

# Friedberg splittings of recursively enumerable sets

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*Abstract*

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A splitting  $A_1 \cup A_2 = A$  of an r.e. set  $A$  is called a Friedberg splitting if for any r.e. set  $W$  with  $W - A$  not r.e.,  $W - A_i \neq \emptyset$  for  $i = 1, 2$ . In an earlier paper, the authors investigated Friedberg splittings of maximal sets and showed that they formed an orbit with very interesting degree-theoretical properties. In the present paper we continue our investigations, this time analyzing Friedberg splittings and in particular their orbits and degrees for various classes of r.e. sets.

## 1. Introduction

Since its beginnings in Gödel's incompleteness theorem, a fundamental issue in recursion theory is to understand the relationship between *algebraic* and *computational* complexity. This is, of course, particularly true in applications of recursion theory such as the word problem (Boone [1], Miller [18], Higman's embedding theorem (Higman [13]), Hilbert's tenth problem (Davis et al. [3], Matijasevic [16]) and degrees of structures (e.g. Feiner [8], Frölich and Shepherdson [9], Metakides and Nerode [17]).

It is therefore not surprising that two of the basic structures of classical recursion theory are  $\mathcal{E}$ , the lattice of recursively enumerable (r.e.) sets (and  $\mathcal{E}^*$

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its quotient modulo finite sets) and  $\mathbf{R}$ , the (Turing) degrees of r.e. sets. After all, the r.e. sets are those that can be effectively listed, and the degrees measure their complexity.

In his classic 1944 paper, Post initiated a programme which, in essence, seeks to understand the relationship of  $\mathcal{E}$  and  $\mathbf{R}$ . In the present paper we continue in this spirit. In particular, we wish to continue our investigations of [5, 6] where we study the behaviour of splittings of r.e. sets under automorphisms of  $\mathcal{E}$ . We remind the reader that r.e. sets  $A_1, A_2$  *split*  $A$  (written  $A_1 \sqcup A_2 = A$ ) if  $A_1 \cup A_2 = A$  and  $A_1 \cap A_2 = \emptyset$ . A splitting is *proper* if both  $A_1$  and  $A_2$  are nonrecursive. The earliest splitting theorem is due to Friedberg [10] who showed that any nonrecursive r.e. set has a proper splitting.

One of the problems in studying  $\text{Aut}(\mathcal{E})$ , the automorphism group of  $\mathcal{E}$ , is the lack of known (definable) orbits. Aside from Soare's [22] hallmark result that maximal sets form an orbit (and some variations) and Harrington's result [24, Ch. XV] that creative sets form an orbit, there were, until recently, no known orbits in  $\text{Aut}(\mathcal{E}^*)$ .

A new direction in such studies was initiated in our [5] where we showed that 'hemimaximal' sets form an orbit. Here if  $P$  is a property of r.e. sets we say a set  $A$  is *hemi- $P$*  if there exists an r.e. set  $A_2$  with  $A_1 \sqcup A_2$  a proper splitting of an r.e. set with property  $P$ . This orbit has a number of very interesting degree-theoretical properties. For instance, all high r.e. degrees contain hemimaximal sets and if  $\mathbf{b} > \mathbf{0}$  there is a hemimaximal set of degree  $< \mathbf{b}$  yet there is also a nonzero degree with no hemimaximal elements below  $\mathbf{b}$ . Furthermore, while there are low<sub>2</sub>-low degrees with no hemimaximal sets, for any degree  $\mathbf{c}$  r.e. in and above  $\mathbf{0}'$  there is a hemimaximal set  $H$  with  $H'$  in  $\mathbf{c}$ . These results refute a number of conjectures about the degrees of definable orbits. For instance, it shows that not all orbits realize degree classes that are closed upwards. Furthermore, while the hemimaximals do not realize all degrees, they do realize all jumps. This is the best known solution to the 'fat orbit' question [24, Ch. XV], which asks if there is a set  $A$  such that  $\{\text{deg}(B) : B \text{ is automorphic with } A\}$  realizes all of the nonzero r.e. degrees. This result takes on a special interest since Harrington [11] has recently shown that the fat orbit question has a negative solution.

One of our key motivations for studying such splitting properties was the old conjecture of Soare:

(1.1) **Conjecture.** If  $A$  is r.e. nonrecursive, then  $A$  is automorphic to a complete set.

Splitting properties are very closely related to (1.1). For instance if we could show that all low r.e. sets were automorphic to complete sets, then all r.e. sets would be too. For take  $A$  to be r.e. nonrecursive and Sacks split  $A = A_1 \sqcup A_2$ . By [23] we know that  $A_1$  and  $A_2$  are both low. Send  $A_1$  to a complete set. Then  $A$  must go to a complete set too.

Harrington and Soare [12] have recently announced that (1.1) fails to hold. Nevertheless, the analysis above remains valid for some classes of sets. Call a set  $A$  half- $\mathcal{P}$  if there is a splitting  $A_1 \sqcup A_2 = A$  such that  $A_1$  has property  $\mathcal{P}$ . We know that there is a complete hemimaximal set, so we conclude that all halphemimaximal sets are automorphic with complete sets. Using this fact, we showed in [5] that all low<sub>2</sub> simple, all simple sets with semilow<sub>1,s</sub> complements, and all d-simple sets with maximal supersets are automorphic with complete sets.

Our starting points for the present paper were the following: we noticed that all splittings of a maximal set are Friedberg ( $f$ -)splittings. This led to the following conjecture [24, Ch. XVI, Question 1.13].

(1.2) **Conjecture.** All Friedberg splittings of (simple) sets are automorphic.

Furthermore, we noticed that Friedberg's [10] original proof of his splitting theorem actually satisfies the following

$$(1.3) \quad (W_e \searrow A) = \infty \Rightarrow W_e \searrow A_1 \neq \emptyset$$

where  $W_e \searrow A = \{z : (\exists t)(z \in W_{e,t} - A, \text{ and } z \in A)\}$ .

Now (1.3) is very reminiscent of the extension theorem of Soare [22], the main tool for constructing automorphisms. We shall call a splitting  $A_1 \sqcup A_2 = A$  satisfying (1.3) a true *Friedberg splitting* (*t-split*). Remember that the usual approach to building automorphisms of  $\mathcal{G}^*$  is to have 2 copies of  $\omega$  and 2 enumerations of r.e. sets  $\{W_e\}_{e \in \omega}$  and  $\{V_e\}_{e \in \omega}$ . So suppose we had  $t$ -splits  $A_1 \sqcup A_2 = A = B_1 \sqcup B_2$ . We wish to map  $A_1$  to  $B_1$ . For each  $W_e$  we build  $\hat{W}_e$  and for each  $V_e$  we build  $\hat{V}_e$  to get the correspondence:

$$W_e \rightarrow \hat{W}_e, \quad \hat{V}_e \leftarrow V_e, \quad A_1 \rightarrow B_1, \quad A_2 \rightarrow B_2.$$

We do so in such a way that the automorphism can be assembled by a back-and-forth argument. Certain obvious conditions must be met. If  $W_e \cap A_1 = * \emptyset$  we must ensure that  $\hat{W}_e \cap B_1 = * \emptyset$  or we lose directly. Our troubles stem from the fact that all of the sets are in a *state of formation*. Hence we cannot know if  $W_e \cap A_1 = \emptyset$  even though perhaps  $(\exists^\infty s)(W_{e,s} \cap A_{1,s} \neq \emptyset)$ . We must build  $\hat{W}_e$ , and are faced with the following problem. Suppose some  $z$  enters  $W_{e,s} - A_s$ . Should we respond by putting some  $\hat{z}$  in  $\hat{W}_{e,s} - A_s$ ? A good candidate here is  $z$  itself. Now if we don't do this we run the risk that  $|W_e \cap \bar{A}| = \infty$  yet  $|\hat{W}_e \cap \bar{B}| < \infty$ . If we do put (say)  $z$  into  $\hat{W}_e$  then while  $z$  later enters  $A_1$ , it may also enter  $B_2$  (not  $B_1$ ). In this way we could get  $W_e \subseteq A_1$  yet  $\hat{W}_e \subseteq B_2$  so  $\hat{W}_e \cap B_1 = \emptyset$ . In this dynamic approach, clearly for one r.e. set  $W_e$  we can avoid this problem by (1.3).

As we will see, in fact, (1.3) is not enough. We will, however, define a 'state' notion of (1.3) and with this a new notion of splitting (an *e-Friedberg splitting* or *e-splitting*) and show

(1.4) **Theorem.** *e-splittings of an r.e. set  $A$  are automorphic.*

In Section 2 we establish (1.4) and some related results on the degrees of e-splittings. We remark that one reason we were interested in t-splittings was the *fat orbit question*. Is there an r.e. set  $A$  such that

$$(1.5) \quad \{\text{deg}(B): B \text{ is automorphic to } A\} = \mathbf{R} - \{\mathbf{0}\}$$

*holds?* We can construct, for instance, a complete r.e. set with t-splittings of all nonzero r.e. degrees. However, we cannot do this with ‘e-split’ in place of ‘t-split’! If  $A_1 \sqcup A_2 = A$  is an e-split of a set of promptly simple degree, then  $A_1$  and  $A_2$  have promptly simple degree too. Moreover, if  $A$  is an r.e. nonrecursive set then there exists a  $\mathbf{b}$  with  $\mathbf{0} < \mathbf{b} < \text{deg}(A)$  such that no e-split of  $A$  has degree  $\mathbf{b}$ . finally, we construct an r.e. set  $A$  with e-splittings of all promptly simple degrees, and hence the existence of another orbit realizing all the promptly simple degrees. These results again take on a lot of interest since, as we mentioned earlier, Harrington [11] has recently shown that (1.5) has a negative solution.

In Section 3 will refute conjecture (1.3). We do this by introducing several new elementary classes of f-splittings of (promptly) simple sets. We delay their precise description till Section 3.

In Section 4 we turn to another conjecture. Do f-creative sets form an orbit? Although we cannot as yet answer this question, we do classify the degrees of f-creative sets as exactly the promptly simple ones. Furthermore, since there are e-creative sets of all promptly simple degrees, there is an orbit in  $\text{aut}(\mathcal{C})$  realizing exactly the promptly simple degrees.

In Section 5 we examine some other hemiproperties. In particular, we give proofs of (generalizations of) results (claimed in [5]) that there are non-halfhemisimple sets yet there are completely halfhemisimple degrees. Notation is standard and follows Soare [24]. All computations, etc. are bounded at stage  $s$  by  $s$ .

## 2. e-splittings

When we try to apply the extension lemma to a t-split we run into problems. The lemma works with *states* not single sets. The most natural approach is to try to satisfy for all states

$$(2.1) \quad |\eta \searrow A| = \infty \Rightarrow \eta \searrow A_i \neq \emptyset$$

(we abuse notation here). By  $\eta \cap A_i \neq \emptyset$  we mean there is some  $z$  of state  $\eta$  in  $A_i$ . Suppose we call a splitting that satisfies (2.1) a *strong f-split* (s-split). It turns out that (2.1) is not enough since we need to know the state of an element on its *entry* into  $A$ . This leads to the notion of an *entry e-state*. This is defined as  $\{i \leq e: x \in W_{i,s} \text{ and } x \in A_s - A_{s-1}\}$ . We write  $\eta \searrow_e A$  (via  $x$  at  $s$ ). Then an

*e*-splitting is one that satisfies

$$|\eta \searrow_e A| = \infty \Rightarrow |\eta \searrow_e A_i| = \infty$$

(for some enumeration of the r.e. sets).

