Degrees of d.c.e. reals

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Abstract

A real is called c.e. if it is the halting probability of a prefix free Turing machine. Equivalently, a real is c.e. if it is left computable in the sense that $L(\alpha) = \{q \in \mathbb{Q} : q \leq \alpha\}$ is a computably enumerable set. The natural field formed by the c.e. reals turns out to be the field formed by the collection of reals of the form $\alpha - \beta$ where α and β are c.e. reals. While c.e. reals can only be found in the c.e. degrees, Zheng has proven that there are Δ_2^0 degrees that are not even n-c.e. for any n and yet contain d.c.e. reals.

In this paper we will prove that every ω-c.e. degree contains a d.c.e. real, but there are $\omega+1$-c.e. degrees and, hence Δ_2^0 degrees, containing no d.c.e. real.

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1 Introduction

The central interest in classical computability theory has been the understanding the computational complexity of sets of *positive integers*, yet, even in the original paper of Turing [15], a central topic is interest in effectiveness considerations for *reals*. Of particular interest to computable analysis (e.g. Weihrauch [16]) and to algorithmic information theory (e.g. Li-Vitanyi [11]), is the collection of *computably enumerable reals.*

A real α is computably enumerable[1] if we can effectively generate it from below. That is, the left cut $L(\alpha) = \{q \in \mathbb{Q} : q \leq \alpha\}$ forms a c.e. set. Equivalently, a real is c.e. if there is a computable sequence of rationals $\{q_i : i \in \mathbb{N}\}$ with $q_{i+1} \geq q_i$ converging to α. If we can effectively compute the radius of convergence, then the real is computable, in the sense that we can compute effectively the n-th bit of its dyadic expansion. Finally, for interest in Kolmogorov complexity, a real is c.e. iff it is the measure of the domain of a prefix-free Turing[2] machine; that is, a halting probability. If M is a universal prefix-free the real obtained by this is called Ω, Chaitin's halting probability, and is definitely *not* computable because it has no such converging sequence. In fact, whilst the easiest way to generate a c.e. real is to take a c.e. *set* W and let $\alpha = .W$, the real whose dyadic expansion if the characteristic function of W, it is not hard to see that Ω is not even in this form.

Various authors have contributed to the study of the collection of c.e. reals. See [3, 4, 5, 6, 7, 16, 17].

However, it is clear the the collection of c.e. reals do not behave well algebraically since, for instance, $1 - \Omega$ is not c.e.. Because of this, in [1], Ambos-Spies, Weihrauch and Zheng investigated the collection of the differences of c.e. reals, $\mathcal{D} = \{\alpha - \beta : \alpha, \beta \text{ c.e. reals }\}$ and proved that $\mathcal{D}$ is closed under the arithmetic operations, and hence forms a field. We call reals in $\mathcal{D}$ d.c.e. reals. The following proposition gives an analytical characterization of d.c.e. reals:

Proposition 1 *(Ambos-Spies, Weihrauch and Zheng [1]) A real number x*

[1] We remark that there have been a host of other names for this class including the *left computable, left computably enumerable, lower semi-computable,* etc reals.

[2] Recall that a Turing machine M is called prefix-free iff for all σ, τ if $\sigma \prec \tau$ and $M(\sigma)\downarrow$, then $M(\tau)\uparrow$. Such machines have measurable domains with the standard measure on 2^ω, being $\mu(\{\alpha \in 2^\omega : \sigma \prec \alpha\}) = 2^{-|\sigma|}$.

is d.c.e. iff there is a computable sequence (x_s) of rational numbers which converges to x such that $\sum_{s\in\mathbb{N}}|x_s - x_{s+1}| \leq c$ for a constant c.

We remark that because of Theorem **??**, we can easily see that the field of d.c.e. reals is in fact a *real closed field.* The proof is simple. If x is d.c.e. and (x_s) converges to x such that $\sum|x_s - x_{s+1}|$ is finite, then the sequence $(\sqrt{x_s})$ converges to $\sqrt{x}$ and the sum $\sum|\sqrt{x_s} - \sqrt{x_{s+1}}|$ is finite too. That is $\sqrt{x}$ is d.c.e..

