13 13

There is no Fat Orbit*

Rod Downey
Mathematics Department
Victoria University
P. O. Box 600
Wellington
New Zealand

Leo Harrington
Mathematics Department
University of California
Berkeley, CA 94720
U. S. A.

December 27, 1995

Abstract

We give a proof of a theorem of Harrington that there is no orbit of the lattice of recursively enumerable sets containing elements of each nonzero recursively enumerable degree. We also establish some degree theoretical extensions.

1 Introduction

Ever since the classic paper of Post [11], it has been recognized that there is an intrinsic interaction between the two fundamental structures of recursion theory $\mathbf{R}$, the uppersemilattice of recursively enumerable (computably enumerable) degrees, and the lattice $\mathcal{E}$ of recursively enumerable sets. A natural approach is to seek to understand the relationship between definability in $\mathcal{E}$ and degrees

*Downey's research supported by Cornell University, an IGC grant from Victoria University and the New Zealand Marsden Fund via grant 95-VIC-MIS-0698 under contract VIC-509. Harrington is partially supported by the NSF via grant DMS92-14048. To appear in the Annals of Pure and Applied Logic.

theoretical properties of $\mathbf{R}$. The is, of course, a persistent intuition that the two are interlinked. A beautiful example of this phenomenon can be found in the recent paper of Harrington and Soare [7] where a elementary property $\mathcal{P}$ is given such that there exist nonrecursive recursively enumerable sets X such that $\mathcal{P}(X)$ holds yet $\mathcal{P}(X)$ cannot hold of a Turing complete recursively enumerable set.

Naturally we would like to completely understand the possible relationships between definability properties of $\mathcal{E}$ and the degrees of the sets realizing these properties. Hallmark results here are the known results on orbits : the (Myhill-) Harrington proof that creative sets form a definable orbit and realize only sets of the complete m-degree (see Soare [14]), Soare's proof in [13] that the maximal sets form an orbit which realizes sets in each high degree by Martin [10], and the Downey-Stob proof that hemimaximal sets, halfs of nontrivial splittings of maximal sets, form an orbit that realizes many degrees including all high degrees, all jump classes and arbitrarily low degrees. (Downey and Stob [3, 4])

All of these theorems seem to point at the idea that the position of a recursively enumerable set in $\mathcal{E}$ *always* has ramifications in the degrees realized by its orbit. Such considerations lead to the natural question of whether there is a recursively enumerable set C that is so "generic" in $\mathcal{E}$ that

$$\{\mathbf{a} : \exists A \in \mathbf{a}(A \text{ is automorphic with } C\} = \mathbf{R} - \{\mathbf{0}\}?$$

Furthermore, can such an orbit be elementarily definable? (e.g. Soare [14], Ch XVI, Question 1.11) Up to the present paper, the best results is the aforementioned result of Downey and Stob who show that the elementary property of being hemimaximal has representatives in all possible jumps. The main goal of the present paper is to give a proof of a theorem of Harrington [6] that there is no such "fat orbit." That is, we prove that for any orbit $\mathcal{O}$ in the automorphism group of $\mathcal{E}$, the lattice of recursively enumerable sets, there is a nonzero recursively enumerable degree $\mathbf{d}$ containing no sets in $\mathcal{O}$. Indeed we shall give an elementary property $\mathcal{S}$ such that

- there is a nonzero recursively enumerable degree $\mathbf{d}$ such that for all recursively enumerable sets A of degree $\mathbf{d}$, $\mathcal{S}(A)$ holds, and
- there is a nonzero recursively enumerable degree $\mathbf{e}$ such that if W is any recursively enumerable set of degree less than or equal to $\mathbf{e}$ then $\mathcal{S}(W)$ fails to hold of W.

Hence the fat orbit question has a very strong negative answer : *no* set of degree $\mathbf{d}$ can be sent to *any* set of degree below $\mathbf{e}$.

A second goal is to develop some some degree theoretical refinements of Harrington's Theorem. In particular, we prove that in the above $\mathbf{e}$ can be high$_2$, and there is an interval $[\mathbf{d}_1, \mathbf{d}_2]$ of recursively enumerable degrees with $\mathbf{d}_2$ high$_2$ and $\mathbf{d}_1$ low such that every recursively enumerable set of degree between $\mathbf{d}_1$ and $\mathbf{d}_2$ has property $\mathcal{S}$. In terms of the jump operator this result is nearly the best possible:

- Certainly $\mathbf{d}$ cannot be high since there are hemimaximals below $\mathbf{e}$ which can be sent into any high recursively enumerable degree. (Downey-Stob [5])
- If $\mathbf{d}_1$ were $\mathbf{0}$ then in the interval there would be hemimaximal sets that could be sent below $\mathbf{e}$.

The remaining possibility is that $\mathbf{e}$ could be high. We leave open the following, a positive solution to which will constitute (in some sense) the best salvage of the fat orbit in terms of the jump operator.

Question. *Given a nonzero* $\mathbf{a}$ *and a high* $\mathbf{b}$ *do there exist* $A \in \mathbf{a}$ *and* $B \in \mathbf{b}$ *such that* A *is automorphic with* B.

Our results might suggest a strong counter fat orbit result saying that the high degrees are very special in terms of orbits and the jump operator, as follows.

Conjecture (False) *There is no REA in* $\mathbf{0}'$ *degree* $\mathbf{a} < \mathbf{0}''$ *such that if* $\mathbf{b}' = \mathbf{c}' = \mathbf{a}$ *then there exist* $B \in \mathbf{b}$ *and* $C \in \mathbf{c}$ *with* B *automorphic with* C.

