Contiguity and Distributivity in the Enumerable Turing Degrees*

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Abstract

We prove that a (recursively) enumerable degree is contiguous iff it is locally distributive. This settles a twenty-year old question going back to Ladner and Sasso. We also prove that strong contiguity and contiguity coincide, settling a question of the first author, and prove that no m-topped degree is contiguous, settling a question of the first author and Carl Jockusch [11]. Finally, we prove some results concerning local distributivity and relativized weak truth table reducibility.

1 Introduction

All degrees and sets in this paper will be enumerable unless otherwise stated. Many of the natural reducibilities occurring in, for instance, effective algebra,

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are stronger than Turing reducibility. For this and many other reasons, one is naturally led to the study of strong reducibilities (see Odifreddi [29],[30]). From the point of view of the Turing degrees, one of the most important of the strong reducibilities is *weak truth table reducibility* introduced by Friedberg and Rogers [20]. Here we recall that A is weak truth table reducible to B ($A \leq_W B$) iff there are a Turing procedure Φ and a computable function φ such that for all x, $\Phi^A(x) = A(x)$ and $u(\Phi^B(x)) \leq \varphi(x)$. (Here $u(\Phi^B(x))$ denotes the *use* of the computation $\Phi^B(x)$, that is, the greatest element used in the computation.)

Many results in the enumerable wtt degrees have important consequences in the enumerable Turing degrees often because of the existence of *contiguous* degrees. A degree $\mathbf{a}$ is called *contiguous* iff it contains a single enumerable weak truth table degree. That is, for all enumerable sets $A, B \in \mathbf{a}$, $A \equiv_W B$. Contiguous degrees were first defined by Ladner [26] and Ladner and Sasso [27]. They have been used extensively in the study of the enumerable Turing degrees in papers such as Ladner and Sasso [27], Ambos-Spies [1], [2], Ambos-Spies and Soare [5], Downey [7], [8], [9], Downey and Jockusch [11], Downey and Welch [19], and Stob [34]. We give one illustration below. Additionally they are important in effective algebra as can be seen from Downey and Remmel [13] and Downey and Stob [15]. For instance, Downey and Remmel proved that if V is a enumerable subspace of V_∞, then the degrees of computably enumerable bases of V are precisely the weak truth table degrees below the degree of V.

A principal reason for the importance of contiguity is the fact that the distributivity of the enumerable weak truth table degrees transfers locally to the contiguous degrees. That is, if a degree $\mathbf{a}$ is contiguous then it satisfies the formula below in the enumerable degrees.

$$\begin{aligned}&\forall \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}(\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a} \wedge \mathbf{b} \leq \mathbf{a} \rightarrow \\ &\quad \exists \mathbf{b}_1, \mathbf{b}_2(\mathbf{b}_1 \cup \mathbf{b}_2 = \mathbf{b} \wedge \mathbf{b}_1 \leq \mathbf{a}_1 \wedge \mathbf{b}_2 \leq \mathbf{a}_2)).\end{aligned}$$

For instance, the way that Ambos-Spies and Soare used contiguity to exhibit infinitely many one-types in the enumerable degrees was to, for each n, construct n degrees $\mathbf{a}_1, ..., \mathbf{a}_n$, each of which bounds no minimal pairs and each pair of which forms a minimal pair (i.e., if $i \neq j$ then $\mathbf{a}_i \cap \mathbf{a}_j = \mathbf{0}$), and such that all the joins $\cup_{j \in P(\{1,...,n\})} \mathbf{a}_j$ are contiguous. The point is that by contiguity, the degrees $\mathbf{a}_1, ..., \mathbf{a}_n$ are exactly the maximal complemented degrees below $\mathbf{a}_1 \cup ... \cup \mathbf{a}_n$. (This actually is an elaboration of an earlier argument of Stob [34].)

Ever since the classic paper by Ladner and Sasso [27] where contiguity was used to investigate the anticupping property in the enumerable degrees, it has been an open question whether *local distributivity* defines contiguity. The main goal of the present paper is to verify the conjecture of Ambos-Spies (and others) that local distributivity does indeed define contiguity:

Theorem 1.1 *A degree* $\mathbf{a}$ *is contiguous iff* $\mathbf{a}$ *satisfies the formula below.*

$$\forall \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}(\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a} \wedge \mathbf{b} \leq \mathbf{a} \rightarrow \\ \exists \mathbf{b}_1, \mathbf{b}_2(\mathbf{b}_1 \cup \mathbf{b}_2 = \mathbf{b} \wedge \mathbf{b}_1 \leq \mathbf{a}_1 \wedge \mathbf{b}_2 \leq \mathbf{a}_2)).$$

As a corollary to the technique employed in the proof of Theorem 1.1, we are also able to solve a question of the first author. In [8], Downey introduced the notion of a *strongly contiguous* degree. This is an enumerable degree in which all the (not necessarily enumerable) sets are of the same weak truth table degree. Downey raised the question of whether strong contiguity and contiguity coincided. We observe that the proof of Theorem 1.1 actually works with "strongly contiguous" in place of "contiguous." As a consequence we obtain the following

Corollary 1.2 *A degree is strongly contiguous iff it is contiguous.*

Another corollary is obtained through the work of Ambos-Spies and Fejer [4], Corollary 2.2. Recall that an enumerable set A is said to have the strong universal splitting property iff for all enumerable degrees $\mathbf{b}$ and $\mathbf{c}$ with $\mathbf{b} \cup \mathbf{c} = \deg(A)$, there exist enumerable disjoint sets $B \in \mathbf{b}$ and $C \in \mathbf{c}$ such that $B \sqcup C = A$. Ambos Spies and Fejer proved that if local distributivity defined contiguity, then the following was true.

Corollary 1.3 *An enumerable degree contains a set with the strong universal splitting property iff it is contiguous.*

Finally, using some unpublished work of Gasarch and Kummer, we are also able to solve a question of Downey and Jockusch [11]. In [11], Downey and Jockusch define an enumerable degree $\mathbf{a}$ to be m-topped iff there is an enumerable set A of degree $\mathbf{a}$ such that for all enumerable sets $B \leq_T A$, $B \leq_m A$. For instance, $\mathbf{0}'$ is m-topped by K. In [11] Downey and Jockusch proved that there are nonzero incomplete m-topped degrees and in fact Downey and Shore [14] have proven that an enumerable degree is low_2 iff it is bounded by an incomplete m-topped degree. Here we solve a question from [11] by proving

Corollary 1.4 *No m-topped degree is contiguous. In fact, no tt-topped degree is contiguous.*

Aside from the use of local distributivity, contiguous degrees have been used in a number of other applications. The basic paper Ladner and Sasso, [27], provides a nice example of this. First Ladner and Sasso prove that all nonzero elements of $\mathbf{W}$ (the set of enumerable wtt-degrees) have the anticupping property: for all $\mathbf{a} \neq \mathbf{0}$, there is a nonzero $\mathbf{b} \leq \mathbf{a}$ with $\mathbf{b} \neq \mathbf{0}$ and such that for all $\mathbf{c} \neq \mathbf{a}$, $\mathbf{c} \cup \mathbf{b} \neq \mathbf{a}$. Now if we apply this result from $\mathbf{W}$ to a contiguous degree, we get a *transfer* result: there exist enumerable T-degrees with the anticupping property. Indeed since each nonzero enumerable degree has a contiguous nonzero predecessor (Ladner and Sasso [27]), it follows that degrees with the anticupping property are downward dense in $\mathbf{R}$. In fact, Downey [9] has shown that similarly using strongly contiguous degrees enables one to get an easy proof of an extension of the Slaman-Steel/Cooper result that there are enumerable T-degrees $\mathbf{a}$ with enumerable predecessors that cannot be cupped to $\mathbf{a}$ in the Turing degrees (the so-called "strong anticupping property.") Notice that our result has the interesting consequence that

- *Every contiguous degree has the strong anticupping property.*

We will prove these results at the end of Section 4.

We remark that often $\mathbf{W}$ is used in applications as it can be much easier to perform constructions in $\mathbf{W}$ (relying on the bounded use) and then transfer the result to $\mathbf{R}$, than to perform the construction directly in $\mathbf{R}$.

The only problem with all the above is that many of the applications only give local structure theory for low_2 degrees since, for instance, the only known locally distributive degrees are contiguous and hence low_2.

The second goal of the present paper is to demonstrate that it is possible to use contiguity-like transfer principles "higher up" by looking at a relativized form of *wtt*-reducibility. In doing so, we follow some unpublished ideas of the first author [17], Theorem 7.1, as well as prove several new results. We provide proofs of some claims in Downey-Stob [17], together with answering a key open question from there. In particular, our main results are to prove that while degrees contiguous over some lesser one (and hence distributive over some lesser one) are dense in the enumerable degrees, they are nontrivially so, in the sense that there do exist enumerable degrees that are nondistributive over all lesser enumerable degrees.

The proof of Theorem 1.1 is a finite injury argument, although it is nevertheless rather complex and employs many ideas from the tree method. The proofs of the results on relativized *wtt*-reducibility use $\mathbf{0}'''$ arguments. We use terminology and notation consistently with Soare [33]. It is assumed that the reader is thoroughly familiar with standard tree arguments. We use the following conventions. First if we append $[s]$ to a parameter this denote its value at the end of stage s. We will denote the use of a procedure by the corresponding lower case Greek letter. (Hence the use of $\Phi^A(x)[s]$ would be $\varphi(x)[s]$.) All uses are monotone in argument and stage number, and computations with use t can only halt *after* stage t. All computations obey the hat convention.

2 The Requirements and the Intuition for the proof of Theorem 1.1

In this section we adopt the "Chicago convention." That is, we use letters at the beginning of the Greek and English alphabet for functionals and sets constructed by *us*, and letters from the latter half of the alphabets for objects constructed by our *opponent*.

Suppose that W is an enumerable set and U a set with a Turing functional Φ such that $\Phi^W = U$. Then we claim that we can build enumerable sets A_0, A_1, and B together with functionals Γ_0, Γ_1, Γ, and Δ so that

$$\Gamma_0^W = A_0 \wedge \Gamma_1^W = A_1 \wedge \Gamma^{A_0 \oplus A_1} = W \wedge \Delta^W = B,$$

and so as to satisfy all the requirements below.

$$R_{\vec{\Psi},\vec{\Xi}} : V_0 = \Psi_0^B \wedge V_1 = \Psi_1^B \wedge B = \Psi^{V_0 \oplus V_1} \wedge \Xi_0^{A_0} = V_0 \wedge \Xi_1^{A_1} = V_1 \rightarrow$$

$$(\exists \text{ wtt } \Lambda)[\Lambda^W = U].$$

That is, we can build a degree-theoretical splitting A_0, A_1 of W and a set $B \leq_T W$ such that if we cannot beat all possible degree-theoretical splittings V_0, V_1 of B then we can witness the fact that $U \leq_W W$ (via Λ). Figure 1 might be useful in visualizing the reductions and sets.