However, Zheng has shown that the class of d.c.e. reals is not closed under computable monotone functions.

Say that a real δ is proper d.c.e. if δ is d.c.e. and does not equal to any c.e. real. An easy example of a proper d.c.e real is found by taking δ as a binary expansion of a proper 2-c.e. set, or a proper d.c.e. set, since a 2-c.e. set can be regarded as the difference of two c.e. sets. We recall the definition of the boolean algebra of n-c.e. sets.

Definition 2 (Ershov [8, 9]) The *difference hierarchy* is defined as follows:

(i) A set $A \subseteq \omega$ is called *n-computably enumerable* (n-c.e., for short), if there is a computable function f such that for all $x \in \omega$,

  (a) $f(x, 0) = 0$,

  (b) $\lim_s f(x, s) \downarrow= A(x)$, and

  (c) $|\{s + 1 \mid f(x, s) \neq f(x, s + 1)\}| \leq n$.

(ii) A set $A \subseteq \omega$ is *α-computably enumerable* (α-c.e., for short) relative to a computable system $\mathcal{S}$ of notations for α if and only if there is a *partial computable* (p.c.) function f such that for all k, $A(k) = f(k, b)$, where b is the $\mathcal{S}$-least notation x such that $f(k, x)$ converges.

By Definition 2, a set $A \subseteq \omega$ is ω-c.e. if and only if there are two computable functions $f(x, s), g(x)$ such that for all $x \in \omega$,

(a) $f(x, 0) = 0$,

(b) $\lim_s f(x, s) \downarrow= A(x)$, and

(c) $|\{s + 1 \mid f(x, s) \neq f(x, s + 1)\}| \leq g(x)$.

The following simple fact is from Arslanov [2] .

Proposition 3 *(Arslanov [2]) Let A be any ω-c.e. set. Then there exists an ω-c.e. set $B \equiv_T A$ and a computable functions $f(x, s)$ such that for all $x \in \omega$,*

(a) $f(x, 0) = 0$,

(b) $\lim_s f(x, s) \downarrow = B(x)$, *and*

(c) $|\{s + 1 \mid f(x, s) \neq f(x, s + 1)\}| \leq x$.

Here is a simple proof of this fact, which was surely known before it was stated by Arslanov. Suppose that A is ω-c.e. with at most $g(x)$ mind changes for each x. Without loss of generality we can suppose that g is an increasing computable function. Now define B by putting $g(x) \in B$ iff $x \in A$. Then $B \equiv_m A$ and B is ω-c.e. via the identity function.

It is easy to see that for any n-c.e. set A, the binary expansiony of A is a d.c.e. real. The converse is not true. Zheng [18] constructed a d.c.e. real not contained in any ω-c.e. degree.

Additionally, in section 2, we prove that any ω-c.e. degree contains a d.c.e. real.

Theorem 4 *Let* **a** *be any ω-c.e. degree, then* **a** *contains a d.c.e. real.*

In view of Zheng's result, and that of Ho [10] that every Δ^0_2 real is the limit of a computable sequence of rationals, it even seemed reasonable that perhaps *every* Δ^0_2 degree contained a d.c.e. real. The answer is no. In section 3, we construct a Δ^0_2 set (indeed, an $\omega + 1$-c.e. set) not Turing equivalent to any d.c.e. real.

Theorem 5 *There are Δ^0_2 degrees containing no d.c.e. real.*

We remark that the method of proof for Theorem 5 is new and may well have other applications.

Our notation and terminology are standard and generally follow Soare [14]. For a set A, we use A_s to denote the set of elements in A at the end of stage s. For a given partial computable (p.c.) functional Φ, and a set A, the use function φ^A is bounded by stages. We also assume that if $\varphi^{A_s}_s(x)$ converges, and $y < x$, then $\varphi^{A_s}_s(y)[s]$ also converges. Seta are identified with their characteristic functions. Finally, Δ denotes symmetric difference.

2 Every ω-c.e. degree contains a d.c.e. real

The proof of Theorem 4 is seperated into two parts. First, we prove that if the bounding function does not increase too fast, then the binary expansionary of the corresponding set is a d.c.e. real.

