

# EFFECTIVELY COMPACT SPACES

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ABSTRACT. We give a systematic technical exposition of the foundations of the theory of effectively compact metric spaces. We discover several new characterizations of effective compactness and apply these characterizations to prove new results in computable analysis and effective topology. We also apply the technique of effective compactness to give new and less combinatorially involved proofs of known results from the literature. Some of these results do not have effective compactness or compact spaces in their statements, and thus these applications are not necessarily direct or expected.

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## 1. INTRODUCTION

1.1. **Compactness.** Compactness plays a central role in classical analysis. We don't have space to talk about all the applications of compactness in analysis, but refer the reader to the survey [52] for a detailed discussion. Compactness allows us to have an intrinsic connection between the infinite and the discrete finite instances if problems. In the words of Hewitt [52],

- "A great many propositions of analysis are:
- trivial for finite sets;
  - true and reasonably simple for infinite compact sets;
  - either false or extremely difficult to prove for noncompact sets."

If the reader thinks about any basic course in analysis, they will be struck about how many elementary theorems about real analysis rely on compactness in some form or another. When we turn to analysis on more general spaces, again compactness plays a central role. Classically, compact metric spaces are perhaps the most well-understood separable spaces after the discrete ones. Metrizable compact manifolds and compact topological groups have been studied extensively.

1.2. **Goals.** The concern of this paper is computable analysis; specifically computability aspects of Polish spaces (complete separable metric spaces). In this paper we will hope to achieve three goals.

- We will give a unified and smooth account of uses of compactness in computable analysis. This will involve the *unification* of a number of disparate approaches offered by many authors down through the years. We will describe and improve the fundamental techniques associated with effective compactness that are scattered throughout the literature.
- We will apply this machinery to prove several new results. This includes a characterization of recursive profinite groups in terms of effective compactness, a new computation of Čech cohomology, and a computable version of the Hausdorff-Alexandroff Theorem (saying that every non-empty compact Polish space is the continuous image of  $2^\omega$ ). We will also give new simplified proofs of some known results (more details below).
- We will offer an answer to the following question:

"What is the 'correct' notion of computability for a compact Polish(able) topological space?"

In contrast with the situation for computable *discrete* algebra, the situation for Polish spaces is not clear *especially if we are willing to view them up to homeomorphism*.

1.3. **An historical context.** Whilst the roots of computable analysis go back to Borel [10], and others who were well-aware of an intuitive notion of "explicit" or algorithmic procedures in analysis. In his seminal paper [127], Alan Turing gave the first universally accepted definition of a computable function. Turing used this definition to solve the Hilbert's *Entscheidungsproblem*. But Turing also introduced computable analysis on  $[0, 1]$ . He analysed computable functions on the field of computable real numbers. He defined a real  $\xi$  to be computable if there is a Turing machine that, on input  $i$ , outputs a rational  $r = \frac{m}{n}$  such that  $|\xi - r| < 2^{-i}$ . This approach was pursued especially in Russia by Markov and his school, culminating in Aberth's book [1]. Strangely, Turing's definition of a computable function (i.e. on the computable reals) is now usually referred to as *Markov computability*. Our paper lies in the tradition of what has become known as "type 2" computable analysis. This tradition goes back to the work of Grzegorzczuk [47–49] and Kleene [71]. In this approach, we view effective functions as those as computable *operators* that are not restricted to computable reals and work for arbitrary reals. Avigad and Brattka [4] give an excellent overview of the development of computable analysis from the work of Turing.

The majority of early research was restricted to computability on the real line and in  $\mathbb{R}^n$ . In these spaces, the rationals and the tuples of rationals can be used to define computable points. This idea can be extended to more general spaces, as follows. For an abstract Polish space, we fix a dense sequence and require that we have a distance

function which is a computable on the dense set. For example, as above, we could use rational polynomials in  $C[0, 1]$ . (Precise definitions will be found in Section 2.) Within this setting, computability-theoretic aspects of metric and normed spaces have also been studied for many decades, some of the earlier references include [99, 109, 110].

Computability-theoretic aspects in the Euclidean and the totally disconnected ultrametric cases, particularly Cantor space  $2^\omega$ , have been of central importance. The Euclidean case – in the sense that the space is actually a subset of  $\mathbb{R}^n$  for some  $n$  – is treated in Pour-El and Richards [111], Ko [73], Braverman and Yampolsky [18], and Weihrauch [130]. The ultrametric case – and more specifically the study of effectively closed subsets of Cantor space – is a well-developed classical subject of computability theory with surveys such as [22, 23]. This subdivision is of course a bit artificial since Cantor space can be viewed as a subset of  $[0, 1]$ , but the metric will no longer be an ultrametric.

In the past decade or two there has been an increasing interest in the computability-theoretic aspects of abstract metric spaces. The central questions in such investigations include:

- When does a space admit a computable presentation, and in what sense is it computable?
- Can we compute certain invariants or objects associated with this space (e.g., the space of probability measures on the space), and if ‘yes’ then in what sense?
- Can we establish computable analogues of the classical topological results?
- Can we classify computable points in computable topological spaces?
- Can we classify computably presentable spaces in a given class? etc.

The computability-theoretic study of abstract metric and topological spaces is developing hand in hand with work on reverse mathematics [97], algorithmic randomness [55], enumeration degree theory [106], and (to some extent) effective descriptive set theory [96]. The notion of a computable presentation of a space is central in such investigations. Many classical spaces such as  $2^\omega$  and  $L^2[0, 1]$  are equipped with a ‘natural’ computability structure which is usually *fixed*; the theory is then developed for the fixed computability structure. The two classical texts [111, 130] essentially take this approach. Even though both books talk about ways to compare different computability structures on a fixed space, the space under consideration is usually equipped with a few ‘natural’ computability structures that can be compared. For example, in  $C[0, 1]$  one could use rational polynomials or, alternatively, rational piecewise linear functions; these turn out to be equivalent (in a rather strong sense). But of course, not every space has a computable presentation simply from cardinality-theoretic observations. Can we describe those spaces that do admit computable presentations, at least from some common classes? For instance, for which compact Polish  $K$  does the Banach space  $C(K)$  have a computable presentation? What about the space of probability measures on  $K$ ? etc.