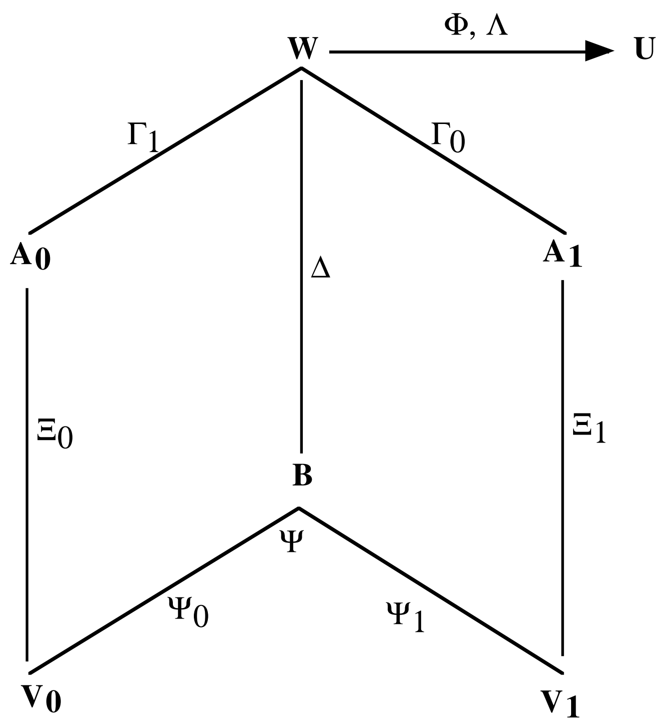

Figure 1: Reductions for the Definability Theorem

Suppose that we meet all the requirements $R_{\vec{\Psi},\vec{\Xi}}$. Suppose that W is not contiguous. Then we choose any enumerable set $W \in \mathbf{a}$ for which there is some set $U \leq_T W$ yet $U \not\leq_W W$; we must construct a splitting $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{a}$ (with $\mathbf{a}_i = \deg(A_i)$) and an enumerable degree $\mathbf{b} = \deg(B) \leq \mathbf{a}$ that cannot be split by any $\mathbf{b}_i = \deg(V_i)$ with $\mathbf{b}_i \leq \mathbf{a}_i$, for $i = 0, 1$. Therefore, $\mathbf{a}$ is not locally distributive.

We now turn to the basic module and the inductive strategies. We will assume that $U \not\leq_W W$. The strategies work independently and by the standard finite injury method (with potential infinite outcome only if the hypothesis $U \not\leq_W W$ is refuted). Thus if R acts and R' has lower priority than R then we will initialize R'. Thus it will suffice to give the construction for a single requirement. We shall assume that we are given enumerations of the sets and functionals sufficiently fast so that at each stage $\ell(s+1) > \ell(s)$ where $\ell(s)$ is the *length of agreement between U and Φ^W at stage s*. In the nomenclature of Soare [33], every stage is (Φ^W, U)-expansionary. Notice that this means that at each stage we will update the functionals $\vec{\Gamma}$ and Δ. If these uses are increased we automatically lift them to be large. We will automatically keep $\delta(z) > \gamma(z)$ for all z.

2.1 The basic 0-module

We begin by describing the strategy for a requirement $R_{\vec{\Psi},\vec{\Xi}}$ trying to achieve $\Lambda^W(0) = U(0)$. We call this the *basic 0-module*. It will be modified later to live with the other modules devoted to $R_{\vec{\Psi},\vec{\Xi}}$. The basic 0-module consists of the following steps.

Step 1. Pick a fresh large number $x_0 > \varphi(0)$ targeted for B. Reset x_0 if $W\lceil(\varphi(0)+1)$ changes before we get to Step 2 below. (Note that we always have $\delta(x_0) > x_0 > \varphi(0)$.)

Step 2. Wait for the $(\vec{\Psi}, \vec{\Xi})$-*length of agreement* $L(s)$ to exceed x_0. This parameter is defined by

$$\begin{aligned} L(s) \quad = \quad & \max\{x \mid \forall y < x(B_s(y) = \Delta^W(y)[s] = \Psi^{V_0 \oplus V_1}(y)[s] \wedge \\ & V_0\lceil(\psi(y)+1) = \Xi_0^{A_0}\lceil(\psi(y)+1) = \Psi_0^B\lceil(\psi_0(y)+1) \wedge \\ & V_1\lceil(\psi(y)+1) = \Xi_1^{A_1}\lceil(\psi(y)+1) = \Psi_1^B\lceil(\psi_1(y)+1))\}. \end{aligned}$$

(Hence, in particular, $\Psi^{V_0 \oplus V_1} = 0 = B(x_0)[s]$ $\vec{\Xi}$- and $\vec{\Psi}$-correctly).

Step 3. Define $\Lambda^W(0) = U(0)$ with use $\lambda(0) = \varphi(0)$. (Note that since Λ is to be a wtt-reduction we cannot change $\lambda(0)$ after we define it.) Restrain $A_1\lceil((\xi_1\psi(x_0)[s]+1)$. (We will call numbers $\leq \xi_1\psi(x_0)[s]$ 0-*small*. The A_1-restraint prevents $V_1\lceil(\psi(x_0)+1)$ from changing.)

Step 4. Wait for $W\lceil(\varphi(0)[s]+1)$ to change at some $y \leq \varphi(0)[s]$ at a stage $t > s$. (Now $\Lambda^W(0)$ is undefined till the next $(\vec{\Psi}, \vec{\Xi})$-expansionary stage.) The construction, while it is waiting for Step 4, will begin the 1-module.

Step 5. (Use Lifting) Put $\gamma(y)[t]$ into A_0 (because of the A_1-restraint in Step 3 which puts 0-small numbers into A_0 and not A_1) and lift $\gamma(y)$ above $\varphi(0)[s]$ and $\xi_1\psi(x_0)[s]$. (This means also that $\delta(x_0) > \xi_1\psi(x_0)[s]$, since $\delta(x_0) > \gamma(x_0) > \gamma(y)$, because $x_0 > y$.) Maintain the $A_1\lceil(\xi_1\psi(x_0)[s]+1)$-restraint.

Step 6. (Recovery) Wait for the $(\vec{\Psi}, \vec{\Xi})$-length of agreement $L(v)$ to exceed x_0 again at a stage $v > t$.

Step 7. (Switch sides) Set $\Lambda^W(0) = U(0)$ with the same use. Now restrain both $A_1\lceil(\xi_1\psi(x_0)[s]+1)$ and $A_0\lceil(\xi_0\psi(x_0)[v]+1)$, giving the A_1-restraint higher priority. (These restraints mean that if a number is 0-small we must put it into A_0 but if it is 0-*medium* (i.e., in the interval $(\,\xi_1\psi(x_0)[s], \xi_0\psi(x_0)[v]\,]$ then we must put it into A_1. We start the 1-module here if it is not already going.)

Step 8. (Resolution) Wait for $W\lceil(\delta(x_0)[s]+1)$ to change. If $U(0)$ changes at a stage $q > v$, then there are two possibilities: If $\Lambda^W(0)[q]$ is undefined because a 0-small number entered W, then we correct $\Lambda^W(0)$ at the next $(\vec{\Psi}, \vec{\Xi})$-expansionary stage and stop the 0-module. Otherwise, the number to enter W below $\varphi(0)$ must be 0-medium. In this case, we put x_0 into B. (The point here is that either we have permanently diagonalized the requirement $R_{\vec{\Psi},\vec{\Xi}}$ at x_0, or at a stage $q' > q$ some 0-small number must enter W below $(\xi_1\psi(x_0)[s]$ and hence below $\lambda(0)$. In this latter case we can correct $\Lambda^W(0)[q']$.)

Step 9. (Correction) If $U(0)$ changes and a 0-small number enters W then correct $\Lambda^W(0)$.

2.2 Analysis of the basic 0-module

The basic 0-module has the following possible outcomes, all of which are finitary:

A) It is permanently stuck at Step 2 or at Step 6, waiting for the $(\vec{\Psi}, \vec{\Xi})$-length of agreement to exceed x_0. The we satisfy the overall requirement $R_{\vec{\Psi},\vec{\Xi}}$, and the overall restraint is 0 or $\xi_1\psi(x_0)[s]$, respectively.

B) The 0-module is permanently stuck at Step 4 or at Step 8, waiting for $\Phi^W(0)$ to become undefined. Then $\Lambda^W(0) = \Phi^W(0) = U(0)$, and the overall restraint is $\xi_1\psi(x_0)[s]$ or $\max\{\xi_1\psi(x_0)[s], \xi_0\psi(x_0)[v]\}$, respectively.

C) The 0-module diagonalizes in Step 8 but gets stuck permanently at Step 9, waiting for 0 to enter U and a 0-small number to enter W. Then we have $\Psi^{V_0 \oplus V_1}(x_0) \downarrow = 0 \neq 1 = B(x_0)$, satisfying the overall requirement $R_{\vec{\Psi},\vec{\Xi}}$. (Note here that $(V_0 \oplus V_1)\lceil(\psi(x_0)+1)$ can change only when $B\lceil(\max\{\psi_0\psi(x_0), \psi_1\psi(x_0)\}+1)$ changes *and* a 0-small number enters $A_0 \cup A_1$ *in the same interval between consecutive* $(\vec{\Psi}, \vec{\Xi})$*-expansionary stages.*

The reader should recognize that since the A_1-restraint has higher priority than the A_0-restraint, numbers z entering W with $\gamma(z)$ below $\xi_1\psi(x_0)[s]$ cause us to change the A_0-restraint from $\xi_0\psi(x_0)[v]$ to $\xi_0\psi(x_0)[v']$ at the next $(\vec{\Psi}, \vec{\Xi})$-expansionary stage v'. Fortunately each time this nasty event occurs we get to make $\Lambda^W(0)$ undefined also till stage v'. Hence we can change the A_0-restraint at most $\xi_1\psi(x_0)[s]$ many times (since we reset uses to be large). Furthermore, note that if originally $\gamma(y)$ entered W in Step 4, then the later injury of the A_0-restraint in Step 5 by some $\gamma(y')$ entering W as just described with $\gamma(y')$ below $\xi_1\psi(x_0)[s]$, *must have* $y' < y$ *because of the way we move the* γ*-uses.* This comment is important in what follows.

2.3 The problems with two numbers

Now suppose that we try to implement the above module for the argument 1 of Λ. The places where we would begin the 1-module would be anywhere where the 0-module might get stuck without actually winning the overall requirement (i.e., under outcome B). The natural places are thus while waiting in Step 4 (namely, waiting for $W\lceil(\varphi(0)[t]+1)$ to change) and in Step 8 which is where we are waiting for 0 to enter U.

Notice that it does not really matter for the 0-module which side we decide to preserve and thus define 0-small numbers. (That is, $\xi_0\psi(x_0)[s]$ would do as well, by interchanging all the 0's and 1's.) Now while we are beginning the 1-module, until we actually define $\Lambda^W(1)[s]$ we will initialize the 1-module each time a 0-small or 0-medium number enters $A_0 \cup A_1$. What would then be a good side for the 1-module to choose to preserve? Suppose that we begin the 1-module in Step 4, and we get to define $\Lambda^W(1)$ before Step 4 is invoked for 0; then it is clear that we ought to use $\xi_1\psi(x_1)[s]$ as 1-small, since both 0 and 1 basically wish to achieve the same goals. In fact this case really causes no problems.

On the other hand, suppose that the 0-module has reached Step 8 before we get to define $\Lambda^W(1)[s]$ or even begin the 1-module. Now 0 is asking us to try to preserve both sides if we can, and most particularly, preserve the A_1-side. Note that we make no progress on the 0-module if some 0-medium number enters $A_0 \oplus A_1$ if that number is bigger than $\delta(x_0)$. All that happens is that the 0-module simply directs us to put it into the A_1-side to preserve $A_0\lceil(\xi_0\psi(x_0)[v]+1)$. However, if we were to begin the 1-module on the A_0-side attempting to preserve the A_1-side as we did for the basic 0-module, then now we could be in real trouble.

Perhaps we have enumerated some numbers into A_0 already, changing $\xi_0\psi(x_1)[s]$. Now the 0-module's direction to put the 0-medium, but unhelpful, numbers into A_1 would also change $\xi_1\psi(x_1)[s]$, and now we have the situation that *both* sides are wrong with respect to $\lambda(1)[s]$.

This problem can be avoided by beginning the 1-module on the A_1-side, and letting $\xi_0\psi(x_1)[s]$ take the role of $\xi_1\psi(x_0)[s]$ in the basic 0-module. That is, based on the assumption that 0 has finished acting on numbers $y' < y$, we will begin by preserving A_0 on $\xi_0\psi(x_1)[s]$. (Here y is the number of Step 4 for the 0-module, which has caused us to switch.) This allows 1 to live with 0 provided that no number $y' < y$ enters W causing us to enumerate a number with $\gamma(y')$ below $\xi_1\psi(x_0)[s]$.

