On Genericity and Ershov's Hierarchy*

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Abstract

It is natural to wish to study miniaturisations of Cohen forcing suitable to sets of low arithmetic complexity. We consider extensions of the work of Schaeffer[9] and Jockusch and Posner[6] by looking at genericity notions within the Δ_2 sets.

Different equivalent characterisations of 1-genericity suggest different ways in which the definition might be generalised. There are two natural ways of casting the notion of 1-genericity: in terms of sets of strings and in terms of density functions, as we will see here. While these definitions coincide at the first level of the difference hierarchy, they turn out to differ at other levels. Furthermore, these differences remain when the remainder of the Δ^0_2 sets are considered.

While the string characterization of 1-genericity collapses at the second level of the difference hierarchy to 2-genericity, the density function definition gives a very interesting hierarchy at level ω and above. Both of these results point towards the deep similarities exhibited by the n-c.e. degrees for $n \geq 2$.

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1 Introduction

Genericity is the central notion of forcing, the main engine in set theory. It has also been used in complexity theory, distributed systems, computability theory, topology and many other areas.

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Odifreddi [7] provides a brief outline of the history of the study of genericity. Genericity was first seen in the literature in the 1950s, when Spector [10] and Friedburg [5] introduced it for the sake of controlling the jump of a set. It was later introduced into set theory by Cohen, who used it to prove the independence of the Axiom of Choice and of the Continuum Hypothesis. There are now an enormous number of applications.

The usual notion of genericity is too coarse to classify Δ^0_2 constructions. In this report we examine several miniaturisations suggested by different versions of density which are classically equivalent. We find it useful to study properties of generic sets of low complexity because the results tend to involve manipulations of correspondingly low complexities, yet are likely to be applicable to generic sets of high complexity.

The sets examined in this report fall on intermediate levels between 1-genericity and 2-genericity. The principal mechanism we use to define these intermediate genericities is to replace Σ_1 sets with various Δ^0_2 sets in the conditions used to characterise 1-generic sets. It is clear that the sets thus formed should fall somewhere in between the 1-generic sets and the 2-generic sets, but the precise computability results determined and reported here are frequently surprising.

Initially we will focus on traditional characterisations of genericity, showing how they might be expanded to give intermediate notions. Later, we will see how these notions interact with each other and with the established calibration points of the 1-generic and 2-generic sets in terms of computability. Most of the proofs shown are constructive, not only demonstrating that various classes of set are possible but showing how they might be built.

Informally, we concentrate upon the *string* characterization of genericity of Jockusch and Posner, and the equivalent one in terms of *density functions*. Principally we are concerned with the number of mind changes approximating the sets and functions in terms of the difference/Ershov hierarchy.

We show that for the string definition, everything collapses. If we are generic for 2-ce sets of strings then we are already generic for Σ_2 sets of strings, and are hence 2-generic.

For the density function approach, following Downey, Jockusch, and Stob [2], and Schaeffer [9], we get a very interesting hierarchy. We begin with 1-genericity, and then get a hierarchy within the total computable functions controlling the number of mind changes. In particular the notion of *pb*-genericity of Downey, et al. is refined. Beyond this we find levels ω, $\omega+1$, which are distinct, then collapse for all $n\omega+k$ for $k \geq 1$, with a distinct new level at ω^2.

It will be interesting to see how this material relates to Shaeffer's material on dynamic notions.

Some of this material formed part of the first author's Honours project at Victoria University.

We use standard notation, and let $\leq_L$ denote lexicographic ordering. We let z_n denote the n-th string in length/lex ordering. We let $\sigma \prec \tau$ denote that σ is a prefix (initial segment) of τ, and, as usual, λ denotes the empty string. For a string S, we let $S \restriction x$ be read as "the string consisting of the first x characters of S".

2 Preliminaries

We will be looking at functions computable in $\mathbf{0}'$. Thus it is natural to begin with the limit lemma.

Lemma 2.1 (Shoenfield) *$A \leq_T 0'$ iff there is a total computable function $f(x,y)$ such that*

1. *$f(x,y) \in \{0,1\}$ for all x, y*
2. *$\lim_s f(x,s)$ exists for all x*
3. *$\lim_s f(x,s) = \chi_A(x)$ for all x*

The Shoenfield limit lemma clearly allows us to define notions such as g-computable enumerability below $0'$.

Definition 2.2 *A set A is* g**-ce** *for some total function g if it has a computable approximation f_A such that $|\{s : f(x,s) \neq f(x,s+1)\}| \leq g(x)$ for all x.*

There are sets computable in $0'$ that are neither n-r.e. for any natural number n nor g-ce for any total computable function g. The **Ershov hierarchy**, based on ordinals, fills in the remainder of the Δ^0_2 sets. Recall that a notation for ordinals provides a computable ordering on a computable set of numbers, where 0, successors and limit ordinals can be computably distinguished and the predecessor and successor functions are computable.

Following ideas of Epstein, Haas and Kramer [3], and Ershov [4], we say that a function $f \leq_T 0'$ is α-ce iff for some notation for α, the number of changes for each $f(x)$ is bounded by some ordinal $\beta \leq \alpha$.

So, what does it mean for a function f to be $(n\omega + k)$-ce?

- If $n = 0$, then f is just k-ce as per the difference hierarchy
- If f is ω-ce then it is g-ce for some total computable g.
- It is probably simplest to illustrate the remaining cases with a concrete example, say a function f that is $2\omega + 3$-ce.

  In essence, f being $2\omega + 3$-ce means that the value of $f(x)$ can change at most three times before the ordinal giving the number of mind changes for $f(x)$ drops below the second limit point, to $\omega + j$ for some natural number j, and then at most j times before the ordinal drops again below the first limit point to some other natural number k whereupon $f(x)$ can change at most k more times. These transitions are illustrated in Figure 1. The exact values taken by j and k are determined by computable functions depending on f and x.

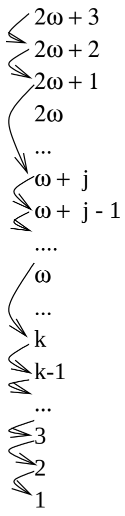

Figure 1: Number of mind changes permitted for the $2\omega + 3$-ce function f

Theorem 2.3 *A function $f \leq_T 0'$ iff f is ω^2-ce for some notation.*

Proof: This proof has a similar structure to a related proof of Epstein, Haas and Kramer [3], which in turn follows a proof of Ershov [4].
Suppose a function f is ω^2-ce. ω^2 is well-ordered, and each time the value of $f(x, \_)$ changes for any x the ordinal bounding the number of possible remaining changes for $f(x, \_)$ must decrease, so the number of changes possible for f must always be finite. Therefore, by the limit lemma, f is computable in $0'$.
Suppose conversely that a function g is computable in $0'$. Then by the limit lemma we know that the number of times $g(x, \_)$ changes for any x will be finite. We use this fact to construct a notation for ω^2 such that ω^2 bounds the number of changes for all $g(x, \_)$.
We firstly define a computable set of pairs

$$Y = \{\langle x, s\rangle : s = 0 \vee g(x, s) \neq g(x, s-1)\}$$

We say $\langle x, s\rangle \leq \langle y, t\rangle$ iff $x \leq y$ or $x = y \wedge s \geq t$. Note that for any $\langle x, s\rangle$ there will therefore be only finitely many $\langle y, t\rangle \in Y$ with $\langle y, t\rangle \leq \langle x, s\rangle$.
It behooves us to construct a predecessor function for successor ordinals and a fundamental sequence for each limit ordinal. We use $F_{\langle a,b\rangle}$ to denote a fundamental sequence that we build in stages for limit ordinal $\langle a, b\rangle$, with $F_{\langle a,b\rangle}(n)$ denoting the nth element in the sequence.
Now for all x:
At stage 0 declare $\langle x, 0\rangle$ a limit ordinal with $F_{\langle x,0\rangle}(0) = \langle x-1, 0\rangle$. From this point there are two possibilities for the value of $g(x, s)$ at each stage s.

1. $g(x, s) = g(x, s-1)$, then for each limit ordinal $\langle x, t\rangle$ so far declared in the construction for x (that is, each $\langle x, t\rangle \in Y$ such that $t < s$) continue building the fundamental sequence by setting $F_{\langle x,t\rangle}(s) = F_{\langle x,t\rangle}(s-1) + 1$.
2. $g(x, s) \neq g(x, s-1)$, then find the smallest limit ordinal $\langle x, t\rangle$ so far declared

in the construction for x (that is, the $\langle x,t\rangle \in Y$ with the greatest $t < s$) and interpolate a new limit ordinal $\langle x,s\rangle$ between $\langle x,t\rangle$ and $F_{\langle x,t\rangle}(s-1)$. Then set
$F_{\langle x,t\rangle}(s) = \langle x,s\rangle + 1$
$F_{\langle x,s\rangle}(s) = F_{\langle x,t\rangle}(s-1) + 1$ and $F_{\langle x,s\rangle}(u) = \langle x-1,0\rangle$ for all $0 \le u < s$
$F_{\langle x,v\rangle}(s) = F_{\langle x,v\rangle}(s-1) + 1$ for all $\langle x,v\rangle \in Y$ such that $v < t$
In this construction, the limit points consist of the $\langle y,t\rangle \in Y$ such that $\langle y,t\rangle \le \langle x,0\rangle$. There are finitely many such $\langle y,t\rangle$ for all $\langle x,0\rangle$, so there will always be finitely many limit points in the notation, each with ω successor ordinals between it and the next limit point.
Whether or not a given ordinal $\langle a,b\rangle$ is a limit ordinal can be computably determined by checking whether $g(a,b) = g(a,b-1)$. If it is a limit ordinal its computable fundamental sequence is $F_{\langle a,b\rangle}$ and if not then its predecessor ordinal is $\langle a,b\rangle - 1$.
The requirements for a notation for ω^2 bounding the number of mind changes for g are clearly met. □

3 1-genericity

There are two natural ways of casting the notion of 1-genericity: in terms of strings and in terms of density functions. Both definitions turn out to coincide at the first level of the difference hierarchy (as shown in Lemma 5.1) but differ at other levels (as shown by Theorems 4.2 and 5.4).