To attack these and similar questions we will often have to depart from classical computable analysis (that deals with fixed ‘natural’ computable presentations) and use methods of computable topology. Although we can point at earlier initiatives such as, e.g., [65–67], [102, 103], and [123–125], most of the related work in computable topology is more recent and includes [54, 68, 76, 78, 128, 131]. Computable topology is notorious for its zoo of various notions of computability for a topological space. In contrast with effective algebra [3, 37] where all standard notions of computable presentability had been separated more than half-a-century ago (e.g., Feiner [38]), some of the key notions of computable presentability in topology have been separated only very recently [8, 51, 54, 82]; these results will be discussed in detail later in the paper.

In the recent years there has been a tendency to focus on the three main notions of computable presentability of a (compact) Polish space: a computable metric space, a computable topological space, and an effectively compact space. Our paper is focused on one of these three important notions of computable presentability, namely *effective compactness* that is defined and discussed below. Effective compactness is clearly restricted to compact Polish spaces. Nonetheless, we will see that the notion and the techniques associated with it have far-reaching applications in computable analysis that are not restricted to compact spaces.

**1.4. Effective compactness.** Recall that we mentioned that the notion of a computable Polish space, or a computably metrized space, seems to be the most well-established notion of computable presentability for a Polish(able) space. The early classical works on computable metric spaces include Ceitin [21] and Moschovakis [95]. A Polish space is *computable* or *computably metrized* if there is a complete, compatible metric  $d$  and a dense subset of *special* or *ideal* points  $(x_i)_{i \in \omega}$  of the space such that  $d(x_i, x_j)$  are computable reals uniformly in  $i$  and  $j$ . (The metric  $d$  is usually assumed to be complete. If we view spaces up to isometry, we fix the metric; if we study them up to

homeomorphism then we assume  $d$  is compatible.) However, the issue is that in a computably metrized compact space, we do not necessarily have *computable* access to its finite covers.

Classical uses of compactness do not need an understanding of how finite covers are obtained. For classical purposes, it is sufficient that the finite covers exist. Thus, when we consider computability aspects of compact spaces, it is natural to quantify what we mean by this. There are many definitions of a space being effectively compact throughout the literature. Remarkably, as we prove below, they – as well as some new useful ones – are all equivalent. For instance, Mori, Tsujii, and Yasugi [94] say that a computably metrized space is *effectively compact* if there is a computable function which takes  $n$  and produces a finite  $2^{-n}$ -cover of the space by open balls centred in special points and having rational radii. (Such balls are called basic.)

The notion has proven to be extremely useful, and the techniques associated with effective compactness tend to be elegant. Indeed, it is not uncommon that a tedious and technical proof in computable analysis becomes transparent and compact (pun intended) after a thoughtful application of effective compactness.

In spite of the usefulness of effective compactness and its numerous applications in the literature, it seems there is no “standard” reference that would contain the most fundamental results and techniques associated with effective compactness. Even though these are some excellent papers and Ph.D. theses written on related subjects (e.g., [60, 93, 105]), it is very difficult to find a detailed and systematic exposition of many fundamental aspects of the theory, let alone a self-contained and more or less complete one. Many results and proofs are scattered throughout the literature. Some of the most standard references cover only the Euclidean or ultrametric special cases or are missing proofs. As a result, it seems that some fundamental facts about effectively compact spaces keep being rediscovered over and over again. Proofs of some other results in the literature (including some recent ones) can be significantly simplified via choosing a more careful set-up in which effective compactness can be used to simplify combinatorics. It seems that some of the standard techniques associated with effective compactness are not necessarily uniformly known, and perhaps even that the theory itself is a bit under-appreciated. Thus, as mentioned above, our *first main goal* of this article is to fill this apparent gap in the literature, at least partially. Once we accumulate enough techniques and develop new ones, we will apply this machinery to prove new results and improve known proofs; this is our *second main goal*. Recall that our *third goal* is to try to suggest a correct notion of computable compact Polish space. The potential candidates for the ‘correct’ computability notion include: a computable topological space, a computably metrized space, and effectively compact space, and some other perhaps more exotic notions – such as a right-c.e. metrized space – that can be found in the literature and some of which will be mentioned later. We suggest that the following might be true:

Effective compactness is the right notion of computability for compact Polish spaces.

Even if the reader will disagree with this thesis after looking at the results that we present here, the definition of effective compactness is certainly *robust*. More formally, Theorem 1.1, below, contains seven equivalent formulations of effective compactness some of which are *new*. Many of the applications that we discuss in this article – perhaps most notably the recently discovered effective Stone and Pontryagin dualities – strongly suggest that our thesis should not be dismissed even if we view spaces up to homeomorphism. We will also explain why all three standard definitions of computable presentability for a compact Polish space – effectively compact, computably metrized, computable topological – differ up to homeomorphism.

Before we proceed, we should admit that giving a complete and comprehensive survey of the existing literature and results on the subject is *not* among the main goals of this article, but nonetheless we will provide many useful references. This is not a survey paper in the usual sense, it is mainly a technical semi-survey paper with many new results, and it should be treated as such. We also chose to spread further discussion and references to the literature throughout the paper (where it is relevant) rather than to write a giant introduction.

**1.5. The Main Theorem.** The following theorem will be proven over the subsequent sections. The fundamentals of effective compactness theory will be developed simultaneously with the proof. We will discuss each clause of the theorem in detail shortly; for now we note that (iii), (iv), and perhaps most notably (vii) are *new*.