Now we get to the final problem. All of this is fine if no number $y' < y$ enters W causing us to enumerate a number with $\gamma(y')$ below $\xi_1\psi(x_0)[s]$. But what happens if such a y' *does* enter W? $\gamma(y')$ is 0-small so it *must* be put into A_0, if we are to preserve $A_1\lceil(\xi_1\psi(x_0)[s]+1)$.

Perhaps we have already played a number into A_1 changing $\xi_1\psi(x_1)[t]$ but now we are also changing A_0 on $\xi_0\psi(x_1)[t]$. Again $\lambda(1)$ is in bad shape with respect to both uses.

2.4 The solution to the final problem, and the refined inductive strategies

One key point with the problem outlined above is that after y' enters W we can get to make $\delta(x_0)$ equal to $\delta(x_1)$ and both large. This means that we can ensure that henceforth if we can put x_1 into B then equally we could put x_0 into B. This single observation proves to be our salvation, as we now see.

Let us make $\delta(x_0) = \delta(x_1)$ at stage t. We say that this makes 0 and 1 *combined*. Now suppose that even after the next $(\vec{\Psi}, \vec{\Xi})$-expansionary stage neither 0 nor 1 has entered U. Then we can still win on 1 as follows. Suppose that 1 enters U, then although there seems to be no way to win with x_1 we *can* put x_0 into B since we have made $\delta(x_0) > \varphi(1)[t]$. This means that we either get a global win on the requirement, or some number $y'' < y'$ must enter W forcing a number below $\xi_0\psi(x_0)[s]$ into A_0. This number would allow us to correct $\Lambda^W(1)$.

However, there is a flaw in all of this reasoning. What happens if even later 0 enters U? Now we have used up x_0 and we can no longer use it for resolving $\Lambda^W(0)$.

The main problem with the above is that we began our 1-module based on a false premise: that no more 0-small numbers would enter W. The key observation is that once we have passed Step 4 (which caused us to switch anyway), the only numbers that can cause injury to 1 are those numbers below y. There are fewer than $\varphi(0)[s]$ (the original value at the stage we defined $\lambda(0)$) such numbers. In fact this statement is clearly true about 0's effect on all numbers $i > 0$. 0-small numbers can only injure them in this way $\varphi(0)[s]$ many times.

The Modification. The modification to the basic module is to pick not one x_0 but $\varphi(0) + 1[s]$ many x_0's (we will call them $x_{0,j}$), initializing the set if $\varphi(0)$ changes before we see a $(\vec{\Psi}, \vec{\Xi})$-expansionary stage with the length of agreement above all $x_{0,j}$. We ensure that each is chosen to exceed the use of the predecessor, so that, provided that they enter in *reverse* order, the entry of $x_{0,t}$ will not affect the set up for $x_{0,t'}$ for $t' < t$.

Now suppose that we have any number of i-modules above 0, based upon the assumption that 0 has passed Step 4, and will no longer cause A_0-enumeration. If a number $y' < y$ enters W causing A_0-enumeration, then we would get to reset $\delta(0)$ to be the same as all of the $\delta(i)$ for such i. 0 is now combined with all such

i. For such i, 0 now takes over the role of resolver. If i enters U then we can put the largest unused $x_{0,j}$ into B. If this does not diagonalize the requirement then some number below $\varphi(0)[s]$, the original value, must enter W. But notice that there are only $\varphi(0)[s]$ many such numbers and hence by the pigeonhole principle, we don't use up all the $x_{0,j}$'s.

Notice that there is no problem with the inductive strategies for k above 0. These strategies are begun with the assumption that the $(k-1)$-module has finished, and at what place the $(k-1)$-module finished (i.e., after Step 4 or before Step 4). This finishing place tells the k-module which side to initially preserve. If it is the case that any of the i-modules for $i < k$ are currently trying to preserve both sides, and a small i-number enters W then immediately we combine i and k and i takes over the role of resolving k as well. The only way that i needs to preserve both sides is that it has passed Step 4 and hence it has enough followers to cover k. If later some number controlled by some $i' < i$ enters W then either the i'-module has not passed Step 4, or there is some i''-module with $i' \leq i'' < i$ which has passed Step 4 and will take over i's job and hence k's job.

Finally notice that we did not need U being enumerable here. Even if it is only Δ^0_2, the fact that we are using $\varphi(n)$ changes to control the construction and the fact that W is enumerable, means that the construction will still succeed. This final observation gives Corollary 1.2.

3 The Construction for Theorem 1.1

In addition to the activity for the $R_{\vec{\Psi},\vec{\Xi}}$-requirements described below, we assume background activity for ensuring the correctness of $\vec{\Gamma}$ and Δ by extending their domains at each stage (which is always assumed (Φ^W, U)-expansionary). This includes redefining $\vec{\Gamma}$ and Δ at old arguments (when previous computations have become destroyed) and always picking a large new use but keeping previous uses unless specifically stated otherwise. This also includes correcting $\Gamma^{A_0 \oplus A_1}$ (at any argument x when x enters W while $\Gamma^{A_0 \oplus A_1}(x) \downarrow = 0$) by enumerating $\gamma(y)$ into A_0 or A_1, depending on which is restrained by higher-priority restraint. Finally, we will always assume that $\delta(y) > \gamma(y)$ for all $y \in \omega$.

Since the action for each $R_{\vec{\Psi},\vec{\Xi}}$-requirement is finitary (unless $U \leq_W W$) we will only describe the construction for a single requirement and assume that any action for one requirement respects the restraints of all higher-priority requirements and initializes all lower-priority requirements. At any stage, the highest-priority requirement that requires action will act. As usual in these constructions, we assume that all parameters remain unchanged unless explicitly redefined, and that all these parameters are measured at the current stage unless specified otherwise.

The action for each requirement $R_{\vec{\Psi},\vec{\Xi}}$ is carried out by n-modules (for $n \in \omega$), each working to define the wtt-reduction Λ^W at argument n. The 0-module starts first; and the n-module may start the $(n+1)$-module. (At each stage, the n-modules (try to) act in increasing order of n if they have been started already.)

The n-module proceeds as follows:

Step 1. Set $k_n = \varphi(n)$. If $W\lceil(\varphi(n)+1)$ changes, or the m-module (for some $m < n$) acts before the n-module reaches Step 3, or computations for the length of agreement for previously chosen $x_{n,j}$ are destroyed, then start the n-module over at Step 1. Perform the following substeps for $j = 0, ..., k_n - 1$.

Substep 0. Pick a fresh number $x_{n,0}$ larger than any previously seen.

Substep $j+1$**.** Wait for the $(\vec{\Psi}, \vec{\Xi})$-length of agreement to exceed x_{n,k_j}. Pick a fresh number $x_{n,j+1}$ larger than any previously seen.

Step 2. Wait for the $(\vec{\Psi}, \vec{\Xi})$-length of agreement to exceed x_{n,k_n} at a stage s_n, say.

Step 3. Set $\Lambda^W(n) = U(n)$ with use $\lambda(n) = \varphi(n)[s_n]$. If $n = 0$ then set $i(n) = 1$; if $n > 0$ and the $(n-1)$-module has not yet reached Step 5 then set $i(n) = i(n-1)$; otherwise set $i(n) = 1 - i(n-1)$. (The parameter $i(n)$ marks the A-side *first* restrained by the n-module.) Set the $A_{i(n)}$*-restraint of the n-module* as $r^{A_{i(n)}}(n) = \xi_{i(n)}\psi(x_{n,k_n})[s_n]$ and *restrain* $A_{i(n)}\lceil(r^{A_{i(n)}}(n)[s_n]+1)$ by, at any future stage s, directing into $A_{1-i(n)}$ all the numbers in the interval $(\,R(n)[s], r^{A_{i(n)}}(n)[s_n]\,]$ wanting to enter $A_0 \cup A_1$. (Here $R(n) = \max\{r^{A_i}(m) \mid m < n \wedge i \leq 1\}$.) Start the $(n+1)$-module.

From now on, if $A_{i(n)}\lceil(r^{A_{i(n)}}(n)[s_n]+1)$ changes (by the restraint of the m-module for some (least) $m < n$) then we proceed to Step 10 as soon as the $(\vec{\Psi}, \vec{\Xi})$-length of agreement exceeds x_{n,k_n}.

Step 4. Wait for some number $y_n \leq \varphi(n)[s_n]$ to enter W at a stage $t_n > s_n$, say.

Step 5. Put $\gamma(y_n)$ into $A_{1-i(n)}$ and lift $\gamma(y_n)$ above $\xi_{i(n)}[s_n]$.

Step 6. Wait for the $(\vec{\Psi}, \vec{\Xi})$-length of agreement to exceed x_{n,k_n} again.

Step 7. Reset $\Lambda^W(n) = U(n)$, necessarily with the same use $\lambda(n)[s_n]$.

Step 8. Wait for one of the following:

(a) $\Lambda^W(n)$ becomes undefined: Then go back to Step 6.

(b) $\Lambda^W(n) \downarrow \neq U(n)$ and the $(\vec{\Psi}, \vec{\Xi})$-length of agreement exceeds x_{n,k_n} at a stage $u_n > t_n$, say: Proceed to Step 9.

Step 9. Put $x_{n,j}$ into B and $\delta(x_{n,j})$ into $A_{i(n)}$ (for the greatest $j \leq k_n$ such that currently $x_{n,j} \notin B$, which is possible by Lemma 4.1(iv)). Set the $A_{1-i(n)}$-*restraint of the n-module* as $r^{A_{1-i(n)}}(n) = \xi_{1-i(n)}\psi(x_{n,k_n})[u_n]$ and *restrain* $A_{1-i(n)}\lceil(r^{A_{1-i(n)}}(n)[u_n]+1)$ by, at any future stage s, directing into $A_{i(n)}$ all the numbers in the interval $(\max\{R(n)[s], r^{A_{i(n)}}(n)[s_n]\}, r^{A_{1-i(n)}}(n)[u_n]]$ wanting to enter $A_0 \cup A_1$. (This $A_{1-i(n)}$-restraint will later be canceled if $A_{1-i(n)}\lceil(r^{A_i(n)}(n)+1)$ changes.) Go back to Step 6.

Step 10. Since $A\lceil(\delta(x_{m,0})+1)$ has changed we may reset the n-module's parameters as follows (as will be shown in Lemma 4.1(vi) and Lemma 4.2(v)). We set

$$\begin{aligned}
\delta(x_{m,j}) &= \delta(x_{n,k_n}) \text{ (for } j \leq k_m),\\
k_n &= k_m,\\
x_{n,j} &= x_{m,j} \text{ (for } j \leq k_m),\\
i(n) &= i(m),\\
r^{A_i}(n) &= r^{A_i}(m) \text{ (for } i \leq 1, \text{ if defined)},\\
s_n &= s_m,\\
t_n &= t_m \text{ (if defined)},\\
u_n &= u_m \text{ (if defined)},\\
y_n &= y_m.
\end{aligned}$$

The m- and n-modules are now in exactly the same position (by Lemma 4.2(vi)); so the n-module starts off at exactly the step at which the m-module currently is. Both work in the same way (including the second paragraph of Step 3), the only difference being that the m-module reacts to $U(m)$-changes and the n-module to $U(n)$-changes. (We say the modules are *combined.*)

4 The Verification

We will show, in a sequence of lemmas, that the construction described above is possible as stated (i.e., that the parameters can be reset, the functionals redefined, and the numbers enumerated as stated) and that, assuming $U \not\leq_W W$, the overall action for each requirement $R_{\vec{\Psi},\vec{\Xi}}$ is finite (i.e., eventually stops and imposes only finite restraint).

We will establish some facts separately for each n-module, first under the assumption that Step 10 is never carried out, and then in the general case.