As our first result we prove

(2.2) **Theorem.** *If  $A_1 \sqcup A_2 = B_1 \sqcup B_2 = A$  are two *e*-splittings of  $A$ , then there is an effective automorphism  $\Phi$  of  $\mathcal{G}^*$  with  $\Phi(A_1) = A_2$ .*

**Proof.** This follows by the version of the extension lemma given in [5]. Specifically, we recall that if  $\{X_e\}_{e \in \omega}$ ,  $\{Y_e\}_{e \in \omega}$  are recursive arrays of r.e. sets, then the *e*-state  $v(x)$  of  $x$  with respect to these arrays is triple  $\langle e, \sigma, \tau \rangle$  where  $\sigma$  is the *e*-state of  $x$  with respect to  $\{X_e\}_{e \in \omega}$  and  $\tau$  with respect to  $\{Y_e\}_{e \in \omega}$ . Also,  $v_{e,s}(x)$  is the approximation to  $v(x)$  at stage  $s$ .

Given full *e*-states  $v = \langle e, \sigma, \tau \rangle$  and  $v' = \langle e, \sigma', \tau' \rangle$ ,  $v \leq v'$  if  $\sigma \subseteq \sigma'$  and  $\tau \supseteq \tau'$ . (The relation  $\leq$  is pronounced 'is covered by'.)

Suppose that a simultaneous enumeration of the r.e. sets  $A$  and  $\{U_e\}_{e \in \omega}$  is given. For an *e*-state  $v$  measured with respect to  $\{U_e\}_{e \in \omega}$ , we define the sets

$$v \searrow_e A = \{x \mid (\exists s)[x \in A_{at.s} \wedge v(e, x, s) = v]\}.$$

Then in [5], the authors established the following lemma.

(2.3) **Lemma.** *Let  $A$  and  $B$  be infinite r.e. sets and  $A_1, A_2$  and  $B_1, B_2$  form splittings of  $A$  and  $B$  respectively. Suppose that  $\{U_n\}_{n \in \omega}$ ,  $\{\tilde{V}_n\}_{n \in \omega}$ ,  $\{\hat{U}_n\}_{n \in \omega}$ ,  $\{V_n\}_{n \in \omega}$  are recursive arrays of r.e. sets and that there is a simultaneous enumeration of a recursive array including all the above such that  $A_i \searrow \tilde{V}_n = \emptyset = B_i \searrow \hat{U}_n$ , for all  $n$  and  $i$ . Furthermore, suppose that for each  $i, i = 1, 2$ ,*

$$(A \vee)[v \searrow_e B_i \text{ infinite}] \Rightarrow (\exists v')[v \leq v' \wedge v \searrow_e A_i \text{ infinite}]$$

and

$$(A \vee)[v \searrow_e A_i \text{ infinite}] \Rightarrow (\exists v')[v' \leq v \wedge v \searrow_e B_i \text{ infinite}].$$

Then there are r.e. sets  $\hat{U}_n$  extending  $\tilde{U}_n$  and  $\tilde{V}_n$  extending  $\hat{V}_n$  such that for each  $i$  and for each full *e*-state  $v$ ,

iff

*infinitely many elements of  $A_i$  have *e*-state  $v$  with respect to  $\{\hat{U}_e\}_{e \in \omega}$ ,  $\{\tilde{V}_e\}_{e \in \omega}$ .*

*iff*

*infinitely many elements of  $B_i$  have *e*-state  $v$  with respect to  $\{\hat{U}_e\}_{e \in \omega}$ ,  $\{\tilde{V}_e\}_{e \in \omega}$ .*

To prove (2.2) it is natural to take  $U_e = W_e$  and  $V_e = W_e$ . Then whenever  $x$  appears in  $V_{e,s} - A_s$  we put  $x$  into  $\hat{V}_e$ . It is then clear that the definition of *e*-splitting is precisely what is needed to satisfy the hypothesis of (2.3) and hence we can extend to an automorphism  $\Phi$  taking  $A_i$  to  $B_i$ .  $\square$ (2.2)

We remark that the notion of e-split is not invariant as we now see. First, if  $A_1 \sqcup A_2$  is a t-splitting of  $A$  then both  $\bar{A}_1$  and  $\bar{A}_2$  are semilow. (Recall that  $\bar{C}$  is semilow if  $\{e: W_e \cap \bar{C} \neq \emptyset\} \leq_T \emptyset'$ .) To see this let  $A_1 \sqcup A_2$  be a t-split of  $A$ . Define

$$g_i(e, s) = \begin{cases} 1 & \text{if } (\exists x)(x \in W_{e,s} - A_{i,s}), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that as  $A_1 \sqcup A_2 = A$  is a t-split,  $\lim_s g(e, s) = g(e)$  exists and  $g(e) = 1$  if  $W_e \cap \bar{A}_1 = \emptyset$ . Hence, by the limit lemma,  $\bar{A}_1$  is semilow. The relevance of this is the following

(2.4) **Theorem** (Downey, Jockusch, Lemma and Stob (unpublished)). *If  $A$  is hh-simple, then there is a splitting  $A_1 \sqcup A_2 = A$  with neither  $A_1$  nor  $A_2$  semilow.*

**Proof.** See Downey-Stob [7].  $\square$

(2.5) **Corollary.**  *$e$ -,  $t$ -,  $s$ -split are not invariant.*

**Proof.** Let  $H_1 \sqcup H_2 = H_3 \sqcup H_4$  be hemimaximal with  $H_1, H_2$  a t-split (and so semilow) and  $H_3, H_4$  not semilow.  $\square$

In fact an e-split measured relative to some enumeration of the r.e. sets is but a special case of measuring relative to a skeleton. Recall that a recursive collection of r.e. sets  $\{X_e\}_{e \in \omega}$  is called a *skeleton* if  $(\forall e)(\exists i)(W_e =^* X_i)$ . All of our splitting notions can be generalized to skeletons. To indicate these more general notions we append a  $*$ . Hence an  $e^*$ -split  $A = A_1 \sqcup A_2$  is one such that for some enumeration of some skeleton  $\{X_e\}_{e \in \omega}$  we have that for all  $\eta$

$$|\eta \restriction_e A| = \infty \Rightarrow |\eta \restriction_e A_i| = \infty.$$

We thus have

(2.6) **Theorem.** *If  $A_1 \sqcup A_2 = A$  and  $B_1 \sqcup B_2 = A$  are  $e^*$ -splits of  $A$  then there is an automorphism  $\Phi$  of  $\mathcal{E}$  with  $\Phi(A_1) = B_1$ .*

We remark that (2.6) generalizes our proof in [5] that hemimaximal sets form an orbit, since the proof there actually shows that hemimaximal sets are  $e^*$ -splits relative to some skeleton and this is preserved under the proof of Soare's [22] theorem that maximal sets form an orbit in  $\text{Aut}(\mathcal{E}^*)$ .

Later we will see that  $e^*$ -splits are also noninvariant. Despite the noninvariance of these notions, they are quite useful in generating automorphisms. They are also related to other well-studied notions as we shall now see.

Recall that a simple set  $A$  is called *promptly simple* if there is a recursive  $f$  such that for all  $e$ ,

$$|W_e| = \infty \Rightarrow (\exists^\infty s, x)(x \in W_{e,at,s} \cap A_{p(s)}).$$

Along similar lines we can call a splitting  $A_1 \sqcup A_2$  of a promptly simple set  $A$  a *prompt splitting* if for all  $e$ ,

$$|W_e \searrow A| = \infty \Rightarrow (\exists^{\infty} s, x)(x \in W_{e,at,s} \cap A_{i,p}(s)).$$

**(2.7) Theorem.** (i) *If  $A$  is a promptly simple set, then  $A_1 \sqcup A_2 = A$  is a prompt splitting iff  $A_1 \sqcup A_2 = A$  is an e-splitting.*

(ii) *If  $A$  has promptly simple degree and  $A_1 \sqcup A_2 = A$  is an e-split, then both  $A_1$  and  $A_2$  have promptly simple degree.*

**Proof.** (i) ( $\Rightarrow$ ) suppose  $A_1 \sqcup A_2 = A$  is a prompt splitting. We need an enumeration of the r.e. sets so that for all states  $\eta$ ,

$$(2.8) \quad \text{if } |\eta \searrow_e A| = \infty \quad \text{then} \quad |\eta \searrow_e A_i| = \infty.$$

So for any state  $\eta$  (with the standard enumeration) if we see some  $x \in \eta - A_s$  we enumerate  $x$  into a test set  $V_\eta$ . By the slowdown lemma [24, Ch. XIII, 1.5], there is an index  $h(\eta)$  and a set  $W_{h(\eta)}$  so that  $V_\eta = W_{h(\eta)}$  and  $x$  enters  $W_{h(\eta)}$  at some  $t$  later than  $s$ . At this time see if  $x \in A_{1,p}(t)$ .

If  $|\eta \searrow A| = \infty$  then  $|W_{h(\eta)}| = \infty$  and either  $W_{h(\eta)} \subseteq A$  or  $|W_{h(\eta)} \searrow A| = \infty$ . Thus, to get (2.8), in the former case we suitably slow down the enumeration of those  $W_i$  in state  $\eta$  and in the latter case, we slow down the enumeration of  $A$ .

( $\Leftarrow$ ) Suppose that  $A_1 \sqcup A_2 = A$  is an e-split of  $A$  and  $A$  is promptly simple with witness  $f$ . Further assume that  $f$  is monotone and we can assume enumerations of r.e. sets so that at most one element enters at most one set at one stage. We will define a function  $g$ . At stage  $s$  suppose  $x$  enters  $W_{e,s}$ . Put  $x$  into  $Y_{e,f(s)+1}$  and hence into  $W_{h(e),t}$  for some  $t > s$ . If  $x$  has entered no  $W_{j,u}$  for some  $s \leq u \leq t$  and  $j \neq e$  with  $j < s$ , define  $g(s) = f(t)$ . Otherwise if  $x$  enters  $W_j$  put  $x$  similarly into  $Y_j$ , and hence into  $W_{h(i),t_1}$ , some  $t_1 > t$ . Now we either define  $g(s) = f(t_1)$  or we continue with another  $Y_k$ .

We claim that  $g$  witnesses that  $A_1 \sqcup A_2$  is a prompt splitting. So suppose that  $|W_e \searrow A| = \infty$ . Hence there exist infinitely many  $x, s$  such that  $x \in W_{e,at,s} \cap A_{f(s)}$ . Suppose that  $|W_e \searrow A| = \infty$  yet  $W_{e,at,s} \cap A_{1,g(s)} = \emptyset$ . Then all those  $x \in W_{e,at,s} \cap A_{f(s)}$  which enter  $A$  by stage  $g(s)$  must enter  $A_2$  and not  $A_1$ . By construction, all those remaining elements are lifted into a higher m-state by  $W_{h(e)}$ . It follows that for some m-state  $\eta$ , while  $|\eta \searrow_e A| = \infty$ , we have  $|\eta \searrow_e A_1| = \emptyset$ ; namely for some  $\eta$  with  $\eta(h(e)) = 0$ .

