Reprinted from

ANNALS OF PURE AND APPLIED LOGIC

Annals of Pure and Applied Logic 70 (1994) 113–139

The structure of the honest polynomial m-degrees

Rod Downey[a,1], William Gasarch[b,2], Michael Moses[c,3]

[a] *Department of Mathematics, Victoria University of Wellington, Wellington, New Zealand*
[b] *Department of Computer Science and Inst. for Adv. Comp. Studies, University of Maryland, College Park, MD 20742, USA*
[c] *Department of Mathematics, George Washington University, Washington, DC 20052, USA*

Received 4 October 1993; revised 15 February 1994; communicated by A. Nerode

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ANNALS OF PURE AND APPLIED LOGIC

⊚ The paper used in this publication meets the requirements of ANSI/NISO 239.48-1992 (Permanence of Paper).

Published fifteen times a year 0168-0072/94/$07.00 Printed in the Netherlands

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The structure of the honest polynomial m-degrees

Rod Downey [a,1], William Gasarch [b,2], Michael Moses [c,3]

[a] *Department of Mathematics, Victoria University of Wellington, Wellington, New Zealand*

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Abstract

We prove a number of structural theorems about the honest polynomial m-degrees (denoted by $\mathbf{H}_m$ or hpm-degrees) contingent on the assumption P = NP (or a unary alphabet). In particular, we show that if P = NP (or a unary alphabet), then the topped finite initial segments of $\mathbf{H}_m$ are exactly the topped finite distributive lattices, the topped initial segments of $\mathbf{H}_m$ are exactly the direct limits of ascending sequences of finite distributive lattices, and all recursively presentable distributive lattices are initial segments of $\mathbf{H}_m \cap \mathbf{RE}$. Additionally, assuming $|\Sigma| = 1$, we show that the theory of the hpm-degrees is undecidable. We also show that index sets cannot be minimal. Lastly, we examine an alternative definition of honest m-reduction under which recursive minimal sets can be constructed.

1. Introduction

Homer [11] has shown connections between the P =? NP question and the existence of sets that are minimal with respect to honest Turing reductions (henceforth, we refer to such sets as "hpT-minimal"). Informally, these reductions are polynomial Turing reductions where the strings queried cannot be "short" compared to the input length. Homer showed that if P = NP then there exist hpT-minimal sets. Homer and Long [10,9] have simplified the original construction of Homer, and showed that the P = NP

[1] Supported in part by the United States-New Zealand Cooperative Science Foundation Grant INT 90-20558, in part by the Victoria University of Wellington internal grant committee, in part by Cornell, and in part by M.S.I at Cornell. Email: downey@math.vuw.ac.nz

[2] Supported in part by NSF grants CCR-88-3641 and CCR 90-20079. Email: gasarch@cs.umd.edu

[3] Email: moses@math.gwu.edu

0168-0072/94/$07.00
SSDI 0168-0072(94)00009-R

assumption can be omitted if $|\Sigma| = 1$. They have also established partial converses by showing that several classes of sets cannot be hpT-minimal. Ambos-Spies [2] later simplified and extended their work. Homer, Long, and Ambos-Spies have also showed the existence of a set that is minimal with respect to honest m-reductions.

There are several motivations guiding research in this area. One is that by deriving consequences from the assumption P = NP, we may learn more about the P =? NP problem (ultimately we would like to derive a contradiction, though this seems unlikely). A second motivation is that there may be a converse to the statement "if P = NP then there exists a minimal set" (or some variation of the statement) that yields a statement about honest degrees that is equivalent to P = NP.

Throughout this paper, all results that can be obtained with P = NP as a hypothesis are also true in the context of tally sets (i.e., if $|\Sigma| = 1$). The proofs in the $|\Sigma| = 1$ case are similar to those in the P = NP case and are omitted. The reader should keep in mind that even though a theorem needs P = NP as a hypothesis, there is an analogous result with $|\Sigma| = 1$ that is proven with no assumption about P =? NP.

In this paper, we pursue four goals; although, the bulk of our results aims towards the first two goals.

- We connect the theory of the classical m-degrees with complexity theory:
  (1) Assuming P = NP we prove many structural theorems about the honest polynomial m-degrees (henceforth referred to as the "hpm-degrees"), and the r.e. hpm-degrees. Our proofs use techniques from the theory of the classical m-degrees [13] (surveyed in [23] and [24]). (2) We show, assuming $|\Sigma| = 1$, that the theory of the hpm-degrees is undecidable.
- We show that the classical r.e. m-degrees and the r.e. hpm-degrees are not elementarily equivalent. Our proof uses an interesting variation of Ladner's [16] looking back techniques.
- Several types of sets have been proven nonminimal without any assumption about P =? NP (e.g., semilow sets in [6] and others in [2,11]). We add to this list by showing that index sets are nonminimal.
- We clarify the distinction between honest m-reductions and total honest m-reductions. An m-reduction from A to B is a function $f \in$ P such that $x \in A$ iff $f(x) \in B$. A natural definition of an honest m-reduction would appear to require that f is honest; however, Ambos-Spies has defined an honest m-reduction to be (informally) an honest function f that is allowed to map a string into {YES, NO}. We call the former definition a *total honest m-reduction* (henceforth abbreviated as 'hmto' and denoted $\leqslant_{\rm m}^{\rm h\text{-}to}$) and the latter just an *honest m-reduction* (and denote it $\leqslant_{\rm m}^{\rm h}$). We show that these two reductions differ in an interesting way. Ladner [16] showed that there are no recursive sets that are hpm-minimal. We show, by contrast, that there are recursive sets that are hmto-minimal. Hence Ladner's theorem does not hold for total honest m-reductions. This is of interest also because all minimal degrees constructed so far have been (necessarily) nonrecursive. The recursive hmto-minimal sets are actually *superminimal*, i.e. for *all* B such that $B \leqslant_{\rm m}^{\rm h\text{-}to} A$, $B \equiv_{\rm m}^{\rm h\text{-}to} A$. The existence of superminimal sets suggests that Ambos-Spies' definition of honest m-reduction is the natural one.

In Section 2 we review motivation, definitions, and notation that will be used throughout this paper. In Section 3 we summarize our results more formally. In Sections 4 through 10 we examine the four goals in detail.

2. Motivations, definitions, and notation

Many concepts and techniques of complexity theory are based on similar notions in recursion theory. Often these concepts are later seen to be of interest for reasons independent of their original motivation (e.g. Schöning's definition of high and low sets in NP [27]). The definition of an honest reduction is partially motivated by an attempt to examine an analog of minimal degrees; though it is of independent interest in complexity theory because of the connections to P = NP, examined by Homer [11]. We review the recursion-theoretic motivation.

A nonrecursive set is $\leqslant_{\mathrm{T}}$-minimal if for any set B such that $B \leqslant_{\mathrm{T}} A$, either $B \equiv_{\mathrm{T}} A$ or B is recursive. Spector [29] constructed a $\leqslant_{\mathrm{T}}$-minimal set recursive in $\emptyset''$; later Sacks [26] constructed one recursive in $\emptyset'$. We briefly examine a naive attempt to define a minimal set in the context of complexity theory.

Definition 2.1 (*Attempt*). A set A is $\leqslant^{\mathrm{p}}_{\mathrm{T}}$-minimal if for any set B such that $B \leqslant^{\mathrm{p}}_{\mathrm{T}} A$, either $B \equiv^{\mathrm{p}}_{\mathrm{T}} A$ or $B \in \mathrm{P}$.

There is a problem with this definition: there are no $\leqslant^{\mathrm{p}}_{\mathrm{T}}$-minimal sets. For any recursive $A \notin \mathrm{P}$, by Ladner [16], there exists a set $B \notin \mathrm{P}$ such that $B <^{\mathrm{p}}_{\mathrm{T}} A$; if A is nonrecursive, then Homer [11] has shown that the set

$$B = \{z0^{2^{|z|}} : z \in A\}$$

is not in P (in fact, it is nonrecursive) and $B <^{\mathrm{p}}_{\mathrm{T}} A$. The set B is contrived as the 0's are there only for padding purposes. and consequently in the $B <_{\mathrm{T}} A$ reduction, on input x we ask a question of A that is very short compared to $|x|$. This motivates us to study reductions where the questions asked are not allowed to be too short. We will need two preliminary definitions before defining a useful notion of minimal.

Definition 2.2. Let q be a nondecreasing function. A polynomial oracle Turing machine M is *q-honest* if, for all sets S, and all strings x, if $M^S(x)$ queries oracle S about y, then $q(|y|) \geqslant |x|$.

Definition 2.3. Let A and B be sets. The set B is *honest polynomial Turing reducible to* A (written $B \leqslant^{\mathrm{h}}_{\mathrm{T}} A$) if there is a polynomial q and a q-honest oracle Turing machine $M^{()}$ such that $B \leqslant^{\mathrm{p}}_{\mathrm{T}} A$ by $M^{()}$. The set B is *honest polynomial Turing equivalent to* A (written $B \equiv^{\mathrm{h}}_{\mathrm{T}} A$) if $B \leqslant^{\mathrm{h}}_{\mathrm{T}} A$ and $A \leqslant^{\mathrm{h}}_{\mathrm{T}} B$. Note that $\equiv^{\mathrm{h}}_{\mathrm{T}}$ is an equivalence relation. The equivalence classes are called *honest polynomial Turing degrees* (hpT-degrees).

Note 2.4. Similar concepts have been studied by Machtey [20], Meyer and Ritchie [21] and Young [30].

Definition 2.5. A set $A \notin \mathrm{P}$ is *hpT-minimal* if

$$(\forall B)[B \leqslant_{\mathrm{T}}^{\mathrm{h}} A \Rightarrow (B \equiv_{\mathrm{T}}^{\mathrm{h}} A) \vee (B \in \mathrm{P})].$$

In Ladner's proof, the reduction of B to A is honest (while in Homer's proof it is not). Hence there cannot be a recursive set that is hpT-minimal. As mentioned in the introduction, if either P = NP or $|\Sigma| = 1$, then there is a (necessarily nonrecursive) hpT-minimal set [11,9].

In both Ladner's and Homer's reductions of B to A, on every input at most one query to A is made. In the cases when no query is made, the machine just says YES or NO. This motivates the next definition.

Definition 2.6. Let A and B be sets. The set B is *honest polynomial m-reducible to* A (written $B \leqslant_{\mathrm{m}}^{\mathrm{h}} A$) if there exists a polynomial q and a function $f \in \mathrm{P}$, $f : \Sigma^* \to \Sigma^* \cup \{\mathrm{YES}, \mathrm{NO}\}$, such that for all x :

1. if $f(x) = \mathrm{YES}$ then $x \in A$;
2. if $f(x) = \mathrm{NO}$ then $x \notin A$;
3. if $f(x) \in \Sigma^*$ then ($x \in B$ iff $f(x) \in A$); and
4. if $f(x) \in \Sigma^*$ then $q(|f(x)|) \geqslant |x|$.

Definition 2.7. The definitions of $\equiv_{\mathrm{m}}^{\mathrm{h}}$, hpm-degree, and hpm-minimal are analogous to the definitions of $\equiv_{\mathrm{T}}^{\mathrm{h}}$, hpT-degree, and hpT-minimal, respectively.

This definition of honest m-reduction is not a direct analog of either m-reductions in recursion theory [25] or polynomial m-reductions [19]. This definition is used by Ambos-Spies [2] because by allowing YES and NO as outputs all sets in P are $\leqslant_{\mathrm{m}}^{\mathrm{h}}$-equivalent. We present a definition that appears more natural, but will turn out not to be.