Lemma 4.1 **(n-Module Lemma I)** *Fix $n \in \omega$. Assume that*

(1) Each m-module (for $m < n$) acts at most finitely often, and

(2) The n-module never carries out Step 10.

Then the n-module acts as follows:

(i) There is a stage s' after which the n-module will no longer carry out Step 1; so k_n and $x_{n,0}, \ldots, x_{n,k_n}$ will be defined permanently.

(ii) If the $(\vec{\Psi}, \vec{\Xi})$-length of agreement ever exceeds x_{n,k_n} after stage s' then Step 3 will never be carried out later and the parameter s_n will be defined permanently.

(iii) If $W\lceil(\varphi(n)[s_n] + 1)$ never changes after stage s_n then $\Lambda^W(n) = U(n)$.

(iv) If $W\lceil(\varphi(n)[s_n]+1)$ changes after stage s_n then Step 5 will never be carried out later and the parameters t_n and y_n will be defined permanently.

(v) If $W\lceil(\varphi(n)[s_n]+1)$ changes after stage s_n then the $A_{i(n)}$-restraint of the n-module applies after stage t_n only when $W\lceil y_n$ changes (i.e., each $\gamma(y)$ for $y \geq y_n$ is free to choose A_0 or A_1 as far as this $A_{i(n)}$-restraint is concerned).

(vi) Suppose the n-module enumerates some $x_{n,j}$ at a stage u_n, and the $(\vec{\Psi}, \vec{\Xi})$-length of agreement exceeds x_{n,k_n} at some stage $u' > u_n$. Then $W\lceil y_n$ must have changed between stages u_n and u' (allowing $\Lambda^W(n)$ to be corrected) and the n-module's $A_{1-i(n)}$-restraint must have been canceled.

Proof. (i) Step 1 is performed only in the beginning, and whenever $W\lceil(\varphi(n)+1)$ changes or the m-module (for some $m < n$) acts, both of which is assumed to happen at most finitely often.

(ii) Clear by the construction.

(iii) Since $U(n) = \Phi^W(n) = \Phi^W(n)[s_n] = \Lambda^W(n)[s_n] = \Lambda^W(n)$.

(iv) Immediate by the construction.

(v) Since $r^{A_{i(n)}}$ does not change after stage s_n and $\gamma(y)$ is lifted above $r^{A_{i(n)}}$ at stage s_n for all $y \geq y_n$.

(vi) We have $B(x_{n,j}) = 1 \neq 0 = \Psi^{V_0 \oplus V_1}(x_{n,j})[u_n]$. Since Step 10 is not carried out, we must have $V_{i(n)}\lceil(\psi(x_{n,j})[u'] + 1) = \Xi_{i(n)}^{A_{i(n)}}\lceil(\psi(x_{n,j})[u'] + 1) = \Xi_{i(n)}^{A_{i(n)}}\lceil(\psi(x_{n,j})[u_n] + 1) = V_{i(n)}\lceil(\psi(x_{n,j})[u_n] + 1)$. Thus $V_{1-i(n)}\lceil(\psi(x_{n,j}) + 1)$ must change between stages u_n and u', as must then $A_{1-i(n)}\lceil(r^{A_{1-i(n)}}(n)[u_n]+1)$. But $A_{1-i(n)}$ cannot change on the interval $(\,r^{A_{i(n)}}(n)[s_n], r^{A_{1-i(n)}}(n)[u_n]\,]$ by the n-module's restraint, so $A_{1-i(n)}\lceil(r^{A_{i(n)}}(n)[s_n]+1)$ must change between stages u_n and u'. The claim now follows by (v). $\Box$

Lemma 4.2 **(n-Module Lemma II)** *Fix $n \in \omega$.*

(i) If the n-module is combined with an m-module (for some $m > n$) then $i(n) \neq i(m)$ and $W\lceil y_n$ (and so also $W\lceil(\varphi(n)+1)$ and $W\lceil(\delta(x_{n,0})+1)$) changes at that stage.

(ii) The n-module is combined with an m-module (for some $m \neq n$) at most finitely often, say never after a (least) stage v_n.

(iii) The n-module acts at most finitely often.

(iv) Whenever the n-module wishes to enumerate some $x_{n,j}$ (for $j \leq k_n$) into B, then at least one such $x_{n,j}$ has not yet been enumerated.

(v) Lemma 4.1 holds for the n-module even without the assumptions (1) and (2) (except at the stages when Step 10 is carried out by the n-module).

(vi) After being combined, the n-module and the m-module work with identical parameters, except that the former reacts to $U(n)$-changes and the latter to $U(m)$-changes.

Proof. We proceed by induction on n. Fix n and assume the lemma holds for all n'-modules for $n' < n$.

We first observe that part of (v) must hold, namely that Lemma 4.1(v) and (vi) holds without the assumptions (1) and (2). Note that when Step 10 is carried out by the n-module then the n-module starts behaving exactly like the m-module (for some $m < n$) with the m-module's parameters, so by induction, Lemma 4.1(v) and (vi) also holds then.

Now we are able to prove our lemma for the n-module.

(i) Suppose the n- and m-modules are combined at a stage u'. First assume, for the sake of a contradiction, that $i(n) = i(m)$. Then the n-module must have an $A_{1-i(n)}$-restraint (i.e., an $A_{1-i(m)}$-restraint) at stage u' stemming from a diagonalization attempt at stage u_n, say, and so by Lemma 4.1(vi) for the n-module and the assumption that the $(\vec{\Psi}, \vec{\Xi})$-length of agreement exceeds x_{m,k_m} at stage u', $W\lceil y_n$ must have changed between stages u_n and u'. Thus either Step 10 was carried out by the n-module or its $A_{1-i(n)}$-restraint at stage u' stemming from stage u_n was canceled.

This establishes $i(n) \neq i(m)$. But then the m-module's $A_{i(m)}$-restraint is injured since the n-module's $A_{i(n)}$-restraint applies, using Lemma 4.1(v). The rest now follows by Lemma 4.1(v) and $y_n \leq \varphi(n)[s_n] \leq \varphi(n), x_{n,0}$.

(ii) By induction on n, we may conclude that the n-module is combined at most finitely often with an m-module (for $m < n$), say never after stage t'. Then after stage t', y_n is fixed. The claim now follows by (i) since $W\lceil y_n$ can change at most finitely often.

(iii) Once the n-module is no longer combined with any m-module, it can clearly act at most finitely often.

(iv) We first observe that it suffices to establish (iv) only in the case that the n-module is never combined with an m-module (for some $m > n$), since (iv) follows by induction on n otherwise. In that case, however, by Lemma 4.1(vi), some $x_{n,j}$ is enumerated into B only after $W \lceil y_n$ has changed; so the claim follows by $y_n \le \varphi(n)[s_n] = k_n$.

(v), (vi) This should now be clear by the construction. $\Box$

Lemma 4.3 **($R_{\vec{\Psi},\vec{\Xi}}$-Satisfaction Lemma)** *(i) If an $R_{\vec{\Psi},\vec{\Xi}}$-strategy (consisting of all its n-modules) acts infinitely often (and is injured at most finitely often) then it will build a total wtt-reduction $\Lambda^W = U$.*

(ii) Each $R_{\vec{\Psi},\vec{\Xi}}$-strategy satisfies its requirement unless a higher-priority requirement shows $U \le_W W$.

Proof. (i) Suppose the $R_{\vec{\Psi},\vec{\Xi}}$-strategy acts infinitely often. By Lemma 4.2(iii), each of its n-modules acts at most finitely often. So the $(\vec{\Psi}, \vec{\Xi})$-length of agreement must have infinite limsup. By Lemma 4.1(vi) (and Lemma 4.2(v)), Λ^W must be correct on its domain; and by Step 8(a) of the construction and the fact that the use $\lambda(n)$ is fixed once defined, Λ^W is also total.

(ii) If the hypotheses of requirement $R_{\vec{\Psi},\vec{\Xi}}$ are satisfied then clearly the $(\vec{\Psi}, \vec{\Xi})$-length of agreement has infinite limsup, and the $R_{\vec{\Psi},\vec{\Xi}}$-strategy will act infinitely often (unless some higher-priority strategy acts infinitely often), so (ii) follows by (i). $\Box$

Lemma 4.4 **(Totality Lemma)** *The functionals $\Gamma^{A_0 \oplus A_1}$, Γ_0^W, Γ_1^W, and Δ^W are all total and correctly compute the sets W, A_0, A_1, and B, respectively.*

Proof. By the construction, all these functionals are correct on their domains. Since the uses of Γ_0 and Γ_1 are never increased once defined, Γ_0^W and Γ_1^W are also total.

Fix $y \in \omega$. Since $\gamma(y)$ is increased only when some $y' \le y$ enters W, the use $\gamma(y)$ will settle down eventually, and $\Gamma^{A_0 \oplus A_1}$ is seen to be total. Since the use $\delta(y)$ is only increased in order to ensure $\delta(y) > \gamma(y)$, the same also holds for Δ^W. $\Box$

Lemmas 4.3 and 4.4 now establish Theorem 1.1. $\Box$

5 Some Corollaries

We turn to the proof of Corollary 1.4. We begin with a Lemma of independent interest.

Lemma 5.1 (Gasarch and Kummer, unpublished) *Let $\leq_T^n$ denote n-query Turing reducibility and $\leq_{tt}^n$ n-query tt-reducibility*[1]*. Suppose that A is m-topped and enumerable. Then for any (not necessarily enumerable) set B we have the following.*

(i) If $B \leq_T^n A$ then $B \leq_{tt}^n A$.

(ii) If h is computable, then $B \leq_T^h A$ implies that $B \leq_{tt} A$. Here $\leq_T^h$ denotes Turing reducibility where, upon input n we are allowed $h(n)$ queries of the oracle. (Hence, in particular, if $B \leq_{wtt} A$ then $B \leq_{tt} A$.)

Proof. We give a proof for completeness. For (i), let $B \leq_T^n A$ via Φ. Say that $\Phi^A(x)$ changes its mind at least y times if there exists a sequence $s_1 < ... < s_y$ of stages such that

$$\Phi_{s_j}^{A_{s_j}}(x) \downarrow \neq \Phi_{s_{j+1}}^{A_{s_{j+1}}}(x) \downarrow .$$

Notice that for all x,

- $\Phi^A(x)$ changes its mind at most $2^n - 1$ many times, and
- $C = \{\langle x, m\rangle : \Phi^A(x) \text{ changes its mind at least } m \text{ times}\}$ is enumerable and $C \leq_T A$.

Now $C \leq_m A$ via some computable function f, as A is m-topped. We use C to show that $B \leq_{tt}^n A$. On input x, compute the least s such that

$$\Phi_s^{A_s}(x) \downarrow = b \in \{0, 1\}.$$

Now use f to compute numbers $q_i = f(\langle x, i\rangle)$ such that $\langle x, i\rangle \in C$ iff $q_i \in A$. Now use $q_1, ..., q_{2^n-1}$ and b to generate a $\leq_{tt}^n$ reduction from B to A. [Use binary

[1]That is, $A \leq_{tt}^n B$ means that there is a procedure Φ total on all oracles and only allows n queries on each computation path, and which computes A from B.

search, first see if $q_{2^{n-1}} \in A$. If not then see if $q_{2^{n-2}} \in A$ and if $q_{2^{n-1}} \in A$ then see if $q_{2^{n-1}+2^{n-2}} \in A$, etc.] The proof of (ii) also follows. $\square$

Now we can prove Corollary 1.4 that no m-topped degree is contiguous.

