting argument ensures that $x(\sigma, i) \not\subset A$ (and so there are always markers in $x(\sigma, i)$ not yet in A_s at any $s > s_0$). The remainder of the proof is given by a sequence of lemmas.

(3.15) LEMMA. *Let q and t be σ -stages with $q > t > s_0$ and such that $s(\sigma, i)$ is defined at stage $t + 1$. Then $A_t[mx(\sigma, i)] = A_q[mx(\sigma, i)]$ implies $(W_{e,t} \oplus V_{e,t})[s(\sigma, i)] = (W_{e,q} \oplus V_{e,q})[s(\sigma, i)]$.*

REMARK. Before we begin the formal proof of this lemma, the reader should note that the "guiding principle" in the verification of (3.15) and later lemmata is that although there are lots of numbers (such as markers) which can apparently injure computations due to their addition to A , they are only added to A in response to—and at the same stage as—(smaller) followers being added to A . Markers never *initiate* such "injury"; only followers may do so.

Proof (of 3.15). Let t and q satisfy the hypotheses of the lemma. Then it follows that there exist σ -stages t_1 and t_2 with $s_0 < t_1 + 1 \leq t_2 \leq t$ such $mx(\sigma, i)$ was appointed at stage $t_1 + 1$ and $s(\sigma, i)$ was appointed at $t_2 + 1$. Now at stage $t_1 + 1$ we set $b(\sigma, t_1 + 1)$ to be large and fresh and

in particular $b(\sigma, t_1 + 1) > mx(\sigma, i)$. ($b(\sigma, t_1 + 1)$ is appointed in Step 5 after $x(\sigma, i)$ has been appointed and completely defined.) By definition of σ -stage t_2 will be the least σ -stage with $1(e, s) > b(\sigma, t_1 + 1)$. By choice of s_0 we know that any *follower* appointed at stages v with $t_1 + 1 < v \leq t_2$ must have guess τ where $\sigma \leq_L \tau$ but $\sigma \not\subseteq \tau$ and $\tau \not\subseteq \sigma$. Also, any follower appointed at stage $t_1 + 1$ is in $x(\sigma, i)$ and so is $\leq m(\sigma, i)$. By cancellation in Step 1, this means that there are no followers z with

$$(3.16) \quad mx(\sigma, i) < z \leq P \quad \text{where } P = \max\{u(\Phi_{e,t_2}(A_{t_2}; y)) : y \leq s(\sigma, i)\}.$$

Suppose that (3.15) fails. First we claim that there must be some *follower* z with

$$(3.17) \quad mx(\sigma, i) < z \leq N \quad \text{where } N = \max\{u(\Phi_{e,t}(A_t; y)) : y \leq s(\sigma, i)\},$$

and such that z enters $A_q - A_t$.

After all, N is also a use function so if $A_q[N] = A_t[N]$ then for all $y \leq s(\sigma, i)$ we see $\Phi_{e,t}(A_t; y) = \Phi_{e,q}(A_q; y) = (W_{e,t} \oplus V_{e,t})(y)$. However, if we suppose $A_q[N] \neq A_t[N]$ by construction some *follower* $\leq N$ must enter $A_q - A_t$.

We shall now argue that no such z can exist. Since z is uncanceled at stage t and $z \leq N$, z must have guess $\gamma \supset \sigma$ and must have been appointed at stage $t_3 + 1$ with t_3 a σ -stage and by (3.16), $t_2 \leq t_3 < t$. We claim that since z still exists at stage t , it must be the case that

$$(3.18) \quad A_{t_3}[mx(\sigma, i)] = A_t[mx(\sigma, i)].$$

If (3.18) fails then some *follower* $< mx(\sigma, i)$ enters A after stage t_3 but before stage t . Since any such follower must have higher priority than z , its entry into A would cancel z (as a follower). But then (3.17) would fail since $z \in A_q - A_t$. Hence (3.18) holds. Now since t_3 was a σ -stage, we know

$$(3.19) \quad z > M \quad \text{where } M = \max\{u(\Phi_{e,t_3}(A_{t_3}; y)) : y \leq s(\sigma, i)\}.$$