Recall that a set A is described as **1-generic** if it is Cohen generic for all 1-quantifier statements. This notion is discussed in depth in [6] and also in [7]. In [6], Jockusch gives an equivalent definition:

Definition 3.1 (Jockusch and Posner (1)) *A set A is 1-generic if or all e, either*

1. $(\exists\sigma)(\sigma \prec A \wedge \sigma \in V_e)$, *or*
2. $(\exists\alpha \prec A)(\forall\beta \in V_e)(\alpha \not\prec \beta)$

where V_e is the eth Σ^0_1 set of strings.

We will refer to this as the "string definition" of 1-genericity, which Schaeffer[9] rephrases as A meeting all sets of the form $\{\sigma : \sigma \in V_e \vee (\forall\tau \succeq \sigma)(\tau \notin V_e))\}$.

Schaeffer also sets out a separate notion of genericity based on **density functions**. This characterisation requires some new definitions.

Definition 3.2 (Schaeffer (1))

1. *A function g is a called a* **density function** *if for all σ, $\sigma \preceq g(\sigma)$.*
2. *A function g is called* **strongly ce over a** *(for some* **a***) if it is ce over* **a** *and there exists some approximation f_g to g where $f_g(\sigma,0) = \sigma$ for all σ.*

3. *A set A is said to* **meet** *a density function g if there exists σ such that $g(\sigma) \preceq A$.*

4. *A set A is γ-***generic** *for any set γ of density functions if it meets every density function in γ*

Schaeffer's characterisation of 1-genericity is then:

Definition 3.3 (Schaeffer (2)) *A set is 1-generic if and only if it meets all strongly ce density functions.*

We will refer to a variant of this as the "density definition" of 1-genericity.

In the examination that follows we hope to gain greater insight into the nature of genericity by extending the density and string definitions to the difference hierarchy. This gives new notions of genericity below $0'$.

4 dn-Generic Sets

If we now allow ourselves to recall that "ce" and "1-ce" are also synonymous, a new direction for generalisation suggests itself.

Definition 4.1 *A set B is dn-generic if for all e, either*

1. $(\exists\sigma)(\sigma \prec B \wedge \sigma \in V_e)$, *or*

2. $(\exists\alpha \prec B)(\forall\beta \in V_e)(\alpha \not\prec \beta)$

where V_e is the eth n-ce set of strings.

It is clear that $d1$-genericity is equivalent to 1-genericity. It is likewise clear that for all $m \geq 2$, all m-generic sets are dn-generic for all n:

1. A set generic with respect to all Σ_2 sets of strings is certainly generic with respect to any subset of the set of Δ^0_2 sets of strings, so all 2-generic sets are dn-generic for all n.

2. Any set that is $m+1$-generic is m-generic.

The main result of this section is that $d2$ genericity and 2-genericity coincide!

Theorem 4.2 *Let A be a $d2$-generic set. Then A is 2-generic.*

Proof: In order to prove this, we need to define the concepts of **cover**, **proper cover** and **properly n-ce**.

Definition 4.3
1. *A set S is a* **cover** *of another set A if $(\forall\sigma \prec A)(\exists\tau \in S)(\tau \succ \sigma)$.*
2. *A set S is a* **proper cover** *of another set A if S is a cover of A and $(\forall\sigma \in S)(\sigma \not\prec A)$. A 1-generic set is therefore one with no Σ^0_1 proper cover.*

We need to prove that every $d2$-generic set A satisfies
1. $(\exists\sigma)(\sigma \prec A \wedge \sigma \in S)$, or
2. $(\exists\sigma \prec A)(\forall\tau \in S)(\sigma \not\preceq \tau)$
for all Σ^0_2 sets of strings S.
We prove this by contradiction. Suppose there exists some $d2$-generic set A and Σ_2 set of strings S such that S properly covers A. Because S is Σ^0_2, we can specify that a string σ is in S iff $(\exists x)(\forall y)(\phi_S(\sigma, x, y))$ where ϕ_S is some total computable predicate.
We can regard a string α as being in S_t for some stage t if $(\exists t_0 < t)(\forall s < t)(\phi_S(\alpha, t_0, s))$ holds, and as not being in S at stage t' if $(\forall t_1 \leq t')(\exists s' \leq t')(\neg\phi_S(\alpha, t_1, s'))$ holds. If α is actually a member of S, there must be some finite stage at which α enters S and does not subsequently leave.
We will construct a 2-ce set S' as a series of stages S'_t with respect to two requirements for all binary strings σ:
$R_\sigma : (\sigma \in S) \Longrightarrow (\exists\tau \succeq \sigma)(\tau \in S')$
$N_\alpha : (\alpha \in S') \Longrightarrow (\exists\beta \preceq \alpha)(\beta \in S)$
We say R_σ requires attention at stage t if $\sigma \in S_t$ and $(\forall\tau \in S'_t)(\tau \not\succeq \sigma)$.
We say N_α requires attention at stage t if $\alpha \in S'_t$ and $(\forall\beta \in S_t)(\beta \not\preceq \alpha)$.
Note that at any stage t, S'_t is finite and so these conditions are computable.

- Initially set $S'_0 = \emptyset$, by setting $S'_0(\sigma) = 0$ for all σ.
- At stage t act on all R_α, N_β requiring attention with $\alpha, \beta \leq t$.
  - For the sake of R_α find the shortest $\sigma \in \{\alpha^\frown 0^n : n \geq 0\}$ such that $(\forall s < t)(S'_s(\sigma) = 0)$. Set $S'_t(\sigma) = 1$.
  - For the sake of N_β set $S'_t(\beta) = 0$.

$S' = \lim_t S'_t$ is thus a 2-ce set of strings. But now we have a contradiction, because S' must also be a proper cover of A, which cannot be the case since A is $d2$-generic. It must therefore be the case that no such Σ_2 set S properly covering any $d2$-generic set A can exist. $\square$

This result is somewhat counter-intuitive. The 2-ce sets are of relatively low complexity compared to the Σ^0_2 sets, yet string-definition genericity with respect to the 2-ce sets is already exactly as powerful as string-definition genericity with respect to the Σ^0_2 sets.

5 nb-generic sets

It is clear from the discussion of dn-genericity in the previous two sections that examining a different but equivalent definition of 1-genericity can indicate new directions in which it can be extended.

We now look at a new example of this phenomenon, based on density functions. Building on work of Downey, Jockusch and Stob [2] about pb-genericity, Schaeffer [9] proved a number of results involving various levels of genericity and their relationship to different sets of density functions. In particular, he shows that every set that meets every ce density function is 1-generic. We can show that the converse of this is also true.

Lemma 5.1 *Every 1-generic set meets all ce density functions.*

Proof: by contradiction. Suppose not. Then there exists a 1-generic set A and ce density function f such that A does not meet f, $(\forall\sigma)(f(\sigma) \not\prec A)$
Consider the proposition

$$P = (\forall\alpha \prec A)(\exists\beta \succeq \alpha)(|t : f(b,t) \neq f(b,t+1)| = 1)$$

Suppose P is false, so $(\exists\alpha \prec A)(\forall\beta \succeq \alpha)(|t : f(b,t) \neq f(b,t+1)| = 0)$. Then we could define a Σ^0_1 set of strings $V = \{\sigma : (\exists\tau)(f(\tau,0) = \sigma \wedge \tau \succeq \alpha)\}$. By the string definition of 1-genericity, A meets $\{\sigma : \sigma \in V \vee (\forall\tau \succeq \sigma)(\tau \notin V)\}$ and must do so via the first clause of the disjunction because every prefix of A will have an extension in V. But V is a subset of the range of f, so that means A meets f.
Conversely, suppose P is true. Then define a a Σ^0_1 set of strings
$W = \{\sigma : (\exists\tau,t)(f(\tau,t) \neq f(\tau,t+1) \wedge f(\tau,t+1) = \sigma)\}$. Again by the string definition of 1-genericity, A meets $\{\sigma : \sigma \in W \vee (\forall\tau \succeq \sigma)(\tau \notin W)\}$ and must do so via the first clause of the disjunction because every prefix of A will have an extension in W. But W is a subset of the range of f and so A meets f.
Therefore, A and f cannot exist, so every 1-generic set meets all ce density functions. □

Schaeffer proved that a set is 1-generic if and only if it meets all strongly ce density functions. What Lemma 5.1 shows us is that it is not necessary to require that the density functions be strongly ce.

