Computability, Definability and Algebraic Structures*

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1 Introduction

In a later section, we will look at a result of Coles, Downey and Slaman [16] of pure computability theory. The result is that, for *any* set X, the set

$$\{\deg(F)' : X \text{ is computably enumerable relative to } F\}$$

always has a least member. While we now know that this theorem has an easy proof, it is nonetheless quite surprising and has consequences for the theory

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of abelian groups. In these notes I hope to lead the reader into understanding this connection, and framing this result in the context of computability considerations yielding insight into classical algebra, and, in particular, coding in algebraic structures.

In the first section, we will look at computability questions for algebraic structures, and in particular the sensitivity needed for the *presentations* of such structures. In the second section we shall examine the question of assigning measures characterizing, at least to some extent, the algorithmic complexity of the relevant structures. The main tool here will be Turing degree, although many other classification tools are possible. In section 3 we will look at presentations of *isomorphism types*. In section 4 we shall look at jump degrees of isomorphism types, reflecting possible codability. This leads into the problem for torsion free abelian groups. In the 5th section, we will look at the least jump theorem, and its connection with enumeration degrees. In section 6, we turn to analysis, where we will look at some material about presenting real numbers. Finally in the last section we offer some thoughts on what constitutes a good question on effective mathematics, and offer some questions we feel are in this category.

The plan is to highlight a number of themes and open questions. While some of the material is familiar, much is new. I hope that the paper will cohere with and rather complement the material in my earlier survey [19], as well as the the long articles of Remmel [87] and the author [20] and many other articles in the Handbook of Recursive Mathematics [30] and the book by Rosenstein [96].

I assume that the reader is familiar with the rudiments of computability/recursion theory. Hopefully the reader will have met a priority argument, but certainly be familiar with the standard coding tricks that reduce computability theory to the study of partial functions on the $\mathbb{N}$. (And reduces complexity theory to studying langauges in Σ^*.) $\langle , \rangle$ denotes a standard pairing function computably and bijectively mapping $\mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$. It is monotone in both variables. All uses etc, are bounded by s at stage s. Otherwise, notation is standard and follows, for instance, Soare [103].

2 Presentations

The central classification tool in classical algebra such as group theory or linear algebra is *isomorphism*. Two structures are regarded as the *same* if they are isomorphic. This does not seem reasonable if one is concerned with trying to understand the algorithmic content of algebra. That is, if one views algebraic structures through computational eyes, and asks questions such as "if I am *given* a structure in such an *online* or *computable* manner so that I can perform the basic operations computably, then what else can I perform computably?"

A good example is that of linear orderings. What is a reasonable definition of a computable linear ordering? Well, a linear ordering is a pair consisting of a set A and a binary relation $\preceq$, which obeys the usual ordering axioms. So a reasonable definition for a *computable* linear ordering would be for the universe to be a computable set, usually identified with $\mathbb{N}$, upon which the order relation is a computable relation: given $a, b \in \mathbb{N}$, we can decide which of $a \prec b, b \prec a$, or $a = b$ holds.

Now classically we have students think of an ordering of type ω as being $\{0, 1, 2, 3, ...\}$, with the usual order. But this is also isomorphic to an ordering listing $12 \prec 1 \prec 1000 \prec 10^{500} \prec 2 \prec ....$ Any permutation of $\mathbb{N}$ gives rise to an ordering of type ω in this way. So there are $2^{\aleph_0}$ differing *presentations* of orderings of type ω. Here for a given structure $\mathcal{A}$, a presentation will be some structure isomorphic to $\mathcal{A}$. The degree of such a presentation $\mathcal{B}$ is simply the degree of its open diagram. In the case of a linear ordering of $\mathbb{N}$, this is simply the degree of the ordering relation. It is obvious that many of the $2^{\aleph_0}$ presentations of ω will have differing computability properties since there are only countably many Turing machines.

However, not completely obvious is the fact that differing computable presentations can have differing computability properties. For instance, the *adjacency* relation $A(x, y)$ on an ordering A is defined to hold on a pair x, y if $x \prec y$ and for all z, it is not the case that $x \prec z \prec y$. The following is an easy example, taken from the folklore.

Observation 2.1 *There is a computable ordering $(A, \prec)$ of type ω for which the adjacency relation is not computable.*

Proof. We begin with $A_0 = \{2x : x \in \mathbb{N}\}$ ordered in the usual way. We devote the pair $4e, 4e+2$ to the satisfaction of the e-th requirement $\mathcal{R}_e$ below.

$$\mathcal{R}_e : \varphi_e(\cdot, \cdot) \text{ is not an adjacency function for } A.$$

Here φ_e denotes the e-th binary partial computable function. It will have output 0 or 1 with 0 meant to represent nonadjacent, and 1 adjacent. We will add at most one point in the middle of the currently adjacent pair $4e, 4e+2$. To this end we set aside the numbers $\{\langle e, s\rangle : s \in \mathbb{N}\}$ for the sake of $\mathcal{R}_e$. The action is that, while $\mathcal{R}_e$ is not declared satisfied, if we see a stage $s > e$ where $\varphi_{e,s}(x, y) \downarrow= 1$, we declare $\mathcal{R}_e$ as satisfied and put $4\langle e, s\rangle + 1$ between $4e$ and $4e + 2$. This clearly makes sure that φ_e cannot be the adjacency function for A, since either we never need to act for it because it never tells us that $4e, 4e + 2$ is an adjacency, or it does so tell us, and we make it wrong by adding a point between them. Since we declare it satisfied, we do this at most once. Hence each point has at most finitely many predecessors. Finally the ordering A so constructed is computable since we decide the fate of even points at the beginning and to decide if $4x + 1$ is in A, and where, we simply compute the pair e, s such that $\langle e, s\rangle = x$. Then if $s \leq e$, $4x + 1 \notin A$, otherwise $4x + 1$ is in A iff we put it in between $4e$ and $4e + 2$ at stage s, and no other. $\square$

A few comments on the "wait and see" proof above. First it admits many variations. For instance, we could define as, say, Cenzer and Remmel [12], an ordering to be *polynomial time* if the universe is $\{0, 1\}^*$ (there are other variations) and the relation $x \prec y$ is polynomial time in the length of x and y. The argument above can easily be modified to show that there is a polynomial time ordering where the adjacency relation is not computable!

Additionally, notice that the argument demonstrates that one can have computable orderings which are isomorphic yet not computably isomorphic (since the computability of the adjacency relation would be preserved by a computable isomorphism). This is not always the case. Define a computable ordering to be *computably categorical* if every computable ordering isomorphic to it is computably isomorphic to it.

Theorem 2.2 (Remmel [86], Goncharov [40]) *A computable linear ordering is computably categorical iff it has a finite number of adjacencies.*

A direct proof of the Remmel-Goncharov theorem works by observing that the usual Cantor back and forth categoricity of the rationals can be effectized for the "if" direction, and by a finite injury priority argument based on guessing and killing adjacencies for the hard direction. We remark that the argument is as easy as the one above in the case that we have a computable list of adjacencies. However this is not always the case.

Theorem 2.3 (Downey and Moses [27]) *There is a computable linear ordering A such that for any computable linear ordering B isomorphic to A, the adjacency relation on B has Turing degree $\mathbf{0}'$, the degree of the halting problem.*

So we have seen that for some order types there can be many computable presentations, and for some only one. One might ask what else is needed to specify a structure up to computable isomorphism.

In the case of ω adjacencies are enough.

Theorem 2.4 (essentially a special case of Moses [79]) *Suppose that two linear orderings of type ω each have computable adjacency relations. Then the orderings are computably isomorphic.*

Proof The proof is fairly easy. Suppose that A and B are the relevant orderings. Without loss of generality, let a_0, b_0 be the least members of each. Using complete search, and the computability of the adjacency relation, one can find the successor of a_0 in A and the successor of b_0 in B, and then map each to the other, etc. $\square$

Ash and Knight (e.g. [1, 65, 66]) have formulated various extensions of the conditions of Theorem 2.4 which give decidability conditions guaranteeing the computable categoricity of classes of computable well orderings or discrete orderings. As one would expect, the conditions connect the finite iterations of the "condensation" operator (two points become identified if they are a finite distance apart) with infinitary decidability conditions. Similar considerations arise in proof theory in connection with uniqueness of paths through Kleene's $\mathcal{O}$.

Turning to general presentability questions, already we can formulate two open questions.

Question 2.5 *Classify, by order type, the linear orderings that have computable presentations, or show that this is impossible.*

Question 2.6 *Does there exist a linear ordering that has presentations of each nonzero Turing degree, but no computable presentation?*

We finish this section with some comments about Questions 2.5 and 2.6.

Regarding 2.6, Wehner [111] and Slaman [98] have each constructed examples of relational structures which have presentations in each degree except zero. Hirschfeldt, Khoussainov, Shore and Slinko [48] have found several canonical reductions allowing results on presentability to be transferred to other structures such as groups, rings, integral domains and the like. Thus (for example) they can construct examples of groups that have presentations in each nonzero degree but are not computably presentable. This reflects earlier work on Borel reducibility by, for instance, Friedman and Stanley [37] (see also Kechris [50].) The Hirschfeldt et. al. methods don't apply to orderings. One notable result here is due to Miller.

Theorem 2.7 (Miller [77]) *There is a linear ordering with presentations in each Δ_2^0 degree except* $\mathbf{0}$.

Harizanov suggests that one can define the *spectrum* of a structure $\mathcal{A}$ as the degrees that $\mathcal{A}$ has presentations in. Similarly for a class of degrees $\mathcal{C}$, Miller defined the *$\mathcal{C}$-spectrum of $\mathcal{A}$* to consist of the intersection of the spectrum of $\mathcal{A}$ with $\mathcal{C}$. So the Theorem above can be rephrased as the Δ_2^0 spectrum of a linear ordering can be each degree except $\mathbf{0}$.