However this attractive conjecture fails. To see this, Cooper [1] and independently Shore [12] constructed a jump class of noncappable degrees. In particular there is a Σ_2 set $A <_T \emptyset''$ such that for all recursively enumerable sets C, if $C' \equiv_T A$, then C has promptly simple degree. Therefore, if $C' \equiv D' \equiv A$, then the degrees of both C and D are promptly simple. Therefore $C \equiv_T \widehat{C}$ and $D \equiv_T \widehat{D}$ for some recursively enumerable sets $\widehat{C}$ and $\widehat{D}$ which are both promptly simple and with semilow complements. By Maass [9], there is an automorphism of the lattice of recursively enumerable sets taking $\widehat{C}$ to $\widehat{D}$.

It is not clear if a weakened form of the false conjecture might work for double jump classes. (One cannot get this from our results since "double jump inversion" does not work by the work of Soare and Stob [15].) Thus we leave open the question below.

Question. Is there an n such that for every n'th jump class $\mathcal{Q}$ there are degrees $\mathbf{a}$ and $\mathbf{b}$ such that for no $A \in \mathbf{a}$ and $B \in \mathbf{b}$ is A automorphic with B?

One final consequence to our degree theoretical results is that are no hemimaximal sets in the interval $[\mathbf{d}_1, \mathbf{d}_2]$, answering a question of Downey and Stob concerning the distribution of hemimaximal degrees.

These degree theoretical refinements use known techniques for making high$_2$ degrees adapted from Downey, Lempp and Shore [2] and methods from Downey-Stob [5]. These techniques are taken wholesale and hence we do not give the whole construction but merely discuss the various strategies in detail.

Notation is standard and follows Soare [14]. All sets will be recursively enumerable unless otherwise specified. We use the lower case Greek letter to denote the use of a functional, which is taken to be monotone in both arguments. We thank Peter Cholak and Mike Stob for helpful discussions.

2 Harrington's Theorem

Let $\mathcal{S}(A)$ denote the following property.

$$(\exists C)(\forall X \subseteq C)[A \cup X \supseteq C \to (\exists B \subseteq X)[A \cup B \supseteq X \wedge (\forall Y)[A \cup Y = \omega \to |(X \cap Y) - B| = \infty]]$$

Lemma 2.1 (Harrington [6]) *There is a degree* $\mathbf{d}$ *such that any set* A *of degree* $\mathbf{d}$ *has* $\mathcal{S}(A)$.

Proof. We delay the finite injury proof until Section 4. $\square$

Lemma 2.2 (Harrington [6]) *There is a nonrecursive set* E *such that if* $A \leq_T E$ *then* $\neg\mathcal{S}(A)$.

Proof. We get to construct $E, X_{e,i}$ and $Y_{e,i,j}$, while our opponent gets to play $\Phi, C_{e,i}$ and $B_{e,i,j}$. We meet the requirements below.

$$\mathcal{P}_e \ : \ \overline{E} \neq W_e.$$

$$\mathcal{N}_{e,i,j} \ : \ \Phi_e^E = A \to [[X_{e,i} \supseteq (C_{e,i} \cup A) \wedge (B_{e,i,j} \subseteq X_{e,i}) \wedge (A_e \cup B_{e,i,j}) \supseteq X_{e,i}] \to$$
$$A_e \cup Y_{e,i,j} = \omega \wedge |(X_{e,i} \cap Y_{e,i,j}) - B_{e,i,j}| < \infty].$$

The $\mathcal{P}_e$ are met in the standard Friedberg way via followers or making E simple. The $\mathcal{N}$ are met by infinitary negative restraints as follows. For the sake of clarity we will drop all subscripts. Now given Φ and some C, we will have numbers enter X only after they enter C. If n enters C at some stage s, then n is restrained from X until a stage s' where all $x < s$ have entered A or there is a Φ^E computation certifying that $x \notin A_{s'}$[1]. If s' is reached then if $n \notin A_{s'}$, then we put n into X. It is then immediate that if $\Phi^E = A$ then $X \cup A \supseteq C$.

Thus the key is the negative restraints on E involved in the building of the special set Y. The strategy for this restraint works in a cyclic fashion and the current cycle ends when we have a stage t where 2.1 and 2.2 below both hold.

2.1. $(C_t \cap Y_t) \supseteq (A_t \cap B_t)$.

2.2. $(X_t \cap Y_t) \subseteq B_t$.

Y-enumeration. Now we will promise that the only time we perform Y enumeration is at the end of a cycle, at a stage t where 2.1 and 2.2 both hold. At such a stage and only at such a stage, we will put into Y all numbers $\leq t$ not yet in Y and not in A *unless they are in* $C - B$.

Comment. Notice that if there are infinitely many cycles with $\Phi^E = A$, we know that $A \cup X \supseteq C$. Thus if we suppose that $(A \cup B) \supseteq X$, it can only be that $C - (A \cup B) = \emptyset$. *Therefore, all numbers not in A are either directly put into Y or*

- *first enter C,*
- *then later enter B,*

[1]As usual with such things, we if we have a requirement predicated on a computation $\Xi^Z = G$, and the length of agreement exceeds p at some stage s. then we do not allow G to change unless some number enters Z below the use ξ after stage s.