**Theorem 1.1.** *For a computably, completely metrized Polish space  $M$ , the following are equivalent:*

- (i) Given  $n$ , we can effectively compute (the finite set of parameters describing) a finite  $2^{-n}$ -cover  $K_n$  of  $M$  by basic open balls.
- (ii) We can effectively enumerate all finite basic open covers (each given at once as a finite set of parameters) of the space.
- (iii) Same as (i), but additionally in each finite cover  $n$  we can uniformly decide whether two basic open balls  $B, B' \in K_n$  intersect.
- (iv) In the notation of (iii), we can additionally uniformly decide (non)emptiness of intersection for any  $B, B' \in \bigcup_{n \in \omega} K_n$ , but balls may have merely computable radii (and, thus, are not necessarily basic).
- (v) There is a computable sequence of computable reals  $(\epsilon_n)_{n \in \omega}$  such that  $\epsilon_n \leq 2^{-n}$  so that, for every  $n$ , we can compute the maximal number of points in the space that are at least  $\epsilon_n$ -far from each other.
- (vi)  $M$  is computably homeomorphic to a computable closed subset of the Hilbert cube.
- (vii)  $M$  is a computable surjective image of  $2^\omega$ .

As we will note later, it is easy to see that (ii) is also equivalent to the approach that is standard throughout reverse mathematics (Simpson [119]): there exists an enumeration functional that, given a countable cover composed of basic open balls outputs (an index of) some finite subcover of the cover. This approach is perhaps the most familiar one to a working mathematician, while (i) is rather an effective analogy of total boundedness. Of course, a complete metric space is compact iff it is totally bounded, but this elementary fact is perhaps not quite as well-known as the standard definition of compactness.

In view of this theorem, we have the following definition, which is a characterization established in this paper:

**Definition 1.2.** A Polish space is called *effectively compact* if it satisfies (the conditions in) Theorem 1.1.

The equivalence of (i) and (ii) is well-known, and usually one of the two is taken as the standard definition of an effectively compact space. See Theorem 3.3.

It seems that (iii) or something similar might be also known, but perhaps in some other form; e.g., see [54] for some related informal discussions. Also, Section 8.4 of [60] contains a discussion of spiritually similar results. However, we were unable to find any proof of the equivalence of (i) and (iii) anywhere in the literature. For a proof, see Theorem 3.10.

As far as we know, (iv) is new. Characterization (iv) will be very useful in several applications greatly reducing the combinatorial complexity of proofs in many cases. It will be especially useful in computing cohomology groups of spaces, but some further (perhaps, less expected) applications will also be presented. This equivalence is stated and then proven in Theorem 3.13.

Iljazovich [58] was the first to discover the equivalence of the fifth formulation (v) with the standard definition; it has recently been re-discovered in [107]. Its significance is the remarkable fact that effective compactness is an isometric invariant of the space. In other words, effective compactness is not a property of some specific nice presentation, but it is an intrinsic property that holds for *all* isometric presentations. The result will appear as Theorem 3.18.

The sixth version (vi) is well-known, we are not sure who was the first to observe its equivalence to effective compactness. See Theorem 3.33 for a proof.

Finally, the last item (vii) of the theorem is an effective version of the classical Hausdorff-Alexandroff Theorem; see Theorem 3.37. It seems to be new, although one important special case of it was discovered by Day and Miller [27] several years ago. Also, Couch, Daniel, and McNicholl [26] proved the result for the special case of closed subspaces of  $\mathbb{R}^n$ . Interestingly, in Remark 3.23 of his large unpublished survey “Algorithmic randomness, martingales and differentiability” Jason Rute refers to this property (being a computable image of  $2^\omega$ ) as being a stronger version of effective compactness<sup>1</sup>. And indeed it may seem at first glance that it should be stronger. Our new computable Hausdorff-Alexandroff theorem will be rather useful in several applications that will be discussed in due course. We also give two substantially different proofs of the result, one using (iv) discussed above, and the other one using  $\Pi_1^0$  classes and the Hilbert cube. Both proofs serve as a fine illustration of the techniques that we develop in the paper.

<sup>1</sup>As of 2022, the survey is still available at the personal homepage of Jason Rute.

The study of effectively compact spaces is very closely related to the investigation of effectively closed subsets of computably metrized Polish spaces, especially when the set happens to be computable closed. As we will see in Proposition 3.26, under some mild restrictions computable closed sets can be viewed as effectively compact spaces, and vice versa by *(vi)* of the theorem above. We cite [60] for an excellent recent survey. The cited survey however does not really have many proofs or proof sketches, so we felt that including proofs should be a good idea; this is done in Subsection 3.4 that includes the necessary facts that we will need in the present paper.

Another classical and closely related subject in computability theory is the area of  $\Pi_1^0$  classes. This area can be viewed as a special case of the theory of effectively closed sets but restricted to  $2^\omega$ . Of course, more can be said about  $2^\omega$  than about an arbitrary space. Unlike the theories of effectively compact spaces and computable closed sets, there is no shortage of excellent surveys and papers about  $\Pi_1^0$ -classes (e.g., [24, 25, 28]), and the draft of a book “Effectively Closed Sets  $\Pi_1^0$  Classes” by Cenzer and Remmel that is available online (as of early 2022). Thus, we will not include many proofs, we just state a few results in Subsection 2.6 that we will need, and will give references.

These three topics – effectively compact spaces, computable closed sets,  $\Pi_1^0$ -classes – are closely related, and no firm line can be drawn between them.

**1.6. Summary of applications.** We now discuss several applications (of effective compactness) that can be found in Section 4. We are mainly focused on the applications that are either new, or give new proofs of known results. We also mention several applications that are very recent and are related to our research interests. We also pose several open questions.