Definition 5.2 (Density Definition of 1-genericity) *A set S is 1-generic if and only if it meets all ce density functions.*

In the same way as we did for the dn-generic sets, we now generate an extension of the density definition of 1-genericity by generalising the 1-ce density function to n-ce density functions:

Definition 5.3 *A set A is nb-**generic** iff for all n-ce density functions g, A meets g.*

It is clear that $1b$-genericity and 1-genericity are the same.

5.1 The nb-generic hierarchy is not proper

As with the dn-generic sets, and for much the same reasons, it appears intuitively that the nb-generic sets might form a proper hierarchy of inclusion. Also as with the dn-generic sets, this does not turn out to be the case. In fact, the nb-generic sets do not form even a bounded hierarchy.

Theorem 5.4 *All 1-generic sets are nb-generic for all n.*

PROOF: by induction, with the base case already demonstrated in Lemma 5.1. Assume all 1-generic sets are kb-generic for $1 \leq k \leq n$, consider $k = n+1$. We prove by contradiction that there does not exist a $(n+1)$-ce density function g and 1-generic set S such that S does not meet g. For suppose otherwise. Then consider the proposition

$$P = (\forall \sigma \prec S)(\exists \tau \succ \sigma)(|\{t : g(\tau, t) \neq g(\tau, t+1)\}| = n+1)$$

Suppose P is false. Then $(\exists \sigma \prec S)(\forall \tau \succ \sigma)(|\{t : g(\sigma, t) \neq g(\sigma, t+1)\}| \leq n)$. Consider the function:

$$g'(\gamma, t) = \begin{cases} \sigma \restriction (|\sigma| - 1)\widehat{\ }(1 - \sigma(|\sigma|) & \text{if } \gamma \prec \sigma \\ \gamma & \text{if } \gamma \not\prec \sigma \wedge \gamma \not\succeq \sigma \\ g(\gamma, t) & \text{otherwise} \end{cases}$$

g' is clearly n-ce. Yet, by its definition, g' is not met by S, contradicting the induction hypothesis.
Conversely, suppose P is true. Then we can define a set of strings W such that $s \in W$ iff $(\exists t_1 < t_2 < \ldots < t_{n+1}, \sigma)(g(\sigma, t_1) \neq g(\sigma, t_2) \neq \ldots \neq g(\sigma, t_{n+1}) = s)$. W is Σ^0_1 and all strings σ in S have an extension $\tau \succ \sigma$ such that $g(\tau)$ is in W. A is 1-generic and thus meets the set $\{\sigma : \sigma \in W \vee (\forall \tau \succeq \sigma)(\tau \notin W))\}$. But this cannot be via the second clause of the disjunction, since we have already shown that every prefix of A must have an extension in W, so it must be the case that some string $\sigma \prec A$ is a member of W, which by the definition of W means that $(\exists \gamma)(g(\gamma) \prec A)$ which means that A meets g — contradiction, it was incorrect to assume that A and g existed. □

5.2 Non-triviality of result

This result would be trivial if it were the case that
$(\forall m, n)(\{\{\text{ra } g\} : g \text{ a n-re density function}\} = \{\{\text{ra } f\} : f \text{ a m-re density function}\})$.
We can show that this is not the case.

Lemma 5.5 *For all n there is a $n+1$-ce density function whose range is not the range of any n-ce density function.*

PROOF: For all n we can enumerate the n-ce density functions in a straightforward way as shown in Figure 2. If $\varphi_i(\alpha, 0)$ halts then it does so at some finite stage t and for all $s \geq t$, $f_{\langle i,s \rangle}(\alpha, \_)$ will be simply a slowed down version of $\varphi_i(\alpha, \_)$ up to and including the nth mind-change.
Given this enumeration, we can diagonalise to construct a $(n+1)$-ce density function g by finite injury such that the range of g is not the range of any n-ce density function. In order to do this, we satisfy for all e
$R_e : (\exists \tau \preceq S_e)((\tau \in \text{ra } g \wedge \tau \notin \text{ra } f_e) \vee (\tau \notin \text{ra } g \wedge \tau \in \text{ra } f_e))$
where f_e is the eth ce density function and $S_e = 0^{e+1}\widehat{\ }1$
A priority ordering exists on R_e. We say R_e requires attention at stage s if
$(\forall \tau \preceq S_e)(f_e(\tau, s-1) = g(\tau, s-1))$
Now the construction is simple:

1. Set $g(\sigma, 0) = \sigma$ for all σ.

input: γ, i, s, n
$k = 0$
$t = 0$
$j = 0$
if $\varphi_i(\gamma, 0)$ halts in s steps **then**
  $f_{\langle i,s\rangle}(\gamma, 0) = \varphi_i(\gamma, 0)$
else
  $f_{\langle i,k\rangle}(\gamma, 0) = \gamma$
repeat
$k = k + 1$
if $|l < k - 1 : f_{\langle i,s\rangle}(\gamma, l)! = f_{\langle i,k\rangle}(\gamma, l + 1)| = n$ **then**
  $f_{\langle i,s\rangle}(\gamma, k) = f_{\langle i,s\rangle}(\gamma, k - 1)$
else if $g_i(\gamma, t)$ does not halt in j steps **then**
  $j = j + 1$
  $f_{\langle i,s\rangle}(\gamma, k) = f_{\langle i,s\rangle}(\gamma, k - 1)$
else $f_{\langle i,s\rangle}(\gamma, k) = g_i(\gamma, t)$
  $t = t + 1$
  $j = 0$

Figure 2: An algorithm for generating $f_{\langle i,s\rangle}$ from φ_i the ith partial computable function, $s \in \mathbb{N}$.

2. At each stage $s > 0$, choose the highest priority requirement R_e to require attention. If $f_e(S_e, s-1) = S_e$ then set $g(S_e, s) = S_e \hat{\ } 1$ else set $g(S_e, s) = S_e$. In both cases, for all $\tau \neq S_e$ set $g(\tau, s) = g(\tau, s - 1)$.

The diagonalisation ensures that the range of g differs from the range of each n-ce density function in turn, so the range of g is not the range of any n-ce density function. $\square$

Verification Lemmas

Lemma 5.6 *For all e, R_e acts finitely often and is met.*

Proof: by induction on e. Suppose for the sake of induction that we have reached the stage s_0 such that $(\forall s > s_0)(\forall j < e)R_j$ does not require attention at stage s.

Then claim R_e requires attention at most $n + 1$ more times. It can only require attention at any stage s if $(\forall \tau \preceq S_e)(f_e(\tau, s - 1) = g(\tau, s - 1))$. In particular this requires that $f_e(S_e, s - 1) = g(S_e, s - 1)$. By priorities, every time R_e requires attention after stage s it will be selected for satisfaction. By the construction, every time R_e is selected for satisfaction $g(S_e, s)$ is set such that
$g(S_e, s) \neq f_e(S_e, s - 1)$, so R_e can only subsequently require attention if
$(\exists t > s - 1)(f_e(S_e, t) \neq f_e(S_e, s - 1))$. The maximum number of times

R_e can require attention after stage s is thus $1 + |\{t > s-1 : f_e(S_e,t) \neq f_e(S_e,t-1)\}| = n+1$ because f_e is n-ce. □

Lemma 5.7 $(\forall\tau)(|\{s : g(\tau,s) \neq g(\tau,s+1)\}| \leq n+1)$.

PROOF: follows from the proof of Lemma 5.6. R_e only requires attention at stage s when $f_e(S_e,s-1) = g(S_e,s-1)$, and if it has priority and receives attention then $f_e(\tau,s) \neq g(\tau,s-1)$ so it can only require attention at a later stage if $(\exists t > s-1)(f_e(S_e,t) \neq f_e(S_e,s-1))$. The number of times the approximation to $f_e(S_e)$ changes is at most n, so the number of times that R_e will act to change the approximation to $g(S_e)$ is at most $n+1$. For τ not equal to any S_e, the approximation does not change at all. □

Comment 5.8 *The range of an n-ce density function is ω-ce for all n, with an upper bound on the number of membership changes for some string σ of $|\sigma|n$ for even n, $|\sigma|(n-1)+2$ for odd n.*

6 Extending the density definition

Theorem 5.4 shows us that when the density definition of genericity is extended to the difference hierarchy, the sets thus generated collapse into the 1-generic sets. A natural extension of this is to continue the examination at higher levels of the Ershov hierarchy.

We will continue using the same naming scheme, so that a set S is called $(n\omega+k)b$-generic if and only if it meets all $(n\omega+k)$-ce density functions. Epstein, Hass and Kramer[3] have shown that ω^2 is a limit in that every Δ^0_2 set is ω^2-ce, so there is no need to consider higher complexities than this.