- *then later enter Y at the end of the current cycle.*

In any case we see that $A \cup Y = \omega$ when all the relevant hypotheses are met and there are infinitely many cycles. Furthermore notice that at the end of the cycle the new numbers entering Y are either already in B or not yet in C and hence after Y enumeration, 2.1 and 2.2 still hold.

Now the cycle acts after stage t as follows.

- No E-restraint is imposed at all until a stage t_1 is found where some element of Y enters X *before* entering B. **Comment :** *Notice that since numbers enter C only at e-expansionary stages, we know that Φ^E is currently looking correct on these numbers and hence A restraint is possible upon them via E-restraint.* These elements entering X from Y before B are the only dangerous elements : their enumeration into A would be fatal, as our opponent would keep them out of B forever. Let J denote the collection of elements of Y not in A_{t_1}. Then let v denote the use of the E-computation witnessing that they are not in A. We must preserve $E \upharpoonright v$. We will preserve $E \upharpoonright v$ until the end of the cycle.

- Now we must argue that eventually 2.1 and 2.2 will hold.

- Now Y will not change during the cycle and hence as it is fixed until the end of the cycle. Hence, assuming that the cycle does not end, either $(C_q \cap Y_q) \subseteq (A_q \cup B_q)$ at some stage $q > t_1$ or we get a global win since either $\Phi^E \neq A$ or $A \cup B \not\supseteq X$. Therefore assuming that the cycle fails to end, there is a stage p such that for all stages $p' > p$, 2.1 holds at stage p'.

- So suppose that 2.2 fails to hold eventually. Then there is some $n \in Y$ violating 2.2. Note that $n \notin C$ at the end of the last cycle since we know that if a number x is in $Y \cap C$ at the end of the last cycle then either $x \in B$ or $x \in A \backslash X$ and in this case $x \notin X$ by construction. ($(A \backslash X) \cap X = \emptyset$.) Therefore $n \notin X$ at then end of the last cycle. Thus n enters X during the present cycle and $n \notin B$ since we are assuming that n is the witness to 2.2's failure. Now by restraints and the way we define X, we see that $n \notin A \backslash X$. Therefore since n enters X at a stage $\geq t_1$, $n \notin A$, since $n \notin A_{t_1}$ and E-restraint is then imposed to preserve J. Therefore if the cycle remains forever, then either $\Phi^E \neq A$ or $n \in (X - (A \cup B))$.

- It follows that there all be a stage $t_2 > t_1$ where 2.1 and 2.2 both hold. From this we see (by the comment) that $A \cup Y = \omega$ and, by 2.2, if $(A \cup B) \supseteq X$, then $X \cap Y \subseteq B$. Therefore $\mathcal{N}_{e,i,j}$ is met.

Coherence.

Coherence for this argument is a straightforward Π_2 tree of strategies argument. The outcomes of the basic $\mathcal{N}_{e,i,j}$ module are ∞ and f : either there are infinitely many cycles or there are only finitely many. Now Y-enumeration is performed at the end of cycles but we build Y locally for the sake only of $\mathcal{N}_{e,i,j}$. X enumeration is predicated only on $\Phi^E = A$ and depends on C-enumeration, but can occur at each Φ-expansionary stage. This of course has no effect on a node α devoted to solving $\mathcal{N}_{e,i,j}$ and thus defining Y_α. Therefore the only injury to α is by Friedberg requirements of higher priority and if α is on the true path then this injury is finite. Thus the only need for the tree is to coordinate the restaints precisely as with a minimal pair argument. Such details are now absolutely standard and are left to the reader. □

Theorem 2.3 (Harrington [6]) *There is no fat orbit, and indeed there are nonzero degrees* **d** *and* **e** *such that no set of degree* **d** *can be taken by an automorphism of $\mathcal{E}$ to a set of degree $\leq$* **e**.

Proof. The theorem follows by Lemmas 2.1 and 2.2. □

3 Making E high$_2$.

In this section we will append a $[s]$ to a parameter P to denote its state at stage s. For the sake of this section we need a little more care with the construction of the previous section. We will have nodes τ devoted to measuring $\Phi_e^E = A_e$, nodes ρ devoted to building $X_{e,i}$ in response to $C_{e,i}$, and finally nodes α devoted to building $Y_{e,i,j}$ in response to $B_{e,i,j}$. In the style of Soare [14], we will write $\tau(\alpha) = \tau(\rho) = \tau$, $e(\tau) = e$, etc. Now if we are to make E nonlow then we know that something infinitary must happen on the true path and "guessing" η-correct computations will be necessary.

The key tension is between the nodes of type ρ and those of type α. X-enumeration occurs essentially ρ stages, but in response to X-enumeration, we impose (lower priority) α-restraint. Notice that α cannot delay X-enumeration since for a fixed ρ there is a tree of α for many Y_α. And ensuring that $X \supseteq (C \cup A)$ is ρ's goal.

It is instructive to consider the effect of trying the canonical infinitary action : making E high. So suppose that we needed to meet the single limit requirements of coding in the jump some say Tot. The standard tool would be to use the thickness requirements of the form

$$\mathcal{T}_q : E^{(q)} =^* Q^{(q)},$$

where Q is some piecewise recursive set. Now we could certainly ask that X enumeration occur only at ρ-correct stages, and not just τ-correct stages. (Therefore ρ restraint would also involve all nodes η devotes to $\mathcal{T}$-type requirements with $\tau \subset \eta \subset \rho$.)