Lemma 6 *Let A be any ω-c.e. set, and f, g be two functions given in Definition 2. If $\sum_{x\in\mathbb{N}} g(x)\cdot 2^{-x}$ is bounded, then $.A$ is a d.c.e. real.*

Proof: Let c be a constant such that $\sum_{x\in\mathbb{N}} g(x)\cdot 2^{-x} \leq c$. W.l.o.g., suppose that $|A_{s+1}\Delta A_s| \leq 1$ for any s. Then $\{0.A_s : s \in \mathbb{N}\}$ is a computable sequence of rational numbers converging to $0.A$ and

$$\sum_{s\in\mathbb{N}} |0.A_s - 0.A_{s+1}| = \sum_{s\in\mathbb{N}} \{2^{-x} : x \in A_{s+1}\Delta A_s\} = \sum_{x\in\mathbb{N}} 2^{-x}\cdot g(x) \leq c.$$

By Propostion 1, $0.A$ is d.c.e.. □

Now we combine Lemma 6 with Propostion 3 to give a proof of Theorem 4.

Proof of Theorem 4: Let **a** be any ω-c.e. Turing degree and $A \in$ **a** be an ω-c.e. set. By Propostition 3, there is an ω-c.e. set B Turing equivalent to A and a computable function f satisfying (a)-(c) in Propostition 3. Since $\sum_{n\in\mathbb{N}} n\cdot 2^{-n} \leq 2$ is bounded, by Lemma 6, $0.B$ is a d.c.e. real. Therefore, **a** contains a d.c.e real. □

We remark here that Theorem 4 can also be proved in a constructive way. That is, given A as an ω-c.e. set, we can construct two c.e. reals α, β such that $\alpha - \beta$ is Turing equivalent to A. The main idea of the construction is as follows.

Let f, g be two computatable functions satisfying the following conditions:

(a) $f(x, 0) = 0$,

(b) $\lim_s f(x, s) \downarrow= A(x)$, and

(c) $|\{s : f(x, s) \neq f(x, s+1)\}| \leq g(x)$.

W.l.o.g., suppose that for any $s \in \mathbb{N}$, $|\{x : f(x, s) \neq f(x, s+1)\}| \leq 1$.

Let $A_s = \{x : f(x, s) = 1\}$. By our assumption on f, for any $s \in \mathbb{N}$, $|A_s\Delta A_{s+1}| \leq 1$.

Define a function h such that $h(n) = \sum_{i<n}(g(n) + 2)$. Obviously, h is computable. We describe the approximations of α and β as follows.

First, let $\alpha_0 = \sum_{n\in\mathbb{N}} 2^{-h(n)}$, $\beta_0 = 0$. $\alpha_{s+1}, \beta_{s+1}$ are defined as follows:

$$\alpha_{s+1} = \begin{cases} \alpha_s + 2^{-(h(n+1)-1)} & \text{if } A_s = A_{s+1} \cup \{n\}; \\ \alpha_s & \text{otherwise.} \end{cases}$$

$$\beta_{s+1} = \begin{cases} \beta_s + 2^{-(h(n+1)-1)} & \text{if } A_{s+1} = A_s \cup \{n\}; \\ \beta_s & \text{otherwise.} \end{cases}$$

Then $\alpha = \lim_{s\to\infty} \alpha_s$ and $\beta = \lim_{s\to\infty} \beta_s$ are both c.e. and hence $\alpha - \beta$ is d.c.e.. It is easy to see that for any n and s, the following hold:

$$\begin{aligned} A_s(n) = 0 &\iff (\alpha_s - \beta_s)\restriction[h(n), h(n+1)) = 100\cdots00 \\ A_s(n) = 1 &\iff (\alpha_s - \beta_s)\restriction[h(n), h(n+1)) = 011\cdots11. \end{aligned}$$

Therefore,

$$\begin{aligned} A(n) = 0 &\iff (\alpha - \beta)\restriction[h(n), h(n+1)) = 100\cdots00 \\ A(n) = 1 &\iff (\alpha - \beta)\restriction[h(n), h(n+1)) = 011\cdots11. \end{aligned}$$

$\alpha - \beta$ is Turing equivalent to A.