Actually this is related to earlier work on the degree spectrum of a relation R on a model $\mathcal{A}$. That is, the possible degrees of R in various computable presentations of $\mathcal{A}$. The notion was coined by Harizanov, who constructed models with, for instance, two element degree spectra. Notice that if we consider the adjacency relation on a linear ordering, then by the Downey-Moses result, we get the following.

Corollary 2.8 *There is a computable ordering A such that the degree spectrum of the adjacency relation has a single element $\mathbf{0}'$.*

Actually, the degree spectrum of the adjacency relation can be fairly rich.

Theorem 2.9 (Downey) *There is a computable linear ordering A for which the degree spectrum of the adjacency relation consists of every nonzero computably enumerable degree.*

Proof This proof uses a technique of Downey and Knight [24]. Let Q be a linear ordering. Define $\gamma(Q) = (\eta+2+\eta)\cdot Q$. That is, we replace each point of Q by a copy of the rationals (η) followed by a single adjacency followed by the rationals. Notice that the collection of adjacencies of $\gamma(Q)$ is isomorphic to Q. Downey and Knight [24] proved that Q has a presentation of degree $\mathbf{a}'$ iff $\gamma(Q)$ has a presentation of degree $\mathbf{a}$. In particular, if Q is chosen to be the linear ordering of Miller's theorem which has presentations $Q_{\mathbf{b}}$ of each computably enumerable degree $\mathbf{b}$ then certainly $\gamma(Q)$ is a computable linear ordering with no computable presentations $\widehat{Q}$ with computable adjacency relation. (To see this suppose that $\widehat{Q}$ is a computable presentation of $\gamma(Q)$ with computable adjacency relation. From this one can construct a computably enumerable subset of $\widehat{Q}$ with the same ordering as $\widehat{Q}$ isomorphic to Q. This is easily seen to be isomorphic to a computable ordering, a contradiction.)

To finish the proof we need to look into the guts of the proof of the Downey-Knight proof. Let Q be given and suppose that Q is a computably enumerable linear ordering of degree $\mathbf{b}$. Via Cantor's proof, we can consider Q as a Π_1^0 subset of the rationals with the standard ordering. We will build $\gamma(Q)$ as a computable ordering so that the the degree of the adjacency relation in $\gamma(Q)$ is the same as the degree of Q. We can computably embed the standard copy of $\mathbb{Q}$ into a computable copy of $(\mathbb{Q} + 2 + \mathbb{Q}) \cdot \mathbb{Q}$, where we regard Q as being a subset of the last copy of $\mathbb{Q}$. Thus we can regard the x-th rational as being identified with a canonical adjacency in $\gamma(\mathbb{Q})$. Note that if $Q = \cap_s Q_s$ is a computable approximation to Q, then $x \in Q$ iff $\forall s(x \in Q_s)$.

The idea is simple. We do nothing for x unless we see a stage s such that $x \notin Q_s$. At such a stage s we put a copy of $\mathbb{Q}$ in the successivity representing x. (Strictly speaking we would have a infinite list of fresh numbers to represent rationals we add, and we would be doing everything with finite subsets, but this obscures the idea.) Clearly $\gamma(Q)$ so constructed is computable, and the set of adjacencies of this copy of $\gamma(Q)$ has, in fact, the same m-degree as Q. This gives the result. □

An interesting question is to look at the possible spectra a linear ordering can have. We know, by a result of Knight [59], that if L has a presentation of degree $\mathbf{a}$ then it has one of each degree above $\mathbf{a}$.

Another example of a structure closely related to orderings is that of boolean algebras. while it is possible to prove things directly, it is nice to extract results for boolean algebras from results for linear orderings via the following technical device.

Theorem 2.10 (The Representation Theorem, folklore after Stone) *Every countable boolean algebra is isomorphic to an interval algebra $Intalg(L)$ of a linear ordering of the same degree. (Recall that $Intalg(L)$ is the algebra generated by the finite unions of left closed right open intervals of the ordering L.)*

Proof. This is just an effective version of Stone's theorem. We represent a given B as an algebra of sets. Thus let $B = \cup_s B_s$ with $B_0 = \{0, 1\}$, and $B_{s+1} - B_s = \{b_s\}$. We will define at each stage a subalgebra $\widehat{B_s}$ containing B_s.

Define L_0 to be the ordering with two points labelled 0 and 1. (Then $Intalg(L_0)$ consists of the two sets $\emptyset$ and $[0, 1)$.) So we have the induced mapping g_0 with $0 \mapsto \emptyset$ and $1 \mapsto [0, 1)$. At stage $s + 1$ we will have a set of $Atoms_s = \{a_{s_1}, ..., a_{s_n}\}$ listing the atoms of the subalgebra of $\widehat{B_s}$ together with the linear ordering $L_s = 0, x_{s_1}, ..., x_{s_n} = 1$ so that $g_s(a_{s_j}) \mapsto [x_{s_{j-1}}, x_{s_j})$ inducing an isomorphism from the subalgebra $\widehat{B_s}$ to $Intalg(L_s)$.

At stage $s + 1$ if b_s is in $\widehat{B_s}$ we need to do nothing. Otherwise, for each $a = a_{s_i}$ such that b_s *splits* a (i.e. both $a \wedge b_s$ and $a \wedge \overline{b_s}$ are nontrivial), add a new point y to L_{s+1} between $x_{s_{j-1}}$ and x_{s_j} to split the interval $[x_{s_{j-1}}, x_{s_j})$ into $[x_{s_{j-1}}, y) \cup [y, x_{s_j})$. Naturally we map $a_{s_j} \wedge b_s$ to one of them, say, $[x_{s_{j-1}}, y)$ and map $a_{s_j} \wedge \overline{b_s}$ to $[y, x_{s_j})$. Note that this generates two new atoms for $\widehat{B}_{s+1}$, and of $Intalg(L_{s+1})$. Let $c_1, ..., c_m$ denote the atoms of $\widehat{B}_{s+1}$ below b_s. Clearly we have ensured that the induced map $g(b_s) = g(c_1) \cup ... \cup g(c_m)$ works. The results follows. $\square$

Notice that in the proof of the Representation Theorem we get a $1 - 1$ correspondence between *atoms* in B and *adjacencies* in L. One might well expect that one could have similar results concerning the relation $A(x) =$"x is an atom" and its spectrum, as to the adjacency relation. However, the situation is rather different.

We need the following Lemma.

Theorem 2.11 (Remmel-Vaught [87]) [1] *(i) Suppose that B is any boolean subalgebra of $\mathbb{Q}$ with infinitely many atoms $\{d_i : i \in \mathbb{N}\}$. Let $\widehat{B}$ be the subalgebra of $\mathbb{Q}$ obtained from B by splitting each atom of B a finite number of times in $\mathbb{Q}$ (together with B). (Thus the atoms of $\widehat{B}$ would be $\{e_{i_1}, ..., e_{i_{n(i)}} : i \in \mathbb{N}$ where $d_i = e_{i_1} \vee ... \vee e_{i_{n(i)}}\}$.) Then $\widehat{B}$ is isomorphic to B.*
(ii) (Rephrasing (i) in terms of linear orderings). Suppose that L and $\widehat{L}$ are linear orderings with infinitely many adjacencies, and $g : L \mapsto \widehat{L}$ is an order-preserving embedding such that
(a) if $[x, y]$ is finite in L then $[g(x), g(y)]$ is finite in $\widehat{L}$, and
(b) if $z \in \widehat{L}$ is not in the image of L, then there are x, y in L such that $[x, y]$

[1] In his Thesis, Vaught proved this result for atomic boolean algebras and Remmel extended the result to all countable boolean algebras

is finite and $z \in [g(x), g(y)]$.
Then $Intalg(L) \cong Intalg(\hat{L})$.

Remmel used Theorem 2.11 to prove many of his results on boolean algebras. For instance, to prove that if a boolean algebra has infinitely many atoms then it is not computably categorical, one needs to guess atoms and then split them if the range looks like an atom. So long as the splitting is finite, then you don't leave the isomorphism type. An easier example is to construct a boolean algebra isomorphic to the given one whose atom set has no infinite computably enumerable subset. We refer the reader to Remmel [87] for details.

We have seen that the adjacency relation for a linear ordering can be intrinsically complete, since its spectrum can consist of a single degree $\mathbf{0}'$. This is never the case for the atomicity relation of a boolean algebra.

Theorem 2.12 (Downey [18]) *Every computable boolean algebra is isomorphic to one where the collection of atoms is Turing incomplete.*

This theorem is proven using Theorem 2.11 and a fairly complicated infinite injury priority argument. To demonstrate the technique, we sketch the proof of a much easier result. By the work of Jockusch and Soare [56], we know that there are linear orderings L with presentations of low degree (i.e. $L' \equiv_T \emptyset'$) not isomorphic to a computable linear ordering. This is not true of boolean algebras.