Our list of applications is not even close to being exhaustive, but we will discuss the literature where more results of this sort can be found. Here is a summary:

- (1) In Subsection 4.1 the reader can find several useful standard results most of which are at least half-a-century old. They serve as a mere illustration of some of the basic techniques.
- (2) The next Subsection 4.2 contains an unpublished result of Nies and Melnikov that states that  $\Pi_1^0$  classes can be used to represent isometric isomorphisms between effectively compact spaces. The result is not difficult, but its consequences are fairly powerful; in particular, combined with several standard results about  $\Pi_1^0$ -classes, this method gives elegant and much more ‘compact’ proofs of some results from the literature.
- (3) Subsection 4.3 contains an application of effective compactness to constructing basic sequences in Banach spaces. The application is elementary but is neat. The subtlety is that, in classical Banach space theory, one routinely uses dual spaces and Hahn–Banach Theorem to construct basic sequences (e.g., [20]), but it is known that Hahn–Banach Theorem is not computable [14, 92]. Effective compactness gives a way to circumvent this obstacle. The result is very recent and can be found in Long’s M.Sc. thesis [81].
- (4) The next Subsection 4.4 applies the techniques developed in Section 3 to Stone spaces; these results are very recent and can be found in [51, 54]. For instance, we will see that a Stone space is effectively compact iff it is computably metrizable iff the dual Boolean algebra is computably presentable. Among many applications, we will explain why the isomorphism problem for Stone spaces is  $\Sigma_1^1$ -complete. The result is ‘known’ but it seems it has never been stated in the literature; we include it for future reference. We will also see that these techniques can be used to produce an example of a computable topological Polish space not homeomorphic to any computably metrized space.
- (5) In Subsection 4.5 we prove that a profinite group is recursively presented (in the sense of [79, 121]; to be defined) iff it is effectively compact; this result is new.
- (6) Subsection 4.6 contains a new algorithm for computing Čech cohomology of an effectively compact space. The algorithm is new, but the result itself is not new (though it is very recent [82]). In the subsection we also discuss several applications of Čech cohomology in computable topology.
- (7) Subsection 4.7 applies computability of Čech cohomology established in the previous subsection to produce examples of computably metrized compact spaces that are not *homeomorphic* to any effectively compact space. It is not hard to find a computably metrized space that is not *isometrically* isomorphic to any

effectively compact space (just take the interval  $[0, \Omega]$ , where  $\Omega$  is Chaitin's omega<sup>2</sup> or some other left-c.e. real that codes  $0'$ .) However, the situation becomes more complex if we view spaces up top homeomorphism. The result is not new but is very recent, and the proof that we give is a new combination of modern and classical techniques some of which we introduce in the preceding subsections. Our new proof is perhaps the simplest one known so far.

- (8) Subsection 4.8 contains a new proof of computable universality of  $C[0, 1]$  among computable Polish spaces up to (computable) isometry. The issue is that the standard proofs of universality of  $C[0, 1]$  rely on Hahn–Banach Theorem; as we have already mentioned above, it is not computable in general. Sierpinski [118] suggested a more direct proof that he thought was ‘effective’; however, his proof gives a merely  $0'$ -computable embedding. We use tools of effective compactness to produce a computable embedding of any computable Polish space to the standard presentation of  $C[0, 1]$ . The result is not new ([5]), but the proof that we here give is new. Our new proof is *much* less combinatorially involved than the one in [5]; the latter does not use effective compactness tools. (However, it is not necessarily clear that our proof holds primitively recursively, while the proof from [5] gives a primitive recursive embedding.)
- (9) In Subsection 4.9 we prove that every effectively compact space of finite covering dimension can be computably embedded into a finitely dimensional Euclidean space. This is an improved version of a very recent result of Harrison-Trainor and Melnikov [50] that establishes that there is an arithmetical embedding. Our result is stronger and the technique that we use is different from what has been used in [50]. The new version heavily relies on one of the new characterizations of effective compactness that we prove in the paper. It will allow us to effectivize one of the standard proofs from the classical literature with only minor modifications.
- (10) Subsection 4.10 contains the proof of the fact that, for an effectively compact  $X$ , the space of probability measures  $\mathcal{P}(X)$  on  $X$  is a computable homeomorphic image of  $2^\omega$ . This is known, even though the standard reference [27] does it only for the special case of  $X = 2^\omega$  and via an explicit construction of a computable map from  $2^\omega$  onto  $\mathcal{P}(X)$ . But it is actually easier to establish effective compactness of  $\mathcal{P}(X)$  directly (using covers), and *then* apply (vii) of Theorem 1.1. In this sense, our approach is new. Effective compactness of  $\mathcal{P}(X)$  can be used to show that a compact incomputable group is effectively compact iff it admits a computable Haar probability measure. This is a known result and we will discuss it more fully in Subsection 4.10.
- (11) The final Subsection 4.11 contains several open questions that are related to the material contained in the previous subsections. Most of these questions are directly or indirectly related to compactness.

To make our exposition smoother, we shall often define notions when we need them. The most commonly known basic notions of computable metric space theory can be found in the preliminaries.

## 2. PRELIMINARIES

All of our spaces are Polish (separable and completely metrizable) spaces. Such spaces are also sometimes called Polishable. All spaces are also compact, unless stated otherwise. There will be only very few exceptions towards the end of the paper where the spaces considered will not be compact, namely the Urysohn space and the space of continuous functions on the unit interval  $C[0, 1]$ .

We will almost never consider the empty space, even though it is actually possible to include this rudimental case into our framework. However, many proofs become more uniform and definitions more convenient if we exclude this case.

We remind the reader that a real  $\alpha$  is computable if there exists a computable sequence  $(q_{f(i)} \mid i \in \mathbb{N})$  of rational numbers such that  $|\alpha - q_{f(i)}| < 2^{-i}$ . If we have a computable sequences of rationals but only know that  $q_{f(i)} \rightarrow \alpha$ , but not a computable modulus of convergence, then we would say that  $\alpha$  is a  $\Delta_2^0$ -real; and if the sequence  $q_{f(i)}$  is monotonically increasing (resp. decreasing), then  $\alpha$  is said to be left-c.e. or lower semi-computable (resp. right c.e. or upper semi-computable).