(ii) This is similar to (i) ( $\Leftarrow$ ). Thus suppose  $A = A_1 \sqcup A_2$  is an e-splitting of a set of promptly simple degree. Thus, there is a recursive function  $f$  such that

$$|W_e| = \infty \Rightarrow (\exists^{\infty} x, s)(x \in W_{e,at,s} \quad \text{and} \quad A[x] \neq A_{f(s)}[x]).$$

(See [24, Ch XIII, 1.6].)

Now for each  $e$  we shall build two r.e. sets  $X_e = W_{h(e)}$  and  $Y_e = W_{k(e)}$ . Initially  $X_{e,0} = \emptyset$ . For the first attack, we had a number  $x_{0,0}$  that occurs in  $W_{e,s_{0,0}}$  and put

all those  $z \leq x_{0,0}$  into  $X_{e,s_{0,0}}$  and hence into  $W_{k(e),t_{0,0}}$ . We continue to do this when  $x_{0,i} (>x_{0,i-1})$  occurs in  $W_e$  until by stage  $f(t_{0,i})$  we see  $A$  permit  $[x_{0,i}]$ . At this stage we put all of the numbers  $z \leq x_{0,i} = x_0$  into  $Y_{f(t_{0,i})+1}$  and hence into  $W_{k(e),u_0}$ . Define  $g(s_{0,j}) = f(t_{0,j})$  for  $j < i$  and  $g(s_{0,i}) = f(u_0)$ .

Now for the second, and subsequent attacks, we need a permission in  $A$  via some  $z$  with  $x_{i-1} \leq z \leq x_j$ , before we enumerate all of  $x_{i-1}, \dots, x_j$  into  $Y$  as above.

Thus, we generate a sequence  $x_0, x_1, x_2, \dots$ , and a sequence  $g(s_0), g(s_1), g(s_2), \dots$ . Note that  $g$  is total (and mostly it equals  $f$ ). We claim it witnesses the promptness of  $A_1 \sqcup A_2$ . Suppose that none of the  $z$  that promptly enter  $A$  enter  $A_1$ . Then as with (i) we will have raised the states of the remaining elements via  $Y_e = W_{k(e)}$  and hence contradict the fact this is an e-splitting.  $\square$

One of the reasons that the above is interesting comes from analyzing Maass' [14] theorem that all promptly simple sets with  $\text{semilow}_{1.5}$  complements are effectively automorphic. Such an effective automorphism carries the property of being an e- ( $e^*$ -) split. Hence

**(2.9) Theorem.** *Let  $A$  and  $B$  be promptly simple sets with  $\text{semilow}_{1.5}$  complements. Suppose  $A = A_1 \sqcup A_2$  and  $B = B_1 \sqcup B_2$  are e-splits ( $e^*$ -splits). Then there is an effective automorphism (resp. automorphism) of  $\mathcal{E}^*$  taking  $A_i$  to  $B_i$  for  $i = 1, 2$ .*

We can also use the proof of Cholak et al. [2] to show that if  $A$  is half of an  $e^*$ -splitting of a promptly simple set, then  $A$  is automorphic to a complete set. This last result also follows from (2.7) (ii) and work of Harrington and Soare [12] who showed that a set of promptly simple degree is effectively automorphic with a complete set.

As a sort of converse to (2.7) we have the following.

**(2.10) Theorem.** *There exists an r.e. set  $A$  such that*

- (i) *if  $B \neq_{\text{T}} \emptyset$  there is an s-splitting  $A_1 \sqcup A_2 = A$  of  $A$  with  $B \equiv_{\text{T}} A_1$ , and*
- (ii) *if  $B$  is promptly simple there is an e-Friedberg splitting of  $A$  with  $A_1 \equiv_{\text{T}} B$ .*

**Proof.** (i) We build  $A, \{C_e, D_e\}_{e \in \omega}$  to meet

$$\begin{array}{l}
 R_e: \quad \text{Either } W_e \equiv_{\text{T}} \emptyset \text{ or} \\
 \quad [C_e \sqcup D_e = A \text{ and } C_e \equiv_{\text{T}} W_e \text{ and } (\forall i)(N_{e,i} \text{ and } \hat{N}_{e,i})]. \\
 N_{e,\eta}: \quad |\eta \searrow A| = \infty \Rightarrow |\eta \searrow C_e| \neq 0, \\
 \hat{N}_{e,\eta}: \quad |\eta \searrow A| = \infty \Rightarrow |\eta \searrow D_e| \neq 0.
 \end{array}$$

Note that if we meet all of the above automatically  $A \equiv_{\text{T}} K$ .

*The coding strategy.* If  $x \in W_e$  we put  $\langle e+1, x, z \rangle$  into  $C_e$  for some  $z \leq e+x+1$ . Thus  $W_e \leq_{\text{T}} C_e$  by direct coding.

*Meeting*  $N_{e,\eta}$ . Wait till we see some  $y \in \eta_s \setminus A_s$ , with  $y > \langle \langle e, i \rangle + 1, \langle e, i \rangle + 1, \langle e, i \rangle + 1 \rangle$ . (This is chosen so that we will not use up a block  $\langle j + 1, x, 0 \rangle, \dots, \langle j + 1, x, x \rangle$  by priorities.) We declare  $y$  to follow  $N_{e,\eta}$  and deny it from lower priority requirements. This means  $y$  is not available for coding and is why the  $z$  in the coding strategy is used. Whilst  $N_{e,\eta}$  is not yet met, we wish to pick more followers of  $N_{e,\eta}$ . The next  $y_1 > y$  to follow  $N_{e,\eta}$  will be chosen with  $y_1 \in \eta$  and  $y_1 > \max\{\langle j, x, z \rangle, s\}$  for all  $z \leq x$  if  $y = \langle j, x, \hat{z} \rangle$  and  $\hat{z} \leq x$ . Note that this gives  $y < y_1 < y_2 < \dots$  and each  $N_{e,\eta}$  takes at most one element per block.

We then finish by waiting for a stage  $u$  where  $W_e$  permits  $y_e$  at  $u$ . We can then enumerate such a  $y_e$  into  $C_e$ . (All other entries into  $A$  must enter  $D_e$ .) Note that there is no conflict between  $N_{e,\eta}$  and  $N_{j,\tau}$  but only between  $N_{e,\eta}$  and  $\hat{N}_{e,\tau}$  and between  $N_{e,\eta}$  and the coding strategy. It is easy that a gentle finite injury argument will do the rest.

(ii) To prove (ii), we additionally meet

$R_e$ : Either  $W_e$  is promptly simple via  $\varphi_e$  or there exists an enumeration of the r.e. sets and  $C_e \sqcup D_e = A$  such that  $(\forall i)(N_{e,\eta}$  and  $\hat{N}_{e,\eta})$  where now  $N_{e,\eta} = |\eta \searrow_e A| = \infty \Rightarrow |\eta \searrow_e C_e| \neq 0$  and  $\hat{N}_{e,\eta}$  similarly.

Remember we get to control the enumerations of the r.e. sets for each  $e$ . Now the strategy for  $N_{e,\eta}$  works as follows. We wait till some  $y$  occurs in  $\eta_s - A_s$  and then use  $\varphi_e$  to decide if  $W_e$  ‘promptly permits’  $y$ . Specifically we enumerate  $y$  into some test set  $V_e = W_{h(e)}$  and await the stage  $t$  where  $y$  occurs in  $W_{h(e),at}$ . For this process at stage  $s$  as above  $y$  is declared to be inaccessible to the other requirements until we see if  $W_e$  promptly permits  $y$  by stage  $\varphi_e(t)$ .

The reader should keep in mind that the states that the  $N_{e,\eta}$  for  $e$  fixed work with are controlled by  $\varphi_e(t)$ . That is, for all stages  $u$  with  $s \leq u \leq p$  where  $\varphi_{e,p}(t) \downarrow$  and  $t$  least, we allow no enumeration into any  $W_k$ . Hence for  $e$  fixed, the  $N_{e,\eta}$  and  $\hat{N}_{e,\eta}$  work with sets  $W_k^e$  based on the belief that  $\varphi_e$  is total. It is clear that if  $\varphi_e$  really is not total then we do not need  $R_e$  anyway. If  $\varphi_e$  is not total, then a stage  $p$  as above will occur and hence we can restart the enumerations of the r.e. sets. In this way it can be seen that, with this enumeration, we win all the  $N_{e,\eta}$  and  $\hat{N}_{e,\eta}$  if  $\varphi_e$  really witnesses the prompt simplicity of  $W_e$ .  $\square$

Note that since all r.e. sets can be e-split it follows that not all r.e. degrees containing e-splits of some set are promptly simple. Nevertheless, there are a lot of restrictions on the degrees of e-splits.

(2.11) **Theorem.**  $(\forall A)(\exists \mathbf{b})(\mathbf{b} \neq \mathbf{0} \ \& \ \mathbf{b} < \text{deg}(A) \ \& \ A_1 \sqcup A_2 = A \ \text{with} \ \text{deg}(A_1) \leq_{\tau} \mathbf{b} \Rightarrow A_1 \sqcup A_2 \ \text{is not an e-splitting of } A).$

**Proof.** Let  $A = \bigcup_s A_s$  be a given r.e. nonrecursive set. We build  $B \leq_{\tau} A$  by permitting

$R_e$ : If  $C_e \sqcup D_e = A$  and  $I_e(B) = C_e$  then  $C_e \sqcup D_e$  is not an e-splitting.

Here we work over triples  $\langle C_e, D_e, I_e \rangle_{e \in \omega}$ . For the sake of  $R_e$ , we build sets  $X_e$  and  $Y_e$  again given as  $W_{h(e)}$ ,  $W_{k(e)}$  via the recursion theorem (without loss of generality,  $h(e) > k(e)$ ), with enumerations given by the slowdown lemma. Now, let  $l(e, s)$  be the  $B$ -controllable length of agreement. That is, define

$$l(e, s) = \max\{x: (\forall z < x)(I_{e,s}(B_s; z) = C_{e,s}(z) \sqcup D_{e,s}(z) = A_s(z))\}.$$

Let  $\gamma_{e,s}(z)$  denote the use of  $I_{e,s}(B_s; z)$ . As usual, we let  $I_e(B)$  control the enumeration of  $C_e$ . We aim to meet

$$P_e: \quad \bar{B} \neq W_e.$$

The sets  $X_e = W_{h(e)}$  and  $Y_e = W_{k(e)}$  are used to meet the requirements  $R_e$  as follows. We will argue there is some state  $\eta$  of length  $h(e)$  such that  $|\eta \restriction_e A| = \infty$  yet  $|\eta \restriction_e C_e| = 0$ . This state  $\eta$  will have  $\eta(h(e)) = 1$  but  $\eta(k(e)) = 0$ . This is done by preserving  $B_s[\gamma_{e,s}(l(e, s))]$ , while elements enter  $A$ . That is at a stage where we see  $l(e, s) > z$  we put  $z$  into  $X_e$  and thus into  $W_{h(e),t(s)}$ . Now if we preserve  $B_s[\gamma_{e,s}(t)]$  then if such an element enters  $A$  it can only go into  $D_e$  not  $C_e$ .