Definition 2.8. Let A and B be two sets. The set B is *honest total m-reducible to A* (written $B \leqslant_{\mathrm{m}}^{\mathrm{h\text{-}to}} A$) if $B \leqslant_{\mathrm{m}}^{\mathrm{h}} A$ by a reduction f that cannot map to an element of $\{\mathrm{YES}, \mathrm{NO}\}$. The definitions of $\equiv_{\mathrm{m}}^{\mathrm{h\text{-}to}}$, hmto-degree, and hmto-minimal are similar to those of $\equiv_{\mathrm{T}}^{\mathrm{h}}$, hpT-degree, and hpT-minimal respectively. A set A is *hmto*-superminimal if

$$(\forall B)[B \leqslant_{\mathrm{m}}^{\mathrm{h\text{-}to}} A \Rightarrow B \equiv_{\mathrm{m}}^{\mathrm{h\text{-}to}} A].$$

Note 2.9. In Section 10 we will see that there exist superminimal sets $A \notin \mathrm{P}$. This is somewhat unnatural since even for sets $B \in \mathrm{P}$, we have $B \not\leqslant_{\mathrm{m}}^{\mathrm{h\text{-}to}} A$.

We need a way to effectively represent the set of all $\leqslant_{\mathrm{m}}^{\mathrm{h}}$ reductions.

Notation 2.10. Throughout this paper, for all e, $p_e(n) = q_e(n) = n^e + e$. The reason we use two different notations for the same polynomial is that we think of p_e as a time bound, and q_e as a polynomial for the honesty condition. Let $M_1, M_2, M_3, \ldots$ be a list of all Turing machines, clocked such that M_e runs in time $p_e(n)$; and on an input of length n either outputs a string of length m where $q_e(m) \geqslant n$, or outputs an element of {YES, NO}. For every e, let f_e be the function computed by M_e, and let $V_e = \text{range}(f_e)$. If $A \subseteq \Sigma^*$ then Θ_e^A is the set that is $\leqslant_{\text{m}}^{\text{h}}$-reduced to A by M_e, namely

$$x \in \Theta_e^A \Leftrightarrow (f_e(x) \in A \text{ or } f_e(x) = \text{YES}).$$

For all e, M_e represents an $\leqslant_{\text{m}}^{\text{h}}$ reduction; and every $\leqslant_{\text{m}}^{\text{h}}$ reduction is represented by some M_e.

Notation 2.11. Let $P_1^{()}, P_2^{()}, P_3^{()}, \ldots$ be an effective enumeration of clocked oracle Turing machines, where $n^e + e$ bounds the runtime of $P_e^{()}$. If no oracle is written then the empty set is assumed to be the oracle. If we restrict some P_e to be 0-1 valued then $L(P_e)$ represents the set recognized by P_e.

Notation 2.12. Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be an acceptable programming system (e.g. an effective enumeration of Turing machines). Let W_e be the domain of φ_e, and $W_{e,s}$ be the set $\{x < s \mid \varphi_e(x) \text{ halts in} \leqslant s \text{ steps}\}$.

Convention 2.13. The term 'least string' means the least string in the lexicographic ordering on strings.

Definition 2.14. A *lattice* is a 4-tuple $\mathcal{D} = \langle D, \leqslant_{\mathcal{D}}, \sqcap, \sqcup \rangle$ such that D is a set, $\leqslant_{\mathcal{D}}$ is a reflexive and transitive order on D, $b \sqcap c$ is the greatest lower bound of b, c and $b \sqcup c$ is the least upper bound of b, c (also called 'the join of b and c'). An element $a \in D$ is *join-irreducible* if $a = b \sqcap c \Rightarrow a = b$ or $a = c$. The lattice is *distributive* if $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$ and $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$. The lattice is *topped* if it has a maximum element. If $E \subseteq D$ then there exists a greatest lower bound, and a least upper bound, of E. These are denoted by $\bigsqcap_{e \in E} e$ and $\bigsqcup_{e \in E} e$.

We will use the following standard facts about distributive lattices. They are proven in [24, pp. 555–557].

Lemma 2.15. *Let $\mathcal{D} = \langle D, \leqslant_{\mathcal{D}}, \sqcap, \sqcup \rangle$ be a distributive lattice with ordering $\leqslant_{\mathcal{D}}$. Let the set of join-irreducible elements be $\{a_i \mid i \in I\}$.*

(i) *If $b, c \in D$ and $b \not\leqslant_{\mathcal{D}} c$, then there exists a_i such that $a_i \leqslant_{\mathcal{D}} b$ and $a_i \not\leqslant_{\mathcal{D}} c$.*

(ii) *If $\mathcal{D}$ is not topped then the lattice formed by placing an additional element above all elements of D (and defining $\sqcap$ and $\sqcup$ in the obvious way) is a topped distributive lattice.*

(iii) *If $b = \bigsqcup_{i \in J} a_i$ and $a_k <_{\mathcal{D}} b$ then there exists $j \in J$ such that $a_k <_{\mathcal{D}} a_j$.*

3. Summary of results

Notation 3.1. The partial order which has the hpm-degrees as its underlying set, and $\leqslant_{\text{m}}^{\text{h}}$ as its ordering, is denoted by $\mathbf{H}_{\text{m}}$. The subordering consisting of those hpm-degrees that contain an r.e. set is denoted by $\mathbf{H}_{\text{m}} \cap \mathbf{RE}$.

Definition 3.2. A partial order $\langle X, \leqslant \rangle$ is an *initial segment of* $\mathbf{H}_\mathrm{m}$ if there exists a 1-1 map $\rho : X \to \mathbf{H}_\mathrm{m}$ such that range(ρ) is closed downward under $\leqslant^\mathrm{h}_\mathrm{m}$ and

$$x \leqslant y \Leftrightarrow \rho(x) \leqslant^\mathrm{h}_\mathrm{m} \rho(y).$$

The existence of an hpm-minimal set (assuming P = NP or $|\Sigma| = 1$) can be restated as "if P = NP or $|\Sigma| = 1$ then the two element chain is an initial segment of $\mathbf{H}_\mathrm{m}$". We have obtained extensions along these lines: If P = NP or $|\Sigma| = 1$ then the following hold.

- The topped finite initial segments of $\mathbf{H}_\mathrm{m}$ are exactly the finite distributive lattices (in Section 5).
- The topped initial segments of $\mathbf{H}_\mathrm{m}$ are exactly the direct limits of ascending sequences of finite distributive lattices (in Section 6).
- The topped finite initial segments of $\mathbf{H}_\mathrm{m} \cap \mathbf{RE}$ are exactly the finite distributive lattices (in Section 8).

By Lemma 2.15(ii) these results suffice to classify all initial segments of $\mathbf{H}_\mathrm{m}$.
We have also obtained the following result with no assumption.

- If $|\Sigma| = 1$ then the theory of $\mathbf{H}_\mathrm{m}$ is undecidable.
- There exists an incomplete r.e. degree that has no strong minimal cover (see the end of Section 6 for the definition of strong minimal cover). This implies that $\mathbf{H}_\mathrm{m} \cap \mathbf{RE}$ is not elementarily equivalent to the r.e. m-degrees.
- Index sets cannot be hpT-minimal.
- There exists recursive sets that are hmto-minimal. In fact, there exists such sets of arbitrarily high time complexity.

4. Honest polynomial partitions

In this paper the following scenario will happen often. We are assuming P = NP, there are sets A and B such that $A \leqslant^\mathrm{h}_\mathrm{m} B$ by function f, and we want to show that $B \leqslant^\mathrm{h}_\mathrm{m} A$. If f^{-1} exists then $B \leqslant^\mathrm{h}_\mathrm{m} A$ since f^{-1} will be an honest polynomial reduction (to prove this, use the honesty of f and the P = NP assumption). Unfortunately f^{-1} need not exist. However, we will construct a partition on Σ^* such that for all y there is an element z in the same part as y such that $f^{-1}(z)$ exists. Hence, in some sense, f^{-1} exists when viewed as a mapping on the parts of the partition. The partition will have properties that enable us to prove $B \leqslant^\mathrm{h}_\mathrm{m} A$.

In this section we define honest polynomial partitions and prove several lemmas about them. These lemmas will be the key to obtaining initial segments of the hpm-degrees.

Notation 4.1. If Π is a partition then $\Pi(x)$ is the set of elements in the same part as x, and $\mu\Pi(x)$ is the least element of $\Pi(x)$.

Definition 4.2. Let $B \subseteq \Sigma^*$ be a set in P. Π is an *honest polynomial partition of B* (henceforth, 'hp-partition') if

(a) there exists a polynomial p such that for all $x, y \in B$ one can determine whether $y \in \Pi(x)$ in time $p(|x|)$;

(b) there exists a polynomial q such that for all $x \in B$, $q(|\mu\Pi(x)|) \geqslant |x|$ (this is the polynomial honesty).

p and q are called *the polynomials associated with* Π.

Definition 4.3. If Π is a partition of B and A is a set, then A *respects* Π if for every $x \in B$ either $\Pi(x) \subseteq A$ or $\Pi(x) \subseteq \Sigma^* - A$.

Lemma 4.4 (P = NP). *If* $X \in \mathrm{P}$ *and* Π *is an hp-partition of* Σ^*, *then* $\bigcup_{x\in X} \Pi(x) \in \mathrm{P}$.

Proof. We have

$$z \in \bigcup_{x\in X} \Pi(x) \quad \text{iff} \quad (\exists x)[x \in X \wedge z \in \Pi(x)].$$

By part (b) of the definition of hp-partition there is a polynomial p such that the search for x can be restricted to $|x| \leqslant p(|z|)$. Formally:

$$z \in \bigcup_{x\in X} \Pi(x) \quad \text{iff} \quad (\exists x)[|x| \leqslant p(|z|) \wedge x \in X \wedge z \in \Pi(x)].$$

Hence the question of $z \in \bigcup_{x\in X} \Pi(x)$ reduces to an NP question. Since we are assuming P=NP this question can be determined in polynomial time. □

Lemma 4.5 (P = NP). *Let* $\Sigma^* = B \cup C \cup D$ *be a partition of* Σ^* *such that* B, C, *and* D *are in* P. *Let* A *be a set such that* $C \subseteq A$ *and* $D \subseteq \Sigma^* - A$. *Let* $e \in \mathbb{N}$. *If there exists an hp-partition* Π *of* B *that* A *respects such that, for every* $x \in B$, $\Pi(x) \cap V_e \neq \emptyset$, *then* $A \equiv^{\mathrm{h}}_{\mathrm{m}} \Theta^A_e$.

Proof. Let Π be the partition given in the hypothesis, and let p and q be the polynomials associated with Π. By definition, f_e runs in p_e steps and is q_e-honest.

The following algorithm computes an $\leqslant^{\mathrm{h}}_{\mathrm{m}}$ reduction g of A to Θ^A_e.

ALGORITHM
1. Input(x).
2. If $x \in C$, then output(YES) and halt. If $x \in D$, then output(NO) and halt.
3. Using P = NP find a string y such that $f_e(y) \in \Pi(x)$. (We later show that y with $|y| \leqslant q_e(q(|x|))$ exists, so P = NP can be used.)
4. Output(y).

END OF ALGORITHM

Since Π respects A, for all $x \in B$

$$x \in A \Leftrightarrow \Pi(x) \subseteq A \Leftrightarrow f_e(y) \in A.$$

Since $\Theta^A_e \leqslant^{\mathrm{h}}_{\mathrm{m}} A$ by f_e

$$y \in \Theta^A_e \Leftrightarrow f_e(y) \in A.$$

Combining these two facts yields

$$x \in A \Leftrightarrow y \in \Theta^A_e.$$

Hence g reduces A to Θ_e^A. It remains to show that g is an $\leqslant_m^h$ reduction.