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<sup>2</sup>While it is not central to this paper,  $\Omega$  is the Lebesgue measure of the domain of a universal prefix-free Turing machine (see Downey and Hirschfeldt [29]). It is a “natural” example of a left c.e. real which is not computable, in the same way that the halting problem is a natural example of a c.e. non-computable set.

Much work in computable analysis from recent years has been concerned with the theory of representations [130]. In the type 1, countable case, when we talk about functions acting on, for example, polynomials, we really mean functions acting on numbers or strings “representing” the objects. In the type 2 case, a representation is a way of assigning an infinite string  $\alpha$  in Baire space  $\omega^\omega$  with the object we wish to run algorithms upon; and to do so in a computationally meaningful way. However, in the case of Polish spaces, Cauchy sequences provide a natural and effective way to represent elements. Thus, we stick throughout with the notation as presented in the subsections below.

**2.1. Effective metrizations of Polish spaces.** A Polish space  $(M, d)$  is *right-c.e. presented* or *admits a right-c.e. metric* if there exists a sequence  $(\alpha_i)_{i \in \omega}$  of  $M$ -points which is dense in  $M$  and such that for every  $i, j \in \omega$ , the distance  $d(\alpha_i, \alpha_j)$  is a right-c.e. real, uniformly in  $i$  and  $j$ . (In particular, we always assume that the metric is complete.) More formally, there is a c.e. set  $W \subseteq \omega^2 \times \mathbb{Q}$  such that for any  $i$  and  $j$ ,

$$\{q \in \mathbb{Q} : d(\alpha_i, \alpha_j) < q\} = \{q : (i, j, q) \in W\}.$$

Note that the sequence  $(\alpha_i)_{i \in \omega}$  may contain repetitions; equivalently, it is possible that  $d(\alpha_i, \alpha_j) = 0$  for some  $i, j$ . We call points  $\alpha_i$  from the sequence *special* or *ideal*. For instance, an undirected (simple) graph with the shortest path metric is a right-c.e. metrized space. We will see that  $\Pi_1^0$  classes can also be viewed as right-c.e. metrized spaces.

The definition of a left-c.e. Polish space is obtained from the notion of a right-c.e. Polish space using the notion of a left-c.e. real, mutatis mutandis.

**Definition 2.1.** A Polish space is *computably presented* or, perhaps more descriptively, *computably metrizable* if there is a (complete) metric on the space which is both right-c.e. and left-c.e.

Strictly speaking, a computable or a right-c.e. metrization of a space is a countable object  $(\alpha_i)_{i \in \omega}$ , but we will usually identify a computable metrization  $(\alpha_i)_{i \in \omega}$  of space  $M$  with its completion  $\overline{(\alpha_i)_{i \in \omega}}$ .

**Remark 2.2.** Note that we intentionally did not emphasise whether we consider Polish spaces up to isometric isomorphism or under some other notion of similarity, such as, e.g., quasi-isometry or homeomorphism. Indeed, these will lead to non-equivalent notions. For example, for a real  $\xi$ , the space  $[0, \xi]$  is isometrically isomorphic to a computably metrized space if, and only if,  $\xi$  is left-c.e. However, for any real  $\xi$  this space is homeomorphic to the unit interval  $[0, 1]$  which is of course computably metrizable. In this paper we usually consider Polish spaces under homeomorphism, that is, a Polish space has a right-c.e. presentation if it is *homeomorphic* to the completion of a right-c.e. metrized space. Nonetheless, we will emphasise this in most of the theorems and lemmas that we prove to make sure that there is no conflict of terminology.

**2.2. Computable topological spaces.** There are several definitions of a computable topological space that can be found in Kalantari and Weitkamp [65] and Spreen [123]. We will use the following.

**Definition 2.3** (see, e.g., Definition 2.1 of [75] or Definition 4 of [131]). A *computable topological space* is given by a computable, countable basis of its topology for which the intersection of any two basic open sets (“basic balls”) can be uniformly computably listed. More formally, it is a tuple  $(X, \tau, \beta, \nu)$  such that

- $(X, \tau)$  is a topological  $T_0$ -space,
- $\beta$  is a base of  $\tau$ ,
- $\nu: \omega \rightarrow \beta$  is a computable surjective map, ( $i$  is called an *index* of  $\nu(i)$ ) and
- there exists a c.e. set  $W$  such that for any  $i, j \in \omega$ ,

$$\nu(i) \cap \nu(j) = \bigcup \{\nu(k) : (i, j, k) \in W\}.$$

Let  $(X, \tau, \beta, \nu)$  be a computable topological space. For  $i \in \omega$ , by  $B_i$  we denote the open set  $\nu(i)$ . As usual, we identify basic open sets  $B_i$  and their  $\nu$ -indices. There are many versions of this notion above in the literature; see, e.g., [123]. All of these notions are essentially Definition 2.3 with some extra assumption. For example, one can also additionally require that there is a computable dense sequence  $(x_i)_{i \in \omega}$  such that  $\{\langle i, j \rangle : x_i \in B_j\}$  is computably enumerable. See [103, 123] for *many* other extra assumptions, some of which definitely seem ad hoc. We thus stick with the basic Definition 2.3. Perhaps, the most natural examples of a computable topological Polish space is given by the proposition below.

**Proposition 2.4** (cf. Theorem 2.3 of [75]). *Every right-c.e. Polish space is a computable topological space.*

*Proof.* Let  $(M, d)$  be a right-c.e. Polish space, and let  $(\alpha_i)_{i \in \omega}$  be its sequence of special points. By  $\tau$  we denote the metric topology of  $(M, d)$ . As usual, the base  $\beta$  of  $\tau$  contains basic open balls

$$B(\alpha_i, q) = \{x \in M : d(\alpha_i, x) < q\}, \quad i \in \omega, q \in \mathbb{Q}^+.$$

For  $i \in \omega$  and  $q \in \mathbb{Q}^+$ , we put  $\nu(i, q) = B(\alpha_i, q)$ .