One easy way to ensure the  $R_e$  is met is to combine this with the necessity of meeting infinitely many subrequirements  $\{R_{e,i}: i \in \omega\}$ . We will believe  $R_{e,i}$  is met if  $>i$  elements  $<l(e, s)$  have entered  $A$  and hence  $D_e$  in any state  $\tau$  with  $\tau(h(e)) = 1$  yet  $\tau(k(e)) = 0$ . Note that if  $(\forall i)(R_{e,i})$  then for some state  $\eta$  of length  $h(e)$  we have  $\eta(h(e)) = 1$ ,  $\eta(k(e)) = 0$  yet  $|\eta \restriction_e A| = \infty$  but  $|\eta \restriction_e C_e| = 0$ .

The only problem with the above is if some  $P_k$  of higher priority than  $R_e$  yet of lower priority than  $R_{e,i}$  wishes to enumerate an element into  $B$ .

Thus we have some follower  $x$  permitted by  $A$  at stage  $s$  that we wish to put into  $B$ . To do so immediately would cause us to lose  $B$ -control of  $C_e$ . So that we won't lose  $R_e$  we must make sure that no elements can enter  $C_e$  in a state  $\tau$  with  $\tau(k(e)) = 0$ . Thus, the idea is that we must first raise the length  $h(e)$  states of potentially injurious elements before so that it is irrelevant if they enter  $C_e$ . To do this, for each  $y$  if  $x < \gamma_{e,s}(y)$ , we enumerate  $y$  into  $Y_e = W_{k(e)}$  and await a stage  $t$  where  $y$  enters  $W_{k(e),t}$  before we put  $x$  into  $B_{t+1}$ . Such delay is fine since  $y$  must enter  $W_{k(e)}$  via the recursion theorem.

The details are to then combine the above with the finite injury technique.  $\square$

(2.12) **Problem.** Classify the degrees of e-splits in terms of  $\text{deg}(A)$ .

The solution would seem to lie in some form of relative prompt simplicity.

### 3. Simple sets, p-splits etc.

The results of Section 2 did not answer our original question motivating our investigation: *Are all Friedberg splits of a simple set automorphic?* (see [24, Ch. XVI, Q1.1.3], [5]). One motivation for this question is that f-splits of simple sets

share so many properties. As Downey [4] observed, if  $A_1 \sqcup A_2 = A$  is an f-split of a simple set, then  $(A_1, A_2)$  form a *maximal pair* (that is, if  $W_e \cap V_e = \emptyset$ ,  $W_e \supseteq A_1$  and  $V_e \supseteq A_2$  then  $|W_e - A_1| < \infty$  and  $|V_e - A_2| < \infty$ ). From this it is easy to deduce that  $A_1$  and  $A_2$  are effectively nowhere simple. The reader should recall that an r.e. set  $A$  is called *nowhere simple* [21] if  $|\bar{A}| = \infty$  and for all  $e$ , if  $|W_e - A| = \infty$  then there is an  $i$  with  $W_i \subseteq W_e - A$  and  $|W_i| = \infty$ . A set is called effectively nowhere simple if the index  $i$  can be computed from one for  $e$ . One characterization of effectively nowhere simple sets is that there is an infinite r.e. set  $B$  such that  $B \cap A = \emptyset$  and for all  $e$ , if  $|W_e - A| = \infty$  then  $|W_e \cap B| = \infty$  (Miller and Remmel [20]). Actually it is not difficult to see that any f-split of an r.e. set gives nowhere simple sets.

(3.1) **Theorem.** *Suppose  $A_1 \sqcup A_2 = A = B_1 \sqcup B_2$  are f-splittings. Then*

- (i)  $A_1$  and  $A_2$  are nowhere simple.
- (ii) (with R. Shore) *Further, if  $A_1$  is effectively nowhere simple, then so too is  $B_1$ .*

**Proof.** (i) Let  $W$  be an infinite r.e. set with  $|W - A| = \infty$ . Now, if  $W - A$  is r.e. we are done. If not, then  $W - A$  is not r.e., and hence  $|W \cap A_2| = \infty$ . Thus, in either case  $W$  has an infinite subset disjoint from  $A_1$  and hence  $A_1$  is nowhere simple.

(ii) If  $A_1$  is effectively nowhere simple, then there is an r.e. set  $B$  such that

$$B \cap A_1 = \emptyset \text{ and if } |W - A_1| = \infty \text{ then } |W \cap B| = \infty.$$

We claim  $B - A$  is r.e. If not then as  $A_1 \sqcup A_2 = A$  is an f-split,  $B \cap A_2 \neq \emptyset$ , contradiction. Consequently we let  $C = (B - A) \cup B_2$ . Let  $W$  be an r.e. set with  $|W - B_1| = \infty$ . If  $W - A$  is not r.e., then  $|W \cap B_2| = \infty$  and hence  $|W \cap C| = \infty$ . If  $W - A$  is r.e., let  $V$  be  $W - A$ . Then  $|V - A_2| = \infty$  and hence  $|V \cap B| = \infty$ . But then as  $V \cap A = \emptyset$ ,  $|V \cap (B - A)| = \infty$  and hence  $|V \cap C| = \infty$  so that  $|W \cap C| = \infty$ . Thus  $C$  witnesses the effective nowhere simplicity of  $B_1$ .  $\square$

The reader should also recall that all effectively nowhere simple sets have semiow<sub>1,5</sub> complements, and hence by [14] are all effectively isomorphic to  $\mathcal{G}^*$ . Thus, in particular, all f-splits of simple sets exhibit deep similarities.

Nevertheless, despite these similarities not all s-splits of (even) a simple set need be automorphic.

(3.2) **Definition.** A splitting  $A_1 \sqcup A_2 = A$  is called a *d-splitting* if, for all r.e. sets  $X$ , there is an r.e. set  $Y \subseteq X$  and  $X - A = Y - A$ , such that for all r.e. sets  $W$  if  $W - (X \cup A)$  is not r.e. then  $(W - Y) \cap A_j \neq \emptyset$  for  $j = 1, 2$ .

Note that a d-splitting is an f-splitting by setting  $X = \emptyset$ .

(3.3) **Theorem.** (i) *There is a simple r.e. set  $A$  with  $A_1 \sqcup A_2 = B_1 \sqcup B_2 = A$  such that  $A_1 \sqcup A_2$  is a  $d$ -split and  $B_1 \sqcup B_2$  is an  $s$ -split that is not a  $d$ -split.*

(ii) *Consequently Friedberg splittings of an r.e. set can realize different elementary types.*

**Proof.** We construct  $A_1, A_2, B_1, B_2$  together with auxiliary sets  $Y_e, Q$  and  $M_e$  to meet

$$P_e^j: \quad |W_e \searrow A| = \infty \Rightarrow W_e \cap B_j \neq \emptyset.$$

Actually, the  $P_e^j$  above only makes an  $f$ -split, but it is routine to modify the below to achieve  $t$ -splitting. We stick to the above for simplicity.

$$R_e: \quad Y_e \subseteq X_e, \quad X_e - A = Y_e - A \quad \text{and} \quad (\forall i, j)(R_{e,i,j})$$

$$R_{e,i,j}: \quad (W_i - (X_e \cup A)) \text{ not r.e. implies } (W_i - Y_e) \cap A_j \neq \emptyset.$$

Here  $\{X_e\}$  is an enumeration of all r.e. sets.

$$P_e: \quad |Q - A| \geq e.$$

$$N_e: \quad (W_e \not\subseteq Q) \vee ((W_e - A) \neq (Q - A)) \vee [(M_e - W_e) \cap B_1 < \infty]$$

and  $(\forall i)(N_{e,i})$  where

$$N_{e,i}: \quad |M_e - (Q \cup A)| \geq i.$$

$$D_e: \quad |W_e| = \infty \Rightarrow W_e \cap A \neq \emptyset.$$

*The basic strategies*

*For  $R_{e,i,j}$ .* To meet  $R_{e,i,j}$  whilst  $(W_{i,s} - Y_{e,s}) \cap A_{j,s} = \emptyset$  we wait till we see some  $z$  in  $W_{i,s} \setminus (X_{e,s} \cup A_s)$  and put  $z$  into  $A_j$ , meeting  $R_{e,i,j}$  forever. Note that if no such  $z$  exists then  $|W_e - (X_e \cup A)| < \infty$ .

*For  $P_e^j$ .* We will treat these as 'active requirements' and if we see some  $z \in W_{e,s} \setminus A_s$  with  $z \geq \langle e, j \rangle$  (i.e.,  $z$  unstrained, this reflects the priority) then if  $W_{e,s} \cap B_{j,s} = \emptyset$  we put  $z$  into  $A_{j,s+1}$ .

*For  $P_e$ .* We ensure that  $e$  things are added to  $Q$  and protected from addition to  $A$ .

*For  $D_e$ .* If we see something in  $W_e$ , put it into  $A$ .

*For  $N_e$ .* We attempt to meet  $N_{e,i}$ . If we fail to do so then we will argue that one of  $W_e \not\subseteq Q$  or  $(W_e - A) \neq (Q - A)$  must hold.

*For  $N_{e,i}$ .* We simply put something into  $M_e$  and keep it from  $Q_s \cup A_s$ .

*Conflicts*

The conflicts between the strategies are as follows: First there are no conflicts between  $D_e$  and either  $N_e$  or  $R_e$  since these requirements only wish to put numbers into  $A_j$  and care nothing of the  $B_k$ . There is a conflict between  $N_e$  and  $P_e^i$  though since we may wish to put some  $z$  into  $B_1$  and this  $z$  has been enumerated into  $M_{e,s}$  for the sake of some  $N_{e,t}$ . Assuming  $P_e^1$  has higher priority than  $N_{e,i}$  but

lower 'global' priority than  $N_e$  (this is the only priority ordering that causes difficulties), we overcome this conflict by *squeezing*  $N_e$ . That is, we note that we can put  $z$  into  $B_1$  provided  $z \in W_e$ . Now, if  $W_e - A \neq Q - A$ , we get a global win on  $N_e$  hence the idea is to put  $z$  into  $Q$  first and wait till  $z$  enters  $W_e$  before we put  $z$  into  $A$ . If  $z$  so enters  $W_e$  then we are free to put  $z$  into  $B_1$ .

We now give some formal details, although this is really a relatively straightforward  $\mathbf{0}''$  argument.

Let  $T = 2^{<\omega}$ . Call members of  $T$  *guesses*, with  $\leq_L$  lexicographical ordering. Assign requirements to guesses as follows. Let  $\text{lh}(\sigma) = 8e + i$ . If  $i = 0$ , assign  $\sigma$  to  $D_e$ . If  $i = 1$ , assign  $\sigma$  to  $P_e^1$  and if  $i = 2$ , assign  $\sigma$  to  $P_e^2$ . If  $i = 3$ , assign  $\sigma$  to  $P_e$ . The other assignments are accomplished via lists  $L_k(\sigma)$  for  $k = 1, \dots, 4$  inductively as follows. Initially  $L_k = \omega$ . Make no changes except as follows.

*Case 1:*  $i = 4$ . Assign  $N_e$  to  $\sigma$  for  $e = \mu z (z \in L_1(\sigma))$ . Let  $L_1(\sigma \smallfrown j) = L_1(\sigma) - \{e\}$  for  $j = 0, 1$ . Let  $L_2(\sigma \smallfrown 1) = L_2(\sigma) - \{\langle e, i \rangle\}$ :  $i \in \omega$ . ( $N_e$  is assigned to  $\sigma$  as its primary node.)