3 A Δ^0_2 degree containing no d.c.e. reals

In this section, we prove that not every Δ^0_2-Turing degree contains a d.c.e. real. To this end, we construct a Δ^0_2-set A which is not Turing equivalent to any d.c.e. real. That is, A is constructed to satisfy the following requirements:

$$\mathcal{R}_e : A \neq \Phi_e^{\alpha_e - \beta_e} \ \vee\ \alpha_e - \beta_e \neq \Psi_e^A \tag{1}$$

where $\{\langle \Phi_e, \Psi_e, \alpha_e, \beta_e\rangle : e \in \mathbb{N}\}$ is an effective enumeration of all 4-tuples $\langle \Phi, \Psi, \alpha, \beta\rangle$, Φ, Ψ computable functionals, and α, β c.e. reals. Say that requirement $\mathcal{R}_e$ has priority higher than $\mathcal{R}_{e'}$ if $e < e'$.

A is constructed as a Δ^0_2 set by stages. Let A_s be the approximation of A at the end of stage s. Then $A = \lim_{s\to\infty} A_s$. We now describe a strategy satisfying a single requirement. First we define the length function of agreement for $\mathcal{R}_e$ at stage s as follows:

$$l(e,s)=\max\{x : A_s(x) = \Phi_{e,s}^{\alpha_{e,s}-\beta_{e,s}}(x)\&(\alpha_{e,s}-\beta_{e,s})\restriction\varphi_{e,s}(x) = \Psi_{e,s}^{A_s}\restriction\varphi_{e,s}(x)\},$$

where φ_e is the use function of the functional Φ_e. Our strategy will ensure that $l(e,s)$ is bounded during the construction, and hence $\mathcal{R}_e$ is satisfied.

We first choose a witness x as a big number and wait for a stage s with $l(e,s) > x$. Put x into A, and wait for another stage $s' > s$ with $l(e,s') > x$. If there is no such a stage, then $\mathcal{R}_e$ is satisfied obviously. Otherwise, we have that $\alpha - \beta$ changes below $\varphi_{e,s}(x)$ between stages s and s'. That is, $\Psi_e^A\restriction\varphi_{e,s}(x)$ have changes between s and s'. Note that the only small number enumerated into A between s and s' is x, so by taking x out of A, we recover the computations $\Psi_e^A \restriction \varphi_{e,s}(x)$ to $\Psi_{e,s}^{A_s} \restriction \varphi_{e,s}(x)$, and we have a temporary disagreement between $(\alpha_e-\beta_e)\restriction\varphi_{e,s}(x)$ and $\Psi_e^A\restriction\varphi_{e,s}(x)$. If $(\alpha_e-\beta_e)\restriction\varphi_{e,s}(x)$ don't change later, then by preserving A on $\psi_{e,s}(\varphi_{e,s}(x))$, we will have

$$(\alpha_e-\beta_e)\restriction\varphi_{e,s}(x) \neq \Psi_e^A\restriction\varphi_{e,s}(x),$$

and $\mathcal{R}_e$ is satisfied again. By iterating such a procedure, we put x into A and take x out of A alternatively, trying to get a disagreement between A and $\Phi_e^{(\alpha_e-\beta_e)}$ or between $(\alpha_e-\beta_e)\restriction\varphi_{e,s}(x)$ and $\Psi_e^A\restriction\varphi_{e,s}(x)$. It is easy to check that if $(\alpha_e-\beta_e)\restriction\varphi_{e,s}(x)$ changes only finitely often, then we can get the wanted disagreement eventually, and $\mathcal{R}_e$ is satisfied. However, $(\alpha_e-\beta_e)\restriction\varphi_{e,s}(x)$ may change infinitely often, as pointed out below, even though both $\alpha_e\restriction\varphi_{e,s}(x)$ and $\beta_e\restriction\varphi_{e,s}(x)$ settle down after a stage large enough.