Proof of Corollary 1.4. Suppose that $\mathbf{a}$ is contiguous. Then it is strongly contiguous by Corollary 1.2. Suppose that additionally $\mathbf{a}$ is m-topped. Let B be any set with $B \leq_T A$. Then $B \leq_{wtt} A$ by strong contiguity. Hence $B \leq_{tt} A$ by Lemma 5.1 (ii). Therefore $\mathbf{a}$ is a enumerable Turing degree containing a largest tt-degree. This contradicts Jockusch [22]. $\square$

Actually, we can improve the above a little. Suppose instead that $\mathbf{a}$ is only tt-topped by A. Then in the proof of Lemma 5.1, for each $\langle x, i\rangle$ we could instead compute a tt-condition $\sigma_{f(\langle x,i\rangle)}$ so that

$$\langle x, i\rangle \in C \text{ iff } A \models \sigma_{f(\langle x,i\rangle)}.$$

Again we can use B and the amalgam of the truth tables $\{\sigma_{f(\langle x,i\rangle)} : i = 1, ..., 2^n - 1\}$ to generate a tt-condition showing $B \leq_{tt} A$. Therefore we obtain the following corollary as well:

Corollary 5.2 *No tt-topped enumerable degree is contiguous.*

6 Basics for Relativized Reductions

In this section we shall look at transfer techniques which apply to enumerable degrees that are not necessarily low$_2$. We shall say that A is C-*wtt* reducible to B, symbolically $A \leq^C_{wtt} B$, if there is a procedure Φ computing A from B with a C-computable use. In some instances this definition is trivially satisfied: for instance, if A and B are enumerable then evidently if $A \leq_T B$ then $A \leq^{\emptyset'}_{wtt} B$. The interest comes from looking at the situation where $C <_T A \leq_T B$. The following is taken from Downey and Stob [17] and shows that distributivity relativizes with enumerable witnesses (rather than only enumerable in C as standard relativization would give).

Theorem 6.1 (Downey and Stob [17]) *Let B, C, A_1, and A_2 be enumerable sets with $B \leq^C_{wtt} A_1 \oplus A_2$. Then there exists an enumerable splitting $B_1 \sqcup B_2 = B$ of B with $B_i \leq^C_{wtt} A_i$ for $i = 1, 2$.*

Proof. Suppose that $\Phi_C(A_1 \oplus A_2) = B$ with use bounded by φ^C. Let $\ell(s)$ denote the associated length of agreement. We will assume that we have enumerations of the relevant sets so fast that $\ell(s+1) > \ell(s)$ for all s, at each stage s, $(\exists y)[y \in A_{1,s+1} \oplus A_{2,s+1} - A_{1,s} \oplus A_{2,s}]$ and the use function φ^C is nondecreasing in both argument and stage number. Now, let x be the least number to occur in $A_{1,s+1} \oplus A_{2,s+1} - A_{1,s} \oplus A_{2,s}$. If x is even, corresponding to enumeration into A_1, set $B_{1,s+1} = B_{1,s} \cup (B_{s+1} - B_{1,s})$ and $B_{2,s+1} = B_{2,s}$. If x is odd, set $B_{2,s+1} = B_{2,s} \cup (B_{s+1} - B_s)$ and $B_{1,s+1} = B_{1,s}$.

Clearly $B_1 \sqcup B_2 = B$. We claim that $B_i \leq_{wtt}^C A_i$ for $i = 1, 2$. To compute $B_i(x)$ first compute relative to C the value of $\varphi^C(x)$ and then a stage $s = s(x)$ at which it applies. Now find the least stage $t > s$ with $A_{i,t}\lceil s = A_i \lceil s$. Then $x \in B_i$ iff $x \in B_{i,t}$. □

Theorem 6.1 allows us to extend the notion of local distributivity to a much wider setting in **R**. Again following Downey and Stob [17], we define an enumerable set A to be C-*contiguous* if for all enumerable sets $B \equiv_T A$, $B \equiv_{wtt}^C A$. Similarly, we define an enumerable set to be *strongly* C-contiguous if for all sets $B \equiv_T A$, $B \equiv_{wtt}^C A$. Theorem 6.1 implies the following

Theorem 6.2 (Downey and Stob [17]) *Suppose that* $\mathbf{a}$ *is* $\mathbf{c}$*-contiguous with* $\mathbf{c} < \mathbf{a}$. *Then* $\mathbf{a}$ *is locally distributive over* $\mathbf{c}$. *That is,*

$$\forall \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}((\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a} \& \mathbf{c} \leq \mathbf{b} \leq \mathbf{a}) \rightarrow \\ (\exists \mathbf{b}_1, \mathbf{b}_2)(\mathbf{b}_1 \cup \mathbf{b}_2 = \mathbf{b} \& \mathbf{b}_i \leq \mathbf{c} \cup \mathbf{a}_i \text{ for } i = 1, 2)).$$

Via index sets, we see that if $\mathbf{a}$ is $\mathbf{c}$-contiguous then $\mathbf{a}'' = \mathbf{c}''$. To finish this section, we prove that the notion has importance for the structure of **R** "higher up".

Theorem 6.3 (Downey and Stob [17]) *There is an enumerable degree* $\mathbf{c} < \mathbf{0}'$ *such that* $\mathbf{0}'$ *is (strongly)* $\mathbf{c}$*-contiguous.*

Proof. In Downey [9], the first author constructed a strongly contiguous degree. In relativized form this result reads:

$$(\exists e)(\forall X)(X <_T W_e^X \& (\forall C)(C \equiv_T W_e^X \rightarrow C \equiv_{wtt}^X W_e^X))$$

Now applying the Jockusch-Shore pseudo-jump theorem (Jockusch and Shore [23]), there is an enumerable set X with $W_e^X \equiv_T \emptyset'$. Then $\emptyset'$ is strongly X-contiguous. $\square$

7 The Nontriviality Theorem

The following result which demonstrates that the relative distributivity is a very widespread phenomenon in $\mathbf{R}$, the enumerable Turing degrees.

Theorem 7.1 (Downey) *Suppose that* $\mathbf{a} < \mathbf{b}$. *Then there exists* $\mathbf{c}$ *with* $\mathbf{a} < \mathbf{c} < \mathbf{b}$ *such that* $\mathbf{c}$ *is strongly* $\mathbf{a}$*-contiguous.*

Note that structurally Theorem 7.1 has the following consequence.

Corollary 7.2 *Degrees locally distributive over some lesser one are dense in* $\mathbf{R}$.

The reader should note that Theorem 7.1 makes for an interesting comparison with the result of Downey-Cholak [6]. There it is shown that if $\mathbf{a} < \mathbf{b}$ then there is a $\mathbf{c}$ with $\mathbf{a} < \mathbf{c} < \mathbf{b}$ and such that one cannot embed the 5 element nondistributive modular lattice 1-3-1 into $[\mathbf{a}, \mathbf{c}]$. The interpretation is that if $\mathbf{c}$ is sufficiently close to $\mathbf{a}$ then $[\mathbf{a}, \mathbf{c}]$ "looks distributive." Theorem 7.1 and Corollary 7.2 provide further evidence for the intuition that $\mathbf{R}$ is much more distributive locally than it is globally.

For completeness we give a proof of Theorem 7.1 in the appendix, Section 8. (No proof of Theorem 7.1 has appeared in the literature. Theorem 7.1 was announced in Downey and Stob [17].)

In this section we prove that Theorem 7.1 applies nontrivially in the sense that not all enumerable degrees are contiguous over some lesser one. Indeed we prove, solving a problem left open in [17], the following theorem.

Theorem 7.3 *There is a nonzero enumerable degree that is not locally distributive over any lesser enumerable degree.*

Proof. We build an enumerable set A and auxiliary enumerable sets $A_i = A_i(\Phi, U), B = B(\Phi, U), i = 0, 1$, and we build procedures Γ_i and Δ (both also depending upon Φ, U) as well as procedures Λ_0 and Λ_1 dependent upon the sequence $\langle \Phi, U, \Psi, \Psi_0, \Psi_1, \Xi_0, \Xi_1, V_0, V_1 \rangle$ to satisfy the requirements below.

$$
\begin{aligned}
R_e = R_{\Phi,U} \quad &: \quad \Phi^A = U \rightarrow \Gamma_0^{A \oplus U} = A_0 \wedge \Gamma_1^{A \oplus U} = A_1 \\
&\qquad \wedge \Gamma^{A_0 \oplus A_1 \oplus U} = A \wedge \Delta^{A \oplus U} = B \\
\widehat{R}_{\langle e,j \rangle} = \widehat{R}_{\Phi,U,\vec{\Psi},\vec{\Xi},\vec{V}} \quad &: \quad [\Phi^A = U \wedge \Xi_0^{A_0 \oplus U} = \Psi_0^{B \oplus U} = V_0 \wedge \\
&\qquad \Phi^A = U \wedge \Xi_1^{A_1 \oplus U} = \Psi_1^{B \oplus U} = V_1 \\
&\qquad \wedge \Psi^{V_0 \oplus V_1 \oplus U} = B] \rightarrow \Lambda_0^U = A \vee \Lambda_1^U = A \\
P_e \quad &: \quad A \neq \{e\}
\end{aligned}
$$

Here all sets and procedures not explicitly built by us are built by our opponent and form the subscripts of $\widehat{R}$. For simplicity in this section we use the convention that the corresponding lower case letter to a procedure is the use, and the use is nondecreasing both in stage number and argument. The reader should note that A must work for *all* pairs Φ, U, whereas A_0, A_1 work only for *one* Φ, U. As an aid to the reader, we are again using the "Chicago convention" that objects denoted by letters near the beginning of the alphabet are built by *us* and objects denoted by letters near the end are built by our *opponent*. The reader might find Figure 2 useful in visualizing the relevant reductions.

We now discuss the strategies to meet the requirements above in isolation. For P_e, we use the canonical Friedberg-Muchnik strategy. That is, we pick a follower y, wait for it to be realized (meaning that $\{e\}(y) = 0$), and then put the follower into A.

For $R_{\Phi,U}$, we will as usual have a node τ on the tree measuring the length of agreement between Φ^A and U, and need to enumerate axioms at τ-expansionary stages. Of course, we can only change an output if a number enters the oracle of the procedure below the use. When there is more than one choice, we update as requested by $\widehat{R}$ living in nodes below τ. This will be done in accordance with the strategy below. Furthermore, τ will enumerate $\gamma(y)$ (when a number y enters A) into A_0 or A_1 depending on which set is currently restrained with higher priority by $\widehat{R}$-strategies below σ working with the same U and Φ.

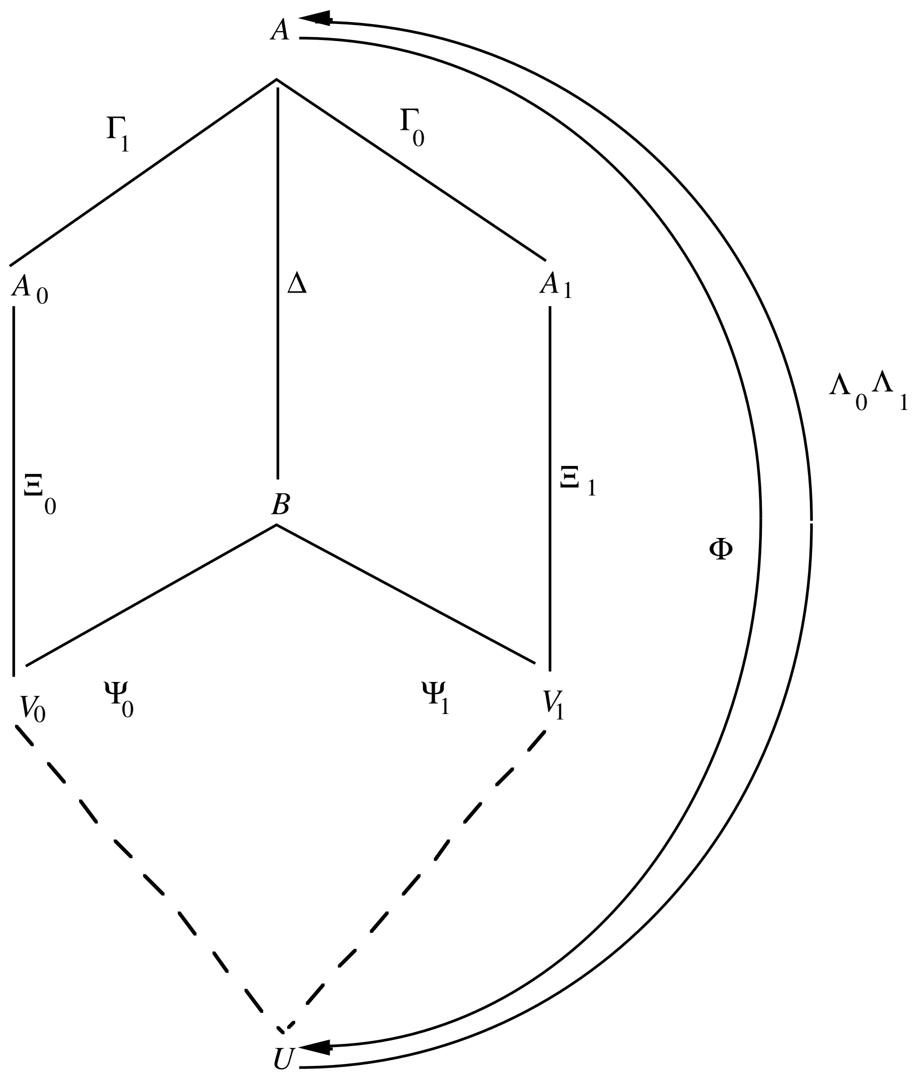

Diagram 1.