Theorem 2.13 (Downey and Jockusch [23]) *Suppose that B is a low boolean algebra. Then B is isomorphic to a computable boolean algebra.*

Proof. Using the proof of Stone's theorem above we can regard $B = Intal(L)$ with L low, and notice that the adjacency relation for L is computable from $\emptyset'$ since L is low. Hence, by the Limit Lemma we can suppose we have a computable approximation $L(x) = \lim_s L_s(x)$ where the fact that a pair is an adjacency or not will be known from some point on. Wlog, we suppose that L has endpoints x_0, x_1. We map $x_i \mapsto y_i$. Now a new point appears between x_0 and x_1. This new point a may or may not be in L, but since L has infinitely many points there is no apparent harm in putting a new point b between y_1 and y_0, and matching. If the point a leaves, then we can find the next least point (by Gödel number) a' and rematch $a' \mapsto b$. The only problem is if we have some interval (a_1, a_2) currently matched to (b_1, b_2). A new point c appears between a_1 and a_2 and we respond with a new point d between the b_i. The point c leaves, but since our order must be computable, d cannot leave. If the $a_i \mapsto b_i$ are good mappings then we don't want to move them. Here is where we get to use lowness. We know that if (a_1, a_2) looks often enough like an adjacency it is one. So we only respond to the appearence of new points between the a_i if it appears that (a_1, a_2) is *not* an adjacency. Either it really is not, in which case we are safe with new elements between the b_i, *or* after a while we will constantly believe that (a_1, a_2) *is* an adjacency, and hence we will have mapped the interval (a_1, a_2) to (b_1, b_2) where (b_1, b_2) contains a finite number of points not in the range of our mapping. Now we can apply the Remmel-Vaught Theorem to get the isomorphism. $\square$

The theorems above suggest the following open questions.

Question 2.14 *What can be said about the possible spectra for the relation "a is an atom" for a boolean algebra B?*

Question 2.15 *What can be said about the possible degrees of boolean algebras not isomorphic to computable ones?*

We remark that, with respect to 2.14, Goncharov modified the coding argument of Feiner [31] to construct a computable boolean algebra whose atoms are intrinsically non-low$_n$ for any $n \in \mathbb{N}$. (See Remmel [87].) Thurber [109] extended the Downey-Knight theorem above to show that every low$_2$ boolean algebra is isomorphic to a computable one, and this has been extended to each low$_4$ boolean algebra by Knight and Stob [53]. New ideas will be needed to show that, as the author conjectures, every low$_n$ boolean algebra is isomorphic to a computable one.

More generally it is very interesting to wonder as to the possible degree spectra of relations on models. There are many possible versions here. Two are (i) the general spectra problem where we look at the possible degrees in *all* models, and (ii) the degrees of the relation in *computable* models.

The current best result for two element degree spectra for computable models is provided by the following result of Hirschfeldt.

Theorem 2.16 (Hirschfeldt [47]) *Let* $\mathbf{a}$ *be any nonzero computable enumerable degree. Then there is a computable structure* $\mathcal{A}$ *and a relation* R *on* $\mathcal{A}$ *whose spectrum is* $\mathbf{0}, \mathbf{a}$.

Amazingly, we don't even know if there is some upper bound on the second degree $\mathbf{a}$ in a two element spectrum containing $\mathbf{0}$. For instance it is completely consistent with our current state of knowledge that there could be a computable structure and a relation with two element spectrum $\mathbf{0}, \mathbf{a}$ such that $\mathbf{a}$ *is not even arithmetical!* [2]

We remark that Hirschfeldt has more generally proven the following regarding two element spectra.

Theorem 2.17 (Hirschfeldt [47]) *Let* $\mathbf{b}, \mathbf{a}$ *be any two computable enumerable degrees. Then there is a computable structure* $\mathcal{A}$ *and a relation* R *on* $\mathcal{A}$ *whose spectrum is* $\mathbf{b}, \mathbf{a}$.

Theorem 2.17 even works for ω-c.e. degrees. We remark that Ash-Cholak-Knight [3] have shown that, in some sense, the Δ_3^0 degrees are the improtant ones. the almost show that if a relation hits all Δ_3^0 degrees then it then it contains all degrees. More precisely, the show the following.

[2]We note that because the set of realizations of a given relation is a Σ_1^1 set, there are partial orderings of c.e. degrees that cannot be degree spectra. Furthermore, we also know that if the degree spectrum contains a nonhyperithmetical degree and a hyperarithmetical degree, it must be uncountable.

Theorem 2.18 (Ash, Cholak, and Knight [3]) *For a relation U on a computable structure $\mathcal{A}$, the following are equivalent.*

(i) For every Δ_3^0 degree $\mathbf{a}$ *there is a computable structure $\mathcal{B}$ and an isomorphism $f : \mathcal{A} \mapsto \mathcal{B}$ such that both f and $f(U)$ have degree* $\mathbf{a}$.

(ii) For every degree $\mathbf{b}$ *there is a structure $\mathcal{B}$ and an isomorphism g : $\mathcal{A} \mapsto \mathcal{B}$ such that both f and $f(U)$ have degree* $\mathbf{b}$.

In his thesis, Hirschfeldt has proven that the inclusion of f in the degree constraints is necessary, since he has constructed, for each n, a computable structure and a relation whose spectrum is exactly the Δ_n^0 degrees. For more on these issues we refer the reader to Hirschfeldt's thesis, Khousainov-Shore [53], or [30].

We remark in passing that these issues are closely related to the *algorithmic dimension* of the structures. The algorithmic dimension is the number of computable isomorphism types of computable models. We have seen that structures such as the rationals can be computably categorical, and hence have dimension 1, and can be like ω which has dimension ∞. Goncharov [39] constructed a computable structure of dimension n for each n with $1 \leq n \leq \infty$. It is also known that if a structure has a computable copy not computably isomorphic to it *but Δ_2^0 isomorphic to it*, then the structure has dimension ∞. Therefore if the dimension is 2, the isomorphism between the models must be non-Δ_2^0. All known examples are Δ_3^0. The following question is open.

Question 2.19 (Khoussainov) *Are there isomorphic computable structures $\mathcal{A}, \mathcal{B}$ of dimension 2, such that the isomorphism between them is not Δ_3^0? How complicated can it be? Can it be not arithmetical?*

We remark that these issues are related to definability in the structures. For instance, a relation on a structure $\mathcal{A}$ is called, following Ash and Nerode [6], *intrinsically* (e.g.) computable, if it is computable in each computable copy of $\mathcal{A}$. Essentially, if the model has enough decidability, then a relation is intrinsically computable iff it is described by a computable $L_{\omega_1,\omega}$ formula.

And similarly, a structure is computably categorical iff it is described by a computable Scott family, provided that it obeys certain decidability conditions. (The reader is directed to the excellent survey [53] for a thorough discussion of these and related matters.)

Turning to Question 2.5, which asks us to classify, by order type, the linear orderings that have computable presentations, or show that this is impossible. The real question is, what is an acceptable classification of the order types? Computability theory can provide an excellent vehicle for demonstrating that no reasonable classification exists by showing that the relevant class is not Borel.

Two nice examples are the following. Slaman and Woodin examined the following question. Given a partial ordering $(P, \preceq)$, a linear extension is a linear ordering $(P, \leq)$ on the same universe P such that $x \preceq y$ implies $x \leq y$. It is not difficult (Rosenstein [96]) to prove that each computable partial ordering has a computable linear extension. It is even true that each dense computable partial ordering has a dense computable linear extension. We remark that it is also classically known that a scattered partial ordering has a scattered linear extension. (Recall that an ordering is called scattered if it does not embed the rationals.) It is unknown if this result effectivises, a question of Rosenstein.

Theorem 2.20 (Slaman and Woodin [99]) *$\{e : W_e$ is a computable partial ordering of $\mathbb{N}$ without a dense linear extension $\}$ is a Σ^1_1 complete set.*

This answers a question of Loś by showing that there is no reasonable classification of the countable partial orderings with dense linear extensions.

A similar example is provided by the theory of difference sets. Recall that a set of integers A is called a difference set iff there is a set B of integers such that $A = D(B)$, where

$$D(B) = \{x - y : y < x \in B\}.$$

Downey, Furedi, Jockusch and Rubel [22] first examined inverting the difference operator effectively. They showed, for instance, that there are computable sets that are the difference set of a co-c.e. set but not the difference set of any computable set. Many other results are proven in [22], including one that has a classical spinoff: every subset of $\mathbb{N}$ is the intersection of two difference sets. The underlying motivating question was due to Ruzsa who, in the 70's, asked for a reasonable classification of the sets that are difference sets. In particular, Rubel reformulated the question to ask if the set was Borel. Again computability theory provides a negative answer.

Theorem 2.21 (Schmerl, [107]) *The set of difference sets is Σ^1_1 complete.*

Schmerl's paper contains many other good things besides, and is beautifully written.

3 Degrees of Isomorphism Types

The natural guess for definition which assigns a degree to an isomorphism type would be the *least* degree of any presentation in that type. For instance, the degree of dense linear orderings without endpoints is clearly $\mathbf{0}$ since we can computably represent $\mathbb{Q}$. Linda Jean Richter was the first author to systematically study the degrees of structures. (Richter [90]) It turns out that some types have degrees and some do not. One example of a structure with a degree is a finitely presented group. If $G = \langle x_1, ..., x_n | y_1, ..., y_m \rangle$ is finitely presented then its degree is simply the degree of its word problem. Note that if H is isomorphic to G then we only need to look at the images of $\{x_1, ..., x_n\}$ to obtain generators of H and through this, H can compute the open diagram of G. Below are some other examples of structures with nontrivial degrees. We obtain these examples using a preliminary theorem of Richter. This theorem encapsulates the so-called "combination method."

Theorem 3.1 (Richter [90, 91]) *Suppose (i) and (ii) below hold for a degree* $\mathbf{a}$ *and a theory T over a finite language L. Then there is a structure of L whose isomorphism type has degree* $\mathbf{a}$.

(i) There is a infinite computable sequence of finite structures $\{A_i : i \in \mathbb{N}\}$ such that A_i is not embeddable into A_j for $i \neq j$.