The only step in the algorithm for g that is not obviously polynomial time is step 3. Since $\Pi(x) \cap V_e \neq \emptyset$, y exists; but we have to show that $|y|$ is bounded by a polynomial. We show that $|y| \leqslant q_e(q(|x|))$. Since f_e is $q_{\bar{e}}$ honest,

$$|y| \leqslant q_e(|f_e(y)|).$$

Since Π is an hp-partition with associated polynomials p and q, and $f_e(y) \in \Pi(x)$,

$$|f_e(y)| \leqslant q(|\mu(\Pi(x))|) \leqslant q(|x|).$$

Since q_e is a strictly increasing function,

$$q_e(|f_e(y)|) \leqslant q_e(q(|x|)).$$

Combining these inequalities we obtain

$$|y| \leqslant q_e(|f_e(y)|) \leqslant q_e(q(|x|)).$$

Lastly we show that g is honest. Let $x \in B$ and $g(x) = y \in \Sigma^*$. Since $f_e(y) \in \Pi(x)$,

$$q(|f_e(y)|) \geqslant |x|.$$

Since p_e bounds the complexity of f_e, $p_e(|y|) \geqslant |f_e(y)|$, hence

$$q(p_e(|y|)) \geqslant q(|f_e(y)|).$$

Combining these inequalities yields $q(p_e(|y|)) \geqslant |x|$. Hence g is $(q \circ p_e)$-honest. □

Lemma 4.6 (P = NP). *Let $\Sigma^* = B \cup C \cup D$ be a partition of Σ^* into three parts which are in* P. *Let T be some set in* P. *Let A be a set such that $C \subseteq A$ and $D \subseteq \Sigma^* - A$. Let $e \in \mathbb{N}$. If there exists an hp-partition Π of B that A respects such that*
(a) *for every $z \in \Sigma^*$ if $f_e(z) \in B$ then $\Pi(f_e(z)) \cap T \neq \emptyset$,*
(b) *for every $y \in B \cap T$, $\Pi(y) \cap V_e \neq \emptyset$,*
then $A \cap T \equiv_m^h \Theta_e^A$.

Proof. $A \cap T \leqslant_m^h \Theta_e^A$ by a modification of the algorithm in Lemma 4.5. During step 3, instead of looking for y such that $f_e(y) \in \Pi(z)$, look for y such that $f_e(y) \in \Pi(z) \cap T$.

$\Theta_e^A \leqslant_m^h A \cap T$ by the following algorithm

ALGORITHM
1. Input(z).
2. If $f_e(z)$ = YES or $f_e(z) \in C$, then output(YES). If $f_e(z)$ = NO or $f_e(z) \in D$, then output(NO).
3. (We know that $f_e(z) \in B$ so, by condition a, $\Pi(f_e(z)) \cap T \neq \emptyset$.) Find $y \in \Pi(f_e(z)) \cap T$. This can be done in polynomial time since we are assuming P = NP and $|y|$ is clearly bounded by a polynomial in $|z|$.
4. Output(y).

END OF ALGORITHM □

5. Embedding finite structures into $\mathbf{H}_\mathrm{m}$

We show that (assuming P = NP) any finite chain is an initial segment of $\mathbf{H}_\mathrm{m}$ (the case of embedding the three-chain, which is simpler, can be found in [4]). We then show that (assuming P = NP) any topped finite distributive lattice is an initial segment of $\mathbf{H}_\mathrm{m}$; moreover, these are exactly the topped finite initial segments of $\mathbf{H}_\mathrm{m}$. Using techniques of [9], the proofs of all theorems in this section could be adapted to the $|\Sigma| = 1$ case, omitting the P = NP assumption.

We use a modification of exptally sets; these sets have been used for constructing minimal honest degrees by Homer, Long, and Ambos-Spies [2,9].

Definition 5.1. Let $g(0) = 1$ and for all $m \geqslant 0$, $g(m+1) = 2^{g(m)}$. A set A is *exptally* if $A \subseteq \{0^{g(m)} \mid m \in \mathbb{N}\}$. Let p be a fixed polynomial. Define

$$E^p = \{0^{g(m)+j} \mid m \in \mathbb{N},\ 0 \leqslant j \leqslant p(m)\}.$$

Sets of the form E^p are called *poly-exptally*. For any fixed m the finite set

$$B^m = \{0^{g(m)+j} \mid 0 \leqslant j \leqslant p(m)\}$$

is called *the* mth *block of* E^p.

Note 5.2. We will later be partitioning E^p by partitioning every block of E^p. The sets that form the partition are called *boxes*. Each block will consist of a finite number of boxes.

Convention 5.3. Modify the machines $P_1, P_2, P_3, \ldots$ so that they are 0-1 valued. Let $L(P_i)$ denote the language recognized by P_i.

Theorem 5.4 (P = NP). *For any r, the r-chain is a finite initial segment of* $\mathbf{H}_\mathrm{m}$.

Proof. It will be simpler, notationally, to show that the $r+1$-chain is a finite initial segment of $\mathbf{H}_\mathrm{m}$.

Let $\langle L, \leqslant_L\rangle$ be the r-chain. We can assume that $L = \{0, \ldots, r\}$ and $b \leqslant_L c$ iff $b \leqslant c$. We show that L is an initial segment of $\mathbf{H}_\mathrm{m}$. Let p be a fixed polynomial such that for all n, $p(n) \geqslant n$.

For $b \in L$ let

$$S_a = \{0^{g(m)+j} \mid m \in \mathbb{N},\ 0 \leqslant j \leqslant p(m),\ j \equiv a \ (\mathrm{mod}\, r)\};$$

$$T_0 = \emptyset; \qquad T_b = S_1 \cup \cdots \cup S_b.$$

We construct $A \subseteq E^p$ such that the sets A^b (indexed by $b \in L$) defined by $A^b = A \cap T_b$ form an initial segment of $\mathbf{H}_\mathrm{m}$ that is isomorphic to L. For all $b, c \in L$ such that $b \leqslant_L c$ we have $A^b \leqslant^{\mathrm{h}}_{\mathrm{m}} A^c$ via

$$f(z) = \begin{cases} \mathrm{NO} & \text{if } z \notin T^b; \\ z & \text{otherwise.} \end{cases}$$

(Formally, to show that this reduction works, use that since $b \leqslant_L c$ we have $\{i \mid i \leqslant_L b\} \subseteq \{i \mid i \leqslant_L c\}$, and hence $T^b \subseteq T^c$.)

We construct $A \subseteq E^p$ in stages to satisfy the following requirements.

Separation Requirements. For every $e \in \mathbb{N}$, for every $b \in \{1, \ldots, r\}$

$R^1_{\langle e,b\rangle}$: f_e is not a reduction of A^b to A^{b-1}.

Discrete Requirements. For every $e \in \mathbb{N}$

R^2_e: $\Theta^A_e \in \mathrm{P}$ or, for some $b, \Theta^A_e \equiv^{\mathrm{h}}_{\mathrm{m}} A^b$.

At the end of each stage s we will have the following:

(a) A set $A_s \in \mathrm{P}$ which is the set of strings committed to A.

(b) A set $\hat{A}_s \in \mathrm{P}$ which is the set of strings committed to $\Sigma^* - A$.

(c) A set $B_s \in \mathrm{P}$ which is the set of strings in E^p that are not committed to A or $\Sigma^* - A$. Formally $B_s = E^p - (A_s \cup \hat{A}_s)$. We will not need to prove that $B_s \in \mathrm{P}$ since if $A_s, \hat{A}_s \in \mathrm{P}$ then clearly $B_s \in \mathrm{P}$. Let $B^m_s = B_s \cap B^m$.

(d) A partition Π_s of E^p such that the following hold.

  (i) Π_s is an hp-partition that respects A_s.

  (ii) For all $z \in E^p$ if $z \in B^m$ then $\Pi_s(z) \subseteq B^m$. This makes Π_s p-honest. Note that Π_s partitions each B^m_s.

  (iii) Let $a \in L - \{0\}$. A box is *a-pure* if it contains some element of T_a and is wholly contained in $\bigcup_{i \geqslant_L a} T_i$ (hence the box has no elements of $\bigcup_{i <_L a} T_a$). Let $N(a,s,m)$ be the number of a-pure boxes contained in B^m_s. We will have $\limsup_{m\to\infty} N(a,s,m) = \infty$.

During the construction, we show inductively that a, b, c, and d all hold. The partitions get coarser and coarser; however, if $z \in A_s$ or $z \in \hat{A}_s$, then for all $t \geqslant s$, $\Pi_t(z) = \Pi_s(z)$. If z enters A_s ($\hat{A}_s$) then we also place all elements of $\Pi_s(z)$ into A_s ($\hat{A}_s$); therefore the condition that Π_s respects A_s will easily be met.

For every $z \in E^p$, there is a stage s such that either $z \in A_s$ or $z \in \hat{A}_s$. The set A is defined as the set of all z that are placed in some A_s. The set A will respect all partitions Π_s.

If at stage $s+1$, A_s ($\hat{A}_s$, Π_s) is not mentioned, then $A_{s+1} := A_s$ ($\hat{A}_{s+1} := \hat{A}_s, \Pi_{s+1} := \Pi_s$). During a stage of the construction A_s ($\hat{A}_s$, Π_s) may change. To avoid notation, whenever A_s ($\hat{A}_s$, Π_s) is mentioned it is meant to be the most recent version of the object.

CONSTRUCTION

Stage 0: $A_0 := \emptyset$. $\hat{A}_0 := \Sigma^* - E^p$. For all z, $\Pi_0(z) = \{z\}$. Clearly (a), (b), (c), (d.i), (d.ii) are satisfied. Since we chose $p(n) \geqslant n$, (d.iii) is satisfied.

Stage $s+1$: There are two cases.

Case 1: $s+1$ *is odd*. Let $s = 2\langle e,b\rangle + 1$. We satisfy $R^1_{\langle e,b\rangle}$. Let z be the shortest element of $B_s \cap T_b$ that is in a b-pure box. (Such a z exists inductively by condition (d.iii).) Note that $z \notin A_s \cup \hat{A}_s$. There are four possibilities.

1. $f_e(z) \in A_s$ or $f_e(z) = \text{YES}$. Let $\hat{A}_{s+1} := \hat{A}_s \cup \Pi_s(z)$.
2. $f_e(z) \in \hat{A}_s$ or $f_e(z) = \text{NO}$. Let $A_{s+1} := A_s \cup \Pi_s(z)$.
3. $f_e(z) \notin (A_s \cup \hat{A}_s \cup \Pi_s(z))$. Let $A_{s+1} := A_s \cup \Pi_s(z)$ and $\hat{A}_{s+1} := \hat{A}_s \cup \Pi_s(f_e(z))$.
4. $f_e(z) \notin A_s \cup \hat{A}_s$ and $f_e(z) \in \Pi_s(z)$. Let $A_{s+1} := A_s \cup \Pi_s(z)$.

In possibilities 1,2, and 3, it is clear that $R^1_{\langle e,b\rangle}$ is satisfied. In possibility 4, note that $z \in A^b$, but since $\Pi_s(z)$ is b-pure, $f_e(z) \notin T_{b-1}$, so $f_e(z) \notin A^{b-1}$; hence $R^1_{\langle e,b\rangle}$ is satisfied.

Case 2: $s+1 = 2e+2$. We satisfy R^2_e. There are two possibilities. Let $NI(b,m)$ be the number of b-pure boxes contained in B^m_s that intersect range$(f_e) = V_e$.

Possibility 1: $(\forall a \in L - \{0\})(\exists n_a)[\limsup_{m\to\infty} NI(a,m) = n_a < \infty]$. Set

$$A_{s+1} := A_s \cup \bigcup_{z \in V_e \cap B_s} \Pi_s(z).$$

Since we are assuming P = NP, $V_e \in$ P. Inductively, Π_s is an honest partition, $B_s \in$ P, and $A_s \in$ P. Hence by Lemma 4.4 $A_{s+1} \in$ P (hence condition a is satisfied). Note that

$$\begin{aligned} z \in \Theta^A_e &\Leftrightarrow (f_e(z) \in A \text{ or } f_e(z) = \text{YES}) \\ &\Leftrightarrow (f_e(z) \in V_e \cap A \text{ or } f_e(z) = \text{YES}) \\ &\Leftrightarrow (f_e(z) \in A_{s+1} \text{ or } f_e(z) = \text{YES}). \end{aligned}$$

Since $A_{s+1} \in$ P we have $\Theta^A_e \in$ P, so the requirement is satisfied. Since $\hat{A}_{s+1} = \hat{A}_s$, $\Pi_{s+1} = \Pi_s$, by the induction hypothesis conditions (b), (c), (d.i), and (d.ii) hold. Since each B^m_s lost fewer than $\sum_{a=1}^{r} n_a$ blocks condition (d.iii) holds.