We prove that the tuple  $(M, \tau, \beta, \nu)$  is a computable topological space. It is sufficient to establish the following: for any  $i, j \in \omega$  and  $q, r \in \mathbb{Q}^+$ , we can (uniformly) effectively enumerate a set  $X \subseteq \omega \times \mathbb{Q}^+$  such that

$$(1) \quad B(\alpha_i, q) \cap B(\alpha_j, r) = \bigcup \{B(\alpha_k, t) : (k, t) \in X\}.$$

Our set  $X$  is defined as follows:  $X$  contains all pairs  $(k, t)$  such that

$$d(\alpha_i, \alpha_k) < q - t \text{ and } d(\alpha_j, \alpha_k) < r - t.$$

Since the space  $(M, d)$  is right-c.e., it is not hard to see that the set  $X$  is c.e., uniformly in  $i, j, q, r$ . If  $(k, t) \in X$ , then by using the triangle inequality, we can easily show that  $B(\alpha_k, t)$  is a subset of  $B(\alpha_i, q) \cap B(\alpha_j, r)$ .

Let  $x$  be an arbitrary point from  $U = B(\alpha_i, q) \cap B(\alpha_j, r)$ . Choose positive rationals  $\epsilon$  and  $\delta$  such that  $\epsilon < q - d(\alpha_i, x)$  and  $\delta < r - d(\alpha_j, x)$ . Since  $U$  is open, we can find  $k \in \omega$  and  $t \in \mathbb{Q}^+$  such that  $x \in B(\alpha_k, t) \subseteq U$  and  $t < \min(\epsilon/2, \delta/2)$ . Then we have

$$d(\alpha_i, \alpha_k) \leq d(\alpha_i, x) + d(\alpha_k, x) < (q - \epsilon) + t < q - \epsilon/2 < q - t.$$

Therefore,  $(k, t)$  belongs to  $X$ , and the set  $X$  satisfies (1). Hence,  $(M, \tau, \beta, \nu)$  is a computable topological space.  $\square$

For instance, every computably metrized Polish space is a computable topological space.

**2.3. The definition of effective compactness.** We have already explained what it means for a Polish space to be computably metrized. We usually assume that all our spaces are Polish metric and non-empty. Recall that a complete metric space  $M$  is compact iff for every  $\epsilon > 0$ , there exists a finite set  $F$  of points such that every point has distance less than  $\epsilon$  to  $F$ . For now, we say that a space is effectively compact if it satisfies (i) of Theorem 1.1:

**Definition 2.5.** A computably metrized space is called *effectively compact* if there exists a computable function that, given  $n$ , outputs the index of a finite tuple of basic open balls of radii  $< 2^{-n}$  that cover  $M$ .

We should explain a bit more carefully all the terms that we use in the definition above. A ball centred in a special point is *basic* if its radius is a rational number. When we consider finite covers, we usually say that we can *compute* a finite cover by basic open balls if we can compute the index of a finite set that codes the indices of the finitely many centres and the rational radii of basic open balls in the cover. This should not be confused with *enumerating* a finite cover, i.e. listing one ball after another in a c.e. fashion.

**Remark 2.6.** Most proofs in this article do not actually need the radii to be rational numbers, but would work with balls of radius a computable real. However, we cannot list computable reals effectively. Therefore, we cannot hope to have an effective base of topology consisting of all basic balls with computable radii and computable points.

Suppose a compact space is computably metrized. How much computational power do we need to make it effectively compact?

**Definition 2.7.** Let  $\mathcal{M} = (M, d, (p_i)_{i \in \mathbb{N}})$  be a compact computable metric space. A *compactness modulus* of  $\mathcal{M}$  is any function that bounds

$$h(n) = \min\{j : \forall i \exists k < j \ d(p_i, p_k) \leq 2^{-n}\}$$

from above. We call  $h$  the least modulus of compactness.

Note that if  $h(n) = j$ , then the  $2^{-n+1}$ -basic open balls centred in  $p_0, \dots, p_j$  cover the space. Since  $d(p_i, p_k) \leq 2^{-n}$  is a  $\Pi_1^0$  condition and the quantifier  $\exists k < j$  is bounded, and the space is compact,  $h$  is computable relative to  $0'$ . It is not difficult to show that there exists a computably metrized compact space in which the least modulus of compactness computes  $0'$ , and indeed, any modulus of compactness as well. As we mentioned in the introduction, the interval  $[0, \Omega]$  defines a computably metrized space that is not isometrically isomorphic to any effectively compact space (by Theorem 3.18), and modulus of compactness for any computable presentation computes  $0'$ .

It what will follow, we will not necessarily need *the* (least) modulus of compactness. Indeed, it is sufficient to calculate *some* (and not necessarily the least)  $j$  such that  $\forall i \exists k < j d(p_i, p_k) \leq 2^{-n}$ . One way to state this would be to require  $j$  to be ‘the first found’ (in some  $\Delta_2^0$  approximation sense) that works. A space is effectively compact if for every  $n$  we can compute *some*  $j$  that works. It is not difficult to manufacture a pathological example of a space where some  $j$  can be computably found for a given  $n$ , but one cannot compute the least such  $j$ . (See the next subsection for a similar counterexample.)

#### 2.4. More about basic open balls.

**Notation 2.8.** For a basic open  $B$ , write  $B^c$  for the basic closed ball with the same centre and radius as  $B$ .

The reader should keep in mind that the space can be very strange, quite unlike  $\mathbb{R}^n$ . For instance, the closure  $\overline{B}$  of  $B$  does not have to be equal to  $B^c$  in general (think of an isolated point in  $B^c \setminus B$ ). Also, in general we cannot decide whether two basic open or closed balls intersect or not, as is illustrated by the example below.