(ii) For each $S \subseteq \omega$, there is a structure A_S such that:
(iia) A_S is a countable structure of T.
(iib) $A_S \leq_T S$.
(iic) A_i is embeddable into A_S iff $i \in S$.

Proof. Fix a degree $\mathbf{a}$ and let D be any set in $\mathbf{a}$. Consider $A_{D\oplus\overline{D}}$. Evidently $A_{D\oplus\overline{D}} \leq_T D$. Now suppose that $B \cong A_{D\oplus\overline{D}}$. Then $i \in D$ iff A_{2i} is embeddable into B, and $i \in \overline{D}$ iff A_{2i+1} is embeddable into B. It follows that $D \leq_T B$ and so the isomorphism type of $A_{D\oplus\overline{D}}$ has degree $\mathbf{a}$. $\square$

Theorem 3.1 has many applications. We list a couple below.

Theorem 3.2 (Richter [90, 91]) *Let $\mathbf{a}$ be any degree.*

(i) There is an abelian group whose isomorphism type has degree $\mathbf{a}$.

(ii) There is a lattice whose isomorphism type has degree $\mathbf{a}$.

(iii) There is a graph whose isomorphism type has degree $\mathbf{a}$.

Proof. They are all basically similar. We do (i). Consider $A_i = \mathbb{Z}_{p_i}$, the cyclic group of order p_i, where p_i denotes the i-th prime. The structure A_S is the standard direct sum of the A_i. Now we apply Theorem 3.1. $\square$

At this stage we should point out that, as with many of these results on structures and their degrees, Theorem 3.2 has several consequences in the degrees. In particular Theorem 3.2 was used by Richter to demonstrate that *If $\mathbf{0}''' \leq_T \mathbf{a}$ and Ψ is a jump preserving automorphism of the degrees then $\Psi(\mathbf{a}) = \mathbf{a}$.*

On the other hand, some structures do not have degrees. For instance we have the following.

Theorem 3.3 (Richter [90]) *If an order type has a degree, that degree is $\mathbf{0}$.*

Proof Sketch To prove Richter's Theorem, Theorem 3.3, we will actually sketch the proof that given a linear ordering L then there is an ordering $\widehat{L}$ isomorphic to L such that the infimum of the degrees of L and $\widehat{L}$ is $\mathbf{0}$; that is, they form a minimal pair. Basically one uses the finite extension method. So for requirement e, we diagonalize against Φ_e, Γ_e who say that *if* $\Phi_e^L = \Gamma_e^{\widehat{L}} = f$, *total, then f is computable.* We will have already specified a finite portion of $\widehat{L}$, and a finite partial isomorphism g_e from an initial segment of L (say, up to and including e) to an initial segment of $\widehat{L}$. Then one asks if there is any way to extend $\widehat{L}_e$ so as to cause a disagreement. (The only proviso here is that the extension must be consistent with the order type. This causes a little bit of concern, and we refer the reader to Richter's paper or Downey [19] for more details.) If we can then we cause a disagreement. If not then one can argue that the common value is computable. □

We remark that the proof of Stone's theorem in the previous section also shows that isomorphism types of boolean algebras don't have nontrivial degrees.

Corollary 3.4 (Richter [90]) *If the isomorphism type of a Boolean algebra has a degree then that degree is* $\mathbf{0}$.

The proof follows by Richter's proof of the linear ordering case. As above, she showed that for any noncomputable linear ordering L there is a an ordering $\widehat{L}$ isomorphic to L such that the infimum of the degrees of L and $\widehat{L}$ is $\mathbf{0}$.

4 Jump Degrees and a Question on Torsion Free Groups

Since orderings and boolean algebras don't have isomorphism types with degrees, Jockusch suggested that this might be because of the number of quantifiers needed to code in such structures. With this in mind he suggested the following definition.

Definition 4.1 *Let $\mathcal{A}$ be any structure. Then the n-*th jump degree *of $\mathcal{A}$ is*

$$\min\{deg(\mathcal{B}^{(n)}) : \mathcal{B} \cong \mathcal{A}\}.$$

The idea is that the measure is coarser than the notion of degree. For linear orderings, we don't seem to get much for $n = 1$.

Theorem 4.2 (Knight [59]) *If a linear ordering has a first jump degree it is $\mathbf{0}'$.*

Knight's theorem is proven by a forcing argument. However, once we get to $n = 2$ all manner of coding is possible.

Theorem 4.3 *Let $n \geq 2$ be any computable ordinal. Let $\mathbf{a} > \mathbf{0}^{(n)}$. Then there are linear orderings L and J such that*

(i) (Ash, Jockusch, Knight [4]) L has n-th jump degree $\mathbf{a}$ and no $n-1$-st jump degree.

(ii) (Downey-Knight [24]) J has n-jump degree $\mathbf{0}^{(n)}$ and no $n-1$-st jump degree.

In the context above, Downey and Jockusch, see [19], began a study of jump degrees of torsion free abelian groups. Recall that $G = (G, +)$ is called torsion free if no element has finite order. Such groups are classically of interest since they are, for instance, precisely the abelian groups that can be ordered. They are of interest computability theoretically as we now see.

We remark that in abelian groups with torsion one can use the Ulm sequence to code a great deal of information. This leads to a well behaved degree and jump degree theory analysed in Lin [71, 72], Smith [108], Goncharov [40], and, for jump degrees, by Oates [82]. However, the structure theory and the computability theory become more subtle if we assume that the group is torsion free.

Khisamiev has studied these objects, and has obtained many fundamental results. Torsion free abelian groups behave very interestingly. For instance, Feiner [31, 32] proved that there are computably enumerably presented linear orderings and boolean algebras not isomorphic to computable ones. This is *not* true for torsion free abelian groups, although it is true for abelian groups in general.

Theorem 4.4 (Khisamiev [59]) *If $G = (G, +, =)$ is a Σ_1 presented torsion-free group then G is isomorphic to a computably presented group.*

In the classical language of combinatorial group theory, every recursively presentable torsion free abelian group is isomorphic to one with a solvable word problem. Khisamiev's Theorem, Theorem 4.4, solves affirmatively a question of Baumslag, Dyer and Miller [7]. This question arose from the study of the integral homology of finitely presented groups, where computably enumerably presented torsion free abelian groups turn out to be the natural objects of study. We also remark that for linear orderings if L is Π^0_1 presented, (that is the ordering relation is Π^0_1) then L is isomorphic to a computably presented ordering.

On the other hand, as we see below, there are Π^0_1 presented torsion free abelian groups not isomorphic to computable ones. This was originally an observation of Khisamiev [59] after Khisamiev and Khisamiev [60]). We give a much simpler proof than the complex Khisamiev and Khisamiev one, and show that G can be chosen to have rank 1. (The rank of a group is defined below.) We then explore the notion of jump degree for rank 1 groups.

We remind the reader of some definitions, etc. from Fuchs [38]. Every countable (X–presentable) torsion free abelian group is isomorphic to an additive (X–presented) subgroup of V_∞, where V_∞ denotes the vector space over $\mathbb{Q}$ with basis $\{e_i : i \in \mathbb{N}\}$ (e_i is the vector that is everywhere zero except for 1 in position i).

If G is isomorphic to a subgroup of $\mathbb{Q}^n$, then the least such n is called the *rank* of the group. The simplest and best understood of the torsion free abelian groups are those of rank 1. These are effectively isomorphic to subgroups of $(\mathbb{Q}, +, =)$.

Consequently we really need only consider subgroups of $(\mathbb{Q}, +, =)$. Let $p_1, p_2, \ldots$ be the primes in increasing order. The p–height $h_p(a)$ of $a \in G$, a subgroup of $\mathbb{Q}$, is defined via

$$h_p(a) = k \text{ if } k \text{ is greatest with } p^k \mid a \text{ (in } G)$$

and

$$h_p(a) = \infty \text{ if } p^k \mid a \text{ for all } k.$$

For $a \in G$ we define the *characteristic* a, $\chi(a)$ via

$$\chi(a) = (h_{p_1}(a), h_{p_2}(a), \ldots).$$

It is not difficult to see that as G has rank 1, if $a, b \in G$ with $a, b \neq 0$ then $\chi(a)$ and $\chi(b)$ are equivalent in the sense that they differ only finitely often. We write $\chi(a) =^* \chi(b)$. We can thus define the *type of* $\chi(G)$ to be the $=^*$ equivalence class of $\chi(a)$ for any $a \neq 0$ in G. The classical theorem of Baer is

Theorem 4.5 (Baer, see e.g. Fuchs [38])) *If G and H have rank 1 then $G \cong H$ iff G and H have the same type.*

Proof. The proof of Theorem 4.5 is easy. In the nontrivial direction take $g \in G$ and $h \in H$ with $\chi(g) =^* \chi(h)$. From this it follows that $mx = ng$ is solvable in G iff $my = nh$ is solvable in H. The isomorphism is given by $x \mapsto y$. $\square$

We remark that, since, for all $x \in G$, such m, n exist, the isomorphism given by the proof above is effective relative to the presentations. We also note that this means torsion free abelian groups of rank 1 are computably categorical in the sense that $G \cong H$ implies $G \cong_c H$, where $\cong_c$ denotes computable isomorphism. The same is in fact true of finite rank torsion free abelian groups: The work of (e.g.) Goncharov [40], Ash–Nerode [6] or direct constructions show the following.