*Case 2:*  $i = 5$ . Assign  $N_{e,i}$  to  $\sigma$  for  $\langle e, j \rangle = \mu z (z \in L_2(\sigma))$  and  $e \in L_1(\sigma)$ . Let  $L_2(\sigma \smallfrown j) = L_2(\sigma) - \{\langle e, i \rangle\}$  for  $j = 0, 1$ .

*Case 3:*  $i = 6$ . Assign  $R_e$  for  $e = \mu z (z \in L_3(\sigma))$  to  $\sigma$ . Let  $L_3(\sigma \smallfrown j) = L_3(\sigma) - \{e\}$ .

*Case 4:*  $i = 7$ . Assign  $R_{e,i,j}$  for  $\langle e, i, j \rangle = \mu z (z \in L_4(\sigma))$  and  $e \in L_3(\sigma)$  and  $j \in \{1, 2\}$  to  $\sigma$ . Let  $L_4(\sigma \smallfrown j) = L_4(\sigma) - \{\langle e, i, j \rangle\}$ .

To indicate  $e$  has been assigned to  $\sigma$  we write  $e(\sigma) = e$ .

In the construction to follow, we work in substages  $t$  of stage  $s$ . We write this as stage  $(s, t)$ . In the construction we will define a string  $\sigma(s, t)$ . We say a stage  $s$  is a  $\sigma$ -stage if  $\sigma \subseteq \sigma(s, t)$  some  $t$ , and  $s$  is a *genuine  $\sigma$ -stage* if  $\sigma = \sigma(s, t)$  for some  $t$ . If  $\text{lh}(\sigma) \equiv 5$  (8), define  $\tau(\sigma)$  to be the unique  $\tau \subseteq \sigma$  such that  $e(\sigma) = e(\tau)$  and  $\text{lh}(\tau) \equiv 4$  (8). Similarly, if  $\text{lh}(\sigma) \equiv 7$  (8), define  $\tau(\sigma)$  to be the unique  $\tau \subseteq \sigma$  such that  $e(\tau) = e(\sigma)$  and  $\text{lh}(\tau) \equiv 6$  (8). If  $\text{lh}(\sigma) \equiv 4$  (8), define a stage  $s$  to be  $\sigma$ -*expansionary* if  $s$  is a genuine  $\sigma$ -stage and for  $e = e(\sigma)$ ,  $W_{e,s} \subseteq Q_s$  and  $W_{e,s} - A_s = Q_s - A_s$ . Note that we need only consider those  $W_e \subseteq Q$  so that we can suppose, without loss of generality,  $(\forall s)(W_{e,s} \subseteq Q_s)$ . If  $\text{lh}(\sigma) \equiv 6$  (8), we say  $s$  is  $\sigma$ -*expansionary* if  $X_{e,s} - A_s \neq Y_{e,s} - A_s$  for  $e = e(\sigma)$ . We append a superscript  $t$  to a parameter to give its value at the end of substage  $t$ . We also use the standard notation of *initializing*. At stage  $s$  we let  $\{a_{i,s} : i \in \omega\}$  list  $\bar{A}_s$ .

### Construction

*Stage*  $(s + 1, 0)$ . If  $W_{0,s} \cap A_s = \emptyset$  and  $(\exists z)(z \in W_{0,s}$  and  $z > a_{e,s})$  put  $z$  into  $A_{s+1} - A_s$ . Declare  $\sigma(s + 1, s + 1) = 0$  and initialize all  $\gamma \in T$  with  $\text{lh}(\gamma) > 1$ . Go to stage  $s + 2$ .

Otherwise, if  $W_{0,s} \cap A_s = \emptyset$  set  $\sigma(s + 1, 0) = 1$ ; if  $W_{0,s} \cap A_s \neq \emptyset$ , set  $\sigma(s + 1, 0) = 0$ .

*Stage*  $(s + 1, t + 1)$ . Adopt the first case to pertain. Let  $\sigma = \sigma(s + 1, t)$ . Let  $e = e(\sigma)$  and  $i = i(\sigma)$ .

Case 1:  $\text{lh}(\sigma) \equiv 0$  (8).

Subcase 1.  $W_{e,s} \cap A_s = \emptyset$  and  $(\exists x)(x \in W_{e,s} \text{ and } x > a_{\text{lh}(\sigma),s} \text{ and } z > r(\tau, s))$  for all  $\tau \leq_L \sigma$ .

Action. Put  $x$  into  $A_{2,s+1}$  and  $B_{2,s+1}$ . Let  $\sigma(s+1, s+1) = \sigma^{\frown} 0$  and initialize all  $\tau \not\leq \sigma(s+1, s+1)$  going to stage  $s+2$ .

Subcase 2. Otherwise. Let  $\sigma(s+1, t+1) = \sigma^{\frown} i$  with  $i = 0$  iff  $W_{e,s} \cap A_s \neq \emptyset$ .

Case 2:  $\text{lh}(\sigma) \equiv 1$  (8).

Subcase 1. There exists  $x \in W_{e,s} - A_s$ ,  $x > a_{\text{lh}(\sigma),s}$ , and  $x > r(\tau, s)$  for  $\tau \leq_L \sigma$ , and  $B_{1,2} \cap W_{e,s} = \emptyset$ .

Action. See if there is a  $\tau \leq_L \sigma$  with  $\text{lh}(\tau) = 4$  (8) and  $x \in M_{\tau,s} - W_{e(\tau),s}$ .

If no such  $\tau$  exists: Put  $x$  into  $B_{1,s+1}$  and  $A_{1,s+1}$ . Set  $\sigma(s+1, s+1) = \sigma^{\frown} 0$  and initialize all  $\gamma \not\leq \sigma^{\frown} 0$ . Go to stage  $s+2$ .

If  $\tau$  exists: Put  $x$  into  $Q_{s+1} - Q_s$ , set  $r(\sigma, s+1) = x$ . Set  $\sigma(s+1, s+1) = \sigma^{\frown} 0$  and initialize all  $\gamma \not\leq \sigma^{\frown} 0$ . Go to stage  $s+2$ .

Subcase 2. Otherwise. Let  $\sigma(s+1, t+1) = \sigma^{\frown} i$  with  $i = 0$  iff  $B_{1,s} \cap W_{e,s} \neq \emptyset$ .

Case 3:  $\text{lh}(\sigma) \equiv 2$  (8).

Subcase 1. There exists  $x \in W_{e,s} - A_s$ ,  $x > a_{\text{lh}(\sigma),s}$ ,  $x > r(\tau, s)$  for  $\tau \leq_L \sigma$  and  $B_{2,s} \cap W_{e,s} = \emptyset$ .

Action. Put  $x$  into  $B_{2,s+1}$  and  $A_{1,s+1}$ .

Subcase 2. Otherwise.

Action. Let  $\sigma(s+1, t+1) = \sigma^{\frown} i$  with  $i = 0$  iff  $B_{2,s} \cap W_{e,s} \neq \emptyset$ .

Case 4:  $\text{lh}(\sigma) \equiv 3$  (8). If  $q(\sigma, e, s)$  is currently undefined, find a large fresh number  $(s, \text{say})$  and define this to be  $q = q(\sigma, e, s)$ . Put  $q$  into  $Q_{s+1} - Q_s$ . Let  $\sigma(s+1, s+1) = \sigma^{\frown} 0$ . Initialize all  $\gamma \not\leq_L \sigma^{\frown} 0$ . Go to stage  $s+2$ . If  $q(\sigma, e, s)$  is defined, set  $\sigma(s+1, t+1) = \sigma^{\frown} 0$ .

Case 5:  $\text{lh}(\sigma) \equiv 4$  (8). If  $s$  is  $e$ -expansory let  $\sigma(s+1, t+1) = \sigma^{\frown} 0$ . Otherwise let  $\sigma(s+1, t+1) = \sigma^{\frown} 1$ .

Case 6:  $\text{lh}(\sigma) \equiv 5$  (8). If  $m(\sigma, e, i, s)$  is not defined, let  $m = m(\sigma, e, i, s) = s$ . Put  $m$  into  $M_{\tau(\sigma),s+1}$ . Let  $r(\sigma, s+1) = s$  and  $\sigma(s+1, s+1) = \sigma^{\frown} 0$ . Initialize all  $\tau \not\leq \sigma^{\frown} 0$ .

Case 7:  $\text{lh}(\sigma) \equiv 6$  (8). Let  $\sigma(s+1, t+1) = \sigma^{\frown} i$  with  $i = 0$  iff  $s+1$  is  $\sigma$ -expansory.

Case 8:  $\text{lh}(\sigma) \equiv 7$  (8). If  $(W_{i,s} - Y_{\sigma,s}) \cap A_{j,s+1} = \emptyset$  and you see some  $x \in W_{i,s} - (A_s \cup Y_{\tau(\sigma),s})$  and  $x > r(\gamma, s)$  for  $\gamma \leq_L \sigma$ , put  $x$  into  $A_{j,s+1}$  and  $B_{2,s+1}$ . Let  $\sigma(s+1, s+1) = \sigma^{\frown} 0$  and initialize all  $\eta \not\leq_L \sigma^{\frown} 0$ . Otherwise, let  $\sigma(s+1, t+1) = \sigma^{\frown} i$  with  $i = 0$  iff  $(W_{i,s} - Y_{\sigma,s}) \cap A_{j,s} \neq \emptyset$ .

To conclude stage  $s$  of the construction, initialize all  $\gamma \not\leq \sigma(s+1, s+1)$  and for any  $\sigma \subseteq \sigma(s+1, s+1)$  with  $\text{lh}(\sigma) \equiv 6$  (8) make  $X_{\sigma,s+1} - A_{s+1} = Y_{\sigma,s+1} - A_{s+1}$  by enumeration into  $Y_{\sigma,s+1}$ . End of construction.

### Verification

The details of the verification are more or less routine so we will be brief. First  $A_1 \sqcup A_2 = B_1 \sqcup B_2 = A$  by force. We always put any  $x$  in  $A$  into one side or the other.

Let  $TP$  denote the true path of the construction, i.e., the leftmost path visited infinitely often. We need to argue that for  $\sigma \subseteq TP$ ,  $\lim_r r(\sigma, s) = r(\sigma)$  exists,  $\lim_s m(\sigma, e, i, s) = m(\sigma, e, i)$  exists,  $q(\sigma, s) = q(\sigma)$  exists and each positive requirement requires attention at most finitely often. This is argued inductively. Suppose the result for  $\sigma^-$  where  $\sigma = \sigma^- \smallfrown i$ . Let, as usual,  $s_0$  be a  $\sigma$ -stage after which we are never above, or to the left of  $\sigma$ . Then if  $\text{lh}(\sigma) \equiv 0$  (8) or  $\text{lh}(\sigma) \equiv 2$  (8) it is clear that we will succeed for  $\sigma$  as there is only a finite restraint. If  $\text{lh}(\sigma) \equiv 1$  (8), then the only reason would fail to meet  $P_\sigma^1$  immediately is that there is a (unique, by construction)  $\tau \subseteq_L \sigma$  with  $\text{lh}(\tau) \equiv 4$  (8) and  $x \in M_{\sigma, s} - W_{e(x), s}$ . This means that, as  $\sigma \subset TP$ ,  $\sigma \supset \tau \smallfrown 0$  and hence there are infinitely many  $\tau \smallfrown 0$ -stages. Then at the next  $\sigma$ -stage  $s_1 \succ s$  we know that since  $x$  will enter  $W_{e(x), s_1}$ , we will be free to add  $x$  into  $B_{1, s_1}$  meeting  $P_\sigma^1$ . It is clear that the  $m(\sigma, e, i, s)$  and  $q(\sigma, s)$  will be defined at the next  $\sigma$ -stage if needed. Finally the  $R_e$  and  $R_{e, i, j}$  are met. If  $\text{lh}(\sigma) \equiv 6$  (8) we know that there will be built a set  $Y_\sigma = X_\sigma$  on  $\bar{A}$  as is forced as the last step of stage  $s$ .