Figure 2: Reductions in the Nontriviality Theorem

7.1 Streaming

Before we can explain the basic $\widehat{R}$ strategy, we need to comment on a feature called *streaming* which we will use and which was first introduced in Downey [10] and Downey and Mourad [12]. This is a technique of the first author that can considerably simplify the combinatorics and notation of certain types of constructions.

Streaming is a way of restraining a set when the overall restraint of a strategy σ may tend to infinity and the strategies below still need to enumerate numbers into that set. Instead of restraining an initial segment of the set, streaming rather "thins out" the set of possible numbers entering that set so as to tightly control which numbers may enter. The stream of a strategy σ is thus defined by induction on the length of the node σ: The empty node has all of ω as its stream (i.e., the set of numbers it can enumerate). Given a strategy σ, a finitary outcome o, and σ's stream, we define the stream of $\sigma\hat{\ }o$ to be the part of σ's stream above σ's restraint (so this is initial-segment restraint). On the other hand, if o is an infinitary outcome of σ, then typically σ will enumerate its stream and select an infinite subset of this stream (enumerated in increasing order) as the stream of $\sigma\hat{\ }o$. Whenever $\sigma\hat{\ }o$ is initialized, its stream is also canceled, and we start enumerating a new version of it.

In our construction at hand, we will use streaming as the restraint for the set A in order to make sure that we can make use of every possible A-change. We cannot use traditional initial-segment restraint since this restraint may tend to infinity if some set U can compute A; but in that case, A still has to be made noncomputable and deal with all the other sets of the form U.

On the other hand, we can afford to use traditional initial-segment restraint on the sets A_0, A_1, and B since these will be irrelevant once we show that U computes A.

7.2 The Basic $\widehat{R}$-Strategy

For a single $\widehat{R}$, we have a strategy that works in "cycles" (m, n) as described below. Cycle $(0, 0)$ will start, and each cycle (m, n) may start cycle $(m, n + 1)$ or $(m + 1, 0)$. Node σ devoted to $\widehat{R}$ will have outcomes s, g_1, g_0, and w in descending order of priority. w will denote the *waiting* outcome (in which case some computation of the opponent's does not recover), g_i will denote the *gap* outcomes (in which case $\Lambda_i^U = A$), and s will denote the *stop* outcome (in which case we will have permanently diagonalized).

The idea is roughly as follows. Each cycle (m,n) picks a fresh witness x (at which it is trying to diagonalize B against $\Psi^{V_0 \oplus V_1}$) and then waits for $\Psi^{V_0 \oplus V_1}(x) = 0$ as well as for computations of V_0 and V_1 from A_0, A_1, and B on its use. Now the cycle tries to lift the use $\delta(x)$ via A and A_0 while restraining $V_0 \oplus V_1$ via A_1 and B. For this, the cycle enumerates one single number y_0 into the stream of $\sigma\hat{\ }g_0$, and waits for this number to be enumerated into A. (While waiting, it starts cycle $(m,n+1)$.) Once y_0 has entered A, the cycle initializes $\sigma\hat{\ }g_0$ and its stream, enumerates a single new number y_1 into the stream of $\sigma\hat{\ }g_1$, and again waits for this number to be enumerated into A. (While waiting, it starts cycle $(m+1,0)$.) When y_1 finally enters, our cycle (m,n) is ready to permanently diagonalize by enumerating x into B while using A_1 to record the A-change and restraining V_0 and V_1 via A_0 and (on what is a now relatively small initial segment of) A_1. There is one extra complication in this: When y_0 or y_1 enters A then this allows a U-change which may destroy $\Psi^{V_0 \oplus V_1}(x)$. So each cycle has to protect against such a U-change by defining parts of reductions Λ_0 and Λ_1 from U to A, where the cycles (m,n) (for each fixed m and all n) collectively try to define (a version of) Λ_0, and the cycles (m,n) (for all m and n) collectively try to define Λ_1.

To be more precise, cycle (m,n) proceeds as follows (the next strategy eligible to act is $\sigma\hat{\ }w$ unless specified otherwise below):

1. Pick a witness $x >$ some number y_0 in σ's stream, both above the restraint of higher-priority $\widehat{R}$-strategies working with the same U and Φ and above the restraint of the cycles preceding cycle (m,n) in the lexicographical ordering. (We write this as "cycles $< (m,n)$.)

2. Wait for $\Psi^{V_0 \oplus V_1 \oplus U}(x) = 0$ and $V_i\lceil(\psi(x)+1) = \Psi_i^{B \oplus U}\lceil(\psi(x)+1) = \Xi_i^{A_i \oplus U}\lceil(\psi(x)+1)$ for $i = 0,1$.

3. Set $\Lambda_0^U\lceil(\delta(x)+1) = A\lceil(\delta(x)+1)$ with the use $\delta(y)$ (for arguments y for which cycles $< (m,n)$ have not defined Λ_0^U) equal to σ's restraint r which we define to be the maximum of $\psi_i\psi(x)$ and $\xi_i\psi(x)$ for $i = 0,1$; restrain $A_1\lceil(r+1)$; start cycle $(m,n+1)$; put y_0 into the stream of $\sigma\hat{\ }g_0$; and let $\sigma\hat{\ }g_0$ be eligible to act for one stage.

4. Wait for y_0 to enter A.

5. At the next τ-expansionary stage (where $\tau \subset \sigma$ is the R-strategy working with the same U and Φ), check whether $U\lceil(r+1)$ has changed. If so then return to Step 2, else proceed to Step 6.

6. Lift $\delta(x)$ above r (using the fact that some number $\leq \gamma(y_0)$ just entered A_0) and also above some $y_1 > r$ in σ's stream.

7. Wait for $\Xi^{A_0 \oplus U}\lceil(\psi(x)+1) = V_0\lceil(\psi(x)+1)$.

8. Discard Λ_0 and all cycles $\geq (m, 0)$ except for (m, n) itself; initialize $\sigma\hat{\ }g_0$; drop the cycle's A_1-restraint; set $\Lambda_1^U\lceil(\delta(x)+1) = A\lceil(\delta(x)+1)$ with the use $\delta(y)$ (for arguments y for which cycles $< (m, n)$ have not defined Λ_1^U) equal to σ's restraint r which we define to be the maximum of the old r and and the current $\psi_i\psi(x)$ and $\xi_i\psi(x)$ for $i = 0, 1$ (note that only $\xi_0\psi(x)$ may have changed here); restrain $A_0\lceil(r+1)$; start cycle $(m+1, 0)$; put y_1 into the stream of $\sigma\hat{\ }g_1$; and let $\sigma\hat{\ }g_1$ be eligible to act for one stage.

9. Wait for y_1 to enter A.

10. At the next τ-expansionary stage (where $\tau \subset \sigma$ is the R-strategy working with the same U and Φ), check whether $U\lceil(r+1)$ has changed. If so then return to Step 7, else proceed to Step 11.

11. Enumerate x into B (using the fact that $y_1 < \delta(x)$ just entered A); stop all cycles of σ; and let $\sigma\hat{\ }s$ be eligible to act from now on (since we have permanently diagonalized).

There are four possible outcomes to this strategy:

w and s: There are only finitely many stages at which any cycle of σ acts: In that case, the last cycle to act is permanently stuck waiting at Step 2 or 7 or at Step 11. So $\hat{R}$ is satisfied, the eventual outcome is w or s, respectively, and the effect on the rest of the construction is finitary.

g_0: There is some fixed m_0 such that the cycles (m_0, n) collectively act infinitely often: We will show later that each cycle can only act finitely often since each cycle can only be injured by finitely many numbers from the streams it enumerates for $\sigma\hat{\ }g_0$ and $\sigma\hat{\ }g_1$. Thus each cycle (m_0, n) eventually is stuck waiting at Step 4 and so defines Λ_0 correctly, satisfying R. The restraint on the sets A_1

and B is infinite but they are no longer needed whereas the effect on A is that the stream of possible numbers to enter is thinned out but still infinite.

g_1: There is no fixed m_0 such that the cycles (m_0, n) collectively act infinitely often but all cycles collectively act infinitely often: Then for each m, there is an n_m such that cycle (m, n_m) is eventually stuck waiting at Step 9, and this cycle is the only one of the form (m, n) stuck at Step 9. So each cycle (m, n_m) defines Λ_1 correctly, thus satisfying R. The restraint on the sets A_0, A_1, and B is infinite but they are no longer needed whereas the effect on A is again that the stream of possible numbers to enter is thinned out but still infinite.

We now turn to some formal details, although they are more or less technique.

7.3 The Lists and the Priority Tree

We generate the priority tree PT via lists $L_1(\alpha)$, $L_2(\alpha)$, and $L_3(\alpha)$ as follows. (Intuitively, these are the sets of indices of requirements of the form P, R, and $\widehat{R}$, respectively, yet to be satisfied by nodes $\supseteq \alpha$.)

Initially, $L_i(\lambda) = \omega$ for $i = 1, 2, 3$ (where λ is the empty node).

Inductively, assume that we are at some node α.

If the length of α is a multiple of 3, then assign R_e to α where e is the the smallest member in $L_2(\alpha)$, and give α the outcomes ∞ and f. For $o \in \{\infty, f\}$, set

$$L_1(\alpha\widehat{\ }o) = L_1(\alpha)$$

and

$$L_2(\alpha\widehat{\ }o) = L_2(\alpha) - \{e\}.$$

Finally, set

$$L_3(\alpha\widehat{\ }f) = L_3(\alpha) - \{\langle e, j\rangle \mid j \in \omega\} \text{ and } L_3(\alpha\widehat{\ }\infty) = L_3(\alpha).$$

If the length of α is congruent to 1 mod 3, then assign P_e to α where e is the smallest member in $L_1(\alpha)$. Give α the outcomes 1 and 0. For $o \in \{0, 1\}$, $i \in \{2, 3\}$, set

$$L_1(\alpha\widehat{\ }o) = L_1(\alpha) - \{e\} \text{ and } L_i(\alpha\widehat{\ }o) = L_i(\alpha).$$

Finally, if the length of α is congruent to 2 mod 3, we will assign α to the first available $\widehat{R}$, where we assume that $e < \langle e, j \rangle$, via the list L_3. Thus, we assign $\widehat{R}_{e,j}$ to α where $\langle e, j \rangle$ is the smallest member in $L_3(\alpha)$. We give α the outcomes s, g_1, g_0, f. Now for $o \in \{s, f\}$ and $i \in \{1, 2\}$, we set

$$L_i(\alpha\hat{\ }o) = L_i(\alpha) \text{ and } L_3(\alpha\hat{\ }o) = L_3(\alpha) - \{\langle e, j \rangle\}.$$

For $o \in \{g_1, g_2\}$ and $i \in \{1, 2\}$, set

$$L_i(\alpha\hat{\ }o) = L_i(\alpha) \text{ and } L_3(\alpha\hat{\ }f) = L_3(\alpha) - \{\langle e, j \rangle \mid j \in \omega\}.$$

As is easily seen, each $\widehat{R}_{\langle e,j \rangle}$-strategy σ has a unique R_e-strategy $\tau(\sigma) \subset \sigma$ above.