Theorem 4.6 *A computable torsion free abelian group is computably categorical iff it has finite rank.*

Let G be a subgroup of $(\mathbb{Q}, +, =)$. Define the *standard type of G $S(G)$* via

$$S(G) = \{\langle i, j\rangle : j \leq \text{ the } i\text{–th member of } \chi(G)\}$$

for some fixed $a \in G$. Note that if G is X–presented, then $S(G)$ is Σ_1^X. Let A be Σ_2^X. Let $\widehat{A} = \{\langle x, 0\rangle, \langle y, 1\rangle : x \in \mathbb{N}, y \in A\}$. Note that $A \equiv_m \widehat{A}$. Then it is easy to construct an X–presentable G with $S(G) = \widehat{A}$. (In fact given any finite type sequence $T = (a_0, a_1, ...)$, with $a_i < \infty$ for all i, which is given by a function which is Σ_1^X there is an X-presentable group $G(T)$ with this type sequence.) It is now easy to see

Theorem 4.7 *There is a rank 1 Π_{n+1}^0–presented torsion free abelian group G not isomorphic to a Π_n^0–presented group.*

Proof. We do the case $n = 0$ and then relativize. Choose A to be Σ_2^0–complete and construct a $\emptyset'$–presentable G with $S(G) = \widehat{A}$. Then by our earlier work we can suppose G is Π_1 and as $S(G)$ is $\Sigma_2^0 - \Sigma_1^0$, by Baer's theorem, G cannot be computably presentable lest $S(G)$ be Σ_1^0. □

We remark that Khisamiev and Khisamiev [60] construct a Π_1^0 computable torsion free abelian group F with a computable basis and a maximal divisible subgroup N of F with F/N in $\Pi_1^0 - \Delta_1^0$. Their construction is much more intricate. We also remark that there is related work by Friedman, Simpson and Smith [35].

What about the degrees of isomorphism types? Using a fairly simple argument, Julia Knight [unpubl.] showed that there exist torsion free abelian groups of arbitrary degree. Downey [19] improved this to rank one groups:

Theorem 4.8 (Downey, after Knight) *Let* $\mathbf{a}$ *be any degree. Then there exists a torsion free abelian group G of rank one with degree* $\mathbf{a}$.

Proof. Let S be any set of degree $\mathbf{a}$. Note that if A is any set with $S \oplus \overline{S} \in \Sigma_1^A$ then $S \oplus \overline{S} \leq_T A$. Let $G = G(S)$ be the rank one group defined via the type sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n = 1$ if $n \in S \oplus \overline{S}$ and $a_n = 0$ otherwise.

Clearly there is a presentation of G computable in S. Conversely, suppose $H \cong G$. Then H has a type sequence $(a_n)_{n\in\mathbb{N}}$. The type sequence of H is Σ_1^H, and hence $S \oplus \overline{S}$ is Σ_1^H. But then $S \leq_T H$. $\square$

Do all torsion free abelian groups possess degrees? This is not the case, even for rank one groups.

Theorem 4.9 (Downey and Jockusch) *(i) There is a rank 1 G (of finite type) with proper* 1*–degree* $\mathbf{0}'$[3]
(ii) More generally, for every degree $\mathbf{a} \geq \mathbf{0}'$ *there is a torsion free abelian group of proper 1-degree* $\mathbf{a}$.
(iii) Similarly, for all degrees $\mathbf{b} \geq \mathbf{0}''$ *there is a torsion free abelian group of rank 1 with proper 2-degree* $\mathbf{b}$.

For a proof, see Downey [19].

The situation becomes very interesting for jump degrees and rank one groups. This is a beautiful example of classical mathematics providing stimulus to classical computability theory.

We say that a group G to have *finite type* if for all i, $|\mathbb{N}^{(i)} \cap S(G)| < \infty$. So for instance, the group of Theorem 4.8 has finite type. That is, G has no elements of infinite height. The following is the key nexus between Baer's theorem and jump degrees.

Observation 4.10 (Downey and Jockusch) *Every torsion free abelian group of finite type and of rank 1 has 1-degree iff the following question has a positive solution.*

- Question. *Is it true that for all sets M, the collection of degrees $\{deg(B') : M$ is computably enumerable in $B\}$ has a least element?*

[3] Here an n-degree, or rather n-th jump degree, is called *proper*, and there is no $n-1$-st jump degree.

A proof of Observation 4.10 can be found in full detail in Coles, Downey, and Jockusch [16]. It is not difficult.

There is a similar version saying that every torsion free abelian group of rank 1 has 2-jump degree iff for all sets X, the set of degrees below has a least member.

$$\{\deg(B)'' : X \text{ is } \Sigma_2^B\}$$

Downey and Jockusch conjectured that the question above has a negative solution. Furthermore they spent much effort trying to find counterexamples. As we will see in the next section, there is a reason for this.

5 Every Set Has a Least Jump Enumeration

The above leads us to consider the sets

$$C(A) = \{X \mid A \in \Sigma_1^X\}, \text{ and,}$$

$$\mathbf{c}(A) = \{\deg(X) \mid A \in \Sigma_1^X\}.$$

This is the sets that A can be enumerated from, and their degrees. More generally, we have seen that there is some interest in the sets

$$\mathbf{c}_\mu^{(n)}(A) = \min\{\deg(X)^{(n)} \mid A \in \Sigma_n^X\}.$$

The question of Downey and Jockusch was open for several years, until it was found that it has a *positive solution*. This is a very surprising result. The original proof was a complicated priority argument. This was later simplified to the forcing argument from Coles-Downey-Slaman [16].

Theorem 5.1 (Coles, Downey, Slaman [16]) *For all X, $\mathbf{c}'_\mu(X)$ exists.*

Proof. Given a set A we want to construct a set F such that A is Σ_1^F and $F' \leq_T X'$ for all sets X with $A \in\in \Sigma_1^X$.

Definition 5.2 *Let $X \subseteq \mathbb{N}$.*

- *The partial order $\mathbb{P}_1^X$ is the set of finite enumerations of subsets of X ordered by extension.*

  So an element of $\mathbb{P}_1^X$ is a function, p say, from $\{0, 1, \dots, k\}$ into X, for some $k \in \omega$.

- *For p and q in $\mathbb{P}_1^X$, we say p extends q in $\mathbb{P}_1^X$ and write $p >_1^X q$, if the graph of p is contained in the graph of q.*

- *We use $\mathbf{1}$ to denote the empty function, and of course $\mathbf{1} \in \mathbb{P}_1^X$.*

We fix a $\mathbb{P}_1^A$ generic enumeration $\mathcal{G}_1$ of A. Clearly A is $\Sigma_1^{\mathcal{G}_1}$ since $y \in A$ if and only if there is an x such that $(x, y) \in \mathcal{G}_1$.

Let F be the forcing relation for $\Pi_1^0(\mathcal{G}_1)$ sentences. That is,

$$F = \{(p, \varphi) \mid p \notin \mathbb{P}_1^A \text{ or } (\varphi \in \Pi_1^0(\mathcal{G}_1) \text{ and } p \nVdash \varphi)\}.$$

We first observe that $A \leq_T F$ as follows. Let e be an index such that $\Phi_e(Y)(x) \uparrow$ for all oracles Y and all $x \in \mathbb{N}$. Let φ be the $\Pi_1^0(\mathcal{G}_1)$ sentence $\forall x(\Phi_e(\mathcal{G}_1)(x) \uparrow)$. Then $(p, \varphi) \in F$ if and only if p is not a forcing condition. Now, let q_n be the constant function from the set $\{0\}$ to n. Then q_n is a forcing condition if and only if $n \in A$. Hence $n \in A$ if and only if $(q_n, \varphi) \notin F$.

Lemma 5.3 *$F \leq_T X'$ for all $X \in \Sigma_1^A$.*

Proof For any p and φ, $(p, \varphi) \in F$ if and only if either

5.1. $p \notin \mathbb{P}_1^A$, or

5.2. there is a proper extension q of p in $\mathbb{P}_1^A$ such that $q \nVdash \varphi$.

Since $\mathbb{P}_1^A \leq_T A$, in the first case we have that $p \in \mathbb{P}_1^A$ is computable in X' for any set X for which A is Σ_1^X. In the second case, we have an existential condition that refers to a finite amount of positive information about A. This condition is Σ_1^X for sets X for which A is Σ_1^X, and hence, for such an X, case 2 is computable in X'. Therefore $F \leq_T X'$ for any X such that A is Σ_1^X. □

Lemma 5.4 *There is a set G such that A is $\Sigma_1^0(G)$ and $G' \leq_T F$.*

Proof. We construct $G \leq_T F$ so that every fact about G' is forced in the sense of $\mathbb{P}_1^A$.

Step 0: Let $p_0 = \mathbf{1}$.

Step $\mathbf{s+1=2e}$:

- If $(p_s, \varphi) \in F$, then let p_{s+1} be the least extension of p_s in $\mathbb{P}_1^A$ such that $p_{s+1} \Vdash \neg\varphi$.
- Otherwise, $(p_s, \varphi) \notin F$. Let $p_{s+1} = p_s$ since $p_s \Vdash \varphi$.

Step $\mathbf{s+1=2e+1}$:

Let x be the least number in A not in the range of p_s. Let p_{s+1} be the least extension of p_s with x in its range.

At the end of the construction define $G = \bigcup_{s\in\omega} p_s$.

Now clearly A is Σ_1^G. Observe that for case 1 of the even stages of the construction, we can find p_{s+1} computably in A. Hence $A \oplus F$ is an oracle that can decide every Π_1^0 condition about G. Therefore $G' \leq_T A \oplus F$.

We have already seen that $A \leq_T F$ and hence, $G' \leq_T F$. Consequently there is a set G such that A is $\Sigma_1^0(G)$ and $G' \leq_T F$. □

We remark that Coles, Downey, and Slaman [16] also used induction to prove a generalization of the theorem for all $n \geq 1$.

Theorem 5.5 (Coles, Downey, Slaman [16]) *For all $n \geq 1$, and for all X, $\mathbf{c}_\mu^{(n)}(X)$ exists.*

We get the corollaries we are after.