**Example 2.9.** *There exists an effectively compact subspace of the unit square such that there is no uniformly computable procedure deciding whether two given basic open or basic closed  $2^{-n}$ -balls intersect.*

To make sure that the non-emptiness of intersection of open balls is undecidable, for every  $n$  create a gadget consisting of two points  $x_n$  and  $y_n$  at distance  $2^{-n} - 2^{-n-2}$  from each other, and also put a third point  $z_n$  at distance exactly  $2^{-n}$  from each of  $x_n$  and  $y_n$ . The point is at the intersection of the  $2^{-n}$ -circles centred at  $x_n$  and  $y_n$ .

- Wait for the  $n$ th potential procedure to declare that  $B(x_n, 2^{-n}) \cap B(y_n, 2^{-n}) = \emptyset$ .
- If this ever happens at some stage  $s$ , take  $m = s + n + 1$  and put a new point  $w_n$  at distance exactly  $2^{-m}$  from  $z_n$  so that  $z_n \in B(x_n, 2^{-n}) \cap B(y_n, 2^{-n})$ .

To make sure that the non-emptiness of intersection of basic closed balls is undecidable, for every  $n$  create a similar gadget, but this time keep  $z_n$  out of the space at every finite stage. Instead, initiate the enumeration of a sequence  $(\xi_{i,n})$  of points in the complement of  $B^c(x_n, 2^{-n}) \cup B^c(y_n, 2^{-n})$  rapidly converging to  $z_n$ , i.e.,  $d(z_n, \xi_{i,n}) = 2^{-n-i}$ . At stage  $s$ , put  $\xi_{s,n}$  into the space.

- Wait for the  $n$ th potential procedure to declare that  $B^c(x_n, 2^{-n}) \cap B^c(y_n, 2^{-n}) \neq \emptyset$ .
- If this ever happens at some stage  $s$ , stop putting points  $\xi_{s,n}, \xi_{s+1,n} \dots$  into the space.

It should be clear that, for each gadget, the diagonalization works. We also note that the gadgets are uniformly effectively compact. We can fit all these gadgets into the unit square and get an effectively compact metric space (of rank 2) with the desired property.  $\square$

Thus, we see that the non-emptiness of set-theoretic intersection of basic open balls is not c.e. in general. We will return to this issue in Subection 3.2 where we will see that there are enough balls for which this property actually is decidable; and hence we can get a characterization of effective compactness where the basic balls used in covers have decidable intersections.

A similar example can be produced to show that inclusion is also not c.e. in general. The following stronger notion is c.e.; it will be very useful throughout the paper. We write  $r(B)$  for the radius of a basic ball  $B$  and we use  $\text{cntr}(B)$  to denote its (distinguished) center.

**Definition 2.10.** A basic open ball  $U$  is said to be *formally included* into a basic open  $W$  if

$$r(U) + d(\text{cntr}(U), \text{cntr}(W)) < r(W).$$

This notion has been around for many decades; see, e.g., [123] where it is called strong inclusion. If the centres and the radii are computable (not necessarily special and rational, respectively), formal inclusion remains c.e. The same can be said about formal  $s$ -disjointness defined as follows.

**Definition 2.11.** Two basic open balls  $U$  and  $W$  are *formally  $s$ -disjoint* if  $r(U) + r(W) < d(\text{cntr}(U), \text{cntr}(W))$  and this can be seen after calculating the radii and the distance with precision  $2^{-s}$ . We say that  $U$  and  $W$  are *formally disjoint* if they are formally  $s$ -disjoint for some  $s$ .

We note that strong inclusion remains c.e. in the context of right-c.e. metric spaces, while strong disjointness remains c.e. in left-c.e. metric spaces.

**2.5. Effectively continuous maps.** Let  $X$  be a computable topological space. For a point  $x \in X$ , its *name* is the set

$$N^x = \{i \in \omega : x \in B_i\}.$$

We say that a map  $f : X \rightarrow Y$  between computable topological spaces  $X$  and  $Y$  is *computable* if there exists an enumeration operator that, given the name of  $x \in X$ , outputs the name of  $f(x) \in Y$ .

An *open name* of an open set  $U \subseteq X$  is a set  $W \subseteq \omega$  such that

$$U = \bigcup_{i \in W} B_i.$$

**Definition 2.12.** Let  $X$  and  $Y$  be computable topological spaces. A function  $f : X \rightarrow Y$  is *effectively continuous* if there is a c.e. family  $F \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$  of pairs of (indices of) basic open sets such that:

- (C1) for every  $(U, V) \in F$ , we have  $f(U) \subseteq V$ ;
- (C2) for every point  $x \in X$  and every basic open  $E$  in  $Y$  such that  $f(x) \in E$ , there exists a basic open  $D$  in  $X$  with  $(D, E) \in F$  and  $x \in D$ .

The elementary fact below is well-known and can be traced back to, e.g., [21]. In this specific form it can be found in [86]. The lemma essentially says that a map is computable if, and only if, it is effectively continuous.

**Lemma 2.13.** *Let  $f : X \rightarrow Y$  be a function between computable Polish spaces. Then the following conditions are equivalent:*

- (1)  $f$  is effectively continuous.
- (2) There is an enumeration operator  $\Phi$  that on input an open name of an open set  $V$  in  $Y$  lists an open name of the set  $f^{-1}(V)$  in  $X$ .
- (3) There is an enumeration operator  $\Psi$  that given the name of a point  $x \in X$ , enumerates the name of  $f(x) \in Y$ .

(We remark that the proof below works for right-c.e. spaces. It also works for computable topological spaces with c.e. formal (strong) inclusion that can be defined abstractly without any reference to a metric; see, e.g., [86, 123].)

*Proof.* (1)  $\Rightarrow$  (2). Suppose  $V = \bigcup_{i \in W} B_i$ . Note that (C2) implies that

$$f^{-1}(V) = \bigcup \{D \in X : (D, E) \in F \ \& \ \exists i \in W \ E \subseteq_{form} B_i\},$$

and thus the name of  $f^{-1}(V)$  can be listed using only positive information about  $W$ , with all possible uniformity.

(2)  $\Rightarrow$  (3). Note that  $B \in N^{f(x)}$  if and only if  $f^{-1}(B)$  contains a basic open set in  $N^x$ .