Possibility 2: $(\exists a \in L - \{0\})[\limsup_{m\to\infty} NI(a,m) = \infty]$. Let b be the largest number such that $\limsup_{m\to\infty} NI(b,m) = \infty$. We intend to set A_{s+1}, $\hat{A}_{s+1}$, and Π_{s+1} such that the following hold.

(a) For every $z \in \Sigma^*$ if $f_e(z) \in B_{s+1}$, then $f_e(z)$ is in the same box as some $y \in T_b$.
(b) For every $y \in B_{s+1} \cap T_b$, $\Pi_{s+1}(y) \cap V_e \neq \emptyset$.

By Lemma 4.6 these two conditions make $A^c = A \cap T_c \equiv^{\mathrm{h}}_{\mathrm{m}} \Theta^A_e$.

We set A_{s+1}, $\hat{A}_{s+1}$, and Π_{s+1} as follows. To satisfy condition (a) above, for every a such that $b <_L a$, place all the elements of every b-pure box that contains an element of V_e into A. Since $\limsup_{m\to\infty} NI(a,m) < \infty$ this will only delete a constant number of a-pure boxes from each B^m_s.

To satisfy condition (b) above is more complicated. First, place all elements of every b-pure box that does not intersect V_e into A. Now, for every m, there are exactly $NI(b,m)$ b-pure boxes contained in B^m_s. Now every b-pure box intersects V_e.

Second we will merge some boxes. Note that if $y \in T_b$ then there exists $a \leqslant_L b$ such that y is in an a-pure box. To ensure that every $y \in B_{s+1} \cap T_b$ is in a box that intersects V_e we will, for every $a \leqslant_L b$, merge (or put into A) every a-pure box with a b-pure box (all of which intersect V_e). Note that the a-pure boxes will remain a-pure, but the b-pure boxes will not remain b-pure. Hence we will set aside some b-pure boxes that will not be merged.

Let $m \in \mathbb{N}$. We describe what to do with the boxes that comprise B_s^m. Order these boxes by the least string in them. For every $d \in L$ and $i \in \mathbb{N}$ let $BOX[d,i]$ be the ith d-pure box (if it exists). Note that if $BOX[b,i]$ exists then it intersects V_e.

For every $a <_L b$ $(a \neq 0)$ we plan to merge some a-pure boxes with b-pure boxes. We need to ensure $\limsup_{m\to\infty} N(b,s+1,m) = \infty$. We plan to merge at most $NI(b,m)/2r$ b-pure boxes with a-pure boxes. Formally we do the following.

(1) Let $U_1 = NI(b,m)/2r$. Let $U = \min\{U_1, N(a,s,m)\}$.
(2) $(\forall i \leqslant U)$ merge $BOX[a,i]$ and $BOX[b,(a-1)U_1+i]$ (the new boxes are a-pure).
(3) $(\forall i > U)$ place all of the of elements of $BOX[a,i]$ into A_{s+1}.

When this process is done at most $|L-\{0\}| \times NI(b,m)/2r \leqslant NI(b,m)/2$ of the b-pure boxes have been merged. Hence there are at least $NI(b,m)/2$ b-pure boxes left.

Using $A_s \in \mathrm{P}$, $\hat{A}_s \in \mathrm{P}$, Π_s polynomial honest, and $\mathrm{P} = \mathrm{NP}$, one can show that $A_{s+1} \in \mathrm{P}$, $\hat{A}_{s+1} \in \mathrm{P}$, and Π_{s+1} is polynomial honest. We now show that, for all a,

$$\limsup_{m\to\infty} N(b,s+1,m) = \infty.$$

For a such that $b <_L a$ there is a constant n such that $N(a,s,m)+n \geqslant N(a,s+1,m)$. For $a <_L b$, $N(a,s+1,m) = \min\{NI(b,m)/2r, N(a,s,m)\}$. For $a=b$, $N(a,s+1,m)$ is at least $NI(b,m)/2$. For all three possibilities, by using $\limsup_{m\to\infty} N(a,s,m) = \infty$ and $\limsup_{m\to\infty} NI(a,m) = \infty$, we easily obtain $\limsup_{m\to\infty} N(a,s+1,m) = \infty$.
END OF CONSTRUCTION

By the comments made during the construction, it is clear that $\{A^b \mid b \in L\}$ form an $(r+1)$-chain of hpm-degrees. □

Theorem 5.5 (P = NP). *The topped finite initial segments of* $\mathbf{H}_\mathrm{m}$ *are exactly the finite distributive lattices.*

Proof. Assume that $\mathcal{D}$ is a topped finite initial segment of $\mathbf{H}_\mathrm{m}$. Then $\mathcal{D}$ is a finite distributive lattice by a proof similar to the same result for the classical m-degrees (see [24, p. 558, Corollary VI.1.10]).

Let $\mathcal{D} = \langle D, \leqslant_\mathcal{D}, \sqcap, \sqcup\rangle$ be a topped finite distributive lattice. We show that $\mathcal{D}$ is an initial segment of $\mathbf{H}_\mathrm{m}$. Let 0 denote the bottom element of $\mathcal{D}$. Let $I = \{1,\ldots,r\}$ be the join-irreducible elements of D (not including the bottom element). We will usually denote an element of I by a.

Let p be a fixed polynomial such that $p(n) \geqslant n$. For $a \in I$ and $b \in D$ let

$$S_a = \{0^{e(m)+j} \mid m \in \mathbb{N}, 0 \leqslant j \leqslant p(m), j \equiv a \pmod r\};$$

$$T_0 = \emptyset; \qquad T_b = \bigcup_{k \leqslant_\mathcal{D} b} S_k.$$

We construct $A \subseteq E^p$ such that the sets A^b (indexed by $b \in \mathcal{D}$) defined by $A^b = A \cap T^b$ form an initial segment of $\mathbf{H}_\mathrm{m}$that is isomorphic to $\mathcal{D}$. For all $b, c \in \mathcal{D}$ such that $b \leqslant_\mathcal{D} c$

we have $A^b \leqslant_m^h A^c$ via

$$f(z) = \begin{cases} \text{NO} & \text{if } z \notin T^b; \\ z & \text{otherwise.} \end{cases}$$

(Formally, to show that this reduction works, use that since $b \leqslant_{\mathcal{D}} c$ we have $\{i \mid i \leqslant_{\mathcal{D}} b\} \subseteq \{i \mid i \leqslant_{\mathcal{D}} c\}$, and hence $T^b \subseteq T^c$.)

We construct $A \subseteq E^p$ in stages to satisfy the following requirements. Let *code* be a 1-1 map from D to $\mathbb{N}$.

Separation Requirements. For $e \in \mathbb{N}$ and $b, c \in D$

$R_{\langle e, code(b), code(c)\rangle}$: If $b \not\leqslant_{\mathcal{D}} c$, then f_e is not a reduction of A^b to A^c.

Discrete Requirements. For every $e \in \mathbb{N}$

R_e^2: There exists a $b \in D$ such that $\Theta_e^A \equiv_m^h A^b$.

Note that if b is the bottom element of the lattice then $\Theta_e^A \in \mathrm{P}$, and if b is the top element then $\Theta_e^A \equiv_m^h A$.

At the end of each stage s we will have the following:

(a) A set $A_s \in \mathrm{P}$ which is the set of strings committed to A.
(b) A set $\hat{A}_s \in \mathrm{P}$ which is the set of strings committed to $\Sigma^* - A$.
(c) A set $B_s \in \mathrm{P}$ which is the set of strings in E^p that are not committed to A or $\Sigma^* - A$. Formally $B_s = E^p - (A_s \cup \hat{A}_s)$. We will not need to prove that $B_s \in \mathrm{P}$ since if $A_s, \hat{A}_s \in \mathrm{P}$ then clearly $B_s \in \mathrm{P}$. Let $B_s^m = B_s \cap B^m$.
(d) A partition Π_s of E^p such that the following hold.
   (i) Π_s is an hp-partition that respects A.
   (ii) For all $z \in E^p$ if $z \in B^m$ then $\Pi_s(z) \subseteq B^m$. This makes Π_s p-honest. Note that Π_s partitions each B_s^m.
   (iii) Let $a \in I$. A box is *a-pure* if it contains some element of T_a and is wholly contained in $\bigcup_{i \geqslant_{\mathcal{D}} a} T_i$ (hence it has no element of $\bigcup_{a \not\leqslant_{\mathcal{D}} i} T_i$). Let $N(a, s, m)$ be the number of a-pure boxes contained in B_s^m. We will have $\limsup_{m \to \infty} N(a, s, m) = \infty$.

During the construction, we show inductively that a, b, c, and d all hold. The partitions get coarser and coarser; however, if $z \in A_s$ or $z \in \hat{A}_s$, then for all $t \geqslant s$, $\Pi_t(z) = \Pi_s(z)$. If z enters A_s ($\hat{A}_s$) then we also place all elements of $\Pi_s(z)$ into A_s ($\hat{A}_s$); therefore the condition that Π_s respects A_s will easily be met.

For every $z \in E^p$, there is a stage s such that either $z \in A_s$ or $z \in \hat{A}_s$. The set A is defined as the set of all z that are placed in some A_s. The set A will respect all partitions Π_s.

If at stage $s+1$, A_s ($\hat{A}_s$, Π_s) is not mentioned, then $A_{s+1} := A_s$ ($\hat{A}_{s+1} := \hat{A}_s, \Pi_{s+1} := \Pi_s$). During a stage of the construction A_s ($\hat{A}_s$, Π_s) may change. To avoid notation, whenever A_s ($\hat{A}_s$, Π_s) is mentioned it is meant to be the most recent version of the object.

CONSTRUCTION

Stage 0: $A_0 := \emptyset$. $\hat{A}_0 := \Sigma^* - E^p$. For all z, $\Pi_0(z) = \{z\}$. Clearly (a), (b), (c), (d.i), (d.ii) are satisfied. Since we chose $p(n) \geqslant n$, (d.iii) is satisfied.

Stage $s+1$: There are two cases.

Case 1: $s+1$ *is odd.* If $s+1$ is not of the form $2\langle e, code(b), code(c)\rangle + 1$ for some $b, c \in D$ then go to stage $s+2$. If $s+1$ is of that form then let b, c be such that $s+1 = 2\langle e, code(b), code(c)\rangle + 1$. We satisfy $R_{\langle e,code(b),code(c)\rangle}$. If $b \leqslant_{\mathcal{D}} c$ then the requirement is satisfied. If $b \nleqslant_{\mathcal{D}} c$ then let $a \in I$ be such that $a \leqslant_{\mathcal{D}} b$ but $a \nleqslant_{\mathcal{D}} c$. (Such an a exists by Lemma 2.15(ii). Note that we are using that $\mathcal{D}$ is a distributive lattice.) Let z be the shortest element of $B_s \cap T_a$ that is in an a-pure box. (Such a z exists inductively by condition (d.iii).) Note that $z \notin A_s \cup \hat{A}_s$. There are four possibilities.

1. $f_e(z) \in A_s$ or $f_e(z) = \text{YES}$. Let $\hat{A}_{s+1} := \hat{A}_s \cup \Pi_s(z)$.
2. $f_e(z) \in \hat{A}_s$ or $f_e(z) = \text{NO}$. Let $A_{s+1} := A_s \cup \Pi_s(z)$.
3. $f_e(z) \notin (A_s \cup \hat{A}_s \cup \Pi_s(z))$. Let $A_{s+1} := A_s \cup \Pi_s(z)$ and $\hat{A}_{s+1} := \hat{A}_s \cup \Pi_s(f_e(z))$.
4. $f_e(z) \notin A_s \cup \hat{A}_s$ and $f_e(z) \in \Pi_s(z)$. Let $A_{s+1} := A_s \cup \Pi_s(z)$.