Corollary 5.6 (Coles, Downey, Slaman [16]) *Suppose that G is a rank 1 torsion free abelian group. Then G has second jump degree. Furthermore, if G has finite type, G has first jump degree.*

Subsequently, Soskov (private communication) found an elegant proof of the above based on the theory of enumeration degrees. (Actually, if one looks closely, the proofs are essentially the same.)

Recall that $A \leq_e B$ iff there is a computably enumerable set W such that for all x,

$$x \in A \text{ iff } \exists u[\langle x, u\rangle \in W \wedge D_u \subseteq B].$$

Here D_u denotes the u-th canonical finite set. It is known that enumeration reducibility can be strongly connected with algebra.

For instance, Charles Miller III (see [46]) proved that given two finitely generated groups G_1 and G_2, G_2 can be embedded into a group finitely presented relative to G_1 iff G_2 is enumeration reducible to G_1. Enumeration reducibility is also the key here.

There is a canonical embedding of the semi-lattice of the Turing degrees into the semi-lattice of the enumeration degrees. The Turing degree of a set A is identified with the enumeration degree of the set $A \oplus \bar{A}$, $d_e(A \oplus \bar{A})$.

For any set A define $A^+ = A \oplus \bar{A}$. A set A is called *total* if $A \equiv_e A^+$. An enumeration degree is total if it contains a total set. So the Turing degrees are the total enumeration degrees. Notice that A is computably enumerable in B iff $A \leq_e B^+$ and $A \leq_T B$ iff $A^+ \leq_e B^+$. So we can define $C(A)$ as

$$C(A) = \{X : X \text{ is total and } A \leq_e X\}$$

Lemma 5.7 (Soskov) *$C(A)$ has an element of least degree iff $d_e(A)$ is total.*

Proof. Clearly if $d_e(A)$ is total, then $d_e(A)$ is the desired least degree. Assume that there exists a $X_0 \in C(A)$, s.t. $X_0 \leq_e X$ for all $X \in C(A)$. Then we have for all total X,

$$A \leq_e X \Rightarrow X_0 \leq_e X.$$

Therefore, by Selman's Theorem (see [104]), $X_0 \leq_e A$ and hence $A \equiv_e X_0$. $\square$

The enumeration jump was defined by Cooper and further studied by K. McEvoy. (See, for example, [73].) It is defined as follows. Given a set A, define

$$K_0^A = \{\langle x, z\rangle : x \in \Psi_z(A)\},$$

where Ψ_z denotes the z-th enumeration operator. Set the enumeration jump of A to be the set $A'_e = (K_0^A)^+$.

MacEvoy [73] studies the basic properties of the enumeration jump. Notice that A'_e is a total set and for total sets the e-jump is enumeration equivalent to the Turing jump.

Lemma 5.8 (Soskov) *For every set A, $d_e(A'_e)$ is the least among the degrees of the elements of*

$$C'(A) = \{X' : X \textit{ is total and } A \leq_e X\}.$$

Proof. By the monotonicity of the e-jump operation, $A'_e \leq X'$ for all $X \in C(A)$. It remains to show that there exists a total X such that $A \leq_e X$ and $A'_e \equiv_e X'$. This follows directly from from [104], Theorem 1.2. $\square$

Soskov also provided a proof of the case for more general $n > 1$. Consider n sets $A_0, \ldots, A_{n-1}$. Define by means of induction the set $P(A_0, \ldots, A_{n-1})$: 1) $P(A_0) = A_0$; 2) $P(A_0, \ldots, A_{k+1}) = P(A_0, \ldots, A_k)'_e \oplus A_{k+1}$.

Let

$$C'(A_0, \ldots, A_{n-1}) = \{X^{(n)} : A_0 \text{ is } \Sigma_1(X), \ldots, A_{n-1} \text{ is } \Sigma_n(X)\}.$$

Theorem 5.9 (Soskov) *$d_e(P(A_0, \ldots, A_{n-1})'_e)$ is the least among the degrees of the elements of the set $C'(A_0, \ldots, A_{n-1})$.*

Proof. Let $P = P(A_0, \ldots, A_{n-1})$. It is sufficient to show that there exists a total X such that $X^{(n)} \equiv_e P'_e$ and A_i is $\Sigma_{i+1}(X)$. Take a total Q such that $P \leq_e Q$ and $P'_e \equiv Q'$. By [104], Theorem 1.2, there exists a total X such that $X^{(n-1)} \equiv_e Q$ and A_i is Σ^X_{i+1}, $i = 0, \ldots, n-1$. Clearly $X^{(n)} \equiv_e P'_e$.

Take $A_0 = \ldots = A_{n-2} = \emptyset$ and $A_{n-1} = A$. Then $P(A_0, \ldots, A_{n-1}) = \emptyset^{(n-1)} \oplus A$. Hence $d_e((\emptyset^{(n-1)} \oplus A)'_e)$ is the least among the degrees of the elements of $C^{(n)}(A)$. □

Soskov observed that his techniques solved a couple of the questions listed at the end of a prelininary version of [16]. One left open was question 2 there, which he interpreted to mean the following.

Question 5.10 *Characterize the sets X for which the enumeration jump is equivalent to the Turing jump.*

6 Computably Enumerable Reals

The results at the end of the previous section demonstrated how reducibilities other than Turing reducibility can yield significant insight into the effective content of classical mathematics. Another example of this phenomenon is the use of Ziegler reducibility, a variation of quasi-reducibility (see Downey, Laforte, Nies [26]), to characterize relative algebraic closure in groups. (See [46].) Yet another example is provided by the degrees of computably enumerable bases for subspaces of V_∞, characterized precisely as the weak truth table degrees below that of the given space. (See Downey and Remmel [28])

In this section we will see yet another example of this phenomenom, and additionally see how this methodology applies to classical analysis.

We begin with a simple question, asked for centuries, *what is a real?*

There are several definitions[4], and we list several below. (We consider reals in $(0,1)$.)

- a has an infinite (e.g.) decimal expansion, $a = .a_1a_2....$
- $a = \Sigma_{n\in A}2^{-n}$, where $A \subset \mathbb{N}$.
- $a = \Sigma_{n\in A}2^{-|n|}$, where $A \subset \Sigma^*$.
- a is the limit of an increasing sequence of rationals.

Now a *computable* real would presumbly be a computable limit of a computable sequence of rationals, using the Dedekind cut definition. More precisely, there is a computable sequence q_i, $i = 1, 2, ...$ of rationals such that for any n we can compute an m such that $|a - q_m| < 2^{-n}$. Now it is not difficult to show that this definition coincides with a having a computable decimal espansion, or $a = \Sigma_{n\in A}2^{-n}$ for a computable subset A of $\mathbb{N}$. (See Rice [88].) However, we would get a little stuck if we also claimed that this is the same as $a = \Sigma_{n\in A}2^{-|n|}$ for some computable $A \subseteq \{0,1\}^*$, as we now see.

Define a *computably enumerable real* to be the limit of a computable sequence of rationals. That is, we have $a = \lim_s q_s$ such that we can compute q_i for all i. But note that although we know that the sequence converges, we don't computably know how fast, at least in general. The following theorem is from Calude, Hertling, Khoussainov and Wang [10]. A set A is called *prefix free* iff for all $\sigma \in A$, and all τ with σ an initial segment of τ, $\tau \notin A$. [5]

Theorem 6.1 (Calude et. al. [10]) *The following are equivalent.*

(i) a is computably enumerable.

[4] These definitions define the relevant real up to equivalence.

[5] Prefix free sets are considered for technical reasons since if a set A is prefix free then, by Kraft's inequality, we know that $\Sigma_{n\in A}2^{-|n|}$ converges. Prefix-free sets are crucial in a proper treatment of (e.g.) randomness, and other complexity issues, for c.e. reals, a fact first realized by Chaitin [11].

(ii) The set $L(a) = \{q \in \mathbb{Q} : q < a\}$ is computably enumerable.

(iii) There is an infinite computably enumerable prefix free set A with $a = \Sigma_{n \in A} 2^{-|n|}$.

(iv) There is a computable prefix free set A such that $a = \Sigma_{n \in A} 2^{-|n|}$.

(iv) There is a computable function $f(x, y)$ of two variables, such that

  (iva) If, for some k, s we have $f(k, s+1) = 0$ yet $f(k, s) = 1$ then there is some $k' < k$ such that $f(k', s) = 0$ and $f(k', s+1) = 1$.

  (ivb) $a = \cdot a_1 a_2 ...$ is a dyadic expansion of a with $a_i = \lim_s f(i, s)$.

(v) There is a computable increasing sequence of rationals with limit a[6].

The reader should note that (iii) implies (iv) because we always have many strings we can potentially add at some stage. The reader should also note that although things coincide for *computable* reals, there are computably enumerable reals b which are not, what we shall call, *strongly* computably enumerable.

Definition 6.2 *A real a is called* strongly *computably enumerable if there exists a computably enumerable $B \subseteq \mathbb{N}$ such that*

$$a = \Sigma_{n \in B} 2^{-n}.$$

The fact that there are c.e. reals that are not strongly c.e. was first noted by Soare [102]. Here is a very quick proof.

To defeat W_e, the e-th c.e. set, we make our real have 1 in the $4e$-th place, and 0 in places around it. We need do nothing unless $4e$ enters W_e. At that stage, we can make the $4e$'th place of *our* real become 0 by making the $4e - 1$-st place become 1.

[6]If one only asks for a limit of a computable sequence of computable reals, then instead of getting the computably enumerable reals one gets reals whose dyadic expansion is computable from the halting problem. ([49])

The motivating question for most of the rest of the section is the following. *How can we present our c.e. real?* Part of the answer is provided by Calude, Coles, Hertling and Khoussainov [9].