Fix i. $(\alpha_e-\beta_e)(i)$ can be changed by changes of $\alpha_e(j)$ or $\beta_e(j)$, where $j > i$. For example, let

$$\alpha_{e,1} = \alpha_{e,2} = 0.101w0, \quad \beta_{e,1} = 0.100w1 \text{ and } \beta_{e,2} = 0.100w0.$$

for some $w \in \{0,1\}^n$ and $n \in \mathbb{N}$. Then we have

$$\alpha_{e,1}-\beta_{e,1} = 0.0010\overbrace{1\cdots1}^{n}1 \quad \text{and} \quad \alpha_{e,2}-\beta_{e,2} = 0.0011\overbrace{0\cdots0}^{n}0.$$

The change of $\beta_e(n+4)$ from 1 to 0 leads to the change of $(\alpha_e-\beta_e)(4)$ from 1 to 0. We call such a change of $\alpha_e-\beta_e$ as a "nonlocal-disturbance". Note that $(\alpha_e-\beta_e)(4)$ can be changed infinitely often by these nonlocal-disturbances since we have infinitely many such ws. Fortunately, if such a "nonlocal-disturbance" happens, then the corresponding segments of $\alpha_e-\beta_e$ will be in quite simple forms. This is summarized below:

Proposition 7 *Let* $\alpha^j = 0.a_1^j a_2^j \cdots a_n^j$, $\beta^j = 0.b_1^j b_2^j \cdots b_n^j$ *and* $\alpha^j - \beta^j = 0.c_1^j c_2^j \cdots c_n^j$ *for* $j = 0, 1$. *If there are numbers* $i < k \leq n$ *such that* $c_i^0 \neq c_i^1$, *and* $a_t^0 = a_t^1$, $b_t^0 = b_t^1$ *for all* $t \leq k$. *Then, there is a* $j \in \{0, 1\}$ *such that*

$$c_i^j c_{i+1}^j \cdots c_k^j = 011 \cdots 1 \quad \& \quad c_i^{1-j} c_{i+1}^{1-j} \cdots c_k^{1-j} = 100 \cdots 0. \tag{2}$$

Now let's turn back to consider $(\alpha_e - \beta_e) \upharpoonright \varphi_{e,s}(x)$. Suppose that both $\alpha_e \upharpoonright \varphi_{e,s}(x)$ and $\beta_e \upharpoonright \varphi_{e,s}(x)$ do not change after a stage large enough, s_1 say, then by Proposition 7, the initial segment $(\alpha_e - \beta_e) \upharpoonright \varphi(x)$ can have only one of two different forms: $0.w011 \cdots 1$ or $0.w100 \cdots 0$ for some fixed binary word w. It leads us to use two-attackers to satisfy $\mathcal{R}_e$, instead of using a single attacker. That is, at stage s', instead of taking x out of A, we put $x - 1$ into A and wait for a stage $s'' > s'$ with $l(e, s'') > x$. At stage s'', we take $x - 1$ and x out of A, and wait for a stage $s''' > s''$ with $l(e, s''') > x$. As a consequence, $A \upharpoonright \psi_e(\varphi_{e,s}(x))$ is recovered to that of stage s. Now we have three uses of $\varphi_e(x)$, i.e., $\varphi_{e,s}(x)$, $\varphi_{e,s'}(x)$, and $\varphi_{e,s''}(x)$. At stage $s''' + 1$, we will have $(\alpha_{e,s'''} - \beta_{e,s'''}) \upharpoonright \varphi_{e,s}(x) = (\alpha_{e,s} - \beta_{e,s}) \upharpoonright \varphi_{e,s}(x)$. As in stage s, we put x into A again. We call the procedure between s and $s''' + 1$ a complete cycle.

Let k be the maximum among $\varphi_{e,s}(x)$, $\varphi_{e,s'}(x)$, and $\varphi_{e,s''}(x)$. Then in a complete cycle $(\alpha_e - \beta_e) \upharpoonright k$ has three different forms. By Proposition 7, in each complete cycle, α_e or β_e must have a change below k. Since α_e, β_e are both c.e., we can assume that after a stage t large enough, $\alpha_e \upharpoonright k$ and $\beta_e \upharpoonright k$ don't change anymore, and therefore, after t, no cycle can be complete. As a consequence, one of the combinations of $A(x - 1)$ and $A(x)$, 00, 01, or 11, satisfies the requirement $\mathcal{R}_e$.