In possibilities 1,2, and 3, it is clear that $R^1_{\langle e,code(b),code(c)\rangle}$ is satisfied. In possibility 4, note that $z \in A^b$. We show that $f_e(z) \notin A^c$. Since $\Pi_s(z)$ is a-pure and $f_e(z) \in \Pi_s(z)$, $f_e(z) \in \bigcup_{i \geqslant_{\mathcal{D}} a} T_i$. Since $c \ngeqslant_{\mathcal{D}} a$, $f_e(z) \notin T_c$ so $f_e(z) \notin A^c$. Hence $R^1_{\langle e,code(b),code(c)\rangle}$ is satisfied.

Case 2: $s+1$ *is even.* Let $s+1 = 2e+2$. We satisfy R^2_e. There are two possibilities. Let $NI(a,m)$ be the number of a-pure boxes contained in B^m_s that intersect range$(f_e) = V_e$.
Possibility 1: $(\forall a \in I)(\exists n_a)[\limsup_{m\to\infty} NI(a,m) = n_a < \infty]$. Set

$$A_{s+1} := A_s \cup \bigcup_{z \in V_e \cap B_s} \Pi_s(z).$$

Since we are assuming P = NP, $V_e \in$ P. Inductively, Π_s is an honest partition, $B_s \in$ P, and $A_s \in$ P. Hence by Lemma 4.4 $A_{s+1} \in$ P (hence condition (a) is satisfied). Note that

$$\begin{aligned} z \in \Theta^A_e &\Leftrightarrow (f_e(z) \in A \text{ or } f_e(z) = \text{YES}) \\ &\Leftrightarrow (f_e(z) \in V_e \cap A \text{ or } f_e(z) = \text{YES}) \\ &\Leftrightarrow (f_e(z) \in A_{s+1} \text{ or } f_e(z) = \text{YES}). \end{aligned}$$

Since $A_{s+1} \in$ P we have $\Theta^A_e \in$ P, so the requirement is satisfied. Since $\hat{A}_{s+1} = \hat{A}_s$, $\Pi_{s+1} = \Pi_s$, by the induction hypothesis conditions (b), (c), (d.i), and (d.ii) hold. Since each B^m_s lost fewer than $\sum_{a=1}^r n_a$ blocks condition (d.iii) holds.

Possibility 2: $(\exists a \in I)[\limsup_{m\to\infty} NI(a,m) = \infty]$. Let

$$J = \{a \in I : \limsup_{m\to\infty} NI(a,m) = \infty\};$$

$$b = \bigsqcup_{\{a \in J\}} a; \qquad J' = \{a \in I : a \leqslant_{\mathcal{D}} b\} - J.$$

We intend to set A_{s+1}, $\hat{A}_{s+1}$, and Π_{s+1} such that the following hold.

(a) For every $z \in \Sigma^*$ if $f_e(z) \in B_{s+1}$, then $f_e(z)$ is in the same box as some $y \in T_b$.
(b) For every $y \in B_{s+1} \cap T_b$, $\Pi_{s+1}(y) \cap V_e \neq \emptyset$.

By Lemma 4.6 these two conditions make $A^b = A \cap T_b \equiv_{\mathrm{m}}^{\mathrm{h}} \Theta_e^A$.

We set A_{s+1}, $\hat{A}_{s+1}$, and Π_{s+1} as follows. To satisfy condition (a) above, for every $a \in I$ such that $a \not\leqslant_{\mathcal{D}} b$ place all the elements of every a-pure box that contains an element of range(f_e) into A. Since $\limsup_{m\to\infty} NI(a,m) < \infty$ this will only delete a constant number of a-pure boxes from each B_s^m.

To satisfy condition (b) above is more complicated. First, for every $a \in J$, place all elements of every a-pure box that does not intersect V_e into A. Now, for every $a \in J$, for every m, there are exactly $NI(a,m)$ a-pure boxes contained in B_s^m.

Second we will merge some boxes. If $y \in T_b$ then there exists $a \in I$ such that $a \leqslant_{\mathcal{D}} b$ and y is in an a-pure box. Note that $a \in J \cup J'$. If $a \in J$ then $\Pi_s(y) \cap V_e \neq \emptyset$ and we are done. Hence we need only look at $a \in J'$. To ensure that every $y \in B_{s+1} \cap T_b$ is in a box that intersects V_e we will, for every $a \in J'$, find a $c \in J$ such that $a \leqslant_{\mathcal{D}} c$, and merge (or put into A) every a-pure box with some c-pure box (all of which intersect V_e). Such a c exists by the Lemma 2.15. Note that the a-pure boxes will remain a-pure, but the c-pure boxes will not remain c-pure. Hence we will set aside some c-pure boxes that will not be merged.

Let $m \in \mathbb{N}$. We describe what to do with the boxes that comprise B_s^m. Order these boxes by the least string in them. For every $d \in D$ and $i \in \mathbb{N}$ let $BOX[d,i]$ be the ith d-pure box (if it exists). Note that for $c \in J$ if $BOX[c,i]$ exists then it intersects V_e.

For every $a \in J'$ $(a \neq 0)$ we plan to merge some a-pure boxes with c-pure boxes. We need to ensure $\limsup_{m\to\infty} N(c,s+1,m) = \infty$. We plan to merge at most $NI(c,m)/2r$ c-pure boxes with a-pure boxes. Formally we do the following.

(1) Let $U_1 = NI(c,m)/2r$. Let $U = \min\{U_1, N(a,s,m)\}$.
(2) $(\forall i \leqslant U)$ merge $BOX[a,i]$ and $BOX[c,(a-1)U_1+i]$ (the new boxes are a-pure).
(3) $(\forall i > U)$ place all of the of elements of $BOX[a,i]$ into A_{s+1}.

When this process is done at most $|L - \{0\}| \times NI(b,m)/2r \leqslant NI(b,m)/2$ of the c-pure boxes have been merged. Hence there are at least $NI(b,m)/2$ c-pure boxes left.

Using $A_s \in \mathrm{P}$, $\hat{A}_s \in \mathrm{P}$, Π_s polynomial honest, and P = NP, one can show that $A_{s+1} \in \mathrm{P}$, $\hat{A}_{s+1} \in \mathrm{P}$, and Π_{s+1} is polynomial honest. We now show that, for all a,

$$\limsup_{m\to\infty} N(c,s+1,m) = \infty.$$

For a such that $c <_{\mathcal{D}} a$ there is a constant n such that $N(a,s,m) + n \geqslant N(a,s+1,m)$. For $a <_{\mathcal{D}} c$, $N(a,s+1,m) = \min\{NI(c,m)/2r, N(a,s,m)\}$. For $a = c$, $N(a,s+1,m)$ is at least $NI(c,m)/2$. For all three possibilities, by using $\limsup_{m\to\infty} N(a,s,m) = \infty$ and $\limsup_{m\to\infty} NI(a,m) = \infty$, we easily obtain $\limsup_{m\to\infty} N(a,s+1,m) = \infty$.
END OF CONSTRUCTION

By the comments made during the construction, it is clear that $\{A^b \mid b \in D\}$ form an initial segment of $\mathbf{H}_{\mathrm{m}}$. □

6. Embedding infinite structures into $\mathbf{H}_\mathrm{m}$

We use the same conventions and notation as in Section 5. We show that (assuming P = NP) any countable linear order with minimal element is an initial segment of $\mathbf{H}_\mathrm{m}$. We then show that (assuming P = NP) any direct limit of ascending sequences of finite distributive lattice is an initial segment of $\mathbf{H}_\mathrm{m}$; moreover, these are exactly the countable initial segments of $\mathbf{H}_\mathrm{m}$. Using techniques of [9], the proofs of all theorems in this section could be adapted to the $|\Sigma| = 1$ case, omitting the P = NP assumption.

Theorem 6.1 (P = NP). *If $\langle L, \leqslant_L \rangle$ is a countable linear ordering with a minimum element, then L is an initial segment of* $\mathbf{H}_\mathrm{m}$.

Proof. This proof is similar in spirit to the proof of Theorem 5.4. We cannot choose the T_b's ahead of time because we do not know what the linear order looks like.

We may assume that L has a maximal element since, if not, we simply append one. Since L is countable, we can (noneffectively) list the elements of L as $\{0,1,2,\ldots\}$ (the order $\leqslant_L$ need not be related to the order $\leqslant$). We assume that 0 is the minimum element and 1 is the maximum element. Let L_s be the numbers $\{0,1,\ldots,s\}$ under the ordering $\leqslant_L$.

We will construct a set A and sets $T_0, T_1, \ldots$ such that the sets A^b (indexed by $b \in L$) defined by $A^b = A \cap T^b$ form an initial segment of $\mathbf{H}_\mathrm{m}$ isomorphic to L. For all $b, c \in L$ such that $b \leqslant_L c$ we have $A^b \leqslant^\mathrm{h}_\mathrm{m} A^c$ via

$$f(z) = \begin{cases} \text{NO} & \text{if } z \notin T^b; \\ z & \text{otherwise.} \end{cases}$$

(Formally, to show that this reduction works, use that since $b \leqslant_D c$ we have $\{i \mid i \leqslant_D b\} \subseteq \{i \mid i \leqslant_D c\}$, and hence $T^b \subseteq T^c$.)

We need $A, T_0, T_1, \ldots$ to satisfy the following requirements.

Separation Requirements. For every $b, c \in \mathbb{N}$

$R^1_{\langle e,b,c\rangle}$: if $c <_L b$ then f_e is not a reduction of A^b to A^c.

Discrete Requirements. For every $e \in \mathbb{N}$

R^2_e: there exists b such that $\Theta^A_e \equiv^\mathrm{h}_\mathrm{m} A^b$.

At the end of each stage s we will have the following:

(a) A set $A_s \in$ P which is the set of strings committed to A.
(b) A set $\hat{A}_s \in$ P which is the set of strings committed to $\Sigma^* - A$.
(c) A set $B_s \in$ P which is the set of strings in E^p that are not committed to A or $\Sigma^* - A$. Formally $B_s = E^p - (A_s \cup \hat{A}_s)$. We will not need to prove that $B_s \in$ P since if $A_s, \hat{A}_s \in$ P then clearly $B_s \in$ P. Let $B^m_s = B_s \cap B^m$.
(d) Sets $T_0, \ldots, T_s$ such that the following hold.

(i) $T_0 = \emptyset$ and $T_1 = E^p$.
(ii) $(\forall b)[T_b \in \mathrm{P}]$.
(iii) $(\forall b, c \leqslant s)[b \leqslant_L c \Rightarrow T_b \subseteq T_c]$
(iv) $(\forall b, c \leqslant s)[b <_L c \Rightarrow \limsup_{m\to\infty} |B_s^m \cap (T_c - T_b)| = \infty]$.

(e) A partition Π_s of E^p such that the following hold.
(i) Π_s is an hp-partition. Π_s respects A_s and, for every $b \leqslant s$, respects T_b.
(ii) For all $z \in E^p$ if $z \in B^m$ then $\Pi_s(z) \subseteq B^m$. This makes Π_s p-honest. Note that Π_s partitions each B_s^m.
(iii) Let $a \leqslant s$. A box is (a,s)*-pure* if it contains some element of T_a and is wholly contained in $\bigcup_{i \geqslant_{\mathcal{D}} a, i \leqslant s} T_i$ (hence it has no element of $\bigcup_{i \leqslant_{\mathcal{D}} a, i \leqslant s} T_i$). Let $N(a,s,m)$ be the number of (a,s)-pure boxes contained in B_s^m. We will have $\limsup_{m\to\infty} N(a,s,m) = \infty$.