If  $\text{lh}(\sigma) \equiv 7$  (8) then we meet  $R_{e, i, j}$  there. This follows since such  $\sigma$  must extend  $\tau \smallfrown 0$  for  $\tau = \tau(\sigma)$ . The construction ensures that  $\sigma$  gets its chance *before* we force

$$X_\sigma \cap \bar{A} = Y_\sigma \cap \bar{A}. \quad \square$$

The result above obviously leads to the improbable suggestion that perhaps all  $d$ -splittings of a (simple) set are automorphic. Again this is not the case. We parallel some result for ‘ $d$ -simple’ sets of Maass et al. [15].

(3.4) **Definition.** Call a splitting  $A_1 \sqcup A_2 = A$  an *inner splitting* if for all r.e.  $B$ , if  $(B - A)$  is not r.e. then there are  $f$ -splittings  $B = B_1 \sqcup B_2 = \hat{B}_1 \sqcup \hat{B}_2$  such that  $B_1 \subseteq A_1$  and  $\hat{B}_2 \subseteq A_2$ .

The reader should note that (3.4) is parallel to the *splitting property* of Maass et al. [15]. The argument [15] shows

(3.5) **Lemma.** *If  $A_1 \sqcup A_2 = A$  is an inner splitting of  $A$ , then both  $A_1$  and  $A_2$  have promptly simple degree.*

Note that if  $A_1 \sqcup A_2 = A$  is an inner splitting and  $A$  is simple, then  $A$  has the splitting property (namely, for all r.e. nonrecursive  $B$ , there is an  $f$ -splitting  $B_1 \sqcup B_2 = A$  with  $B_1 \subseteq A_1$ ). It is not clear if all sets with the splitting property necessarily have an inner splitting.

We can show

(3.6) **Lemma.** *If  $A = A_1 \sqcup A_2$  is an inner splitting, then it is a  $d$ -splitting.*

**Proof.** The argument is along the lines of Maass et al. [15]. Let  $A_1 \sqcup A_2$  be an inner splitting of  $A$ . Given  $X$  let  $X = X_1 \sqcup X_2 = \hat{X}_1 \sqcup \hat{X}_2$  be the  $f$ -splittings of the

definition with  $X_1 \subseteq A_1$  and  $\hat{X}_2 \subseteq A_2$ . Let  $Y = X_2 \cup \hat{X}_1$ . Claim  $Y$  is the desired set. Now  $Y \subseteq X$  is clear. Also  $Y \cap \bar{A} = X \cap \bar{A}$  since  $X_1, \hat{X}_2 \subseteq A$ . Suppose that  $W - (A \cup X)$  is not r.e. If  $W - X$  is r.e., then as  $W - (A \cup X) = (W - X) - A$  is not r.e.,  $|(W - X) \cap A_j| = \infty$  as  $A_1 \cup A_2 = A$  is an f-splitting. (To see that  $A_1 \sqcup A_2 = A$  is an f-splitting, apply the definition of inner splitting with  $A = B$ .) Hence, in particular,  $(W - Y) \cap A_j \neq \emptyset$ . If  $(W - X)$  is not r.e., then  $(W - X_1)$  is not r.e. (as is  $W - \hat{X}_2$ ) and hence  $W \cap X_1$  is infinite. But  $(W \cap X_1) \subseteq (W - Y) \cap A_1$ . Hence  $(W - Y) \cap A_1 \neq \emptyset$ , and similarly  $A_2$ .  $\square$

This allows us to differentiate between d-splittings.

(3.7) **Theorem.** (i) *There is a (promptly simple) r.e. set  $A$  of low degree and d-splittings  $A_1 \sqcup A_2 = B_1 \sqcup B_2 = A$  such that  $B_1 \sqcup B_2$  is inner yet  $A_1 \sqcup A_2$  is not.*  
(ii) *Therefore d-splittings are not enough to guarantee automorphism.*

**Proof.** We build  $A, Y_e, Q, C_e, D_e$ .

$P_e$ :  $|W_e| = \infty \Rightarrow W_e \cap A \neq \emptyset$  promptly.

$R_{e,i,j}$ :  $Y_e \subseteq X_e$  and  $X_e - A = Y_e - A$  and

$|W_i - (X_e \cup A)| = \infty \Rightarrow |(W_i - Y_e) \cap A_j| \neq \emptyset$ .

(This ensures that  $A_1 \sqcup A_2$  is a d-split as  $A$  is simple.)

$M_e$ :  $\bar{Q} \neq W_e$ .

$S_i$ :  $|Q - A| \geq i$ . (This is enough if  $A$  is simple.)

$T_e$ :  $W_e \cup V_e = Q \Rightarrow W_e \notin A$ , or  $W_e$  recursive.

(This says  $(W_e, V_e)$  is not a witness to  $A_1 \sqcup A_2$  being inner.)

$U_e$ :  $|W_e - A| = \infty \Rightarrow C_e \sqcup D_e = W_e$  and  $C_e \subseteq B_1$  and either

$W_e$  recursive or  $(\forall i)(\bar{U}_{e,i})$  where

$\bar{U}_{2\langle e,i \rangle}$ :  $W_i - W_e$  not r.e.  $\Rightarrow W_i \cap C_e \neq \emptyset$ ,

$\bar{U}_{2\langle e,i \rangle+1}$ :  $W_i - W_e$  not r.e.  $\Rightarrow W_i \cap D_e \neq \emptyset$ .

$\bar{U}_e$ : same for  $B_2$  as  $U_e$  was for  $B_1$ .

The proof is quite similar to (3.3) so we shall only sketch the details. The strategies are clear enough. They are:

*For  $P_e$ .* If we see some  $x$  enter  $W_{e,at_s}$  and  $x$  is unrestrained, put  $x$  into  $A_{s+1} - A_s$ . [We are always safe to put  $x$  into  $A_{2,s}$  and either  $B_1$  or  $B_2$ .]

*For  $R_{e,i,j}$ .* As per (3.3).

*For  $M_e$ .* We pick a follower  $q(e, s)$  and if it occurs in  $W_{e,s}$  put it into  $Q_{s+1} - Q_s$  (restraint  $r(e, s)$ ).

*For  $S_i$ .* Keep  $i$  things of  $Q$  out of  $A$ .

For  $T_e$ . We have a length of agreement function  $l(e, s) = \max\{x : (\forall y < x)(W_{e,s} \sqcup V_{e,s} = Q_s)\}$ . Let  $ml(e, s) = \max\{t < s \mid l(e, t)\}$ . For any  $x < ml(e, s)$  we only let  $x$  enter  $A_{1,i+1}$  at a stage where  $l(e, t) > ml(e, t)$ . (Note this is compatible with  $P_e$  since  $P_e$  can put things into  $A$  via  $A_2$ . It will only cause minimal pair type delay to  $R_{e,i,j}$ .)

For  $U_e$ . If  $x$  enters  $W_e$ , we will put  $x$  into  $C_e$  or  $D_e$ , and we must keep  $C_e \subseteq B_1$ . For  $\bar{U}_{2\langle e,t \rangle}$ . If  $C_{e,s} \cap W_{t,s} = \emptyset$  and some  $x = W_{e,s}$  enters  $W_{e,s}$  before  $A_s$ , we would desire to put  $x$  into  $C_{e,s}$  (meeting  $V_{e,i}$  forever) and therefore we need  $x$  to enter  $B_{1,s^{+1}}$ .

As the reader must guess, the only real conflict here is between  $U_e$  and  $T_e$ .  $T_e$  wishes us to wait till numbers enter  $W_e$  or  $V_e$  before we add them to  $A$ , whereas  $U_e$  asks us to build  $C_e$  and  $D_e$  to split  $W_e$ . However, in the usual  $\pi_2$  way a version of  $U_e$  guessing that  $T_e$ 's action is infinite can live with this delay. This simply delays building  $C_e \sqcup D_e = W_e$  till a stage where the relevant elements enter  $W_e$  or  $V_e$ .

Again  $T_e$  is compatible with  $U_e$  as we can enumerate elements into  $A_2$  and  $U_e$  is compatible with  $R_{e,i,j}$  by enumeration into  $D_e$ .

The remaining parts of the argument fit together in the usual way as with (3.3) (but with much detail). Fitting in either lowness or prompt simplicity causes no especial grief.  $\square$

So, we see that the only known simple sets such that all f-splits are automorphic are the ones from [5]: the f-quasimaximal of rank  $n$  ( $n$  fixed). Indeed, they form an orbit. We offer two conjectures here:

- (i) If  $A$  r-maximal then the f-splits of  $A$  are all automorphic.
- (ii) If  $A$  is a simple set such that the f-splits of  $A$  form an orbit, then either  $A$  is r-maximal or quasimaximal of finite rank.

#### 4. Friedberg splittings of creative sets

One conjecture left open by the previous sections is that f-splittings of creative sets form an orbit, where we define  $A$  to be f-creative if  $A$  is half of a splitting of a creative set. In this section, we address the degrees of f-creative sets. First, not all (complete) sets are f-creative.

(4.1) **Theorem** (Also observed by E. Hermann). *If  $A$  is creative, then  $A$  is not f-creative. Indeed,  $A$  is not f-anything.*

**Proof.** There are many proofs. Let  $K \sqcup B = C$  be creative and let  $K = \{\langle x, y \rangle : \phi_x(y) \downarrow\}$ . Let  $\text{id}(x) = x$  be the usual productive function for  $x$ . We build a recursive collection of r.e. sets  $W_{f(x)}$  as follows. (These are given by the recursion

theorem.) Let  $Q = \{\langle f(x), x \rangle : x \in \omega\}$ . If ever there is an  $x$  with  $\langle f(x), x \rangle \in B$ , put  $x$  into  $W_{f(x)}$ . (Causes  $K \cap B \neq \emptyset$ .) If ever  $\langle f(e), e \rangle \in W_e$ , put  $e$  into  $W_{f(e)}$  causing  $W_e \neq Q - K$  and hence  $Q - K$  is not r.e. Thus  $Q - (K \sqcup B)$  is not r.e. but  $Q \cap B = \emptyset$ .  $\square$

**Remark.** This result also follows from the observation that f-splits are nowhere simple, and Shore's observation [21, Proposition 11] that no creative set is nowhere simple.

We can do much better for degrees.

(4.2) **Theorem.** *If  $A$  is  $f$ -creative, then  $A$  has promptly simple degree.*

**Proof.** Let  $K^* = A \sqcup B$  be an f-split of  $K$ . We show  $A$  is of promptly simple degree. We show that  $A$  satisfies the promptly simple degree theorem [24, Ch. XIII, Theorem 1.6]. Here we construct  $E$  and let  $K^* = \{2x : \varphi_x(x) \downarrow\} \oplus E$ .