7.4 The Construction

The construction proceeds in substages $t \leq s$. We will work down the apparent true path TP_s at stage s, making at each substage t a strategy $\alpha \subseteq TP_s$ of length t eligible to act.

We are then left with describing the action of each strategy when it is eligible to act at substage t of stage s. We distinguish cases depending on the type of requirement assigned to the strategy:

Case 1: A P_e-strategy α is eligible to act. Let the streams of $\alpha\hat{\ }1$ and $\alpha\hat{\ }0$ equal the stream of α. Proceed to the first subcase that applies.

Subcase 1.1: α has already enumerated a number into A. Then let $\alpha\hat{\ }1$ be eligible to act next.

Subcase 1.2: α currently does not have a witness y. Then pick an unused witness y in α's stream if possible and let $\alpha\hat{\ }0$ be eligible to act next; else end the stage.

Subcase 1.3: $\{e\}(y) = 0$ for α's current witness y. Then put y into A and let $\alpha\hat{\ }1$ be eligible to act next.

Subcase 1.4: Otherwise. Let $\alpha\hat{\ }0$ be eligible to act next.

Case 2: An $R_{U,\Phi}$-strategy τ is eligible to act. Again let the streams of $\tau\hat{}\infty$ and $\tau\hat{}f$ equal the stream of τ. Denote the length of agreement between Φ^A and U by $l(\tau)$. If s is not τ-expansionary then let $\tau\hat{}f$ be eligible to act next. Otherwise, extend the definitions of $\Gamma_0^{A\oplus U}$, $\Gamma_1^{A\oplus U}$, $\Gamma^{A_0\oplus A_1\oplus U}$, and $\Delta^{A\oplus U}$, setting the use on new arguments to a large number, and lifting the use on old arguments to a large number when requested by $\widehat{R}$-strategies below τ; and let $\tau\hat{}\infty$ be eligible to act next. (For technical reasons, we let the use function δ majorize the use function γ.)

Case 3: An $\widehat{R}$-strategy σ is eligible to act. The strategy works in cycles (m, n). If σ has not been eligible to act (since its most recent initialization) then let cycle $(0, 0)$ act, else let the least cycle (if any) act that can proceed past the current wait. For this cycle, we refer to the description of the basic module in Section 7.2.

At the end of each stage s, we initialize all strategies $> TP_s$ (the longest node eligible to act at stage s).

7.5 The Verification.

It is now not difficult to argue that the construction succeeds similar to the intuitive verification for the basic $\widehat{R}$-module above. Let TP denote the leftmost path visited infinitely often. We argue by simultaneous induction the following.

Lemma 7.4 *Let $\alpha \subset TP$ be a strategy.*

(i) α is initialized at most finitely often.

(ii) α's stream (after α's last initialization) is an infinite computable set, enumerated in increasing order.

(iii) If α is an $\widehat{R}$-strategy then the restraint on A_0, A_1, and B imposed by higher-priority strategies working with the same U and Φ is constant after α's last initialization.

Proof: (i) Routine.

(ii) We proceed by induction on the length of α. The inductive claim is trivial if α is not of the form $\beta\hat{\ }g_i$.

If $i = 1$ then β does not stop (after its last initialization) but has infinitely many cycles stuck at Step 9, each enumerating one number into α's stream in increasing order.

If $i = 0$ then β does not stop or let $\beta\hat{\ }g_1$ act (after α's last initialization) but has infinitely many cycles stuck at Step 4, each enumerating one number into α's stream in increasing order.

(iii) By the definition of the lists, all $\widehat{R}$-strategies $\subset \alpha$ working with the same U and Φ must have outcome s or w along α. $\square$

Lemma 7.5 *Each P-strategy $\alpha \subset TP$ eventually has a permanent witness y and ensures $A(y) \neq \{e\}(y)$.*

Proof: Since α's stream is infinite and no strategy below α is eligible to act until α has a witness, the first half of the claim is clear. The rest is now routine. $\square$

Lemma 7.6 *Let $\tau \subset TP$ be an R-strategy. Then there are three possibilities:*

(i) $\tau\hat{\ }f \subset TP$ and $\Phi^A \neq U$.

(ii) There is an $\widehat{R}$-strategy σ with $\tau \subset \sigma \subset \sigma\hat{\ }g_i \subset TP$ for some $i \in \{0, 1\}$.

(iii) For each j, there is an $\widehat{R}_{e,j}$-strategy σ with $\tau \subset \sigma \subset \sigma\hat{\ }o \subset TP$ for some $o \in \{w, s\}$.

Proof: Routine by the construction and the definition of lists. $\square$

Lemma 7.7 *Let $\sigma \subset TP$ be an $\widehat{R}$-strategy.*

(i) If $\sigma\hat{\ }o \subset TP$ for some $o \in \{s, w\}$ then $\widehat{R}$ is satisfied.

(ii) If $\sigma\hat{\ }g_i \subset TP$ for some $i \in \{0, 1\}$ then $\Lambda_i^U = A$.

Proof: (i) As in the basic module, some cycle of σ must be waiting at Step 2, 7, or 11 permanently, establishing the claim.

(ii) We assume $i = 0$, the other case being similar (and easier). Then starting at some stage s_0, Λ_0 is no longer discarded, and for some fixed m_0, the cycles (m_0, n) collectively define Λ_0^U.

Now for each n, cycle (m_0, n) can be injured (via U) only finitely often, namely only after A-changes via the $n+1$ many y's put into $\sigma\hat{\ }g_0$'s stream by the cycles (m_0, n') for $n' \leq n$. Whenever some such y enters A (via some lower-priority P-strategy) then the cycle (m_0, n') which put this y into the stream will either be able to proceed to Step 6 (contradicting our choice of s_0), or see a U-change allowing the correction of $\Lambda_0^U(n)$. Thus $\Lambda_0^U = A$ as desired. □

The above lemmas now establish Theorem 7.3. □

8 Proof of the Density Theorem 7.1

In this section, for completeness, we will sketch a proof the following density theorem which we restate below.

Theorem 8.1 (Downey) *Suppose that* $\mathbf{a} < \mathbf{b}$. *Then there exists* $\mathbf{c}$ *with* $\mathbf{a} < \mathbf{c} < \mathbf{b}$ *such that* $\mathbf{c}$ *is strongly* $\mathbf{a}$*-contiguous.*

We are given enumerable sets $A <_T B$ and need to construct $C = \cup_s C_s$ to meet the requirements:

$$P_e : \Phi_e(A) \neq C.$$

$$N_e : \Gamma_e(A \oplus C) \text{ total and } \Delta_e(\Gamma_e(A \oplus C)) = A \oplus C \ \rightarrow\ \Gamma_e(A \oplus C) \equiv_{wtt}^{A} A \oplus C.$$

(For this section, it seems more convenient to index reductions via $e \in \omega$. For the N_e we work over all enumerations $\langle \Gamma_e, \Delta_e \rangle$ consisting of two reductions.)

We begin by reviewing the construction of Downey [9] of a strongly contiguous degree. That is the construction with $A = \emptyset$. Let

$$\ell(e, s) = \max\{x : (\forall y < x)(\Delta_{e,s}(\Gamma_{e,s}(A_s \oplus C_s); y) = (A_s \oplus C_s)(y))\}$$

Let m denote the corresponding maximum length of agreement for ℓ. The fundamental idea is the following. At each stage s, when $\ell(e,s) > m(e,s)$, for any number x targeted for C not yet *e-confirmed*, with $\ell(e,s) > x$ we declare x to be e-confirmed and cancel all (followers) y with $x < y < s$ (targeted for C). [At least this is the action with $A = \emptyset$.] Now for all stages $t > s$, we only appoint numbers $\geq t$ as followers. Hence the only numbers we allow to ever enter C below s are $\leq x$, and, indeed are either x or earlier confirmed followers. As usual, followers are appointed "large," that is bigger than any number seen at any preceding stage of the construction. Furthermore we promise that if x enters C then we cancel all numbers (followers) $\geq x$.

- *Note that at stage s, we inductively know that if x_1 and x_2 are followers with $x_1 < x_2$ then x_2 actually exceeds the maximum use of the $\Delta_{e,s}(\Gamma_{e,s}(A_s \oplus C_s); x_1)$ computation.*

Now the idea is that if $\ell(e,s) \to \infty$ then eventually we only enumerate e-confirmed followers into C. Also we promise that at e-expansionary stages (i.e., where $\ell(e,s) > m(e,s)$), we cancel all followers not guessing that $\ell(e,s) \to \infty$. We claim that this makes $\deg(C)$ strongly contiguous.

First $\Gamma_e(A \oplus C) \leq_{wtt} A \oplus C$. To see this given x find the least stage $s = s(x)$ such that some follower $y(x) > x$ is e-confirmed, so that $\ell(e,s) > m(e,s) > y > x$. Now compute the least stage $t > s$ such that $(A_t \oplus C_t)\lceil s = (A \oplus C)\lceil s$, and that $\ell(e,t) > m(e,t)$. We claim that $\Gamma_e(A \oplus C)(x) = \Gamma_{e,t}(A_t \oplus C_t; x)$. By induction, for each e-expansionary stage $s_1 > s$ we know that the only numbers $\leq \gamma_{e,s_1}(A_{s_1} \oplus C_{s_1}; x)$ are followers $\leq s$. [To see this, note that if no number $\leq x$ enters C after stage s we are done since the use is unchanged. If some number $z \leq x$ enters, then the use surely changes, but when the number z enters it cancels all $q \geq z$. Furthermore at the least e-expansionary stage s_2 after the stage at which z enters we would cancel all followers appointed after the stage s_3 at which z entered. (They would have the wrong guess.) In particular at stage s_2 there would be no numbers w left alive with $z \leq w \leq s_2$.] Thus since the only numbers $\leq \gamma_{e,s_1}(A_{s_1} \oplus C_{s_1}; x)$ are followers $\leq x$, we see that $\Gamma_{e,s_1}(A_{s_1} \oplus C_{s_1}; x)$ can only change if some number $\leq x$ and hence $\leq s$ enters C. But then in particular, as $(A \oplus C)\lceil s = (A_t \oplus C_t)\lceil s$, it follows that $\Gamma_e(A \oplus C; x) = \Gamma_e(A_t \oplus C_t; x)$, and hence $\Gamma_e(A \oplus C) \leq_{wtt} A \oplus C$.

To see that $A \oplus C \leq_{wtt} \Gamma_e(A \oplus C)$, given $x \in \omega$, first go to stage x and see if x is a follower. If not then $x \in A \oplus C$ iff $x \in A_x \oplus C_x$. If x is a follower, go to the least stage $s_0 > x$ where $x \in C$, x is canceled, or x is e-confirmed. Assuming the last case, compute the least stage $s_1 > s_0$ where $\Gamma_{e,s_1}(A_{e,s_1} \oplus C_{e,s_1})\lceil s_1 = \Gamma_e(A \oplus C)\lceil s_1$. Then essentially the same reasoning as above will show that $x \in C$ iff $x \in C_{s_1}$, and hence $A \oplus C \leq_{wtt} \Gamma_e(A \oplus C)$.

One organizes the basic module on a Π_2 guessing tree and the construction goes through without any problems.