Again by the reasoning used to establish (3.16) there are no followers g with $mx(\sigma, i) < g \leq M$. Now by (3.18) this means that

$$(3.20) \quad A_{t_3}[M] = A_t[M].$$

This, in turn, implies that $M = N$ (computations are unchanged) and now we obtain the desired contradiction: (3.20) and (3.17) say that $z \leq N = M$ yet (3.18) says $z > M$. The lemma now follows. \square

(3.21) LEMMA. Let t and q be σ -stages with $q > t > s_0$ and $s(\sigma, i)$ defined at stage $t + 1$. Then $(W_{e,t} \oplus V_{e,t})[s(\sigma, i)] = (W_{e,q} \oplus V_{e,q})[s(\sigma, i)]$ implies $A_t[mx(\sigma, i)] = A_q[mx(\sigma, i)]$.

Proof. Suppose the lemma fails. Let t, q be as above. We know that if $A_t[mx(\sigma, i)] \neq A_q[mx(\sigma, i)]$ some (least) follower $z \leq mx(\sigma, i)$ enters $A_q - A_t$. Let $t_1 \leq t$ be the σ -stage where $s(\sigma, i)$ was appointed at stage $t_1 + 1$.

Now there exist σ -stages t_2 and t_3 with $s_0 < t_2 < t_3 \leq t_1$ such that z was appointed at stage $t_2 + 1$ and $t_2 = ls(\sigma, t_3)$. By the same reasoning as (3.16), we know there is no follower g with

$$(3.22) \quad z < g \leq P \quad \text{where } P = \max\{u(\Phi_{e,t_3}(A_{t_3}; y)) : y \leq D\}$$

$$\text{where } D = \max\{u(\Gamma_{e,t_3}(W_{e,t_3} \oplus V_{e,t_3}; q)) : q \leq z\}.$$

(3.22) follows because we reset $b(\sigma, t_2 + 1)$ to exceed z . Now since z is uncanceled at stage t , we know that no follower less than z can have entered A between stages t_2 and t , since the entry of such followers would automatically cancel z . It follows that

$$(3.23) \quad A_t[P] = A_{t_3}[P].$$

But now (3.23) and the definition of P in (3.22) mean that

$$(3.24) \quad (W_{e,t} \oplus V_{e,t})[D] = (W_{e,t_3} \oplus V_{e,t_3})[D].$$

Now as $D \leq s(\sigma, i)$ since $t_3 \leq t_1$, we see that the hypotheses of the lemma mean

$$(3.25) \quad (W_{e,t_3} \oplus V_{e,t_3})[D] = (W_{e,q} \oplus V_{e,q})[D].$$

This means that the Γ_e -computations involved in D in (3.22) are unchanged at stage q , and in particular

$$(3.26) \quad \Gamma_{e,t_3}(W_{e,t_3} \oplus V_{e,t_3}; z) = \Gamma_{e,q}(W_{e,q} \oplus V_{e,q}; z) = 0.$$

Therefore $z \notin A_q$. This specifically contradicts the assumption that $z \in A_q - A_t$, and so establishes the lemma. \square

If we put (3.21) and (3.15) together, summarizing, we see

(3.27) LEMMA. Let t and q be σ -stages with $s_0 < t < q$ and $s(\sigma, i)$ defined at stage t . Then

(i) $(W_{e,q} \oplus V_{e,q})[s(\sigma, i)] \neq (W_{e,t} \oplus V_{e,t})[s(\sigma, i)]$ iff $A_q[mx(\sigma, i)] \neq A_t[mx(\sigma, i)]$.

(ii) If $A_q \neq A_t$, the least numbers to enter $A_q - A_t$ are y and $y + 1$ where y is a follower and for some least $j \leq i$, $y + 1 = mx(\sigma, j)$.

(iii) Suppose $A_q \neq A_t$ and $x(\sigma, i)$ is defined at stage t . Suppose further that for j given by (ii) we have $j < i$. Then there exist at least two numbers $g_i, h_i \neq y, y + 1$ with $g_i, h_i \in x(\sigma, i)$ and $g_i, h_i \in A_q - A_t$.

(iv) The construction does not break down at Step 6 and $Q \sqcup R = *A$.