We construct an array of r.e. sets  $\{V_{e,x} : e, x \in \omega\}$  with indices given by the recursion theorem  $V_{e,x} = W_{f(e,x)}$  such that  $W_{f(e,x)} = \emptyset$  or  $W_{f(e,x)} = \omega$ . We aim to meet

$P_e: \quad |W_e| = \infty \Rightarrow \exists s, x (x \in W_{e,at,s} \text{ and } A_s \text{ permits } g(x) \text{ by } p(s)),$

for some recursive  $p(s)$ .

Here  $g(x)$  is a recursive function.

We will have  $E = \bigcup_e E_e$ . Let  $F_e = \{f(e, x) : x \in W_e\}$ . We build sets  $G_e = F_e \cup E_e$  with  $E_e \subseteq \{2x + 1 : x \in \omega\}$ .

We aim to also meet

$R_e: \quad |W_e| = \infty \Rightarrow E_e - K^* \text{ not r.e.}$

Let

$R_{e,i}: \quad |W_e| = \infty \Rightarrow E_e - K^* \neq W_i.$

The basic strategy for  $R_{e,i}$  is to pick a follower  $z = z(e, i)$  from the odds. Keep  $z$  in  $E_e$  not in  $W_i$  till  $z$  occurs in  $W_i$ . Then put  $z$  into  $K^*$ . However, we will only do this at times allowed by  $W_e$  and only do it for many  $z(e, i)$ . Indeed for  $R_{e,i}$  we will need an infinite collection  $z(e, i, j) : j \in \omega$  of potential followers. Initially all  $z(e, i, j) \in E_e$ .

Let  $g(x) = \max_{e \leq x} f(e, x)$ . The action is the following. Keep  $V_{e,s} = \emptyset$ . If we see  $x$  occur in  $W_e$ , declare  $V_{e,x} = \omega$  and for any  $z = z(e, i, j) \in W_{i,t}$  with  $z \leq g(x)$  put  $z$  into  $E_{e,t}$ . Thus at some stage  $p(t) > t$  we see such  $z$  and  $f(e, x)$  enter  $K$ . If any of these enter  $A$  (i.e.,  $A$  permits  $g(x)$  between  $t$  and  $p(t)$ ), then declare  $P_e$  as met. If not, they all enter  $B$ .

Now suppose  $P_e$  fails. Then  $|W_e| = \infty$ . This means that all the  $R_{e,i}$  are met since we get, infinitely often, opportunities to put  $z$  into  $E_e$  if necessary (note  $z \leq g(x)$

here). But since no number, added to  $K$  for the sake of  $P_e$  or  $R_e$ , can enter  $A$ , we must have  $E_e \cap A = \emptyset$ . But then  $E_e - K^*$  is not r.e. so  $A \sqcup B$  is not an f-split.  $\square$

Conversely, we have

(4.3) **Theorem.** *Let  $\mathbf{a}$  be promptly simple. Then  $\mathbf{a}$  is e-creative.*

**Proof.** Let  $A$  be of promptly simple degree  $\mathbf{a}$  with witness  $f$ . We build  $C \sqcup D = K^*$ , to meet

- R:  $C \leq_{\tau} A$ ,
- $R_\eta$ :  $|\eta \searrow_e K| = \infty \Rightarrow |\eta \searrow_e A| \geq 1$ ,
- $\hat{R}_\eta$ :  $|\eta \searrow_e K| = \infty \Rightarrow |\eta \searrow_e D| \geq 1$  and
- $\hat{R}$ :  $A \tau \geq C$ .

Again, we need to build  $K^* = K \oplus E$  with  $E$  used for coding. This time on the odds we do a construction similar to (2.9) so we only sketch details. Divide  $2\omega + 1$  into  $\omega$  boxes, the  $x^{\text{th}}$  box having  $x + 1$  members. A requirement  $R_\eta$  can only use elements from box  $2\eta + 1$  onwards. The coding is if  $x$  enters  $A$  at stage  $s$ , put the remaining elements of the  $x$ th box into  $K^* \cap C$ .

The  $R_\eta$  ( $\hat{R}_\eta$ ) get to decide the fate of at most  $x$  of these elements (the remainder goes into  $A$ ). They do so exactly as in (2.10). Namely, they use auxiliary sets and the prompt permitting function to see if elements when they first achieve some state  $\eta$  will promptly enter  $K^*$ , and if so we can put in the  $C$  ( $D$ ) if necessary. The remaining details run along the lines of (2.10) but are easier.  $\square$

(4.4) **Corollary.** *There is an orbit of  $\mathcal{G}^*$  consisting of sets of precisely the promptly simple degrees.*

**Proof.** Let  $O$  be the orbit generated by the e-splits of a creative set. Thus  $\text{deg}(O) \supseteq \mathbf{PS}$  by (4.3). If  $C \sqcup D$  is an e-split, it is an f-split and so has promptly simple degree by (4.2). Finally, f-creativity is elementarily definable (Harrington, see [24]).  $\square$

**Remark.** This result can also be proven by constructing in each promptly simple degree a non-hh-simple r.e. set with the splitting property and semilow complement, and apply Maass [14].

It is not clear if the property of being an e-split of a creative set (or indeed an e\*-split) is elementarily definable. We also conjecture that if  $A$  is f-creative then  $\{\text{deg}(B) : B \approx A\}$  runs over all promptly simple degrees.

We now go back to our original question of whether f-creative sets form an orbit. We cannot use d-splits for creative sets.

(4.5) **Theorem.** *If  $A \sqcup B$  is a splitting of a creative set, then it is not a d-split.*

**Proof.** Again via Myhill's Theorem, we need only construct a creative set  $C$  with the property. So we construct  $C$ ,  $Q_e$  and  $M_e$  so that we meet

$R_e$ :  $W_e \sqcup V_e = C$  implies  $W_e \sqcup V_e$  is not a d-split.

That is,

$$R_{e,j}: \quad W_e \sqcup V_e = C \Rightarrow (\exists Q_e)(\forall Y_e)(Y_e \not\subseteq Q_e \vee Y_e - C \neq Q_e - C \\ \vee M_e - (A \cup R_e) \text{ is not r.e.}) \quad \text{and} \\ (M_e - Y_e) \cap W_e = \emptyset \quad \text{or} \quad (M_e) - Y_e \cap V_e = \emptyset.$$

Here we build  $Q_e$  and  $M_e$  and the opponent builds  $\langle W_e, V_e, Y_e \rangle_{e \in \omega}$ . Thus we meet

$R_{e,j,k}$ :  $M_e - (A \sqcup Q_e) \neq W_k$ .

We can encode  $K$  via  $x \in K$  iff  $\langle 0, x \rangle \in C$ , and so  $C$  is creative.

We shall build  $M_e$ ,  $Q_e$  in  $\omega^{(e+1)}$ . The argument is not difficult. To meet  $R_{e,j,k}$  we run as follows. Pick  $z_1 = \langle e + 1, z \rangle$ . Put  $z_1$  into  $M_e$ . If  $z_1$  occurs in  $W_{k,s_1}$ , put  $z_1$  into  $Q_{e,s_1+1}$ . Await a stage  $t$  where  $z_1$  enters  $Y_{e,at}$ . (If no such stage  $t$  occurs, then  $Y_e - C \neq Q_e - C$ .) When  $t$  occurs, put  $z_1$  into  $C$ . The strategies combine in a no injury way.  $\square$

## 5. (Half)hemisimple sets

In this section, we concentrate on hemisimple (and related) sets. Recall we used half-hemisimple sets to show (e.g.) all low<sub>2</sub> simple sets are automorphic to complete sets. We tell that half-hemi-P sets are interesting in their own right and may shed light on various invariant classes. We believe that this is true of several properties  $P$ . The recent results of Harrington and Soare [12] support this belief. We begin this analysis by letting  $P$  be simplicity. We give proofs of (extensions of) claims from [5].

(5.1) **Theorem.** *There exist nonhalfhemisimple sets ( $n$ -sets). Indeed,*

- (i) *all high r.e. degrees contain them;*
- (ii)  $\mathbf{a} > \mathbf{0} \Rightarrow (\exists \mathbf{b} < \mathbf{a})(\mathbf{b} \neq \mathbf{0} \text{ and } \mathbf{b} \text{ contains an } n\text{-set}).$

**Proof.** We build an r.e. set  $A$  and auxiliary sets  $Q$  in stages to meet, for  $e \in \omega$ ,

$Pe$ :  $\bar{A} \neq W_e$ ,

$R_e$ :  $X_e \sqcup Y_e = A$  implies  $Z_e \sqcup X_e$  is not simple, or  $X_e$  is recursive.

Here we work over sets  $\langle X_e, Y_e, Z_e \rangle$  with  $X_e \cap Y_e = \emptyset$  and  $Z_e \cap X_e = \emptyset$ . We meet the  $Pe$  by a Friedberg procedure. We will have a follower  $x$  (of the correct state; this is without the highness requirement) targeted for  $x$ . If  $x$  occurs in  $W_e$ , we put

$x$  into  $A$ . The strategy for the  $R_e$  runs as follows:

*Basic module.* For the sake of  $R_0$  we build a recursive set  $Q_0$ . In this case we begin by building  $A \subseteq \omega^{(0)}$  and let  $Q_0 = \omega^{(1)}$ . If some  $z_1$  occurs in  $Z_0 \cap \omega^{(1)}$  at stage  $s_1$ , we promise that no number  $\leq z_1$  will enter  $X_e$  after stage  $f(s_1)$ ,  $f$  a recursive function.

We can ensure this in one of two ways.

*Way 1: enumeration.* We enumerate all  $y < z_1$ , into  $A_{s_1}$  immediately and wait for such  $y$  to enter  $X_{0,f(s_1)} \cup Y_{0,f(s_1)}$  for some  $f(s_1) > s_1$ . [The existence of  $f$  is predicated on  $X_0 \sqcup Y_0 = A$ .] Note that after stage  $f(s_1)$  since  $z_1 \in Z_0$  and  $Z_0 \cap X_0 = \emptyset$ , no number  $\leq z_1$  can enter  $X_0$ .

*Way 2: nonenumeration.* We promise that for all  $y < z_1$ , if  $y \in A_{s_1}$  then  $y \in A$ . Again since  $z_1 \in X_0$  as  $z_1 \in Z_0$ , this causes us to be able to compute  $f(s_1)$  where  $X_{0,f(s_1)}[z_1] = X_0[z_1]$ .

We remark that, as we will see, the choice of ways is important for degree reasons.

Inductively, assume we have defined  $z_i$  as we did for  $z_1$ . Now we wait for  $z_{i+1} > z_i$  to occur in  $\omega^{(1)} \cap Z_{0,s_{i+1}}$  and using one of the above strategies, causes us to fix  $Z_{0,s_{i+1}}[z_{i+1}]$ .

Now either the module acts infinitely often, so that  $Z_0$  is recursive; or the module acts finitely often, in which case,  $(Q_0) = \omega^{(1)} \cap (Z_0 \cup A) = * \emptyset$ , so that  $Z_e \cup X_e$  is not simple.

The reader should think of the above as attempting to maximize the state of certain elements. We seek to define a *stream* of numbers  $T_0 = \{z_1, z_2, z_3, \dots\}$  in the high 0-stage where we will in the future build our sets. Note that if  $R_0$  acts infinitely often,  $T_0$  is a recursive set.