Turning now to the general case, we have additional problems since the set A is not empty, and is probably not computable. For the requirements

$$P_f : \Phi_f(A) \neq C,$$

we need some sort of Sacks coding strategy (Sacks [31]). We remind the reader of this strategy. The idea is that we have a collection of followers $x = x(f, i, s)$ all targeted for C. The notation is that x is the current "marker" set up to code the atomic fact that "$i \in B$." Let

$$L(f, s) = \max\{z : \forall y < z(\Phi_{f,s}(A_s; y) = C_s(y))\}.$$

A marker for $i+1$ is only picked at a stage when $x(f, i, s)$ is defined and $L(f, s) > x(f, i, s)$. We agree that if $x(f, i, s)$ is defined and $L(f, s) \leq x(f, i, s)$ due to a change in the current A-computation we cancel $x(f, i + 1, s)$, enumerating it into C. (We use the hat convention on all uses.) Note that this means that if $\limsup L(f, s) \to \infty$, yet $\liminf L(f, s) \leq x(f, i, s)$, then $x(f, i + 1, t) \to \infty$ as $t \to \infty$. Moreover this happens computably. (Soare's "Window Lemma.") Finally, while $x(f, i + 1, t)$ is defined and we see i occur in B we promise to put $x(f, i, t)$ into C to try to cause a disagreement. Note that if $\liminf L(f, s) \to \infty$, since we can A-computably recognize if a computation $\Phi_{f,s}(A_s; x(f, i, s))$ is A-correct, it follows that $B \leq_T A$, a contradiction.

For a node γ devoted to P_e, this familiar strategy has for each i, the outcomes (i, ∞) for "unbounded use at $x(e, i, s)$," or (i, f) for "disagreement at $x(e, i, s)$." The former means that the relevant node will encode a computable injury set from the $x(e, j, s)$ for $j \geq i$. The latter means that the node is finitely active. Hence the outcomes for γ will be an ω sequence. And we will have $\gamma\widehat{\ }(0, \infty), \gamma\widehat{\ }(0, f), ..., \gamma\widehat{\ }(i, \infty), \gamma\widehat{\ }(i, f), ...$ on the priority tree.

We turn now to the N_e. The trouble with the N_e is that *A-injury* can occur. (That is, after we have some sort of apparent computation, a relatively small number can enter A (which is more or less out of control) destroying the computation.) The interaction of the A-injury with the N_e is complex, but still essentially familiar. As usual with Π_2 nodes being A-injured we get a $\mathbf{0}'''$ argument. However the injuries are relatively simple. Again we would have followers x waiting to be e confirmed by N_e. However when we e-confirm, we cannot just cancel all $y > x$ since confirmation could be later A-injured. Repeating the cycle infinitely often, x could cause the cancelation of *all* $y > x$. Thus in addition to the usual outcomes of the P_j, we will attach outcomes for the N_e of higher priority. That is if we have some $x(j, i, s)$ which is infinitely often e-confirmed and this e-confirmation is later A-injured, then correspondingly there will be an outcome (e, i, ∞). (Or rather (σ, i, ∞) on the strategy tree.) This outcome corresponds to the fact that $\lim_s x(j, i, s)$ is the witness that we have an A-divergent computation in the hypothesis of N_e and hence at the expense of losing (this version of) P_j we get a *global* win for N_e.

Now all this gives the following timing problem. Imagine a version of N_e at a node σ on the priority tree. Below σ we have nodes τ and γ with γ associated with P_j and τ with P_k. Suppose that both γ and τ extend $\sigma\hat{}\infty$. (As usual, this means that they are guessing that $\ell(e, s) \to \infty$.) Suppose also that τ extends $\gamma\hat{}(i, \infty)$. Now we will, of course, appoint a follower x to τ during a τ and hence $\gamma\hat{}(i, \infty)$-stage. Suppose that we do this and that x is a coding marker for n. Thus $x = x(j, n, s)$. The point is that we may never again have a $\gamma\hat{}(i, \infty)$-stage, but we *need* to build the $\leq^A_{wtt}$ reductions for N_e at σ. Thus in particular, we need to enumerate axioms for x at σ-confirmation stages. Thus we must σ-confirm at the next σ-stage with $\ell(e, s) > x$. (Note here that this is a σ-stage and *not* necessarily a τ-stage.) However it might well be the case that this σ-confirmation is A-incorrect. With a traditional construction, we note that τ is conceivably forever inaccessible since perhaps there is never again a $\gamma\hat{}(i, \infty)$-stage. (Perhaps $\Phi_{j,t}(A_s(x(\gamma, i, s))) \uparrow$ for all stages $t > s$.)

Our solution is to use the *kangaroo* methods of Downey-Stob [18], and "jump" directly from σ to τ. (A different solution to this dilemma can be found in Cholak-Downey [6]. This latter solution involves attaching outcomes for nodes $\eta <_L \beta$ to a node β.) To be precise, suppose that, as above, at some σ-stage we σ-confirm x. But at a later stage, this σ confirmation is A-injured. The we would create a link (σ, τ) and jump directly from σ to τ. At τ we would play the outcome (σ, n, ∞). This draws attention to the fact that x is witnessing a potential global win (by divergence) of the requirement N_σ. Notice that if $\gamma\hat{}(\sigma, n, \infty)$ is leftmost and infinitely often visited this way, then N_e will be met by the divergence at x. Naturally we will need to restart all requirements between σ and τ below the outcome (σ, n, ∞).We can do this via lists in the usual way. There is one problem that all this causes relating to τ-correctness. Between τ and σ there are infinitary nodes such as $\gamma\hat{}(i, \infty)$ which are apparently saying that we should not believe a computation at or below τ until all the relevant numbers they will enumerate clear the relevant uses. These nodes may or may not actually be visited infinitely often and could be totally incorrect advice to τ. This clearly affects the notion of being $\tau\hat{}(\sigma, n, \infty)$-correct.

Accordingly we will attach to the outcome (σ, n, ∞) suboutcomes *corresponding to the* Π_2 *behavior of the nodes between* σ *and* τ. Thus, in the scenario above, there will be nodes $\tau\hat{}(\sigma, n, \infty)\hat{}O_1$ and $\tau\hat{}(\sigma, n, \infty)\hat{}O_2$, with O_1 left of O_2 and saying that $\gamma\hat{}\infty$ puts infinitely many numbers into C and O_2 saying that it only puts finitely many numbers into C. Note that, as with the density theorem of Sacks, A can figure out enough of the construction to decide if, for instance, a confirmation is A-correct or not. This kangaroo methodology is discussed in detail in Downey and Stob [15]. With this modification, the requirements are organized in the usual ways, and the argument goes through in a more or less canonical way. (The full details can be obtained from the first author upon request.)

References

[1] K. Ambos-Spies, Contiguous r.e. degrees, in *Computation and Proof Theory* (eds. E. Börger, W. Oberschelp, M. M. Richter, B. Schinzel and W. Thomas), Springer Lecture Notes in Mathematics No. 1104, Springer-Verlag, Berlin, 1984, pp. 1-37.

[2] K. Ambos-Spies, Antimitotic recursively enumerable sets, Z. Math. Logik Grundlagen Math. **31** (1985), 461-467.

[3] K. Ambos-Spies, D. Ding, and P. Fejer, Embedding lattices preserving 1 below a nonzero recursively enumerable Turing degree, in *Logical Methods,* (eds. J. Crossley, et. al.) Birkhäuser, Boston, 1994, pp. 92-129.

[4] K. Ambos-Spies and P. A. Fejer, Degree theoretical splitting properties of recursively enumerable sets, J. Symbolic Logic **53** (1988), 1110-1137.

[5] K. Ambos-Spies and R. I. Soare, The recursively enumerable degrees have infinitely many one-types, Ann. Pure Appl. Logic **44** (1989), 1-23.

[6] P. Cholak and R. G. Downey, Lattice nonembeddings and intervals in the recursively enumerable degrees, Annals Pure and Applied Logic, **61** (1993), 195-222.

[7] R. G. Downey, Localization of a theorem of Ambos-Spies and the strong antisplitting property, Archiv math. Logik Grundlag. **26** (1987), 127-136.

[8] R. G. Downey, Δ_2^0 degrees and transfer theorems, Illinois J. Math. **31** (1987), 419-427.

[9] R. G. Downey, Subsets of hypersimple sets, Pacific J. Math. **127** (1987), 299-319.

[10] R. G. Downey, A contiguous nonbranching degree, Z. Math. Logik Grund. Math. **35** (1989), 375-383.

[11] R. G. Downey and C. G. Jockusch, Jr., T-degrees, jump classes and strong reducibilities, Trans. Amer. Math. Soc. **301** (1987), 103-136.

[12] R. G. Downey and J. Mourad, Superbranching degrees, *Proceedings Oberwolfach 1989*, Springer Verlag Lecture Notes in Mathematics, 1990, pp. 175-186.

[13] R. G. Downey and J. B. Remmel, Classification of degree classes associated with r.e. subspaces, Ann. Pure Appl. Logic **42** (1989), 105-125.

[14] R. G. Downey and R. A. Shore, Degree theoretical definitions of low_2 recursively enumerable sets, J. Symbolic Logic, to appear.

[15] R. G. Downey and T. A. Slaman, Completely mitotic r.e. degrees, Ann. Pure Appl. Logic **41** (1989), 119-152.

[16] R. G. Downey and M. Stob, Structural interactions of the recursively enumerable W- and T-degrees, Ann. Pure Appl. Logic **31** (1986), 205-236.

[17] R. G. Downey and M. Stob, Splitting theorems in recursion theory, Ann. Pure Appl. Logic **65** (1993), 1-106.

[18] R. Downey and M. Stob, Minimal pairs in initial segments of the recursively enumerable degrees, to appear Israel J. Math.

[19] R. G. Downey and L. V. Welch, Splitting properties of r.e. sets and degrees, J. Symbolic Logic **32** (1986), 137-151.

[20] R. Friedberg and H. Rogers, Jr., Reducibilities and completeness for sets of integers, Z. Math. Logik Grundlagen Math. **5** (1959), 117-125.

[21] M. Ingrassia, *P-genericity for Recursively Enumerable Sets*, Thesis, University of Illinois at Urbana (1981).

[22] C. Jockusch, Relationships between reducibilities, Trans. Amer. Math. Soc. **162** (1969), 229-237.

[23] C. Jockusch and R. Shore, Pseudo-jump operators I : The R. E. case, Trans. Amer. Math. Soc. **275** (1983), 599-609.

[24] A. H. Lachlan, Embedding nondistributive lattices in the recursively enumerable degrees, in *Conference in Mathematical Logic, London, 1970* (ed. W. Hodges), Lecture Notes in Mathematics No. 255, Springer-Verlag, New York, 1972, pp. 149-177.

[25] A. Lachlan, A recursively enumerable degree that will not split over all lesser ones, Ann. of Math. Logic **9** (1975), 307-365.

[26] R. E. Ladner, A completely mitotic nonrecursive recursively enumerable degree, Trans. Amer. Math. Soc. **184** (1973), 479-507.

[27] R. E. Ladner and L. P. Sasso, Jr., The weak truth table degrees of recursively enumerable sets, Ann. Math. Logic **8** (1975), 429-448.

[28] M. Lerman and J. B. Remmel, The universal splitting property, II, J. Symbolic Logic **49** (1984), 137-150.

[29] P. Odifreddi, Strong reducibilities, Bull. (New Series) Amer. Math. Soc. **4** (1981), 37-86.

[30] P. Odifreddi, *Classical recursion theory*, North-Holland, Amsterdam, 1990.

[31] G. Sacks, The recursively enumerable degrees are dense, Annals of Math. (2) **80** (1963), 211-231.

[32] T. Slaman, The density of infima in the recursively enumerable degrees, Annals Pure and Applied Logic, **52** (1991), 155-179.

[33] R. I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, New York, 1987.

[34] M. Stob, Wtt-degrees and T-degrees of recursively enumerable sets, J. Symbolic Logic **48** (1983), 921-930.