During the construction, we show inductively that (a), (b), (c), (d) and (e) all hold. The partitions get coarser and coarser; however, if $z \in A_s$ or $z \in \hat{A}_s$, then for all $t \geqslant s$, $\Pi_t(z) = \Pi_s(z)$. If z enters A_s $(\hat{A}_s)$ then we also place all elements of $\Pi_s(z)$ into A_s $(\hat{A}_s)$; therefore the condition that Π_s respects A_s will easily be met.

For every $z \in E^p$, there is a stage s such that either $z \in A_s$ or $z \in \hat{A}_s$. The set A is defined as the set of all z that are placed in some A_s. The set A will respect all partitions Π_s.

If at stage $s+1$, A_s $(\hat{A}_s, \Pi_s)$ is not mentioned, then $A_{s+1} := A_s$ $(\hat{A}_{s+1} := \hat{A}_s, \Pi_{s+1} := \Pi_s)$. During a stage of the construction A_s $(\hat{A}_s, \Pi_s)$ may change. To avoid notation, whenever A_s $(\hat{A}_s, \Pi_s)$ is mentioned it is meant to be the most recent version of the object.

CONSTRUCTION

Stage 1: $A_0 := \emptyset$. $\hat{A}_0 := \Sigma^* - E^p$. For all x, $\Pi_0(z) = \{z\}$. $T_0 = \emptyset$ and $T_1 = E^p$. Clearly (a), (b), (c), (d), (e.i), (e.ii) are satisfied. Since we chose $p(n) \geqslant n$, (e.iii) is satisfied.

Stage $s+1$: Let $a, b \leqslant s$ be the elements adjacent to $s+1$ such that $a <_L s+1 <_L b$ (since 0 is minimal and 1 is maximal, such a, b exist). Let T_{s+1} be a set such that the following hold.
(1) $T_{s+1} \in \mathrm{P}$, $T_a \subseteq T_{s+1} \subseteq T_b$.
(2) Π_s respects T_b.
(3) $\limsup_{m\to\infty} |B_s^m \cap (T_{s+1} - T_a)| = \infty$.
(4) $\limsup_{m\to\infty} |B_s^m \cap (T_b - T_{s+1})| = \infty$.
Such a T_{s+1} exists by the induction hypothesis on T_a and T_b.

There are two cases.

Case 1: $s+1$ *is odd*. Let $s+1 = 2\langle e,i,j\rangle + 1$. (The function $\langle -,-,-\rangle$ is such that $(\forall b,c,e)[b,c \leqslant 2\langle e,b,c\rangle]$, so T_b and T_c have been defined before stage $s+1$.) We satisfy $R^1_{\langle e,b,c\rangle}$ in this stage. If $b \leqslant_L c$ then the requirement is satisfied and no more action need be taken. Otherwise we do as follows. Let z be the shortest element of $B_s \cap T_b$ that is in an (b,s)-pure box. (Such a z exists inductively by condition (e.iii).) Note that $z \notin A_s \cup \hat{A}_s$. There are four possibilities.

1. $f_e(z) \in A_s$ or $f_e(z) = \text{YES}$. Let $\hat{A}_{s+1} := \hat{A}_s \cup \Pi_s(z)$.
2. $f_e(z) \in \hat{A}_s$ or $f_e(z) = \text{NO}$. Let $A_{s+1} := A_s \cup \Pi_s(z)$.
3. $f_e(z) \notin (A_s \cup \hat{A}_s \cup \Pi_s(z))$. Let $A_{s+1} := A_s \cup \Pi_s(z)$ and $\hat{A}_{s+1} := \hat{A}_s \cup \Pi_s(f_e(z))$.
4. $f_e(z) \notin A_s \cup \hat{A}_s$ and $f_e(z) \in \Pi_s(z)$. Let $A_{s+1} := A_s \cup \Pi_s(z)$.

In possibilities 1,2, and 3, it is clear that $R^1_{\langle e,b,c\rangle}$ is satisfied. In possibility 4, note that $z \in A^b$, but since $\Pi_s(z)$ is (b,s)-pure, and $c <_L b$, $f_e(z) \notin T_c$, so $f_e(z) \notin A^c$; hence $R^1_{\langle e,b,c\rangle}$ is satisfied.

Stage $s+1 = 2e+2$ (Satisfy R^2_e): There are two possibilities. For $i \leqslant s$ let $NI(b,m)$ be the number of (b,s)-pure boxes contained in B^m_s that intersect range$(f_e) = V_e$.

Possibility 1: $(\forall a \in L_{s+1})(\exists n_a)[\limsup_{m\to\infty} NI(a,m) = n_a < \infty]$. Set

$$A_{s+1} := A_s \cup \bigcup_{z \in V_e \cap B_s} \Pi_s(z).$$

Since we are assuming P = NP, $V_e \in$ P. Inductively, Π_s is an honest partition, $B_s \in$ P, and $A_s \in$ P. Hence by Lemma 4.4 $A_{s+1} \in$ P (hence condition a is satisfied). Note that

$$\begin{aligned} z \in \Theta^A_e &\Leftrightarrow (f_e(z) \in A \text{ or } f_e(z) = \text{YES}) \\ &\Leftrightarrow (f_e(z) \in V_e \cap A \text{ or } f_e(z) = \text{YES}) \\ &\Leftrightarrow (f_e(z) \in A_{s+1} \text{ or } f_e(z) = \text{YES}). \end{aligned}$$

Since $A_{s+1} \in$ P we have $\Theta^A_e \in$ P, so the requirement is satisfied. Since $\hat{A}_{s+1} = \hat{A}_s$, $\Pi_{s+1} = \Pi_s$, by the induction hypothesis conditions (b), (c), (d.i), and (d.ii) hold. Since each B^m_s lost fewer than $\sum_{c=1}^{r} n_c$ blocks condition (d.iii) holds.

Possibility 2: $(\exists a \in L_{s+1})[\limsup_{m\to\infty} NI(a,m) = \infty]$. Let b be the largest number such that $\limsup_{m\to\infty} NI(b,m) = \infty$.

The rest of the construction is similar to the construction in Theorem 5.4 except that instead of using '$2r$' we use '$2s$'.

END OF CONSTRUCTION

By the comments made during the construction, it is clear that $\{A^b : i \in L\}$ form an initial segment of $\mathbf{H}_\mathrm{m}$ that is isomorphic to L. □

Theorem 6.2. *The topped initial segments of* $\mathbf{H}_\mathrm{m}$ *are exactly the direct limits of ascending sequences of distributive lattices.*

Proof. We combine the techniques used in Theorems 5.5 and 6.1.

Assume that $\mathcal{D}$ is a topped initial segment of $\mathbf{H}_\mathrm{m}$. Then $\mathcal{D}$ is the limit of an ascending sequence of finite distributive lattice by a proof similar to the same result for the classical m-degrees (see [24, p. 561, Corollary VI.1.13]).

Let $\mathcal{D} = \langle D, \leqslant_\mathcal{D}, \sqcap, \sqcup\rangle$ be the limit of an ascending sequence of finite distributive lattices $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots$. Let $\mathcal{D}_j$ have base set D_i. We show that $\mathcal{D}$ is an initial segment of $\mathbf{H}_\mathrm{m}$. Let the indecomposable elements of D_s be I_s. The elements of I_s are indecomposable in $\mathcal{D}_s$ but might end up not being indecomposable in $\mathcal{D}$. Note that I_s may grow or shrink from stage to stage. Let $I = \lim_{s\to\infty} I_s$. We can assume that the top element of $\mathcal{D}$ is indecomposable (if not then put another element on top of it) and that both the top and bottom element of D are in D_0.

Let $p(n)$ be a polynomial such that $p(n) \geqslant n$. We will construct a set $A \subseteq E^p$ and sets $T_0, T_1, \ldots$ such that the sets A^b (indexed by $b \in \mathcal{D}$) defined by $A^b = A \cap T^b$ form an initial segment of $\mathbf{H}_\mathrm{m}$ that is isomorphic to $\mathcal{D}$. For all $b, c \in \mathcal{D}$ such that $b \leqslant_{\mathcal{D}} c$ we have $A^b \leqslant^\mathrm{h}_\mathrm{m} A^c$ via

$$f(z) = \begin{cases} \mathrm{NO} & \text{if } z \notin T^b; \\ z & \text{otherwise.} \end{cases}$$

(Formally, to show that this reduction works, use that since $b \leqslant_{\mathcal{D}} c$ we have $\{i \mid i \leqslant_{\mathcal{D}} b\} \subseteq \{i \mid i \leqslant_{\mathcal{D}} c\}$, and hence $T^b \subseteq T^c$.)

We construct $A \subseteq E^p$ in stages to satisfy the following requirements. Let *code* be a 1-1 map from D to $\mathbb{N}$.

Separation Requirements. For $e \in \mathbb{N}$ and $b, c \in D$

$R_{\langle e, code(b), code(c)\rangle}$: If $b \nleqslant_{\mathcal{D}} c$, then f_e is not a reduction of A^b to A^c.

Discrete Requirements. For every $e \in \mathbb{N}$

R^2_e: There exists a $b \in D$ such that $\Theta^A_e \equiv^\mathrm{h}_\mathrm{m} A^b$.

Note that if b is the bottom element of the lattice then $\Theta^A_e \in \mathrm{P}$, and if b is the top element then $\Theta^A_e \equiv^\mathrm{h}_\mathrm{m} A$.

At the end of each stage s we will have the following:

(a) A set $A_s \in \mathrm{P}$ which is the set of strings committed to A.
(b) A set $\hat{A}_s \in \mathrm{P}$ which is the set of strings committed to $\Sigma^* - A$.
(c) A set $B_s \in \mathrm{P}$ which is the set of strings in E^p that are not committed to A or $\Sigma^* - A$. Formally $B_s = E^p - (A_s \cup \hat{A}_s)$. We will not need to prove that $B_s \in \mathrm{P}$ since if $A_s, \hat{A}_s \in \mathrm{P}$ then clearly $B_s \in \mathrm{P}$. Let $B^m_s = B_s \cap B^m$.
(d) Sets $T_0, T_1, \ldots$ (the number of T_i is $|\mathcal{D}_s|$) such that the following hold.
  (i) $T_0 = \emptyset$ and $T_1 = E^p$.
  (ii) $(\forall b)[T_b \in \mathrm{P}]$.
  (iii) $(\forall b, c \leqslant s)[b \leqslant_D c \Rightarrow T_b \subseteq T_c]$
  (iv) $(\forall b, c \leqslant s)[b <_D c \Rightarrow \limsup_{m\to\infty} |B^m_s \cap (T_c - T_b)| = \infty]$.
(e) A partition Π_s of E^p such that the following hold.
  (i) Π_s is an hp-partition. Π_s respects A, and, for all $b \leqslant s$, Π_s respects T_b.
  (ii) For all $z \in E^p$ if $z \in B^m$ then $\Pi_s(z) \subseteq B^m$. This makes Π_s p-honest. Note that Π_s partitions each B^m_s.
  (iii) Let $a \leqslant I_s$. A box is (a,s)-*pure* if it contains some element of T_a and is wholly contained in $\bigcup_{i \geqslant_{\mathcal{D}} a, i \leqslant s} T_i$ (hence it has no element of $\bigcup_{a \nleqslant_{\mathcal{D}} i, i \leqslant s} T_i$). Let $N(a,s,m)$ be the number of (a,s)-pure boxes contained in B^m_s. If $a \in I$ then we will have $\limsup_{m\to\infty} N(a,s,m) = \infty$.

During the construction, we show inductively that (a), (b), (c), and (d) all hold. The partitions get coarser and coarser; however, if $z \in A_s$ or $z \in \hat{A}_s$, then for all $t \geqslant s$,

$\Pi_t(z) = \Pi_s(z)$. If z enters A_s ($\hat{A}_s$) then we also place all elements of $\Pi_s(z)$ into A_s ($\hat{A}_s$); therefore the condition that Π_s respects A_s will easily be met.