Downey, Stob and Jockusch[2] define a *pb*-generic set as follows:

Definition 6.1 (Downey, Stob and Jockusch (1))

1. *A function f is pr-**ce** if there exists a primitive recursive g such that for all σ, $|\{s : f_s(\sigma) \neq f_{s+1}(\sigma)\}| \leq g(\sigma)$*
2. *A set A is pb-**generic** if it meets every pr-ce density function.*

The *pr*-ce functions therefore represent a strict subset of the ω-ce functions. It follows that the ωb-generic sets must represent a subset (though not necessarily strict) of the *pb*-generic sets, and that ωb-genericity is at least as complex as *pb*-genericity.

Downey, Stob and Jockusch further show that there are 1-generic sets that are not *pb*-generic, it follows directly that there are 1-generic sets that are not ωb-generic. This gives the first instance of strictly increasing strength we have seen when using the density definition of genericity with sets computable in $0'$.

The reasonable question to ask next is whether the $(n\omega+k)b$-generic sets collapse into the ωb-generic sets in the same way that the nb-generic sets have been shown to collapse into the 1-generic sets. We find that in fact they do not, but that they do collapse into the $(\omega+1)b$-generic sets.

```
input: e, i, α
t = 0
n = 0
changes = 0
p_⟨e,i⟩(α, 0) = α
s = 1
while changes < f_i(α)
    s = s + 1
    if φ_{e,n}(α, t) is in a halting state
      p_⟨e,i⟩(α, s) = φ_e(α, t)
      t = t + 1
      n = 0
      if t > 0 ∧ φ_e(α, t) ≠ φ_e(α, t − 1)
        changes = changes + 1
    else
      p_⟨e,i⟩(α, s) = p_⟨e,i⟩(α, s − 1)
      n = n + 1
repeat
    p_⟨e,i⟩(α, s) = p_⟨e,i⟩(α, s − 1)
    s = s + 1
```

Figure 3: An algorithm for generating $p_{e,i}(\alpha)$ from φ_e the eth partial computable function, f_i the ith primitive recursive function.

6.1 ωb-genericity and pb-genericity

In the introduction to this section we saw that the ωb-generic sets must be a subset of the pb-generic sets simply by their definitions, but were unable to determine directly from the definitions whether they are a strict subset. We can now show that the subset is strict.

Theorem 6.2 *There exists a pb-generic set that is not ωb-generic.*

PROOF: by construction. We construct by finite injury a pb-generic set A and concurrently construct a ω-ce density function g such that A does not meet g and thus is not ωb-generic. As we construct g we will construct a computable function m_g which bounds the mind changes for g.
There is no direct enumeration of the pr-ce density functions. However, we can enumerate $S_P = \{p_{\langle e,i\rangle} : \forall e, i\}$, as shown in Figure 3. It is clear from the construction of $\{p_{\langle e,i\rangle}$ that for all pr-ce density functions h, there exists some j and k such that the range of h is the same as the range of $p_{\langle j,k\rangle}$, and thus that a set that meets all members of S_P is pb-generic. The members of S_P have the added advantage of all being total.

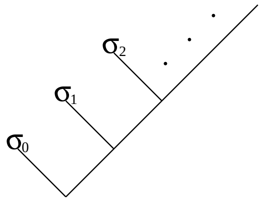

Figure 4: A set of strings where no σ_i is a prefix of any other.

We build A in stages A_s with respect to requirements R_e for all e:
$R_e : (\exists\alpha)(\lim_s p_e(\alpha, s) \prec A) \wedge (\forall\alpha \prec A \upharpoonright e)(|\{s : g(\alpha, s) \neq g(\alpha, s{+}1)\}| \leq m_g(\alpha))$
where p_e is as previously defined and m_g is total and computable. A standard priority ordering exists on R_e.
For the sake of R_e we choose a witness string σ_e that can change over time. We use $\sigma_{e,t}$ to denote the value of the witness for R_e at time t. Initially we set the witnesses so that no witness is a prefix of any other as shown in Figure 4.
We say R_e requires attention at stage s if $p_e(\sigma_{e,s}, s) \downarrow \not\prec A_s$.

- Initially set $A_0 = \lambda$. Set $m_g(\alpha, 0) \uparrow$ for all α and $\sigma_{i,0} = 1^i \frown 0$ for all i. We will keep a "high water mark" h_s denoting the length of the longest string for which m_g is defined at stage s, and set $h_0 = 0$.
- At stage s choose the highest priority R_e to require attention. Let $\tau = f_e(\sigma_{e,s}, s)$.
  - Set $A_{s+1} = \tau \frown 1$.
  - Set $g(\alpha, s+1) = \begin{cases} \tau \frown 0 & \text{if } \sigma_{e,s} \preceq \alpha \preceq \tau \\ g(\alpha, s) & \text{otherwise} \end{cases}$
  - Set $m_g(\alpha, s+1) = \begin{cases} (\Pi_{\sigma_{i,s} \prec \sigma_{e,s}} m_g(\sigma_i, s)) + 1 & \text{if } m_g(\alpha, s) \uparrow \\ m_g(\alpha, s) & \text{otherwise} \end{cases}$
  - If $|A_{s+1}| > h_s$ then set $h_{s+1} = |A_{s+1}|$, else set $h_{s+1} = h_s$
  - Set $\sigma_{i,s+1} = \begin{cases} A_{s+1} \frown 0^{h_{s+1}} \frown 1^i \frown 0 & \text{if } i > e \\ \sigma(i, s) & \text{otherwise} \end{cases}$

In the limit g and m_g will be defined on all inputs, with m_g bounding the number of mind changes possible for g since any $g(\alpha)$ can only be damaged by the action of a requirement with a witness $\sigma \prec \alpha$, and the number of times each of these requirements can act has already been incorporated into $m_g(\alpha)$. In fact under the terms of this construction m_g will be a considerable overestimate of the number of mind changes for g. Thus g is clearly a ω-ce density function and A does not meet it, so A is not ωb-generic. $\square$

6.2 Refining the Previous Result

Notice that in the proof above, the mind change function $m_g(\alpha, s+1)$ is calculated as a product. However, if one thinks about it, this is really only a convenience, and a sum would be enough. That is, in the proof we could have chosen

$$m_g(\alpha, s+1) = \begin{cases} (\Sigma_{\sigma_{i,s} \prec \sigma_{e,s}} m_g(\sigma_i, s)) + 1 & \text{if } m_g(\alpha, s) \uparrow \\ m_g(\alpha, s) & \text{otherwise} \end{cases}$$

This is because, if R_e recieves attention at stage s then we can initialize all lower priority R_j. So the worst case scenario is that in turn each higher priority requirement moves f to the max, but the effect is still additive.

As a consequence, we see that all that is important is that we get enough domination to get the construction to work.

Theorem 6.3 *Suppose that f and g are increasing recursive functions with $g \in \Omega(f)$, $f \notin \Omega(g)$. Then there is a set A which is fb-generic (i.e. it meets all f-ce density functions) which is not gb-generic.*

The other refinement we can prove is that the argument works for degrees. The following is really a routing modification, so we will only sketch the proof.

Theorem 6.4 *There is a pb-generic set A which does not have ωb-generic degree.*

Proof: (Sketch) The argument is a modification of the proof above. The R_e requirements making A *pb*-generic remain the same. Additionally we must build A to satisfy the requirements below.

$N_e : \Phi_e(A)$ total and $\Gamma_e(\Phi_e(A)) = A$ implies $\Phi_e(A)$ is not ωb generic.

Let $\ell(e, s)$ denote the A-controllable length of agreement. Thus

$$\ell(e, s) = \max\{x : \forall y \leq x(\Gamma_{e,s}(\Phi_{e,s}(A_s; y) = A_s(y))\}.$$

Naturally, we regard A as controlling $\Phi(A)$, in the sense that nothing changes once the length is above x unless we change A on the use. The construction is involved, but routine. We consider the situation of a single N_e with many R_j for $j > e$. Now we will be defining g_e in place of g, based on the totality of the e-functionals. (Of course in the full construction, we'd be defining g_σ for the correct node on the true path.)

The basic idea is this. A string σ following R_j may or may not be e-confirmed. When $\ell(e, s) > \sigma$ for the first time, the node σ becomes e-confirmed. The effect is that we initialize all R_k for $k > j$. Now we get to define $m_{g_e}(\tau, s+1)$ for every string τ for which the mind change function is not yet defined, and τ is below $\ell(e, s)$. We again set $m_{g_e}(\tau, s+1)$ to exceed the product of things around at this stage.

Now we cover the strings below the length of agreement in $\Phi_{e,s}(A_s)$ as we did for A_s in the last construction. The crucial thing is that again when some R_j acts it initializes the lower priority requirements. We don't need to modify our cover until the next expansionary stage, where we will again have control of $\Phi_{e,s}(A_s)$ on the strings covered so far. Hence the counting is the same.

The full details are messy but provide no real insight so we omit them.

6.3 Partial Density Functions

In order to carry out the remaining proofs in this section, we need to introduce the notion of partial density functions.

Definition 6.5 *A function f is partial η-dense if:*

1. $f(\sigma) = \lim_s f(\sigma, s)$ *if* $f(\sigma)\downarrow$.
2. $f(\sigma, s)\downarrow \Longrightarrow (\forall t > s)(f(\sigma, t)\downarrow)$
3. $f(\sigma, s)\downarrow \Longrightarrow f(\sigma) \succeq \sigma$
4. *if* $f(\sigma)\downarrow$ *then* η *bounds the number of mind changes for* $f(\sigma, \_)$.