Those authors examined the cut definition of real, and its effective content. They defined a *representation* A of a (c.e.) real a as an increasing sequence of rationals q_i for $i \in A$ with limit a. They asked what types of degrees can A have. Already we have seen thet A can be computable. It is not difficult to prove the following.

Lemma 6.3 (Soare [102], Calude et. al. [9]) *Let $L(a)$ denote the lower cut of a real a. That is, $L(a)$ is the collection of rationals less than a. Then if A is a representation of a, $A \leq_T L(a)$.*

Lemma 6.3 extends earlier work of Soare who examined, in particular, the relationship between $L(a)$ and deg(B) for $a = \Sigma_{n \in B} 2^{-n}$. In [102], Soare observed that $L(a) \leq_T B$ and $B \leq_{tt} L(a)$. However, he also proved that there are strongly c.e. a, as above, with $L(a) \not\leq_{tt} B$.

We remark that c.e. reals play an important part in the theory of randomness. There one studies c.e. reals under a modified form of weak truth table reducibility, called in [15], *Solovay* reducibility. It would take us a little far to motivate and examine this material in detail, but as an example, for *strongly* c.e. reals Solovay reducibility becomes the following. We say $a = \Sigma_{n \in A} 2^{-n}$ reduces to $b = \Sigma_{n \in B} 2^{-n}$ iff there is a constant c and a weak truth table procedure Φ with use φ, such that

- $\Phi^B = A$ and
- for all n, $\varphi(n) \leq c \cdot n$.

The general case is similar but operates on cuts. In an unpublished manuscript, Solovay proposed the reducibility above, and demonstrated that it preserves the "degree of randomness." It is interesting to examine the degree structure, which has a version of the halting problem called Ω, the "halting probability," of Chaitin, that is naturally *wtt*-complete but not *tt*-complete. It is also illuminating that again the relevant reducibility turns out to be stronger that T-reducibility. We refer the reader to Calude et. al. [10, 15, 9] for more details.

Returning to our story, we note that Calude et. al. proved the following very interesting generalization of Lemma 6.3.

Lemma 6.4 (Calude et. al. [9]) *Suppose that A represents a. Then A is an infinite half of a splitting of $L(a)$.*

Proof. Clearly, if A represents a then A must be an infinite c.e. subset of $L(a)$. The thing to note is that $L(a) - A$ is also c.e.. Given rational q, if q occurs in $L(a)$, we need only wait till either q occurs in A or some rational bigger than q does. $\square$

Note that this means that if a is computable then every representation of a is computable. Also note that the proof actually gives that if A represents a, $A \leq_{wtt} L(a)$. The following is easy to show using the technique of the proof above.

Lemma 6.5 (Calude et. al. [9]) *Let A be a representation of a. For subsets B of A, the following are equivalent.*

- *B represents a.*
- *B is half of a splitting of A.*

It would be nice to be able to reverse the above. That is, prove that every splitting of $L(a)$ is a representation of a. Clearly this is false since some splittings won't even be representations.

Calude et. al. [9] did however prove that *(i) a has a representation of degree $deg(L(a))$, and (ii) every representation can be extended to one of degree $deg(L(a))$*. Here is their proof.

Let $q_i \mapsto a$, be a given c.e. sequence of rationals converging to a. We build the splitting of $L(a)$ and the representation of a of degree $\deg(L(a))$ in stages. At stage 0, let $r_0 = q_0$. Suppose that at stage s we have $r_0, ..., r_{n(s)}$ with $q_i = r_{n(i)}$ for each $i \leq s$. At stage s let g_s be the largest Gödel number seen so far. Consider the rationals q with $q_s < q \leq q_{s+1}$ and such that the Gödel number of q is less than or equal to the maximum of g_s and the Gödel number of q_{s+1}. Suppose that there are m such q. Then we let $n(s+1) = n(s) + m$, and let $r_{n(s)+k}$ be those q in increasing order for $k = 1, ..., m$.

The sequence r_i is increasing and converges to a. Hence $\{r_i\} \leq_{wtt} L(a)$. Conversely, to decide if $q \in L(a)$, compute a stage s where g_s exceeds the Gödel number of q. Assuming that q is not yet in $L(a)$, then since $q > r_{n(s)}$, $q \in L(a)$ iff $q \in \{r_i\}$. Note that this is in fact an m-reduction.

Actually, with a little care, we can improve the Calude et. al. [9] result, and get a complete characterization of the representations of a in terms of the weak truth table degrees, and later m-degrees, of splittings of $L(a)$.

Theorem 6.6 (Downey) *The following are equivalent*

- $\mathbf{b}$ *is the weak truth table degree of a splitting of* $L(a)$.
- $\mathbf{b}$ *is the weak truth table degree of a representation of* a.

Proof. To prove Theorem 6.6, we need only show that if $L(a) = C \sqcup D$ is any c.e. splitting of $L(a)$ then there is a representation $\widehat{C} = \{c_i\}$ of a of wtt degree that of C. (Without loss of generality, we suppose that C is noncomputable.) We do this in stages. At each stage s, we assume that we have enumerated C_s and D_s so that $L(a)_s = C_s \sqcup D_s$, where $L(a)_s$ is the collection of rationals in $L(a)$ by stage s, including all those of Gödel number $\leq s$. Additionally we will have a parameter $m(s)$. At stage $s+1$ compute C_{s+1} and D_{s+1}. Find the least rational, $q \in C_{s+1}$, by Gödel number, if any, such that $q > m(s)$.

If no such q exists, set $m(s+1) = m(s)$, and do nothing else.

If one exists, put q into $\widehat{C}_{s+1}$ and reset $m(s+1)$ to be the maximum rational in $L(a)_{s+1}$.

To verify the construction, first note that $\widehat{C}$ is an increasing sequence of rationals. Its limit will be a provided that it is infinite, because of the use of $m(s)$.

First we claim that $m(s) \to \infty$. Suppose not, so that there is an s such that, for all $t \geq s$, $m(s) = m(t)$. Then we claim that C is computable, this being a contradiction. To decide if $z \in C$, go first to stage $s' = s + g(z)$, where $g(z)$ denotes the Gödel number of z. If $z \notin C_{s'}$, then either $z > m(s)$, or $z \in D_{s'}$. In either case, $z \notin C$. Hence $m(s) \to \infty$.

Note that $\widehat{C} \leq_{wtt} C$ because only numbers entering C enter $\widehat{C}$ and can do so only at the same stage.

Finally we see that $C \leq_{wtt} \widehat{C}$. To decide if $q \in C$, find the least stage s such that $s > g(q)$ and $\widehat{C}_s \upharpoonright g(q) + 1 = \widehat{C} \upharpoonright g(q) + 1$.

If $q \leq m(s)$ then as above we can decide if $q \in C$. Otherwise, $q > m(s)$. Suppose that $q \notin C_{s+1}$. We claim that $q \notin C$. Otherwise, consider the stage s_1 where q enters C. Now, if $q > m(s_1 - 1)$ then either q or some even smaller number must enter $\widehat{C}$. Thus it can only be that $q \leq m(s_1 - 1)$. As a consequence, there is some least stage t where $s \leq t < s_1$ where $m(t) < q$ and $m(t+1) \geq q$.

Consider any stage $t' \geq s$ where $m(t') \neq m(t' + 1)$. For an induction, we suppose that $q \notin C_{t'}$. We only reset $m(t')$ because we saw some q' enter $C_{t'} \sqcup D_{t'}$ with $q' > m(t') \geq m(s)$. We put q' into $\widehat{C}_{t'+1}$. Since $\widehat{C}_s \upharpoonright g(q) + 1 = \widehat{C} \upharpoonright g(q) + 1$, we can only have that the Gödel number of q' exceeds $q + 1$. Therefore, it cannot be that q enters C at stage t'. If it did, then either q or some number with Gödel number less than or equal to q would enter $\widehat{C}$ in place of q'. (Remember here, we are considering in $C_u \sqcup D_u$ *all* rationals in the associated lower cut with Gödel number $\leq u$.)

Thus, in particular, at stage $t = t'$, $q \notin C_t$. But if q, or some number with Gödel number below q does not enter $\widehat{C}$ at stage $t+1$, which it cannot, then q is not in C. Hence $C \leq_{wtt} \widehat{C}$. □

The proof above can be improved to give the following.

Theorem 6.7 (Downey) $\mathbf{a}$ *is the m-degree of a c.e. splitting of* $L(a)$ *iff* $\mathbf{a}$ *is the m-degree of a representation of a.*

Proof. The modification is the following. At stage s, when we increase $m(s)$, we put into $\widehat{C}$ not just the least q but, in rational increasing order, all q entering C with Gödel number less than s. Now as we have seen, since $m(s) \to \infty$, $\widehat{C}$ is a representation of a. Moreover, the same argument shows that $C \equiv_{wtt} \widehat{C}$. We claim the reductions are m-reductions.

First $\widehat{C} \leq_m C$. Given q go to a stage s bigger than the Gödel number of q. If q is below $m(s)$ then, as before, we can decide computably if $q \in C$. Else, note that $q \in C$ iff $q \in \widehat{C}$. The same argument shows that $C \leq_m \widehat{C}$. □

We remark that many of the theorems of Calude et. al. [9] now come out as corollaries to the characterization above, and known results on splittings and *wtt* degrees. We refer the reader to, for instance, Downey and Stob [29]. For instance, we get the following.