However, when such X_ρ enumeration occurs, even at a ρ-correct stage, we immediately set up α restraints since we now wish to fix $E \upharpoonright v(\alpha)$ until B_α eats the new X elements. (Remember, this is the whole point of the cycles.) But now there are also infinitary $\mathcal{T}$ requirements at nodes ν with $\rho \subset \nu \subset \alpha$. And our opponent will certainly make sure that the τ computations $\Phi^E = A[s]$ are *not* α-correct and in the sense that they will not be ν-correct. Therefore α's attempt at E-restraint is surely doomed. (The salient feature of the basic construction is that there are only a finite number of finitary Friedberg requirements between ρ and α and we simply allowed such requirements to injure α at will.)

We can overcome these combinatorial difficulties is we make E high$_2$ instead of high. To achieve this goal we use the technology of Downey, Lempp and Shore [2]. Thus we will build a E-recursive functional Λ so that for all x, we meet the requirements below.

$$\mathcal{H}_x \ : \ \lim_m \lim_t \Lambda(E; x, m, t) = Cof(x),$$

where $Cof = \{x : W_x \text{ is infinite}\}$ is the canonical Σ^0_3-complete set. For the sake of the $\mathcal{H}_x$ requirements we will have nodes β $(= \beta(x, m)$, which test if $[m, \infty) \subseteq W_x$. This is a Π_2 test. The outcomes here are ∞ and f. Again the ∞ outcome corresponds to the Π_2 outcome looking correct. β is responsible for defining $\Lambda(E; x, m, t)$ for all t. As pointed out in [2], the principal difficulty is that since Λ must be E-recursive, its definition cannot depend on knowledge of the true path TP. Thus as in [2], we will devote a whole *level* of the tree to collectively define $\Lambda(E; x, m, t)$. We will assume initially that $\Lambda(E; x, m, t)[s] = 0$, and Cof will only ask us to redefine this to be 1.

It is important that the reader note that since we are using double limits, it is permissible for us to be wrong on a finite number of m and a finite number of t for each m. (This is in fact the key distinction from making E high.)

The fundamental idea from [2], is that when we hit β we will redefine all values of $\Lambda(E; x, m, t)[s]$ for $t \leq s$ in accordance with our current picture of Cof. Of course we can only do provided that we are allowed to by higher priority requirements. We could be restrained since correcting $\Lambda(E; x, m, y)$ involves E-enumeration of $\lambda(x, m, t)[s]$. As in [2], in advance we will fix values of $\lambda(x, m, t)$ as fixed coding markers. The main point to keep in mind is that *all* the level of $\mathcal{H}_x$ use the same $\lambda(x, m, t)$ as coding markers and all of them wish to do the same thing. Whether a particular β actually succeeds depends on β. Therefore when we hit β and the Π_2 condition saying that "$x \in Cof$ with witness $m' \leq m$ has again appeared to hold," then we will try with priority β, to put $\lambda(x, m, t)$ into E.

It is convenient to regard the action of α demanding a restraint at a ρ-stage as the creation of a *delay* $d(\rho, \alpha)$. Basically we don't get to remove the restraint until we remove the delay. It is important that the reader remember we do not change Y except at stages when we visit α. Hence we will only change Y at stages when we visit α when there is no restraint. In the high_2 case of this section we will use the Slaman-Soare notion of "links" to represent implications in the following, we remark that Y is not changed if we only access α *not directly but via a link.*

The main idea from [2] is that we should clear the potentially injurious $\lambda(x, m, t)$ from the $E \upharpoonright v$ use before we actually pass ρ.

In more detail, there are several scenarios depending on the relevant priorities of (the requirements associated) with β and ρ. Let $\mathcal{N}$ be the requirement associated with ρ responsible for constructing $X_{e,i}$ and $\mathcal{H}_{x,m}$ the subrequirement of $\mathcal{H}_x = \mathcal{H}$ associated with $\beta = \beta(x, m)$.

The most complex sitaution is where $\mathcal{H}$ is of lower priority than $\mathcal{N}$. Of course this means that there will be no requirements associated with $\mathcal{H}_x$ above ρ. Naturally if $\alpha <_L \beta$ then we simply let α control β. The problem case is where $\rho \subset \beta \subset \alpha$. Now we hit ρ and would like to put some number z from C into X which is not as yet in $B = B_\alpha$. If we do so than when we hit β we might code some $\lambda(x,m,t)$ into E allowing some number into E allowing this number into A. Naturally if $\alpha \supseteq \beta\widehat{\ }f$ then α is guessing that the effect of β is finite and in that case we again proceed as in the basic Harrington construction of the last section. We know that if α is on TP then β's effect will be finite anyway, and hence the basic Harrington argument works.

The problem case is that $\alpha \supseteq \beta\widehat{\ }\infty$. Now α expects that β will code lots of numbers into E. What we do in this sitaution is that when we hit ρ, we *delay X-enumeration* and then circumvent this possible future injury by *immediately* putting $\lambda(x,m,t) < v[s]$ into $E[s]$. The idea is that we wish to clear all $\lambda(x,m,t) < v[s']$ from below the E-use so that our E-restraint will be successful if $\beta\widehat{\ }\infty$ is on the true path TP. The outcomes of a node α in the previous construction were $\{\infty, f\}$. That is, infinitely often the α-restraint was dropped (since B_α recovered) Now to take the new strategy into account we will add one further outcome to the node α namely (ρ,β,τ,∞) with $\infty <_L (\rho,\beta,\tau,\infty)$. This outcome should be interpreted as saying that that β causes the Φ_τ computations to diverge via unbounded use. (Of course we will have many of these outcomes.)