We cannot directly convert the proof of Theorem 5.4 for use in proving properties of sets beyond ωb-genericity because it relies on an appeal to 1-genericity which we would no longer be able to make. In order to make a slightly modified proof, we need the following lemma:

Lemma 6.6 *Suppose a set S is $(\eta + 1)b$-generic. Then for all η-ce partial density functions g, S meets or avoids g.*

PROOF: Consider a η-ce partial density function g. If S is not covered by g then it avoids g trivially. So suppose S is covered by g, that is $(\forall\sigma \prec S)(\exists\tau \succ \sigma)(g(\tau)\downarrow)$

Then we can define a new total density function g':

$$g'(\sigma, t) = \begin{cases} g(\tau, s) & \text{if} \tau \succ \sigma \text{ and } g(\tau, s)\downarrow \text{ and the first to do so} \\ \sigma & \text{otherwise} \end{cases}$$

By our initial assumption, we know for every $\sigma \prec S$ there is a $\tau \succ \sigma$ such that $(g(\tau)\downarrow)$, which means that there is a stage s for which $(g(\tau, s)\downarrow)$. We can therefore find the appropriate value for τ corresponding to each σ by dovetailing, and this process is computable.

Given this value for τ, $g(\tau, s)$ can change at most η times and $g'(\sigma, t)$ can change at most once more for a total of $\eta + 1$ times. g' is thus a $(\eta + 1)$-ce density function and as such must be met by S, yet clearly if S meets g' then it must also meet g, so S meets g. □

6.4 Collapsed Hierarchies in the Density Definition

In the proof of Theorem 5.4 we considered the existence of a cone of strings extending some prefix of a $1b$-generic set S and showed that no matter how many times a 2-ce density function g changed (up to the limit of 2, obviously) on that cone, it was impossible for S to avoid g. This proof was then extended to all n-ce density functions.

The essential character of the proof is the same at this higher level of the Ershov Hierarchy, but the function examined with respect to the cone of strings is not the $(\eta + k)$-ce density function g itself, but on the function that gives the number of mind changes for g.

Theorem 6.7 *Suppose a set S is $(\eta+1)b$-generic for some η. Then S is $(\eta+k)b$-generic for all $k \geq 1$.*

Proof: by induction. Assume all $(\eta+1)b$-generic sets are $(\eta+k)b$ generic for $1 \leq k \leq m$, consider $k = m+1$. We prove by contradiction that there does not exist a $(\eta+m+1)$-ce density function f and $(\eta+m)b$-generic set S such that S does not meet f.
We consider a cone of strings on S as before, with the proposition

$$P = (\forall \sigma \prec S)(\exists \tau \succeq \sigma)(|\{s : f(\tau,s) \neq f(\tau,s+1)\}| = \eta+m+1)$$

Suppose P is false, so $(\exists \sigma \prec S)(\forall \tau \succeq \sigma)(|\{s : f(\tau,s) \neq f(\tau,s+1)\}| \leq \eta+m)$.
As before, we can then define

$$f'(\gamma,t) = \begin{cases} \sigma \upharpoonright (|\sigma|-1)\widehat{\ }(1-\sigma(|\sigma|) & \text{if } \gamma \prec \sigma \\ \gamma & \text{if } \gamma \not\prec \sigma \wedge \gamma \not\succeq \sigma \\ f(\gamma,t) & \text{otherwise} \end{cases}$$

f' is a $(\eta+m)$-ce density function so must be met by S, which means f must be met by S.
Suppose then that P is true. Then we can define a new function $\hat{f}$:

$$\hat{f}(\alpha,s) = \begin{cases} f(\alpha,s) & \text{once } f(\alpha,s) \text{ has changed } m+1 \text{ times} \\ \alpha & \text{otherwise} \end{cases}$$

$\hat{f}$ is a partial, η-ce density function and thus by Lemma 6.6 S meets or avoids it. S cannot avoid $\hat{f}$ because we have already stipulated that every prefix of S has a successor on which f changes $\eta+m+1$ times , so every prefix of S has a successor in the range of $\hat{f}$. If S meets $\hat{f}$ then it must also meet f. □

Corollary 6.8 *For all n,k, a set S is $(n\omega+1)b$ generic iff it is $(n\omega+k)b$ generic (immediate).*

We cannot use this proof to collapse $(n\omega+k)b$-genericity right down into $(n\omega)b$-genericity because ω is not the successor of anything — we cannot express $n\omega$ as $\eta+1$ for any η. At this point we have shown only that between limit points the genericities collapse, we have neither proved nor disproved anything about the relationship between $(kn+1)b$-genericity and $(ln+1)b$-genericity for any $k \neq l$.

To start in this direction, we show that $1b$-genericity is not the same as ωb-genericity, and then that ωb-genericity is not the same as $(\omega+1)b$-genericity

6.4.1 ωb- and $(\omega+1)b$-genericity

The construction proving Theorem 6.9 is based on on requirements that use functions that have been labelled under some enumeration scheme. Recall that any enumeration scheme which includes all total computable functions must also include all partial computable functions. It therefore simplifies matters greatly if, rather than acting on the labelled partial functions $\varphi_x(\_,\_)$ directly, we act on functions $\overline{\varphi_x}(\_,\_)$ defined as follows:

For the sake of $\overline{\varphi_x}(\sigma,\_)$, some σ,

- set $s(0)=0$, $\overline{\varphi_x}(\sigma,0)=\sigma$

- starting from stage 0, at each stage t check whether $\varphi_x(\sigma, s(t))$ halts at stage $t+1$.
  - if it halts, set $\overline{\varphi_x}(\sigma, t+1) = \varphi_x(\sigma, s(t))$ and $s(t+1) = s(t)+1$.
  - if it doesn't halt, set $\overline{\varphi_x}(\sigma, t+1) = \overline{\varphi_x}(\sigma, t)$ and $s(t+1) = t$.

If φ_x is total, $\overline{\varphi_x}$ is simply a slowed-down version of φ_x. If the number of mind changes for any $\varphi_x(\alpha, \_)$ is bounded by $\phi(\alpha)$ for some computable ϕ, then the number of mind changes for $\overline{\varphi_x}(\alpha, \_)$ is bounded by $\phi(\alpha)+1$ which is clearly also computable, so if φ_x is ω-ce then so is $\overline{\varphi_x}$. Additionally, if φ_x is total then $\operatorname{ra} \varphi_x = \operatorname{ra} \overline{\varphi_x}$. Therefore, a set that meets all $\overline{\varphi_y}$ that are ω-ce density functions meets all ω-ce density functions.

If φ_x is not total then it is not relevant to the construction as it cannot be a total ω-ce density function, but $\overline{\varphi_x}$ will not harm the construction in any way. In fact, because $\overline{\varphi_x}$ is always a total density function, it is considerably easier to work with than φ_x directly. By Downey, Jockusch and Stob [2], we know that there are ωb generic sets that are not 1-generic. This is also true at the next level:

Theorem 6.9 *There exists an ωb-generic set S which is not $(\omega+1)b$-generic.*

Proof: We construct an ωb-generic set S and $(\omega+1)$-ce density function f in stages such that S does not meet f.
We work with respect to a set of requirements $R_{\langle e,i\rangle}$ over all e, i:

$$\begin{array}{lll} R_{\langle e,i\rangle}: & & \varphi_e \text{ not total} \\ & \vee & (\exists\sigma)(\lim_s \varphi_e(\sigma, s) \not\succeq \sigma) \\ & \vee & (\exists\alpha)(|\{t : \varphi_e(\alpha,t) \neq \varphi_e(\alpha, t+1)\}| > \varphi_i(\alpha)) \\ & \vee & (\exists\beta)(A \succeq \lim_s \varphi_e(\beta, s)) \end{array}$$

Each $R_{\langle e,i\rangle}$ has its own witness $\sigma_{\langle e,i\rangle}$ which may change with time. We only need to act on $R_{\langle e,i\rangle}$ if $\overline{\varphi_e}(\sigma_{\langle e,i\rangle}, \_)$ has halted on a value that extends $\sigma_{\langle e,i\rangle}$ and $\varphi_i(\sigma_{\langle e,i\rangle})$ has also halted. Until these have both happened we can regard $R_{\langle e,i\rangle}$ as fulfilled by default.
We say $R_{\langle e,i\rangle}$ is *removed from consideration at stage s_1* if

$$\begin{array}{ll} & (\exists s_0, \alpha)(s_0 \leq s_1 \wedge |\alpha| \leq s_1 \wedge \overline{\varphi_e}(\alpha, s_0) \not\succeq \alpha) \\ \vee & (\exists\alpha)(\varphi_i(\alpha)\downarrow \wedge |\alpha| \leq s_1 \wedge |\{t : \overline{\varphi_e}(\alpha, t) \neq \overline{\varphi_e}(\alpha, t+1)\}| > \varphi_i(\alpha)) \end{array}$$

For the sake of $R_{\langle e,i\rangle}$ we choose a witness string $\sigma_{\langle e,i\rangle}$ such that $|\sigma_{\langle e,i\rangle}| \geq \langle e,i\rangle$. This witness can change over time, so we use $\sigma_{\langle e,i\rangle}(t)$ to denote the value of the witness string for $R_{\langle e,i\rangle}$ at time t.
The key realisation in using witnesses for this construction is that no witness $\sigma_{\langle a,b\rangle}(t)$ should be a prefix of any other prefix $\sigma_{\langle c,d\rangle}(t)$ unless $\varphi_b(\sigma_{\langle a,b\rangle}(t))$ halts at stage t so the mind change complexity of $\overline{\varphi_a}(\sigma_{\langle a,b\rangle}(t), \_)$ is known. Initially, therefore, we need to set the witnesses so that no witness is a prefix of any other as in Figure 4.
We say $R_{\langle e,i\rangle}$ *requires attention at stage t* if $\overline{\varphi_e}(\sigma_{\langle e,i\rangle}, t)\downarrow \succeq \sigma_{\langle e,i\rangle} \wedge \varphi_i(\sigma_{\langle e,i\rangle}, t)\downarrow$ and R_e is not removed from consideration or declared satisfied at stage t.
As we build the $(\omega+1)$-ce density function g covering A we need to also construct a ce function m_g which gives the number of mind changes for g.