Proof. (i) simply combines (3.21) and (3.15). (ii) is an easy induction. (iii) is exactly what the mechanics of the construction were set up to ensure, and follows immediately by what happens in Step 4, Case 2 of the construction. Finally (i) implies (iv). \square

(3.28) LEMMA. $W_e \equiv_T Q_\sigma$ and $V_e \equiv_T R_\sigma$.

Proof. We verify that $W_e \equiv_T Q_\sigma$. The fact that $V_e \equiv_T R_\sigma$ is proved *mutatis mutandis*. First $Q_e \leq_T W_e$. To determine $Q_e[z]$ for $z \in \omega$, find the least σ -stage $t > s_0$ with $s(\sigma, z)$ defined at stage t and $W_{e,t}[s(\sigma, z)] = W_e[s(\sigma, z)]$. Direct analysis of Step 6 and (3.27)(i) imply that $Q_{\sigma,t+1}[mx(\sigma, z)] = Q_\sigma[mx(\sigma, z)]$ and so $Q_{\sigma,t+1}[z] = Q_\sigma[z]$ since $z \leq mx(\sigma, z)$.

The reverse direction is more difficult. We claim similarly that if t is the least σ -stage with $t > s_0$ and $Q_{\sigma,t+1}[mx(\sigma, z)] = Q_\sigma[mx(\sigma, z)]$ then $W_{e,t}[s(\sigma, z)] = W_e[s(\sigma, z)]$.

Suppose not. Let $q > t$ be the least σ -stage with $W_{e,t}[s(\sigma, z)] \neq W_{e,q}[s(\sigma, z)]$, and let $t_1 = ls(\sigma, q)$. By (3.27)(i) we see that $A_q[mx(\sigma, z)] \neq A_{t_1}[mx(\sigma, z)]$. There exists some least $g \leq z$ such that

$$(3.29) \quad \begin{cases} A_{t_1}[mx(\sigma, g)] \neq A_q[mx(\sigma, g)] & \text{and} \\ W_{e,t}[s(\sigma, g)] \neq W_e[s(\sigma, g)]. \end{cases}$$

By construction, Step 6, and (3.27)(iii) we know that for at least one $y \in x(\sigma, g)$, we have $y \in Q_{\sigma,q+1} - Q_{\sigma,q}$. But then $Q_{\sigma,t+1}[mx(\sigma, z)] \neq Q_\sigma[mx(\sigma, z)]$ after all. (This is the whole point of the $x(\sigma, g)$ in the construction.) \square

REMARK. We remark that A constructed as above has contiguous r.e. degree (i.e. consists of a single r.e. *wtt*-degree). This follows by (3.27)(i). Not every r.e. set of contiguous degree is USP. Take for example a hypersimple r.e. set of contiguous degree. However, every r.e. contiguous degree does contain an SUSP r.e. set. This is proved in [AF] where

Ambos-Spies and Fejer actually show that an r.e. cylinder of contiguous degree is SUSP. We do not know if every contiguous degree contains a *simple* SUSP r.e. set.

(3.30) COROLLARY. *(S)USP is not invariant under automorphisms of the lattice of r.e. sets.*

Proof. It is well known that there are promptly simple hypersimple r.e. low sets. For example, by [AJSS], the deficiency set of a (low) promptly simple r.e. set is (low and) promptly simple. By Dekker [De], the deficiency set of an r.e. nonrecursive set is hypersimple.

Now let B be a hypersimple low promptly simple r.e. set. By Theorem (3.2) B is non-USP. Let A be an SUSP promptly simple low r.e. set given by (3.11). By Maass [Ma] there is an automorphism of the lattice of r.e. sets taking A to B . □

We remark that Martin's methods (cf. [So1]) of constructing automorphisms appear sufficient to prove the above result directly if we do not wish to use Maass's very powerful result [Ma] and hence to use the full automorphism machinery.

We finally remark that in [Do2], [DS1] and [DW], stronger "nonsplitting" properties of r.e. sets were examined. For example in [DW], Downey and Welch construct r.e. sets A and B with $\emptyset <_T B <_T A$ such that if $A_1 \sqcup A_2 = A$ is an r.e. splitting of A , then $A_1 \leq_T B$ implies $A_1 \equiv_T \emptyset$. A is said to have the *antisplitting property*. It is unclear if (for example) maximal sets have the antisplitting property and this would seem an interesting line of investigation to pursue.

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