Corollary 6.8 *There exist computably enumerable reals* a_i *such that the collection of T-degrees of representations* $R(a_i)$ *have the following properties.*

(i) $R(a_1)$ *consists of every c.e. (m-) degree*

(ii) $R(a_2)$ *forms an atomless boolean algebra, which is nowhere dense in the c.e. degrees.*

(iii) There is a c.e. degree $\mathbf{b}$ *with* $\mathbf{0} < \mathbf{b} < deg(L(a_3))$ *such that if* $\mathbf{x}$ *is the degree of a representation above* $\mathbf{b}$ *then* $\mathbf{x} = deg(L(a_3))$*, and if* $\mathbf{x}$ *is below* $\mathbf{b}$ *then* $\mathbf{x} = \mathbf{0}$.

Proof. See Downey and Stob [29]. □

We also remark that the above has a number of other consequences regarding known limits to splittings. For instance;

Corollary 6.9 *If a c.e. real a has representations in each T-degree below that of $L(a)$ then either $L(a)$ is Turing complete or low$_2$.*

Proof See Downey [21]. $\square$

As our final topic we look at the other form of representing reals. To avoid confusion we have the following definition.

Definition 6.10 *Let $A \subset \{0,1\}^*$.*

(i) We say that A is a presentation *of a c.e. real a if A is a prefix free c.e. set with*

$$a = \Sigma_{n \in A} 2^{-|n|}.$$

(ii) We say that a c.e. set A is a strong *presentation of a if*

$$a = \Sigma_{\sigma \in A} \cdot \chi_\sigma.$$

(Here, the $\cdot\chi_\sigma$ denotes the binary expansion.)

Previously we have seen that a has representations of degree $L(a)$. However, presentations can behave quite differently.

Theorem 6.11 *There is a c.e. real a which is not computable, but such that if A presents a then A is computable.*

Proof Sketch We briefly sketch the proof, details being found in Coles, Downey and Laforte [15]. We only mention the case (i) above. We must meet the requirements below.

$$R_e : W_e \text{ presents } a \text{ implies } W_e \text{ computable.}$$

We build a presentation of A, via the nearly c.e. definition. That is, we have an approximation $a = \cdot a_{0,s}...$ and obey the conditions that if $a_{i,s} = 1$ and

$a_{i,s+1} = 0$ then $a_{j,s+1}$ becomes 1 for some $j < i$. To make a noncomputable, we must lso meet the requirements:

$$P_e: \text{ For some } i,\ \ i \in W_e \text{ iff } a_i = 0.$$

The strategy for P_e is simple. We must pick some i to follow it, and initially make it 1. At some stage s, if we see i enter W_e, then we must make $a_{i,t} = 0$ for some $t \geq s$.

To make this cohere with the R_e we need a little work. First, we need to surround i with some 0's so that we can make it 0 by making the one before, say, 1. However, more importantly, we need to also make sure that for those R_k of higher priority if W_k presents a then W_k is computable.

Associated with R_k will be a current "length of agreement". This is a number $m(k,s)$ such that $a_s - \Sigma_{n \in W_{k,s}} 2^{-|n|} < 2^{-m(k,s)}$. We can assume that $a_s \geq \Sigma_{n \in W_{k,s}} 2^{-|n|}$, for otherwise, we can win by withholding numbers from a.

We promise that once $m(k,s) > d$, then no number of length $\leq d$ can enter W_k.

Now the idea is that when we see some P_e require attention for e bigger than k, if i is smaller than $m(k,s)$, the interesting case, then we wish to put a relatively big number into a, by changing position i for the sake of P_e, yet we wish to not allow numbers of low length to enter W_k.

The idea is to slowly work backwards. So first we will make position $m(k,s) + 1 = 1$ by adding something of length $2^{-m(k,s)+1}$ into A_{s+1}.

We then do nothing until W_k responds by giving us a stage $t > s$ with $a_t - \Sigma_{n \in W_{k,t}} 2^{-|n|} < 2^{-m(k,s)+1}$.

Note that W_k can only change on strings of long length, since we only changed A slightly. Now we repeat, adding another string of the same length $2^{-m(k,s)+1}$ into A_{s+1}. Again we wait for another expansion stage. Note that this next addition changes things at position $m(k,s)$ or earlier. We can continue in this way at most $2^{m(k,s)-i}$ many times till we get to change position i.

Now there are two outcomes. Either at some stage, we don't get recovery, so that W_k does not present A, or W_k responds at each stage and we get a change only on long strings. This means that we can compute W_k.

This is the basic module for a standard Π_2 priority argument, akin to the embedding of 1-3-1. Details can be found in Coles, Downey, Laforte [15]. $\square$

We remark that the proof above is very similar to the embedding of 1-3-1 in the c.e. Turing degrees. The result is related to lattice embeddings since Coles, Downey and Laforte did show the following.

Theorem 6.12 (Coles, Downey, Laforte [15]) *If $L(a)$ has promptly simple degree, then a has a noncomputable presentation.*

We remark that all of this is related to *wtt* degrees. As our final results we offer the following.

Theorem 6.13 (Coles, Downey, Laforte [15]) *Suppose that A presents a c.e. real a. Let $B \leq_{wtt} A$. Then there is a presentation of a of wtt degree B.*

Proof Sketch Suppose that $\Gamma^A = B$ is a wtt reduction with use γ and A is a presentation of a. We suppose that every stage is expansionary.

We build a presentation $C \equiv_{wtt} B$. We suppose that $0 \notin B$. For each $i \leq \gamma(n)$, and each σ with $|\sigma_i| = i$, choose a length $m \geq n$, and strings $\tau_k(\sigma_i)$ of length m for $k \leq p(\sigma_i, n)$ large enough, so that the set of τ's can be chosen to keep C prefix free, and such that there are $2^m - 2^i$ many $\tau_k(\sigma_i)$ of length[7] m.

At stage s assume that there is a unique element n in $B_{s+1} - B_s$. For each $i \leq \gamma(n)$, $|\sigma_i| = i$, if $\sigma_i \in A_{s+1} - A_s$, put $2^{m(\sigma_i,n)-i}$ coding markers $\tau_k(\sigma_i, n)$ of length $m = m(\sigma_i, n)$ into C_{s+1}, causing i to enter $\Sigma_{\tau \in C_{s+1}} 2^{-|\tau|}$.

[7] The idea here is that we can use the τ's collectively to add something of length i to the C-sum.

If $j > \gamma(n)$ and σ_j of length j enters A_{s+1}, put $2^{-\langle \sigma_j, 0\rangle}$ into C.

It is not difficult to argue that the set C works.

First $C \leq_{wtt} B$. To see this, to decide if β enters C compute $\langle \tau, n\rangle$ with $\beta = \langle \tau, n\rangle$. If $n = 0$ compute a stage s where $B \upharpoonright |\tau| = B_s \upharpoonright |\tau|$. Then $\beta \in C$ iff $\beta \in C_{s+1}$. If $n \neq 0$ then since β can only enter at C the stage n enters B we get $C \leq_{wtt} B$.

Conversely, to see that $B \leq_{wtt} C$, to decide if n enters B simply compute a stage s where $C_s \upharpoonright \tau_k(\sigma_i, n) = C \upharpoonright \tau_k(\sigma_i, n)$ for all $i \leq \gamma(n)$, and $k \leq p(\sigma_i, n)$.

It is clear that C presents the real since when i enters the A-sum, it is because some σ_i enters A. At such a stage, we will put enough $\tau(\sigma_i, n)$ into C to cause i to enter the C-sum. □

Corollary 6.14 *Suppose that a is strongly c.e. with $a = \cdot\chi_A$. Then B is the wtt degree of a presentation of a iff $B \leq_{wtt} A$.*

Proof Put $0^{i-1}1 \in B$ iff $i \in A$. This is a presentation with m-degree that of A. Now apply the theorem. □

7 Epilog: What are good questions in computable mathematics?

Some of the first questions asked in effective mathematics must be those of Hilbert and of Dehn. Focusing on Dehn, he asked the famous word, conjugacy, and isomorphism problems for finitely presented groups. These questions gave rise to combinatorial group theory, and are primarily of interest because they give enormous insight into the structure of groups. This is the key. Good questions should give insight into either computability (as in our torsion free result) or need considerable algebraic or analytic insight to solve. We offer with some trepidation a couple below which we feel will fall into this category.

Question 7.1 *A structure is called Σ_n-decidable if one can decide all n-quantifier statements. For each n, is there a finitely presented group which is n but not $n+1$ decidable? Is there a finitely presented group which is not decidable, but is n decidable for each n?*

Question 7.2 (Downey and Kurtz) *For each Π^1_0 class C is there a compuably presented torsion free group with the orderings (up to reversal) in 1-1 correspondence with the members of C?*

Reed Solomon has observed that the answer is no if the group must be abelian. Incidentally, Solomon [100, 101] proved that a computably presented group is isomorphic to a quotient of a computably ordered version of a free group by a computable convex normal subgroup. This theorem has a remarkably difficult proof, going through a classical group ring construction and needing small cancellation theory. It would be interesting to know why this so, and whether this must be the case. Perhaps proof theory can provide the answer. Note that there is no known simple proof that the two generator free group can be ordered.

Question 7.3 (Friedman, Simpson, Smith [35][8]) *Given a Π^0_1 C is there a commutive ring with 1 whose set of prime ideals is in 1-1 correspondence with the member of C?*

Question 7.4 *For each n, classify, by order type, the computable linear orderings with a nontrivial Π^0_n automorphism.*

The answer is known for $n = 0$ (Schwartz), but open for $n = 1$. See Downey [19] for more on this question.

Question 7.5 (Rosenstein) *It is known that a scattered linear partial ordering has a scattered linear extension. Is this computably true? What is its proof theoretical strength?*

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