We describe the whole construction of A below.

Construction of A

During the construction, say that a requirement $\mathcal{R}_e$ *requires attention* at stage $s + 1$ if x_e is defined and $l(e, s) > x_e$. When we initialize a requirement $\mathcal{R}_e$, we undefine all parameters associated with it.

Stage $s = 0$: Do nothing.

Stage $s + 1$: If no requirement requires attention at stage $s + 1$, then choose a least e such that x_e is not defined and define $x_e = s + 2$.

Otherwise, let $\mathcal{R}_e$ be the requirement of the highest priority requiring attention and define

$$A_{s+1}(x_e-1)A_{s+1}(x_e)=\begin{cases}01 & \text{if } A_s(x_e-1)A_s(x_e)=00;\\ 11 & \text{if } A_s(x_e-1)A_s(x_e)=01;\\ 00 & \text{if } A_s(x_e-1)A_s(x_e)=11.\end{cases} \tag{3}$$

Initialize all requirements with lower priority, and declare that R_e *receives attention* at stage $s+1$.

This completes the construction.

We now verify that A constructed above satisfies all the requirements. We only need to prove the following lemma.

Lemma 8 *For any $e\in\mathbb{N}$, $\mathcal{R}_e$ requires and receives attention finitely often.*

Proof: We prove Lemma 8 by induction on e. Assume that, for any $i<e$, $\mathcal{R}_i$ requires and receives attention only finitely often. Let s_0 be the least stage after which no requirement $\mathcal{R}_i$, $i<e$, requires attention. By the choice of s_0, R_e is initialized at stage s'. Let $s_1>s_0$ be the stage at which x_e is defined. Since $\mathcal{R}_e$ cannot be initialized after stage s_0, x_e cannot be canceled afterwards. We prove below that after a stage large enough, $s_2>s_1$ say, $\mathcal{R}_e$ does not require attention anymore, and hence $l(e,s)$ cannot exceed x_e for $s>s_2$, $\mathcal{R}_e$ is satisfied.

For a contradiction, suppose that after stage s_1, there are infinitely many stages $t_0+1, t_1+1, t_2+1, \cdots$ at which $\mathcal{R}_e$ requires attention. Then, at stage t_0+1, we have $l(e,t_0)>x_e$, $A_{t_0}(x_e-1)A_{t_0}(x_e)=00$, $A_{t_0+1}(x_e-1)A_{t_0+1}(x_e)=01$. By the choice of s_0, no requirement with higher priority can put numbers smaller than $\psi_{e,t_0}(\varphi_{e,t_0}(x_e))$ into A. Since all requirements with lower priority are initialized at stage t_0+1, when these requirements receive attention after t_0+1, the numbers they put into A or take out of A are all larger than t_0, and hence larger than $\psi_{e,t_0}(\varphi_{e,t_0}(x_e))$. Therefore, the computations $\Psi^{A_{t_0}}_{e,t_0}(\varphi_{e,t_0}(x_e))$ can only be changed by $\mathcal{R}_e$ itself by changing $A(x_e-1)$ or $A(x_e)$. Thus, by a simple induction, we have for all $n\in\mathbb{N}$,

$$A_{t_0}\upharpoonright\psi_{e,t_0}(\varphi_{e,t_0}(x_e))=A_{t_{3n}}\upharpoonright\psi_{e,t_0}(\varphi_{e,t_0}(x_e))$$

because $A(x_e-1)A(x_e)$ changes always in the order $00 \to 01 \to 11 \to 00$. Therefore,

$$(\alpha_{e,t_{3n}} - \beta_{e,t_{3n}}) \upharpoonright \varphi_{e,t_0}(x_e) = (\alpha_{e,t_0} - \beta_{e,t_0}) \upharpoonright \varphi_{e,t_0}(x_e).$$

This means that the computation $\Phi_{e,t_{3n}}^{A_{t_{3n}}}(x_e)$ is actually the same as that of $\Phi_{e,,t_0}^{A_{t_0}}(x_e)$. Similarly, we can prove that the computation $\Phi_{e,t_{3n+1}}^{A_{t_{3n+1}}}(x_e)$ is the same as that of $\Phi_{e,t_1}^{A_{t_1}}(x_e)$, and the computation $\Phi_{e,t_{3n+2}}^{A_{t_{3n+2}}}(x_e)$ is the same as that of $\Phi_{e,t_2}^{A_{t_2}}(x_e)$.