- Set initial values for the various elements:
  - $A_0 = \lambda$,
  - $g(\beta, 0) = \beta$ and $m_g(\beta, 0) = 1$ for all β
  - $\sigma_{\langle a,b\rangle}(0) = 1^{\langle a,b\rangle\frown}0$ for all a, b.
- At stage s, choose the highest priority $R_{\langle e,i\rangle}$ to require attention.
  Let $\alpha = \overline{\varphi_e}(\sigma_{\langle e,i\rangle}(s))$,
  $n = \varphi_i(\sigma_{\langle e,i\rangle}, t)$,
  $k = \Pi_{\sigma_{\langle a,b\rangle}(s) \prec \sigma_{\langle e,i\rangle}(s)} m_g(\sigma_{\langle a,b\rangle}(s), s)$
  - set $A_{s+1} = \alpha^\frown 1$
  - set $m_g(\gamma, s+1) = \begin{cases} n+k+1 & \text{if } \sigma_{\langle e,i\rangle}(s) \preceq \gamma \preceq \alpha \\ m_g(\gamma, s) & \text{otherwise} \end{cases}$
  - set $g(\gamma, s+1) = \begin{cases} \alpha^\frown 0 & \text{if } \sigma_{\langle e,i\rangle} \preceq \gamma \preceq \alpha \\ g(\gamma, s) & \text{otherwise} \end{cases}$
  - set $\sigma_{\langle d,j\rangle}(s+1) = \begin{cases} \sigma_{\langle d,j\rangle}(s) & \text{if } \langle d,j\rangle \leq \langle e,i\rangle \\ A_{s+1}{}^\frown 1^{\langle d,j\rangle(s)\frown}0 & \text{otherwise} \end{cases}$
  - Declare $R_{\langle e,i\rangle}$ satisfied, declare $R_{\langle d,j\rangle}$ unsatisfied for all $\langle d,j\rangle > \langle e,i\rangle$.

Initially, then, the partial solution A_0 is an empty set and all of the witnesses σ_x are independent as shown in the diagram. When some $R_{\langle y,z\rangle}$ is acted on at stage t, A_{t+1} is routed through $\overline{\varphi_y}(\sigma_{\langle y,z\rangle}(t), t)$ and values for $g(\sigma_{\langle y,z\rangle}(t), \_)$ through $g(\overline{\varphi_y}(\sigma_{\langle y,z\rangle}(t), t), \_)$ are changed and a number of mind changes for those values committed to.
We note that $g(\tau, \_)$ for $\sigma_{\langle y,z\rangle}(t) \preceq \tau \preceq \overline{\varphi_y}(\sigma_{\langle y,z\rangle}(t), t)$ can be affected not only by changes in $\overline{\varphi_y}(\sigma_{\langle y,z\rangle}(t_1), t_1)$ at some time $t_1 > t$ where $\sigma_{\langle y,z\rangle}(t_1) = \sigma_{\langle y,z\rangle}(t)$ but also by changes in $\overline{\varphi_a}(\sigma_{\langle a,b\rangle}(t_2), t_2)$ at some time $t_2 > t$ if $\sigma_{\langle a,b\rangle}(t_2) \preceq \sigma_{\langle y,z\rangle}(t), t)$. It is for this reason that we only allow a witness α to be a prefix of another witness β if the mind change complexity of $g(\alpha, \_)$ is known and therefore can be incorporated into the mind change complexity of $g(\beta, \_)$ when required.
Suppose now that some $R_{\langle w,x\rangle}$ of higher priority than $R_{\langle y,z\rangle}$ requires attention for the first time at stage $t_0 > t$. By the construction, A_{t_0+1} will be routed through $\sigma_{\langle w,x\rangle}(t_0)$, which is not a prefix of any other witness at that stage. $A_{t'}$ for $t' > t_0$ will remain routed through this string unless some even higher priority requirement acts for the first time at a later stage. In either case, the values for $g(\alpha, \_)$ which had been changed at earlier stages cannot be changed again because no future witness will ever be a prefix of α. In this way we can commit to a constraint on mind changes for $g(\beta)$ at the first stage $g(\beta)$ is changed for all β, so g is $(\omega+1)$-ce and not met by A, hence A is ωb-generic but not $(\omega+1)b$-generic. □

We remark that one can again use standard methods, as before, to show.

Corollary 6.10 *The is a set which is ωb-generic which is not of $(\omega+1)b$-generic degree.*

6.4.2 Collapse of hierarchy for remaining $n\omega + k$ levels

Having shown that that $1b$-genericity is not the same as ωb-genericity, and that ωb-genericity is not the same as $(\omega + 1)b$-genericity, we wish to extend to more general results.

It transpires that once a set is $(\omega + 1)b$-generic it is sufficiently powerful to be $(n\omega + k)b$-generic for all $n \geq 2, k \geq 0$.

Before carrying out the necessary proofs, it is helpful to examine how the mind changes for a $(n\omega+k)$-ce function might be formally bounded. It is easiest to think of a new bound being applied every time the function (applied to some input) drops through a limit point, with the number of changes then subject to that bound until a subsequent limit point (and thus new bound) occurs.

A very natural parallel can then be drawn with the functions in the difference hierarchy, which can change values a finite number of times up to a natural number bound. Can we use an n-ce function to bound the number of mind changes for an $(n + 1)\omega$-ce function? Yes, we can, by constraining the number of mind changes between limit points in the $(n + 1)\omega$-ce function to correspond with the output values of the n-ce function.
Unfortunately, this leaves us with the question of how to bound mind changes for $(n\omega+k)$-ce functions where $k \neq 0$. We can do this most easily by introducing a subset of the n-ce functions which we will call the k-origin-n-ce functions.

Definition 6.11 *A function f is k-origin-n-ce for some $k, n \in \mathbb{N}$ iff* $(\forall \alpha)(|\{s : f(\alpha, s) \neq f(\alpha, s + 1)\}| \leq n \wedge f(\alpha, 0) = k)$

Now we can use a k-origin-n-ce function to bound any $(n\omega + k)$-ce function.

Theorem 6.12 *All $(\omega+1)b$-generic sets are $(n\omega+k)b$-generic for all $n, k \in \mathbb{N}$.*

Proof: very similar to the proof for Theorem 6.7. We use contradiction within induction.
Suppose for all $1 < m \leq n - 1$ all $(\omega + 1)b$-generic sets are $(m\omega + k)b$-generic for all k, so meet all $(m\omega + k)$-ce density functions for all k. Claim then that all $(\omega + 1)b$-generic sets are $(n\omega + k)b$-generic for all k.
Suppose not. Then there exists some $(\omega + 1)b$-generic set A and a $(n\omega + l)$-ce density function g (some l) such that A does not meet g. Let m_g be the (l-origin-n-ce) function giving the number of mind changes for g.
Then consider the proposition

$$P = (\forall \sigma \prec A)(\exists \tau \succeq \sigma)(|\{s : m_g(\tau, s) \neq m_g(\tau, s + 1)\}| = n)$$

That is, do all prefixes of A have an extension that undergoes mind changes potentially dropping through n limit points as allowed by m_g having n mind changes?
Suppose P is true. Then define a function $\hat{g}$:

$$\hat{g}(\alpha,s) = \begin{cases} g(\beta,s) & \text{once } m_g(\beta,s) \text{ has changed } n-1 \text{ times where } \beta \succeq \alpha \text{ and is the first to do so} \\ \alpha & \text{otherwise} \end{cases}$$
$$m_{\hat{g}}(\alpha,s) = \begin{cases} m_g(\beta,s) & \text{once } m_g(\beta,s) \text{ has changed } n-1 \text{ times where } \beta \succeq \alpha \text{ and is the first to do so} \\ 1 & \text{otherwise} \end{cases}$$