Of course there are, as usual, two versions of  $R_1$ , and two of  $P_1$ . A version of  $P_1$  guessing that  $R_0$  acts only finitely often, chooses a follower in  $\omega^{(0)}$ . One guessing that  $R_0$  acts infinitely often, chooses one in  $T_0$ . Now a version of  $R_1$  guessing that  $R_0$  acts only finitely often, uses  $Q_1 = \omega^{(2)}$  as above. The only difference here is that in Way 1,  $R_1$  is only allowed to enumerate into  $A$  numbers under its control. Namely, if it sees  $u_1$  in  $Q_1 \cap Z_{1,2}$ , it can enumerate all  $y \leq u_1$  into  $A$  with  $y$  not following  $P_0$ ,  $P_1$  and  $y \notin \omega^{(1)}$ . This is fine since this version of  $R_1$  'knows' that nothing in  $\omega^{(1)}$  will enter  $A$ .

The version of  $R_1$  guessing that  $R_0$  acts infinitely often, uses  $T_0$  as its universe. It begins with a recursive bijection  $g: \omega \rightarrow T_0$  and uses  $g(\omega^{(2)})$  as its  $Q_1$ .

It is clear that the above is fairly standard and an  $e$ -state construction does the rest. We leave the details to the reader.

Now we turn to degrees. We begin with (ii). To get (ii) we use permitting and Way 2 (nonenumeration). Thus we need a set of followers  $x_0, x_1, \dots$ , devoted to satisfying  $R_e$ . Once these get the right state, they form a recursive set and so we eventually get a permission in the usual way.

(i) This is more difficult, but still fairly standard. We must achieve two goals: coding and high permitting. Let  $H$  be a given high r.e.  $e$ -dominant set. (We assume the reader is familiar with high permitting.) We basically need to know

that for an  $e$ -state  $\sigma$  we can get ‘almost all’ permission. This is the same as Martin permitting in a maximal set construction. To achieve coding, initially we set aside coding markers in  $\omega^{(0)}$  (the  $P_j$  are not needed). The  $x$ th location moves from  $\omega^{(0)}$  to  $\omega^{(1)}$  only if  $\omega^{(1)}, \dots, \omega^{(j-1)}$  appear disjoint from  $A$ , the coding location is not in  $\omega^{(1)}, \dots, \omega^{(j-1)}$  and the module  $R_j$  has acted (say)  $j$ -times since the last time  $R_1, \dots, R_{j-1}$  acted. The fact that we use Way 2 implies that  $A$  can comprehend if a marker moves. Note that if  $R_k$  for  $k < j$  acts then the coding location can move back to  $\omega^{(k)}$ . The fact that we use high  $H$ -permission means  $H$  can decide when a marker is stable so that  $A \leq_T H$ . The fact that coding occurs means that  $H \leq_T A$ .  $\square$

One question left open by the above is

(5.2) **Question.** Does jump inversion hold? That is, if  $\mathbf{a}$  is r.e. in and above  $\mathbf{0}'$ , is there an  $n$ -set  $A$  with  $A' \in \mathbf{a}$ ?

Certainly not all r.e. degrees contain  $n$ -sets as we now see in our final result.

(5.3) **Theorem.** (i) *There exists a (low) completely halfhemisimple degree  $\mathbf{a}$ . That is, an r.e. degree  $\mathbf{a} \neq \mathbf{0}$  such that if  $A$  is an r.e. element of  $\mathbf{a}$ , then  $A$  is halfhemisimple.*

(ii)  $(\forall \mathbf{a} \neq \mathbf{0})(\exists \mathbf{b} < \mathbf{a})(\mathbf{b}$  is completely halfhemisimple).

**Proof.** We build r.e. sets  $A, Q_e, R_e$  and  $H_e$  and our opponent builds  $\Phi_e, V_e, \Gamma_e$  and  $W_e$  for  $e \in \omega$ . We must satisfy the requirements:

$$M_e: \quad \Phi_e(A) = V_e \quad \text{and} \quad \Gamma_e(V_e) = A \quad \text{implies} \quad Q_e \cup R_e = V_e.$$

$$M_{e,i}: \quad \Phi_e(A) = V_e \quad \text{and} \quad \Gamma_e(V_e) = A \quad \text{implies} \quad \bar{Q}_e \neq W_i.$$

$$D_{e,i}: \quad \Phi_e(A) = V_e \quad \text{and} \quad \Phi_e(V_e) = A \quad \text{and} \quad |W_i| = \infty$$

implies  $W_i \cap (Q_e \cup H_e) \neq \emptyset$ .

We additionally ensure that  $Q_e \cap H_e = \emptyset$  and  $Q_e \cap R_e = \emptyset$ . Note that as  $Q_e$  is not recursive, this must make  $\underline{Q_e \cup H_e}$  coinfinite, and hence simple, where at each stage  $s, \{b_{e,i,s} : i \in \omega\}$  lists  $\underline{Q_{e,s} \cup H_{e,s}}$ .

Now the argument to follow is finite injury and hence it will suffice to describe the basic modules. Dropping the ‘ $e$ ’ subscript we must first ensure that

$$(5.4) \quad \text{if } \Phi(A) = V \quad \text{and} \quad \Gamma(V) = A, \quad \text{then} \quad Q \sqcup R = V.$$

To achieve (5.4), we monitor  $V$ -changes at  $e$ -expansionary stages. Thus let  $l(\mathbf{a}) (= l(e, s))$  denote the current  $A$ -controllable length of agreement in (5.4). That is,

$$l(s) = \max\{x : \forall y < x (\Gamma_s(V_s; y) = A_s(y) \quad \text{and} \\ (\forall z)(z < u(\Gamma_s(V_s; y)) \rightarrow \Phi_s(A_s; z) = V_s(z))\}.$$

We let  $ml(s) = \max\{l(t) : t < s\}$  and  $ls(s) = \max\{0, t : t < s \text{ and } l(t) > ml(t)\}$ . We say  $t$  is expansionary if  $l(t) > ml(t)$ . Now at each expansionary stage  $s$  we will enumerate  $V_s - V_{rs(s)}$  into either  $Q$  or  $R$ . We should put this into  $Q$  whenever possible. As usual, we regard  $A$  as controlling  $V$  and don't allow  $V$ -changes unless  $A$  changes below the relevant use. Thus if  $l(s) > x$ ,  $V[\gamma_s(x)]$  can change only if  $A[\varphi_s(\gamma_s(x))]$  changes, where  $\gamma_s$  and  $\varphi_s$  denote the use functions of  $T_s$  and  $\Phi_s$  respectively. Note that the only 'conflict' here is that once  $x$  enters  $H$  if  $x$  later appears in  $V$ , then we must put  $x$  into  $V$  (to keep  $Q \cap R = \emptyset$ ).

The main idea of the construction is not to get too keen in putting things into  $H$ . In particular, we must wait for 'setups' to occur for all the higher priority requirements before we really attend  $D_{e,i}$ .

Each time a requirement  $M_{e,i}$  or  $D_{f,j}$  acts it initializes all lower priority ones and  $M_{e,i}$  resets its restraint to be  $s$ . The cycle for a single  $M_{e,i}$  is the following.

*Step 1.* Assume  $M_{e,i}$  has been initialized for the last time by higher priority  $M_{f,j}$  and  $D_{p,q}$ . At the next  $e$ -expansionary stage we define a marker  $\lambda(e, i, s_1) = \lambda(s_1)$  to be  $s$ . (As usual  $s$  exceeds all uses, etc. seen so far.)

*Step 2.* Wait till  $l(s_2) > \lambda(s_1) = \lambda(s_2) = \lambda$  while between Step 1 and Step 2 all  $M_{e,j}$  and  $D_{e,j}$  for  $j > i$  have been frozen, and remain so until we complete a setup for  $M_{e,i}$ .

At this stage  $s$ , we choose a follower  $x = x(e, i, s)$  targetted for  $A$ . Note that

$$(5.5) \quad x > u = \max\{\varphi_s(z) : z < \lambda\}.$$

Again note we initialize all lower priority requirements (who can hence only choose  $y > z$  to add to  $A$ ).

It follows that, if all the requirements of higher priority than  $M_{e,i}$  have ceased activity by  $s$ , then

$$(5.6) \quad A_{s_2}[u] = A[u]$$

and hence, by monotonicity of the uses,

$$(5.7) \quad V_{s_2}[\lambda] = V[\lambda].$$

*Step 3.* Wait till  $s_3 > s_2$  with  $l(e, s_3) > x$ .

Again, as explained earlier, as  $R_{e,i}$  acts, we initialize all lower priority requirements again. We now declare  $M_{e,i}$  as *active* with a *complete setup*, and unfreeze all the  $M_{e,j}$  and  $D_{e,j}$  for  $j > i$ . The point is that, unless we do anything else,

$$V_{s_3}[\hat{u}] = V[\hat{u}], \quad \text{where } \hat{u} = u(T_{s_3}(V_{s_3}; x))$$

(since  $\hat{u} < s_3$  and by initialization).

*Step 4.* There occurs a stage  $s_4 > s_3$  with  $\bar{Q}_{s_4}[\hat{u}] = W_{i,s_4}[\hat{u}]$ . The key claim is that

$$(5.8) \quad V_{s_4}(z) = Q_{s_4}(z) \quad \text{for all } z \text{ with } \lambda \leq z < \hat{u}.$$

To see that (5.8) holds, first we use fresh numbers and initialization each time  $M_{e,i}$  acts. Furthermore, the only reason we put a number in  $W_e$  not into  $Q_e$  (note the  $e$ -subscript) is for a  $D_{e,j}$ . Such  $D_{e,j}$  for  $j > i$  have been *frozen* until the setup for  $M_{e,i}$  was complete. Thus all such  $z$  have gone into  $Q_e$  not  $H_e$ .

Thus we can win by adding  $x$  into  $A$  initializing (and not changing  $A[x-1]$ ). This will cause a change in  $V(z)$  for some  $\lambda \leq z < \hat{u}$ . By (5.8) we can put this change into  $Q_{e,t}$  for the least  $e$ -expansionary stage  $t > s_4$ .

The  $D_{e,i}$  are played in the obvious way: At  $e$ -expansionary stages, when not frozen by some  $R_{e,j}$  for  $j < i$ ,  $D_{e,i}$  can enumerate an unrestrained element into  $H_e$ .

The details consist of combining the above strategies via the finite injury method.  $\square$

Since the argument can obviously be made to permit, we see

(5.9) **Corollary.** *If  $\mathbf{a} > \mathbf{0}$  there is a  $\mathbf{b} < \mathbf{a}$  that is completely halfhemisimple.*

The exact (jump) classification of the completely halfhemisimple degrees eludes us. We know it is a subset of  $\bar{H}_1$  and contains members in  $L_1$ . We can make a mild contribution to this question by observing that the technique of making a degree  $m$ -topped (Downey–Jockusch) can be used to show

(5.10) **Theorem.** *There is a  $\text{low}_2$ -low<sub>1</sub> completely halfhemisimple  $\mathbf{a}$ .*

We do not give a Proof of (5.10) since the technique is essentially the same as that employed in the construction of an  $m$ -topped degree with no hemimaximal sets of [6]. It is unclear what such degrees have to do with completely halfhemisimple degrees. The dynamics involved in their construction seem remarkably similar (and combine easily).

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