When we do this the idea is that we create and traverse a link (ρ,α) and play the outcome (ρ,β,τ,∞). The next time we hit ρ, say at stage s_1, we again know that the Φ^E computations are ρ-correct, and we see if it is yet safe to enumerate into X_ρ and then impose α-restraint and not be circumvented by β. Notice that the argument s that whose use we are trying to clear is unchanged although the use $v[s_1] > v[s]$ because of $\lambda(x,m,t)$-enumeration at stage s, the last time we were here. Of course if now B_α contains all the elements we wish to put into X then we are okay since no restraint wishes to be imposed by α.

Our opponent will of course do no such thing. If we hit ρ and there are no markers $\lambda(x,m',t')$ associtated with β below $v[s_1]$ then we will be safe, we will enumerate into X, remove the link and impose α-restraint. In general, from some point onwards, if α is on TP then this restraint will be successful.

However, if we hit ρ and again we find that some $\lambda(x,m',t')$ is below $v[s_1]$ then again we will enumerate such potentially injurious numbers into E, traverse the link and play the outcome (ρ,β,τ,∞). Now if it is the case that *every time* we hit ρ we traverse the link to α then the node $\alpha\widehat{\ }(\rho,\beta,\tau,\infty)$ will be on the true path. But then it can only be that $\Phi^E(y)\uparrow$ for some $y \le s$. This then gives a *global win* on $\mathcal{N}_e$.

There are two costs to this win.

3.1. We might have caused $\lim_t \Lambda(x, m, t) = 1$ although if ρ were not around, the "correct" outcome of β should be f.

3.2. There might be nodes ν with $\rho \subset \nu \subset \alpha$ which are permanently linked over. (In the terminology of Downey-Stob [5] ν might be on the true path but not on GTP, the *genuine* true path.)

Now 3.1 is not really a problem since we have achieved a "global win" on $\mathcal{N}$ (and every requirement predicated on $\Phi^E = A$) at the expense of injuring $\mathcal{H}_x$. In the standard $\mathbf{0}'''$ way we could simply restart $\mathcal{H}_x$ below $\alpha\widehat{\ }(\rho, \beta, \tau, \infty)$. The so-called finite injury along the true path feature.

The reader should note that the only way that α could be visited infinitely often and $\beta\widehat{\ }\infty$ not be on the genuine true path, is that from some point on there is a permanent link from ρ to α. Also note that if $\alpha\widehat{\ }\infty$ is on GTP then essentially $\beta\widehat{\ }\infty$ is *correct*.

Also 3.2 is solvable by known techniques which were introduced in Downey-Stob [5], and have been employed in several other places such as Downey-Lempp-Shore [2]. The problem is that we must have a notion of ν-correct computations for nodes on the genuine true path. Now a node ν below $\alpha\widehat{\ }(\rho, \beta, \tau, \infty)$ needs to be able to look up and "know" which nodes apparently putting an infinite amount into E are truly doing this. (For instance, we might have $\xi\widehat{\ }\infty \subseteq \nu$ and we need to know if the construction puts $\lambda(x(\xi), m(\xi), t(\xi))$ into E.) The solution to this problem, introduced in [5], is to attach a finite set of outcomes to $\alpha\widehat{\ }\infty$. For example, we attach outcomes where each outcome of a finite set of nodes of the form $\xi\widehat{\ }\infty \subseteq \alpha$ ordered in any fashion so that if $F \subset F'$ then $F' <_L F$. The appropriate suboutcome of $\alpha\widehat{\ }(\rho, \beta, \tau, \infty)$ is played whenever we visit $\alpha\widehat{\ }(\rho, \beta, \tau, \infty)$ and the nodes of F have all have their markers enumerated since the last time we were at $\alpha\widehat{\ }\infty$. In this way we see that 3.2 can also be dealt with by standard techniques. (See Downey-Stob [5] for more details.)

The next sitaution we need to examine is that $\mathcal{H}_x$ has higher priority than $\mathcal{N}$ and hence there will be nodes β' devoted to $\mathcal{H}_x$ *above* ρ despite the fact that β is below ρ. Now the first sitaution is that ρ is below some $\xi\widehat{\ }\infty$ devoted to $\mathcal{H}_x$ on GTP. Now in this sitaution β has only one outcome ∞ since it is wishing to dump all of its row into E. Now the argument is even easier. When we hit ρ it will be a stage when $\xi\widehat{\ }\infty$ looks correct and hence if there is a link from ρ to σ it is the case that $\xi\widehat{\ }\infty$ would desire us to put β's markers into E. Therefore in this sitaution, if ρ is on GTP then the extra enumeration at ρ cannot injure $\mathcal{H}_x$

at all.

The final sitaution is that ρ is not below any $\xi\hat{\ }\infty$ devoted to solving $\mathcal{H}_x$ on GTP (but $\mathcal{H}_x$ still has higher priority than $\mathcal{N}$). In this case we will *not* allow ρ to assert control over $\beta\hat{\ }\infty$ since we will actually ensure that it is only possible for there to be a link from ρ to α iff for no β associated with $\mathcal{H}_x$, is it the case that $\rho \subset \beta\hat{\ }\infty \subset \alpha$.