$m_{\hat{g}}$ is thus a 1-origin-1-ce function, it governs $\hat{g}$ which is thus a $(\omega+1)$-ce density function, so A must meet $\hat{g}$. But the range of $\hat{g}$ is a subset of the range of g, so if A meets $\hat{g}$ it must also meet g. If it doesn't meet g, it can't meet $\hat{g}$ and thus can't be $(\omega+1)b$-generic.
Then suppose instead that P is false, so
$(\exists\sigma \prec A)(\forall\tau \succeq \sigma)(|\{s : m_g(\tau,s) \neq m_g(\tau,s+1)\}| < n)$.
Then for some prefix of A all extensions of this prefix must undergo mind changes dropping through fewer than n limit points because m_g changes fewer than n times. Let σ represent this prefix, then define a new function g':

$$g'(\gamma,t) = \begin{cases} \sigma \restriction (|\sigma|-1)\widehat{\ }(1-\sigma(|\sigma|) & \text{if } \gamma \prec \sigma \\ \gamma & \text{if } \gamma \not\prec \sigma \wedge \gamma \not\succeq \sigma \\ g(\gamma,t) & \text{otherwise} \end{cases}$$
$$m_{g'}(\gamma,t) = \begin{cases} 0 & \text{if } \gamma \prec \sigma \\ 0 & \text{if } \gamma \not\prec \sigma \wedge \gamma \not\succeq \sigma \\ m_g(\gamma,t) & \text{otherwise} \end{cases}$$

$m_{g'}$ is thus 0-origin-$(n-1)$-ce, so g' must be $(n-1)\omega$-ce, so by the induction hypothesis A must meet g'. But the definition of g' is such that A can only be meeting g' via the portion of g''s range that derives from g, because the remainder of ran g' is explicitly designed in such a way that it cannot be a prefix of A.
Therefore either A does not meet g', contradicting the $(\omega+1)$-genericity of A, or A meets g', contradicting the claim that A does not meet g. □

6.5 Sets below $0'$

We have already seen that a $2d$ generic is already 2-generic and hence, in particular, not below $\mathbf{0}'$. What of our new intermediate sets?

Lemma 6.13 *There exists an $(\omega+1)b$-generic set that is computable in $0'$.*

PROOF: We use a slight modification of the standard oracle construction for a 1-generic set. We meet requirements $R_{\langle e,i\rangle}$ for all $e, i \in \mathbb{N}$:

$$\begin{array}{lll} R_{\langle e,i\rangle}: & & \varphi_e \text{ not total} \\ & \vee & \varphi_i \text{ not total} \\ & \vee & (\exists\sigma)(\lim_s \varphi_e(\sigma,s) \not\succeq \sigma) \\ & \vee & (\exists\alpha)(|\{t : \varphi_e(\alpha,t) \neq \varphi_e(\alpha,t+1)\}| > \varphi_i(\alpha)+1) \\ & \vee & (\exists\beta)(A \succeq \lim_s \varphi_e(\beta,s)) \end{array}$$

We build A by finite extension, such that $A = \lim_s A_s$, initially have $A_{-1} = \lambda$. At stage $\langle e,i\rangle$ act to satisfy $R_{\langle e,i\rangle}$ as follows:

- Ask whether $\varphi_i(A_{\langle e,i\rangle-1})\downarrow$.

- NO — then $R_{\langle e,i\rangle}$ is satisfied because φ_i is not total, go on to the next stage.
- YES — then calculate $n = \varphi_i(A_{\langle e,i\rangle -1})$.

- Ask whether $|\{t : \varphi_e(A_{\langle e,i\rangle -1}, t) \neq \varphi_e(A_{\langle e,i\rangle -1}, t+1)\}| > n+1$.
  - YES — then $R_{\langle e,i\rangle}$ is satisfied.
  - NO — then find the value of $\lim_s \overline{\varphi_e}(A_{\langle e,i\rangle -1}, s)$, set $\sigma = \lim_s \overline{\varphi_e}(A_{\langle e,i\rangle -1}, s)$

- Set $A_{\langle e,i\rangle} = \sigma$.

A clearly meets all $(\omega+1)$-ce density functions, so is $(\omega+1)b$-generic. We were able to construct it with $0'$ as an oracle, so $A \leq_T 0'$, proving that there exists an $(\omega+1)b$-generic set below $0'$. □

Lemma 6.14 *No $\omega^2 b$-generic set is computable in $0'$.*

Proof: by contradiction. Suppose there exists some $\omega^2 b$-generic set A such that $A \leq_T 0'$. Then by the Shoenfield Limit Lemma, $A = \lim_s A_s$, that is $x \in A$ iff $\lim_s A_s(x) = 1$, where $\lim_s A_s(x)$ exists for all x and ranges over $\{0,1\}$.
Then build a Δ^0_2 function g, defined via an approximation such that $g(\sigma) = \lim_s g(\sigma, s)$.

$$g(\gamma,t) = \begin{cases} \gamma & \text{if } t = 0 \text{ or } \gamma \not\prec A_t \\ \gamma \upharpoonright (|\gamma|-1)^\frown(1 - A_t(|\gamma|)) & \text{otherwise} \end{cases}$$

By this definition, $g(\gamma, \_)$ must come to a limit for all γ, because $\lim_s A_s(x)$ exists for all x, in particular the answer to question of whether or not $\gamma \prec A_t$ must eventually come to a limit because $\lim_t A_t(x)$ exists for $1 \leq x \leq |\gamma|$.
If $\lim_s g(\gamma, s)$ exists for all γ, then g must be Δ^0_2 by the Shoenfield Limit Lemma. But g is not met by A, which contradicts the $\omega^2 b$-genericity of A.
It cannot therefore be the case that A is both $\omega^2 b$-generic and computable in $0'$. □

Corollary 6.15 *There is a $(\omega+1)b$-generic set that is not $\omega^2 b$-generic.*

Proof: immediate from Lemmas 6.13 and 6.14. □

6.6 $\omega^2 b$-genericity

Lemma 6.14 shows us that there is no $\omega^2 b$-generic set below $0'$. Last time we saw a set of generic sets that did not contain a set below $0'$, it was the $d2$-generic sets. The $d2$-generic sets turned out to be equivalent to the 2-generic sets, is this the case for $\omega^2 b$-generic sets as well? The answer turns out to be no.

Theorem 6.16 *There exists an $\omega^2 b$-generic set which is not 2-generic.*

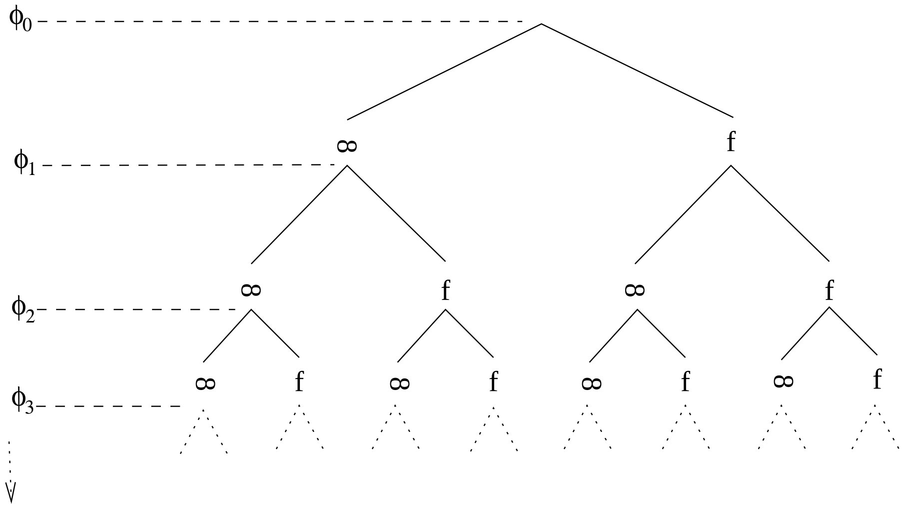

Figure 5: The tree T.

PROOF: This theorem is proved by an infinite injury construction.
We construct an $\omega^2 b$-generic set A and a 2-ce set of strings $S = \{\tau_i : i \in \omega, \tau_i = \lim_s \tau_{i,s}\}$ that is a proper cover of A. By Theorem 4.2, it will follow that A is not 2-generic.
We need to introduce the infinite injury technique because for any given density function g with computable approximation $g(\_,\_)$, there is no way to computably determine whether the value of the approximation changes finitely often and thus whether g needs to be met by A.
For this argument we consider a tree $T = \{\infty, f\}^{\infty}, \infty <_L f$ of possible states for the set of all functions, as shown in Figure 5. Any given function changes either finitely or infinitely often. As the construction progresses various branches of the tree begin to seem more or less likely.
We will take $\{\varphi_i(e,s) : i \in \omega\}$ to be a listing of all slowed down partial computable approximations to density functions, such that

- $(\forall i, \sigma, s)(\varphi_i(\sigma, s) \uparrow \vee \varphi_i(\sigma, s) \succ \sigma)$
- $(\forall i, \sigma, s)(\varphi_i(\sigma, s) \downarrow \Longrightarrow (\forall s' > s)(\varphi_i(\sigma, s') \downarrow))$

A sketch of an algorithm for generating φ_i from ϕ_i (the ith partial recursive function) is in Figure 6.
We could make the functions in this listing total by initially setting `last result` to take the same value as the input, instead of setting it to diverge.
We wish to meet, for all e, the requirements

$$
\begin{array}{lll}
R_{\langle e,i \rangle}: & & \varphi_e \text{ not total} \\
& \vee & (\exists \sigma)(\lim_s \varphi_e(\sigma, s) \prec A) \\
& \vee & (\exists \beta)(\lim_s (\beta, s) \text{ does not exist})
\end{array}
$$

```
input: σ
s = 0
t = 0
n = 0
lastresult = ↑
repeat
    t = t + 1
    execute φ_{i,n}(σ, s)
    if φ_{i,n}(σ, s) is in a halting state and extends σ
        φ_i(σ, t) = φ_{i,n}(σ, s)
        last result = φ_i(σ, t)
        s = s + 1
        n = 0
    else
        φ_i(σ, t) = last result
        n = n + 1
```

Figure 6: An algorithm for generating a slowed down version of ϕ_i.