For every $z \in E^p$, there is a stage s such that either $z \in A_s$ or $z \in \hat{A}_s$. The set A is defined as the set of all z that are placed in some A_s. The set A will respect all partitions Π_s.

If at stage $s+1$, A_s ($\hat{A}_s$, Π_s) is not mentioned, then $A_{s+1} := A_s$ ($\hat{A}_{s+1} := \hat{A}_s, \Pi_{s+1} := \Pi_s$). During a stage of the construction A_s ($\hat{A}_s$, Π_s) may change. To avoid notation, whenever A_s ($\hat{A}_s$, Π_s) is mentioned it is meant to be the most recent version of the object.

CONSTRUCTION

Stage 1: $A_0 := \emptyset$. $\hat{A}_0 := \Sigma^* - E^p$. For all x, $\Pi_0(z) = \{z\}$. $T_0 = \emptyset$ and $T_1 = E^p$. Clearly (a), (b), (c), (d), (e.i), (e.ii) are satisfied. Since we chose $p(n) \geqslant n$, (e.iii) is satisfied.

Stage $s+1$: We first create sets T_x for elements of $x \in D_{s+1} - D_s$. We will do this for each x in turn in some arbitrary order. We will let E denote the elements of $D_{s+1} - D_s$ that we have already dealt with. Initially $E = \emptyset$.

For every $x \in D_{s+1} - D_s$ do the following:

(1) Let

$$\begin{aligned} SMALL &= \{a \in D_s \cup E \mid c <_D x\}, \\ LARGE &= \{b \in D_s \cup E \mid x <_D c\}, \\ INC &= (D_s \cup E) - (SMALL \cup LARGE), \\ E &= E \cup \{x\}. \end{aligned}$$

(2) Let T_x be a set such that the following hold.
- $T_x \in$ P.
- T_x respects the partition Π_s.
- $\bigcup_{a \in SMALL} T_c \subseteq T_x \subseteq \bigcup_{b \in LARGE} T_x$.
- $T_x \cap \bigcup_{c \in INC} T_c = \emptyset$.
- If $x \in I_{s+1}$ and $a \in SMALL$ then $\limsup_{m\to\infty} |B_s^m \cap (T_x - T_a)| = \infty$.
- If $b \in I_{s+1}$ then $\limsup_{m\to\infty} |B_s^m \cap (T_b - T_x)| = \infty$.

Such a T_x exists by induction hypothesis which is condition (e).

The rest of this construction is virtually identical to Theorem 5.5 using D_s (I_s) instead of D (I).

END OF CONSTRUCTION □

Having characterized exactly which finite and countable lattices are initial segments of $\mathbf{H}_m$, our next goal is to examine uncountable structures. It is here that the similarity between $\mathbf{H}_m$ and the m-degrees might fail.

Definition 6.3. Let $\langle X, \leqslant \rangle$ be any partial order. An element $x \in X$ has a *$\leqslant$-minimal cover* if there exists $y \in X$ such that $x < y$ and

$$(\forall z \in X)[z < y \Rightarrow z \not\geqslant x].$$

An element $x \in X$ has a *strong* $\leqslant$*-minimal cover* if there exists $y \in X$ such that $x < y$ and

$$(\forall z \in X)[z < y \Rightarrow z \leqslant x].$$

The first step towards characterizing the uncountable structures that are initial segments of the (classical) m-degrees is showing that every m-degree has a strong $\leqslant$-minimal cover. An analogous theorem in $\mathbf{H}_{\mathrm{m}}$ is not known. Moreover, the question of whether or not $\mathbf{H}_{\mathrm{m}}$ is elementarily equivalent to the m-degrees is open.

7. The unary theory of $\mathbf{H}_{\mathrm{m}}$ is undecidable

Theorem 7.1. *Assume* $|\Sigma| = 1$. *The theory of* $\mathbf{H}_{\mathrm{m}}$ *is undecidable.*

Proof. Let $\mathcal{D}$ be any finite distributive lattice. We claim that there exists hpm-degrees **a** and **b** such that the hpm-degrees between **a** and **b** form a partial order isomorphic to $\mathcal{D}$.

The proof of Theorem 5.5 can easily be modified to show that if $|\Sigma| = 1$ then that any finite distributive lattice can be embedded in the hpm-degrees. Let B be the top element mapped to. Let **a** be the 0 hpm-degree and **b** be the hpm-degree of B. Hence the claim is established.

The rest of the proof is similar to the proof that the theory of the Turing degrees is undecidable [12] (see [17, pp. 136–137]). We include it for completeness.

Let $\mathcal{L}$ be the language consisting of all the usual logic symbols, and the additional symbol $\leqslant$. Any sentence in $\mathcal{L}$ is a sentence about partial orders. Let DL be the set of all sentences of $\mathcal{L}$ that are true in all distributive lattices; let *FINITE* be the set of all sentences of $\mathcal{L}$ that are true in all finite distributive lattices. Ershov and Taitslin [7] showed that there is no recursive set R such that $DL \subseteq R \subseteq FINITE$.

We show that if the theory of $\mathbf{H}_{\mathrm{m}}$ is decidable then such a recursive set R exists. Let $\theta(x, y)$ be the formula $\mathcal{L}$ that says that the set of elements z such that $x \leqslant z \leqslant y$ form a distributive lattice. Let ψ be any sentence of $\mathcal{L}$. Let $\psi'(x, y)$ be the formula obtained from ψ by restricting all quantifiers in ψ to elements z such that $x \leqslant z \leqslant y$. Let ψ'' be the sentence $(\forall x, y)[\theta(x, y) \Rightarrow \psi'(x, y)]$. Let $H = \{\psi \mid \mathbf{H}_{\mathrm{m}} \models \psi''\}$. Since, by the claim above, every finite distributive lattice is between some two hpm-degrees, $DL \subseteq H \subseteq FINITE$. Hence H cannot be recursive, so the theory of $\mathbf{H}_{\mathrm{m}}$ must be undecidable. □

8. The structure of $\mathbf{H}_{\mathrm{m}} \cap \mathbf{RE}$

We examine the structure $\mathbf{H}_{\mathrm{m}} \cap \mathbf{RE}$. The r.e. m-degrees and $\mathbf{H}_{\mathrm{m}} \cap \mathbf{RE}$ resemble each other in the same way the m-degrees and $\mathbf{H}_{\mathrm{m}}$ do; however, they are *not* elementarily equivalent. We first prove a theorem about the resemblance, and then about the difference.

Lachlan [14,24] showed that the topped finite initial segments of the r.e. m-degrees are exactly the finite distributive lattices. We show this holds for $\mathbf{H}_{\mathrm{m}} \cap \mathbf{RE}$.

Theorem 8.1 (P = NP). *The topped finite initial segments of* $\mathbf{H}_{\mathrm{m}} \cap \mathbf{RE}$ *are exactly the finite distributive lattices.*

Proof sketch. Combine the techniques of Theorem 6.1 with the e-state construction of a maximal r.e. set (see [28] for an e-state construction in recursion theory, and see [2] for a modification used in complexity theory). The proof resembles Lachlan's proof cited above. □

Lachlan [15,24] showed that every incomplete r.e. m-degree has an r.e. strong minimal $\leqslant_m$-cover (see the end of Section 6 for a definition). We show that the analogous theorem for $\mathbf{H}_{\mathrm{m}} \cap \mathbf{RE}$ does not hold, which shows that the r.e. m-degrees are not elementarily equivalent to $\mathbf{H}_{\mathrm{m}} \cap \mathbf{RE}$.

Theorem 8.2. *There exists an incomplete r.e. degree that has no strong* $\leqslant_{\mathrm{m}}^{\mathrm{h}}$*-minimal cover.*

Proof sketch. We construct r.e. sets $A, B, Q_1, Q_2, \ldots$ to satisfy the following requirements:

$$P_e: \quad \neg(B \leqslant_{\mathrm{m}}^{\mathrm{h}} A \text{ by } M_e),$$
$$N_e: \quad A <_{\mathrm{m}}^{\mathrm{h}} W_e \Rightarrow (Q_e <_{\mathrm{m}}^{\mathrm{h}} W_e \text{ and } Q_e \not\leqslant_{\mathrm{m}}^{\mathrm{h}} A).$$

The N_e requirements ensure that A has no strong $\leqslant_{\mathrm{m}}^{\mathrm{h}}$-minimal cover. Since the degree P has a strong minimal cover, this also establishes that $A \notin \mathrm{P}$. The P_e requirements ensure that A is incomplete.

We break N_e into two subrequirements some of which may entail an infinite number of requirements.

Requirement R_e^1: $A <_{\mathrm{m}}^{\mathrm{h}} W_e \Rightarrow Q_e <_{\mathrm{m}}^{\mathrm{h}} W_e$.

The condition $Q_e \leqslant_{\mathrm{m}}^{\mathrm{h}} W_e$ will hold since Q_e will be constructed by a Ladner-style "looking back technique", [1,16,18] so Q_e will be "W_e with holes in it". Hence $\neg[Q_e <_{\mathrm{m}}^{\mathrm{h}} W_e]$ is equivalent to $Q_e \equiv_{\mathrm{m}}^{\mathrm{h}} W_e$, and R_e^1 is equivalent to the following statements:

$$\neg[Q_e <_{\mathrm{m}}^{\mathrm{h}} W_e] \Rightarrow \neg[A <_{\mathrm{m}}^{\mathrm{h}} W_e];$$
$$[Q_e \equiv_{\mathrm{m}}^{\mathrm{h}} W_e] \Rightarrow \neg[A <_{\mathrm{m}}^{\mathrm{h}} W_e];$$
$$[Q_e \equiv_{\mathrm{m}}^{\mathrm{h}} W_e] \Rightarrow \neg[A <_{\mathrm{m}}^{\mathrm{h}} Q_e];$$
$$[Q_e \equiv_{\mathrm{m}}^{\mathrm{h}} W_e] \Rightarrow [(\forall i)\neg[A \leqslant_{\mathrm{m}}^{\mathrm{h}} Q_e \text{ by } M_i] \vee [A \equiv_{\mathrm{m}}^{\mathrm{h}} Q_e].$$

To satisfy R_e^1, it suffices to satisfy the following requirements

$$N_{\langle e,i\rangle}: \quad \neg(A \leqslant_{\mathrm{m}}^{\mathrm{h}} Q_e \text{ by } M_i).$$

Requirement R_e^2: $A <_{\mathrm{m}}^{\mathrm{h}} W_e \Rightarrow Q_e \not\leqslant_{\mathrm{m}}^{\mathrm{h}} A$.

To satisfy this requirement, it suffices to satisfy the following statements, which we henceforth refer to as R_e^2:

$$Q_e \leqslant_{\mathrm{m}}^{\mathrm{h}} A \Rightarrow W_e \leqslant_{\mathrm{m}}^{\mathrm{h}} A.$$

The requirements P_e and $N_{\langle e,i\rangle}$ are satisfied by diagonalization, which may involve putting elements into A, or restraining elements from entering A. Requirement R_e^2 is satisfied by trying to code W_e into A. These requirements appear to conflict. The conflict is *not* resolved by priority. Instead, when diagonalizing, we try to make Q_e look like W_e so that the $Q_e \leqslant_m^h A$ hypothesis of requirement R_e^2 will help to get $W_e \leqslant_m^h A$.

We order the requirements by $P_0, N_{\langle 0,0\rangle}, P_1, N_{\langle 0,1\rangle}, \ldots$. This is the order they will be acted on. R_e^2 is not in the ordering because it will not be satisfied by an overt action.

We attempt to code W_e into A by trying to put $\langle 1^e, \tau\rangle$ into A as soon as τ enters W_e.