This is again injury down the true path. It is not reasonable for ρ to injure $\mathcal{H}_x$ given the respective priorities. Hence what we do is that below any node of the form $\beta\hat{\ }\infty$ we will restart a new version ρ' of ρ where we will be building a new $X_{\rho'}$. It therefore follows that in the situation where $\mathcal{H}_x$ has higher priority than ρ if we hit ρ and there is a link from α up to ρ then again this is all okay with $\mathcal{H}_x$ since there are no *infinitary* outcomes associated with $\mathcal{H}_x$ between ρ and α.

Now the only problem with this is the fact that we might permanently link over some β whose "true outcome" should be ∞ but this node is not visited more than a finite number of times. The trick is to make sure that immediately below $\alpha\hat{\ }(\rho, \beta, \tau, \infty)\hat{\ }F$ for all F there is a node β' devoted to solving $\mathcal{H}_x$. This node could have a finite number of markers restrained only. If it has the infinite outcome then it will restart ρ anyway.

The remaining details are to implement the strategies above on a priority tree as in for instance, Downey-Stob [5]. This is essentially routine and we refer the reader there for details. $\square$

4 Proof of Lemma 2.1, and turning d into a big interval.

4.1 Lemma 2.1

Our first task is to give a proof of Lemma 2.1. Recall that the property $\mathcal{S}(A)$ was the following.

$(\exists C)(\forall X \subseteq C)[A\cup X \supseteq C \rightarrow (\exists B \subseteq X)[A\cup B \supseteq X \wedge (\forall Y)[A\cup Y = \omega \rightarrow |(X\cap Y) - B| = \infty]]$

Recall that Lemma 2.1 was the following statement.

- *There is a degree* $\mathbf{d}$ *such that any set* A *of degree* $\mathbf{d}$ *has* $\mathcal{S}(A)$.

To prove Lemma 2.1, we build sets D, A_e, C_e and $B_{e,i}$ in stages to satisfy the requirements $\mathcal{R}_{e,j}$ below.

$$\mathcal{R}_{e,i} \ : \ \Gamma_e^D = A_e \wedge \Delta_e^{A_e} = D \rightarrow (\forall j)\mathcal{R}_{e,i,j}, \text{ where}$$
$$\mathcal{R}_{e,i,j} \ : \ X_i \subseteq C_e \wedge X_i \cup A_e \supseteq C_e \rightarrow$$
$$(B_{e,i} \subseteq X_i \wedge A_e \cup B_{e,i} \supseteq X_i \wedge X_i \cup Y_j = \omega \rightarrow |(X_i \cap Y_j) - B_{e,i}| = \infty).$$

Let $\ell(e,s)$ denote the D-controllable length of agreement between D and A_e. That is

$$\ell(e,s) = \max\{x : \forall y \leq x[\Delta_{e,s}^{A_{e,s}} = D_s(y) \wedge (\forall z \leq \delta_{e,s}(y)[\Gamma_{e,s}^{D_s}(z) = A_{e,s}(z)]]\}$$

Naturally we regard D as indirectly controlling A_e and hence once $x \leq \ell(e,s)$ then $A_{e,s} \restriction \delta_{e,s}(x)$ is fixed unless we change D_s on some argument $\leq \gamma_{e,s}(\delta_{e,s}(x))$. We break the $\mathcal{R}_{e,i,j}$ into infinitely many subrequirements of the form $\mathcal{R}_{e,i,j,k}$ which are the same as the $\mathcal{R}_{e,i,j}$ except that they assert that $|(X_i \cap Y_j) - B_{e,i}| \geq k$ instead of $|(X_i \cap Y_j) - B_{e,i}| = \infty$. Clearly if we meet all the $\mathcal{R}_{e,i,j,k}$ for all k then $\mathcal{R}_{e,i,j}$ will be met too.

We meet the requirements $\mathcal{R}_{e,i,j,k}$ using the finite injury method. It will essentially suffice to describe the strategy for a single requirement. Thus we will drop the subscript "e" from the sets and functionals. Clearly if $\ell(e,s) \not\rightarrow \infty$ via finite limsup we are done and since we are using a finitary methods we can thus without loss of generality suppose that $\ell(e,s) \rightarrow \infty$. Furthermore there is an easy win on $\mathcal{R}_{e,i,j}$ if we ever see $X_i \backslash C \neq \emptyset$. This is because we control C and we can thus win $\mathcal{R}_{e,i,j}$ by simply restraining any element of $X_{i,s} - C_s$ from entry to C thereby negating one of the hypotheses of $\mathcal{R}_{e,i,j}$. (For instance, we will do this if we see a win for $\mathcal{R}_{e,i,j}$ with priority $\langle e,i \rangle$.) Therefore, without loss of generality we will suppose that $|X_i \backslash C| = 0$. The cycle for a single $\mathcal{R}_{e,i,j,k}$ requirement is a follows.

4.1. Pick a follower d targeted for D which is large. Do this at an e-expansionary stage. (Namely when $\ell(e,s)$ exceeds $m = m\ell(e,s)$ which denotes the maximum of $\ell(e,t)$ for $t < s$.)

4.2. Freeze $D \upharpoonright d$ and C until a stage s_1 where $\ell(e, s_1) > d$. The region $J = [\delta_s(m)+1, \delta_{s_1}(d)]$ now becomes $\mathcal{R}_{e,i,j,k}$'s *critical region*. At stage s_1 $\mathcal{R}_{e,i,j,k}$ continues to restrain $C \upharpoonright \delta_{s_1}(d)$ as well as $D \upharpoonright s_1$ but otherwise imposes no further restraint. (This it why the other requirements can be met.)