$N_e : (\lim_s \tau_{e,s} \succ A \upharpoonright e) \wedge (\lim_s \tau_{e,s} \not\prec A)$
In other words, if φ_e is a Δ^0_2 density function then A must meet it.
In the basic module, for the sake of each R_e we have two witnesses. σ^f_e witnesses A meeting φ_e (via $\lim_s \varphi_e(\sigma^f_e, s) \prec A$) and σ^∞_e witnesses φ_e not being a Δ^0_2 function (via $\lim_s(\alpha, s)$ not existing). If φ_e changes finitely often A will route through $\lim_s \varphi_e(\sigma^f_e, s)$ and thus through σ^f_e, otherwise it will route through σ^∞_e. The basic module for R_0 in isolation is as follows:

- Choose σ^f_0 and σ^∞_0 such that $\sigma^\infty_0 <_L \sigma^f_0$ (for example, $\sigma^f_0 = 111$ and $\sigma^\infty_0 = 000$)
- Wait for the first stage s such that $\varphi_0(\sigma^f_0, s) \downarrow$. If this never occurs, it indicates that φ_e is not total and R_e is automatically satisfied without any action being necessary.

  Suppose then that a stage t does occur such that $\varphi_{0,t}(\sigma^f_0, 0) \downarrow = \alpha$. Then we set $A_t = \alpha^\frown 1$ and $\tau_{0,t} = \alpha^\frown 0$.
- Now go into a loop.
  - Wait for a stage t' such that $\varphi_0(\sigma^f_0, t') \neq \varphi_0(\sigma^f_0, t'-1)$. If such a stage occurs, we are forced to acknowledge the possibility that it is because $\lim_s \varphi_0(\sigma^f_0, s)$ does not exist. Therefore, set $A_{t'} = \sigma^{\infty\frown}_e 1$ and $\tau_{0,t'} = \sigma^{\infty\frown}_e 0^n$ where n is the smallest value for which $(\forall t_0 \leq t')(\tau_{0,t_0} \neq \tau^{\infty\frown}_e 0^n)$. This is necessary because S must be 2-ce

– However, it is also possible that this was the last change for $\varphi_0(\sigma_0^f, \_)$, in other words $\varphi_0(\sigma_0^f, \_)$ does come to a limit and therefore A should be routed through $\varphi_0(\sigma_0^f, t')$. For this reason, set $A_{t'+1} = \varphi_0(\sigma_0^f, t')^\frown 1$ and $\tau_{0,t} = \varphi_0(\sigma_0^f, t')^\frown 0^n$ where n is the smallest value for which $(\forall t_0 \leq t')(\tau_{0,t_0} \neq \tau_e^{f\frown} 0^n)$.

The modules are integrated into a full construction. Note that any function whose approximation changes infinitely often will cause its associated R_e (and thus N_e) to require attention infinitely often as well. It is therefore necessary to act on more than one requirement at any given stage, or else a single non-Δ_2^0 function could potentially block all lower priority requirements from ever being satisfied.
The full construction requires some additional definitions. We define α-stage and α-expansionary by induction on τ and thence define TP_s.

Definition 6.17 *1. Every stage s is a λ-stage.*
2. A stage s is α-expansionary if $\varphi_{|\alpha|}(\sigma^f_{|\alpha|}(s), s) \neq \varphi_{|\alpha|}(\sigma^f_{|\alpha|}(s), t)$
Where t is the last α-stage before s, or 0 if there is no previous α-stage.
3. If s is a α-stage and s is α-expansionary, then s is a $\alpha^\frown\infty$-stage. If s is a α-stage and s is **not** *α-expansionary, then s is a $\alpha^\frown f$-stage.*

Definition 6.18 *TP_s is the unique string α of length s such that s is a α-stage.*

TP_s represents what we believe to be true about the first s functions at stage s. It is a finite path through T that we say has been "visited" at stage s. As s increases, longer and longer paths in T will be visited in this way.
We initiate the construction with

- $\tau_{i,-1} \uparrow$ for all i
- $\sigma_e^\infty(-1) = 0^e$, $\sigma_e^f(-1) = 1^e$ for all e

We say R_e requires attention at stage s if $e \leq s$ and $TP_s(d) \neq TP_{s-1}(d)$ for some $d \leq e$.
At some points we will need to initialise requirements. For the purposes of this construction, initialising N_e at stage s will entail setting $\tau_{e,s} \uparrow$ and initialising R_e at stage s will entail setting $\sigma_e^f(s) = A_s{}^\frown 1^e$ and $\sigma_e^\infty(s) = A_s{}^\frown 0^e$.
While some requirements may need to be initialised at a particular stage s, others may only need to be maintained. For the purposes of this construction, maintaining N_e at stage s will mean setting $\tau_{e,s} = \tau_{e,s-1}$ and maintaining R_e at stage s will mean setting $\sigma_e^f(s) = \sigma_e^f(s-1)$ and $\sigma_e^\infty(s) = \sigma_e^\infty(s-1)$.
The construction operates via a calculation of TP_s at each stage s. We calculate TP_s by building it up from the inductive definition of an α-stage while modifying low priority witnesses if required.

- Find the shortest β for which s is a β-stage but $s-1$ is not a β-stage.

  CASE 1: No such β is found with $|\beta| \leq s-1$. Then calculating TP_s is a simple matter of appending ∞ to TP_{s-1} if s is TP_{s-1}-expansionary and appending f to TP_{s-1} otherwise. In this case we need to act only on behalf of R_s and N_s at this stage. Set $\gamma = TP_s$.

CASE 2: Some such $\beta \preceq TP_{s-1}$ is found. Then it becomes necessary to reroute the partial solution A_s and consequently initialise R_e and N_e requirements of lower priority, **including those that are associated with longer initial segments of** TP_s. Set $\gamma = \beta$.

- Now go into a loop to build the remainder of TP_s while concurrently satisfying R_e and N_e for all $e \leq s$ and constructing A_s. The construction of A_s is done in stages using a variable *str* which is assigned to A_s at the end of the loop. We also treat γ as a variable of the loop, at the end of the loop γ will have the value of TP_s. A third loop variable, δ, is used merely to clarify the construction.

  Note that for $i < |\gamma|$ we have R_i and N_i already satisfied by induction from stage $s-1$ and cannot damage this satisfaction by any of the actions in this loop, so we only need to act to satisfy $R_{|\gamma|}$ and $N_{|\gamma|}$ on each iteration.

  Initially let $\sigma^f_{|\gamma|}(s) = \sigma^f_{|\gamma|}(s-1)$ and $\sigma^\infty_{|\gamma|}(s) = \sigma^\infty_{|\gamma|}(s-1)$.

  - If $\gamma(|\gamma|) = f$ then $\delta = \varphi_{|\gamma|}(\sigma^f_{|\gamma|}(s))$ else $\delta = \sigma^\infty_{|\gamma|}(s)$. Set $str = \delta\widehat{\ }1$ and $\tau_{|\gamma|,s} = \delta\widehat{\ }0^n$ where n is the smallest value for which $(\forall t_0 \leq t')(\tau_{0,t_0} \neq \tau^\infty_e\widehat{\ }0^n)$.
  - If $|\gamma| = s$, stop here.
  - If s is a γ-expansionary stage set $\gamma = \gamma\widehat{\ }\infty$, otherwise set $\gamma = \gamma\widehat{\ }f$.
  - Set $\sigma^f_{|\gamma|}(s) = str\widehat{\ }111$ and set $\sigma^\infty_{|\gamma|}(s) = str\widehat{\ }000$.

- Set $A_s = str$ and $TP_s = \gamma$.

- Maintain all N_e and R_e for $e \leq s$ where $\sigma^f_e(s)$ and $\sigma^\infty_e(s)$ have not already been defined by this construction.

- Initialise all N_e and R_e for $e > s$.

Definition 6.19 *The True Path (TP) is the leftmost (lexicographically) branch of T visited infinitely often. It represents the "true" state of the world in the limit.*

In the limit of this construction TP is the leftmost path through T visited infinitely often, and the true value of A is the leftmost set visited infinitely often as A_s. This A will meet all ω^2-ce functions and thus be $\omega^2 b$-generic. The set S is certainly by the construction a proper cover of A, and equally certainly is 2-ce, so A is not 2-generic. □

Formal verification of this proof will be done in later work.

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