The sets Q_e are defined by a Ladner-style looking back construction. In particular, a 0-1 valued polynomial time function $f(-,-)$ with domain $\mathbb{N} \times \mathbb{N}$ in unary form will be constructed, and the sets $\{Q_e\}_{e=1}^{\infty}$ will be defined by

$$\sigma \in Q_e \Leftrightarrow (f(e,|\sigma|) = 1 \text{ and } \sigma \in W_e).$$

The construction is slowed down to allow f to be computed in polynomial time.

At the end of each stage s, we will have the following:

(a) A_s, the strings committed to A, and $\hat{A}_s$, the strings committed to $\Sigma^* - A$;

(b) for each e, the values of $f(e,0), f(e,1), \ldots, f(e,s-1)$ (but no more);

(c) implicit information about the sets of type Q_e, since we know some values of $f(e,n)$.

If at stage $s+1$, A_s ($\hat{A}_s$, B_s, $\hat{B}_s$) is not mentioned, then $A_{s+1} := A_s$ ($\hat{A}_{s+1} := \hat{A}_s$, etc.).

CONSTRUCTION

Stage 0: $A_0 := \emptyset$, $\hat{A}_0 := \emptyset$. $B_0 := \emptyset$, $\hat{B}_0 := \emptyset$. For all x, $f(e,n)$ is undefined (hence nothing is known about Q_e). For all e and b both P_e and $N_{\langle e,i\rangle}$ are labeled NOT SATISFIED.

Stage $s+1$: First, we work on the R_e^2 requirements for $e \leqslant s$. For all $e \leqslant s$, if there is a string $\sigma \in W_{e,s+1} - W_e$ and $\langle 1^e, \sigma\rangle \notin \hat{A}_s$, then place $\langle 1^e, \sigma\rangle$ into A_{s+1}. Since $W_{e,s}$ is defined as the set of elements of $\{0, 1, \ldots, s\}$ that $M_{e,s}$ halts on, computing $W_{e,s}$ (with $e \leqslant s$) can be done in time polynomial in s.

Second, we work on the least P_e or $N_{\langle e,i\rangle}$ type requirement that is not satisfied.

Case 1: The requirement is P_e. For all i, set $f(i,s) = 1$ (this is done to help the requirements R_i^2). For s steps, do the following: try to find an x such that (i) $x \notin B_s \cup \hat{B}_s$, (ii) either $M_e(x) \in \{\text{YES}, \text{NO}\}$ or $M_e(x) \in A_s \cup \hat{A}_s$, and (iii) if $M_e(x) \in \Sigma^*$ and is of the form $\langle 1^i, \sigma\rangle$, then $f(i,|\sigma|) = 1$ (this will enable $Q_e \leqslant_m^h A$ to help obtain $W_e \leqslant_m^h A$).

Such an x must exist since M_e is honest and $A_s \cup \hat{A}_s$ is finite (though we may not find x at this stage). If such an x is found, then

(a) if $M_e(x) = \text{YES}$ then $\hat{B}_{s+1} := \hat{B}_s \cup \{x\}$;

(b) if $M_e(x) = \text{NO}$ then $B_{s+1} := B_s \cup \{x\}$; and

(c) if $M_e(x) \in \Sigma^*$ then $B_{s+1} := B_s \cup \{x\}$ and $\hat{A}_{s+1} := \hat{A}_s \cup \{M_e(x)\}$.

In either case, declare P_e SATISFIED.

Case 2: The requirement is $N_{\langle e,i\rangle}$. Set $f(e,s) = 0$. For all $i \neq e$ set $f(i,s) = 1$ (this is done to help the requirements R_i^2). For s steps do the following: try to find an $x =$

$\langle 0, 1^{s+1}\rangle$ such that (i) $x \notin A_s \cup \hat{A}_s$, and (ii) $M_e(x) \in \{\text{YES}, \text{NO}\}$ or $f(e, |M_e(x)|) = 0$. Such an x must exist since M_e is honest, $A_s \cup \hat{A}_s$ is finite, and if we work on this requirement for quite some time then a long consecutive set of values of $f(e, \text{-})$ are being set to 0. (The x need not be found at this stage.) If such an x is found then

(a) if $M_e(x) = \text{YES}$ then $\hat{A}_{s+1} := \hat{A}_s \cup \{x\}$;
(b) if $M_e(x) = \text{NO}$ then $A_{s+1} := A_s \cup \{x\}$; and
(c) if $M_e(x) \in \Sigma^*$ then $A_{s+1} := A_s \cup \{x\}$ and since $f(e, |M_e(x)|) = 0$ we know that $M_e(x) \notin Q_e$.

In any case we declare $N_{\langle e,i\rangle}$ to be satisfied.
END OF CONSTRUCTION

For all e ($\langle e, i\rangle$), the work done on requirements P_e ($N_{\langle e,i\rangle}$) eventually leads to it being satisfied since the x that is being sought must exist.

We show that for all b, R_i^2 is satisfied. First, we note that when a string σ enters $W_{i,s}$ and is *not* coded into A, then since $\langle 1^i, \sigma\rangle$ is restrained from A, $f(i, |\sigma|) = 1$. Hence, assuming $\sigma \in W_i$, if $f(i, |\sigma|) = 0$, then the string $\langle 1^i, \sigma\rangle$ was placed into A.

If $Q_i \leqslant_{\rm m}^{\rm h} A$ by h, then we obtain $W_i \leqslant_{\rm m}^{\rm h} A$ as follows:

Given σ, compute $f(i, |\sigma|)$. If $f(i, |\sigma|) = 1$, then ($\sigma \in Q_i$ iff $\sigma \in W_e$). Hence $\sigma \in W_i$ iff $h(\sigma) \in A$. If $f(i, |\sigma|) = 0$, then ($\sigma \in W_e$ iff $\langle 1^i, \sigma\rangle \in A$). □

It is an open question whether, in the above theorem, we can replace 'strong minimal cover' with 'minimal cover'. This is of interest because if there exists a set A such that A has no minimal cover, then the theorem "There exists a minimal hm-set" would not relativize to A. (For more on honesty and relativization, see [8].) It is known (see [5]) that there is an r.e. set A which is incomplete with respect to wtt-reductions, and has no r.e. minimal cover in the $\leqslant_{\rm T}^{\rm h}$ degrees.

It is also unknown which infinite structures can be embedded in $\mathbf{H}_{\rm m} \cap \mathbf{RE}$. The analogous question for the r.e. m-degree has been answered: every effective distributive lattice can be embedded [3,22].

9. Index sets are not hpT-minimal

We show index sets are nonminimal. To do this, we need a convention about our programming system.

Convention 9.1. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ denote an acceptable programming system such that the s-m-n functions [25] are computable in polynomial time. This is reasonable if the machine model is similar to real programs, in which case the s-m-n functions merely replace a read statement with an assignment statement.

Definition 9.2. A set A is an *index set* if whenever φ_i and φ_j are the same partial recursive function, then either $i, j \in A$ or $i, j \notin A$.

The proof of the following theorem resembles the proof of Rice's Theorem [25].

Theorem 9.3. *If A is a nontrivial index set (i.e. $A \neq \emptyset$, $A \neq \Sigma^*$), then A is not hpT-minimal.*

Proof. Let $a \in A$ and $b \notin A$. Let C be a recursive set such that $C \notin \mathrm{P}$ and φ_i be a recursive function that computes the characteristic function of C. Let z be such that

$$\varphi_z(x,y) = \begin{cases} \varphi_a(y) & \text{if } x \in C; \\ \varphi_b(y) & \text{if } x \notin C. \end{cases}$$

Since we assume the s-m-n functions to be polynomial-time computable, we have a function $f \in \mathrm{P}$ such that for all x and y, $\varphi_z(x,y) = \varphi_{f(z,x)}(y)$.

$$x \in C \Rightarrow \varphi_{f(z,x)} \equiv \varphi_a \Rightarrow f(z,x) \in A.$$
$$x \notin C \Rightarrow \varphi_{f(z,x)} \equiv \varphi_b \Rightarrow f(z,x) \notin A.$$

The function f is polynomial time computable and honest; therefore, $C \leqslant_{\mathrm{m}}^{\mathrm{h}} A$. Since C is recursive and A is not recursive, it is not the case that $A \leqslant_{\mathrm{T}} C$. Hence, $C <_{\mathrm{m}}^{\mathrm{h}} A$ and thus, A is not an hpT-minimal set. □

10. A recursive superminimal set

The sets constructed in Section 4, and in all of the honest minimal degree literature, are nonrecursive. This is necessary since Ladner's Theorem says that recursive sets cannot be hpm-minimal. However, Ladner's Theorem does not say anything about hmto-minimal sets. Total honest m-reductions differ from honest m-reductions in a significant way. In the cases where an honest m-reduction maps a string to either YES or NO, the honesty condition does not come into play. By contrast, in the case of total honest m-reductions, the honesty condition always comes into play. Thus, intuitively, these reductions are different from each other. We prove this by showing that there exists a recursive set A, $A \notin \mathrm{P}$, that is hmto-minimal, and in fact is superminimal (i.e. for any $B \leqslant_{\mathrm{m}}^{\mathrm{h\text{-}to}} A$, $B \equiv_{\mathrm{m}}^{\mathrm{h\text{-}to}} A$.) We see this result as unnatural, which is evidence that hmto-reductions are unnatural.

Definition 10.1. Let g be as defined in Section 5 (first definition of that section). Let $B^m = \{x : g(m) \leqslant |x| < g(m+1)\}$. A set A is *blocktype* if it is the union of sets of the form B^m. Note that, for all m, if any element of B^m is in A, then all elements of B^m are in A (hence if some element of B^m is not in A, then no element of B^m is in A).

Lemma 10.2. *If A is blocktype and $C \leqslant_{\mathrm{m}}^{\mathrm{h\text{-}to}} A$, then $A \leqslant_{\mathrm{m}}^{\mathrm{h\text{-}to}} C$.*

Proof. Let $C \leqslant_{\mathrm{m}}^{\mathrm{h\text{-}to}} A$ by a total honest f. Let p and q be polynomials such that p bounds the runtime of f, and f is q-honest.

The reduction $A \leqslant_{\mathrm{m}}^{\mathrm{h\text{-}to}} C$ is as follows: on input y, find m such that $y \in B^m$, and then output $0^{q(g(m))}$.

We show

$$y \in A \Leftrightarrow f(0^{q(g(m))}) \in A \Leftrightarrow 0^{q(g(m))} \in C.$$

The second equivalence holds because f is a reduction of C to A. The first equivalence will follow if we show that $f(0^{q(g(m))}) \in B^m$, since A is blocktype.

Since f is computable in time p,

$$|f(0^{q(g(m))})| \leqslant p(q(g(m))) < 2^{g(m)} = g(m+1).$$

Since f is q-honest,

$$q(|f(0^{q(g(m))})|) \geqslant q(g(m)).$$

Therefore, $|f(0^{q(g(m))}|) \geqslant g(m)$. Hence,

$$g(m) \leqslant |f(0^{q(g(m))})| < g(m+1)$$

therefore, $f(0^{q(g(m))}) \in B^m$. □

Theorem 10.3. *Given any recursive function T, there exists a recursive set A such that $A \notin \mathrm{DTIME}(T(n))$ and A is hmto-minimal.*

Proof. It is easy to construct a blocktype set that is not in $\mathrm{DTIME}(T(n))$. By the above Lemma, that set is hmto-minimal. □

The techniques in this section can also be used to show that Ladner's proof cannot be extended to finite-to-1 reductions (honesty is not needed).

Acknowledgements

We would like to thank Tim Long for extensive discussions. We would also like to thank Amihood Amir, Valentina Harizanov, David Kamishlian, Clyde Kruskal, and Arun Sharma for proofreading and discussion. The second author would like to thank Carolyn Brophy for always being honest, though not always being in polynomial time. The first author was partially supported by a United States-New Zealand Cooperative Foundation grant. The second author was partially supported by NSF grants MCS 80-05386, MCS 82-00269, and MCS 80-10707.

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