4.3. Wait for a stage s_2 such that $A_{s_2} \cup Y_{j,s_2} \supseteq \{0, ..., \delta_{s_1}(d)\}$. *If such a stage s_2 does not occur then $A \cup Y_j \neq \omega$ and hence we win $\mathcal{R}_{e,i,j}$.* At this stage s_2 we declare that $\mathcal{R}_{e,i,j,k}$ is *active.*

4.4. Put all of the interval $[\delta_s(m)+1, \delta_{s_1}(d)]$ into C but still continue to freeze $D \upharpoonright s_1$.

4.5. Wait for a stage s_3 where $X_{i,s_3} \supseteq (C_{s_3} \cup A_{s_3}) \upharpoonright s_1$. *If such a stage s_3 does not occur then we win $\mathcal{R}_{e,i,j}$.*

4.6. When s_3 occurs then put d into D_{s_3} but continue to restrain $D \upharpoonright d-1$.

4.7. Wait for the next e-expansionary stage s_3. Notice that some number n, say, from $(C_{s_1} - A_{s_1}) \cap [\delta(m)+1, \delta s_1(d)]$ must have entered A_{s_3}. Furthermore notice that $n \in Y_{j,s_3}$ since we knew in Step 4.3 that $A \cup Y_{j,s_2} \supseteq \{0, ..., \delta_{s_1}(d)\}$ and since $A_{s_1} \upharpoonright \delta_{s_1}(d) = A_{s_2} \upharpoonright \delta_{s_1}(d)$ by D-restraint, we know that, in particular, n must have entered Y_j since it was not in A. Our action is to put into B_{i,s_3} the least collection of elements to cause $A_{s_3} \cup B_{i,s_3} \supset C_{s_3}$ for arguments less than or equal to $\ell(e, s_3)$.

The key point is that n will not enter B_i and hence we have increased the value of $|(X_{i,s_3} \cap Y_{j,s_3}) - B_{i,s_3}|$ by one since stage s, the beginning of the cycle. In this way we force $|X_i - B_i| \to \infty$. Notice that we only put elements into C in response to Y_j gaining new elements. Moreover, whenever we put elements into C provided that X_i responds by making $X_i \cup A \supseteq C$ (locally), we will later ensure that $B_i \cup A \supseteq X_i$. Of course, for any $i' \neq i$, while we are attacking some $\mathcal{R}_{e,i,j,k}$ we simply make $B_{i'} = X_{i'}$ locally with no effect on $\mathcal{R}_{e,i,j,k}$. Thus the requirements cohere exactly by as standard application of the bounded injury priority method.
$\square$

4.2 Making D into a big interval.

In this subsection we prove that we can replace **d** by an interval with a low bottom D_1 and a high$_2$ top D_2. Specifically, we need to build D_1, D_2, A_e, C_e and

B_e, i to meet the requirements below.

$$\mathcal{R}_{e,i} : \Gamma_e^{D_1} = A_e \wedge \Delta_e^{A_e} = D_2 \to (\forall j)\mathcal{R}_{e,i,j}, \text{ where}$$
$$\mathcal{R}_{e,i,j} : X_i \subseteq C_e \wedge X_i \cup A_e \supseteq C_e \to$$
$$B_{e,i} \subseteq X_i \wedge A_e \cup B_{e,i} \supseteq X_i \wedge A_e \cup Y_j = \omega \to |(X_i \cap Y_j) - B_{e,i}| = \infty.$$

Of course this generates new requirements $\mathcal{R}_{e,i,j,k}$ as in the last subsection. We similarly modify the $\ell(e,s)$ function to get the D_1, D_2-controllable length of agreement:

$$\ell(e,s) = \max\{x : \forall y \leq x[\Delta^A = D_1 \wedge (\forall z \leq \delta(y))[\Gamma^{D_2} = A[s]]]\}$$

where now we drop e subscripts for clarity.

The set D in the previous construction played *two* roles. We used D first to *freeze* a critical region J of A by freezing $D \restriction \delta(d)[s_1]$. Here we will modify things so that D_1 will do the freezing. Second, when a stage s_2 occurred when $A \cup Y[s_2]$ ate all of $\{0, ..., \delta(d)[s_1]\}$ we put things into C maintaining $D(D_2)$ restraint until X recovered. Then the role of D switched because we put d into D (in our new case D_1) to *force a change* in A allowing us to leave something our of B.

It remains to prove that we can make D_1 low, $D_1 \leq_T D_2$, and D_2 high$_2$. The first two are easy with $D_1 \leq_T D_2$ by simple permitting. We make D_2 high$_2$ using virtually the same technique as we did in the last section for making E high$_2$. Therefore we will only sketch the details.

To construe the construction as a tree argument we will use similar notation as will the last section for the nodes. This notation will hopefully reflect the symmetry of the situation. Now we will have nodes τ devoted to measuring $\ell(e,s)$ and to building C_e, nodes ρ devoted to building $B_{e,i}$ in response to $X_{e,i}$, and finally nodes α devoted to solving $\mathcal{R}_{e,i,j,k}$. the nodes α will form cones with tops α' which reflects the overall priority of $\mathcal{R}_{e,i,j}$. Again we will have the $\mathcal{H}_x$ and $\mathcal{H}_{x,m}$ nodes β of Section 3.