Let $k = \max\{\varphi_{e,i}(x_e) : i \leq 3\}$. Choose an n large enough such that $l(e, t_n) > x_e$, and

$$\alpha_{e,t_n} \upharpoonright k = \alpha_{e,t} \upharpoonright k \quad \& \quad \beta_{e,t_n} \upharpoonright k = \beta_{e,t} \upharpoonright k$$

for any $t \geq t_n$. W.o.l.g., suppose that $n = 3m$ for some m. Then $A_{t_n}(x_e - 1)A_{t_n}(x_e) = 00$, $A_{t_{n+1}}(x_e-1)A_{t_{n+1}}(x_e) = 01$, and $A_{t_{n+2}}(x_e-1)A_{t_{n+2}}(x_e) = 11$. By our choices of t_n, t_{n+1}, t_{n+2}, we have

$$\Phi_{e,t_n}^{\alpha_{e,t_n}-\beta_{e,t_n}}(x_e-1)\Phi_{e,t_n}^{\alpha_{e,t_n}-\beta_{e,t_n}}(x_e) = 00, \quad \text{and}$$

$$\Phi_{e,t_{n+1}}^{\alpha_{e,t_{n+1}}-\beta_{e,t_{n+1}}}(x_e-1)\Phi_{e,t_{n+1}}^{\alpha_{e,t_{n+1}}-\beta_{e,t_{n+1}}}(x_e) = 01.$$

This implies that

$$(\alpha_{e,t_n} - \beta_{e,t_n}) \upharpoonright \varphi_{e,t_n}(x_e) \neq (\alpha_{e,t_{n+1}} - \beta_{e,t_{n+1}}) \upharpoonright \varphi_{e,t_{n+1}}(x_e)$$

and hence $(\alpha_{e,t_n} - \beta_{e,t_n}) \upharpoonright k \neq (\alpha_{e,t_{n+1}} - \beta_{e,t_{n+1}}) \upharpoonright k$. By Lemma 7, there exists a binary word w such that $(\alpha_{e,t_n} - \beta_{e,t_n}) \upharpoonright k$ takes one of the forms $0.w100\cdots0$ and $0.w011\cdots1$, and $(\alpha_{e,t_{n+1}} - \beta_{e,t_{n+1}}) \upharpoonright k$ takes the other one. Assume that $(\alpha_{e,t_n} - \beta_{e,t_n}) \upharpoonright k$ takes the form $0.w100\cdots0$ and $(\alpha_{e,t_{n+1}} - \beta_{e,t_{n+1}}) \upharpoonright k$ takes the form $0.w011\cdots1$. By the same argument, since $(\alpha_{e,t_{n+1}} - \beta_{e,t_{n+1}}) \upharpoonright k$ takes the form $0.w011\cdots1$, we know that $(\alpha_{e,t_{n+2}} - \beta_{e,t_{n+2}}) \upharpoonright k$ takes the form $0.w100\cdots0$. Thus,

$$(\alpha_{e,t_{n+2}} - \beta_{e,t_{n+2}}) \upharpoonright k = (\alpha_{e,t_n} - \beta_{e,t_n}) \upharpoonright k,$$

and hence

$$00 = \Phi_e^{\alpha_e-\beta_e}(x_e-1)[t_n]\Phi_e^{\alpha_e-\beta_e}(x_e)[t_n] = \Phi_{e,}^{\alpha_e-\beta_e}(x_e-1)\Phi_e^{\alpha_e-\beta_e}(x_e) = 11.$$

A contradiction. Therefore, after stage t_n, $\mathcal{R}_e$ can require (and hence receive) at most two more times

This ends the proof of Lemma 8. □

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