The construction is actually a little easier. The basic idea is that when we hit a α node, we

- Pick a follower $d = d_\alpha$ impose restraint on $D_2 \restriction s$ and link back up to ρ (where C is enumerated.)

- Now we wait for the next ρ stage s_1 where we get to impose restraint on $\delta(d)[s_1]$, as well as $D_2 \upharpoonright s$. We impose this restraint with priority $\alpha'(\alpha)$ which is the top of the tree of requirements devoted to collectively solving $\mathcal{R}_{e,i,j,k}$ above α and below ρ.

- As with the basic construction the plan is then to wait for Y_α to respond, then put the interval $[\delta(m), \delta(d)]$ into C, and at the next ρ stage we remove all $\delta(d)[s_1]$ restraint (but keep the s restraint) and put d simultaneously into into D_2 and D_1.

- At the next ρ stage we will correct B_ρ but be able to meet $\mathcal{R}_{e,i,j,k}$ at α. We can leave something out of B_ρ which is in $X_\rho \cap Y_{\alpha'}$.

Now again the construction must live with the potentially infinitary β devoted to $\mathcal{H}_x$ living on the tree.

Again if the global priority of the requirement $\mathcal{R}$ associated with ρ is higher than $\mathcal{H}_x$ then when we need to add a new outcome $(\rho, \beta, \tau, \infty)$, as in Section 3. (I.e. if $\rho \subset \beta\widehat{\ }\infty \subseteq \alpha$). Again when we hit ρ if there are potentially injurious β numbers below the ρ-use then we again pre-empt β and just enumerate them into D_2.Notice that this gives in fact a potentially a win on all $\mathcal{R}$ based on $\ell(e, s)$. (For if we traverse the link at all future stages and play the outcome $(\rho, \beta, \tau, \infty)$ then in fact $\delta_e(d) \to \infty$.) Precisely the same two problems as we saw in Section 3 occur and we circumvent them in the same way. (Restarting $\mathcal{H}_x$ and a set of suboutcomes for $(\rho, \beta, \tau, \infty)$ for the $TP - GTP$.)

On the other hand if we hit ρ and all of the potentially injurious $\beta\widehat{\ }\infty$ coding markers have cleared the $\ell(e, s) > d$-use then we can further D_2, C-restrain, (this time $\delta(d)[s_1]$), now with priority α'. If this restraint fails then α' is not on the true path. If it succeeds and and we see that $Y_{\alpha'}$ eats $[0, ..., \delta(d)[s_1]]$ then we would put the interval as above into C and reimpose restraint with priority ρ. (Now we are waiting for a X_i response.) Now we just follow the module as outlined above.

Similarly if the global priority of $\mathcal{H}_x$ is higher than ρ then we can proceed as in Section 3. The argument goes through virtually verbatim.

Finally note that we can easily make D_1 low since we only put things into D_1 at step 4.6 of the basic module. (And when we hit α' here.) But this can be interrupted. If we hit α' and some lowness requirement has requested that D_1 be left alone, then we can simply put nothing into D_2 but put all the appropriate numbers into $B_{\alpha'}$. This injures $\mathcal{R}_{e,i,j,k}$ at α but wins another lowness requirement. $\square$

References

[1] S. B. Cooper, "A jump class of non-cappable degrees," J. Symbolic Logic, Vol. 54 (1989) 324-353.

[2] Rod Downey, Steffen Lempp, and Richard Shore, "Highness and bounding minimal pairs," Mathematical Logic Quarterly, Vol. 39, 1993, 475-491.

[3] Rod Downey and Mike Stob, "Automorphisms of the lattice of recursively enumerable sets : orbits," Advances in Mathematics, Vol. 92, 1992, 237-265.

[4] Rod Downey and Mike Stob, "Jumps of hemimaximal sets,"Zeitschrift für Math. Logik Grundlagen Math., Vol 37, 1991, 113-120.

[5] Rod Downey and Mike Stob,"Minimal pairs in initial segments of the recursively enumerable degrees," submitted.

[6] Leo Harrington, *There's no fat orbit*, typewritten notes, (e-mail announcement) 26 July 1991.

[7] Leo Harrington and Robert Soare, "Post's program and incomplete recursively enumerable sets," Proceedings of the National Academy of Science, U. S. A., Vol. 88, 1991, 10242-10246.

[8] Leo Harrington and Robert Soare, "Definable properties of the lattice of recursively enumerable sets," to appear.

[9] Wolfgang Maass, "Variations on promptly simple sets," J. Symbolic Logic, Vol. 50 (1985) 138-148.

[10] D. Martin, "Classes of recursively enumerable sets and degrees of unsolvability," Zeitschrift für Math. Logik Grundlagen Math., Vol. 12, 1966, 295-310.

[11] Emil Post, "Recursively enumerable sets of positive integers and their decision problems," Bulletin of the American Math. Society, Vol. 50, 1944, 284-316.

[12] Richard Shore, "A noninversion theorem for the jump operator," Annals of Pure and Applied Logic, Vol. 40 (1988) 277-303.

[13] Robert Soare,"Automorphisms of the lattice of recursively enumerable sets, part 1 : maximal sets," Annals of Mathematics, Vol. 100, 1974, 80-120.

[14] Robert Soare,*Recursively Enumerable Sets and Degrees,* Springer-Verlag, Berlin, 1987.

[15] Robert Soare and Michael Stob, "Relative recursive enumerability," in *Proceedings Herbrand Symposium,* (ed. J. Stern) North Holland, (1982) 299-324.