(3)  $\Rightarrow$  (1). Define a collection  $F$  of pairs  $(D, E)$  of (indices of) basic open sets in  $X \times Y$  as follows. Fix a basic open  $E$  in  $Y$ . Enumerate all basic open  $D$  in  $X$ , and for each such  $D$ , enumerate all finite collections  $D, A_1, \dots, A_k$  of basic open sets (in  $X$ ) such that  $D \subseteq_{form} \bigcap_{i \leq k} A_i$  (meaning that  $D$  is formally contained in each  $A_i$ ). Feed these finite collections to  $\Phi$  and wait for some  $E$  to be enumerated in the output. When  $E$  is enumerated (if ever), put  $(D, E)$  into  $F$ .

We claim that  $F$  defined above satisfies (C1) and (C2). We check (C1). If  $(D, E) \in F$  then  $f(D) \subseteq E$ . Indeed, suppose  $d \in D$ . There exists a sequence  $D, A_1 \dots A_k$  such that  $\Phi^{\{D, A_1, \dots, A_k\}}$  enumerates  $E$ . Recall  $D \subseteq_{form} \bigcap_i A_i$  implies  $D \subseteq \bigcap_i A_i$ , thus for any  $d \in D$  the sequence listed by  $\Phi^{N^d}$  will contain  $E$ , and therefore  $f(D) \subseteq E$ . We now check (C2). Fix  $x \in X$  and a basic open  $E \ni f(x)$ . We must show that for some basic open  $D \ni x$ ,  $(D, E) \in F$ . By assumption,  $\Phi^{N^x}$  enumerates  $N^{f(x)}$  that contains  $E$ . Suppose  $E$  is listed with use  $A_1, \dots, A_k$ . Since the  $A_i$  all contain  $x$ , there exists a basic open  $D \ni x$  that is formally included into their intersection. Since the operator uses only positive information about its oracle, it will list  $E$  on input  $\{D, A_1, \dots, A_k\}$  as well, and thus  $(D, E)$  will be enumerated into  $F$ .  $\square$

**2.6.  $\Pi_1^0$ -classes.**  $\Pi_1^0$  classes will be important for some of the work to follow. Thus we give a brief reminder of the basic definitions and results.

We fix the standard computable presentation of  $2^\omega$  under the usual shortest common initial segment ultra-metric. The space  $2^\omega$  can be viewed as  $[2^{<\omega}]$  the set of infinite paths through the complete binary tree, so points are paths. A closed subset  $C$  of  $2^\omega$  is called a  *$\Pi_1^0$  class* if we can computably enumerate the basic clopen sets whose union make up the complement of  $C$  in  $2^\omega$ . That is,  $C$  is the set of paths through a computable subtree of  $2^{<\omega}$ .  $\Pi_1^0$  classes

are also defined in Baire space  $\omega^\omega$  where they are called *computably bounded*  $\Pi_1^0$  classes  $C$ , although in practice the computably bounded is usually understood. Here we ask that  $C = [T]$ , where  $T$  is a computable subtree of  $\omega^{<\omega}$ , that is, there is a computable function  $f : \omega^{<\omega} \rightarrow \omega^{<\omega}$ , where  $f(\sigma)$  bounds the number of immediate extensions of  $\sigma$  ( $T$  is a computably branching tree). It is not difficult to show that if  $C$  is a computably bounded  $\Pi_1^0$  class in this latter sense, there is a  $\Pi_1^0$  class  $\hat{C}$  in  $2^\omega$  whose members are in effective 1-1 correspondence (and indeed have the same  $m$ -degree) as those of  $C$ . (This statement will be stated formally and verified later in the paper.) Thus we usually need only concern ourselves with  $\Pi_1^0$  classes in  $2^\omega$ .

A moment's thought reveals that in computable mathematics  $\Pi_1^0$  classes occur everywhere. One of the fundamental correspondences are  $\Pi_1^0$  classes and degrees of theories, pioneered by Jockusch and Soare in the early 1970's [62–64], and even earlier by Kreisel [77] and Shoenfield [117]. That is,  $\Pi_1^0$  classes effectively correspond to (completions of) axiomatizable theories under Stone duality.

We shall need the following elementary fact:

**Fact 2.14.** An isolated point in a  $\Pi_1^0$ -class is computable.

Thus, for instance, if a  $\Pi_1^0$  class is countable, it must have a computable point. Another well-known but less elementary fact that we will refer to is the following result.

**Theorem 2.15** (The Low Basis Theorem [64]). *A non-empty  $\Pi_1^0$  class contains a member  $P$  of low Turing degree, that is  $P' \equiv_T \emptyset'$ .*

Since we are concerned with effective compactness and  $\Pi_1^0$  classes are co-c.e closed subsets, it seems reasonable to see what effective compactness means in this context. In the context of  $\Pi_1^0$  classes, effective compactness seems an often re-discovered concept, under various names. Let  $C = [T]$  be a  $\Pi_1^0$  class. Of course there are many computable  $T \subseteq 2^{<\omega}$  for which  $C = [T]$ , but for each we can look at  $\text{Ext}(T) = \{\sigma \in T \mid \exists \alpha \in 2^\omega (\sigma \alpha \in C)\}$ , the set of *extendible* nodes of  $T$ . If  $C = [T] = [\hat{T}]$  then  $\text{Ext}(T) = \text{Ext}(\hat{T})$ , so this set is an invariant. We can associate a degree  $\text{deg}_T(\text{Ext}(T))$ , as the Turing degree of  $C$ . We will call this  $\text{deg}(C)$ , and the reader should note that since  $\text{Ext}(T)$  is a co-c.e. set  $\text{deg}(C)$  is c.e..

The situation of when  $\text{deg}(C) = \mathbf{0}$  is of interest to us, and this has been given several names historically, *recursive*, *recursively closed*, *decidable*, e.g. LaRoche [80] and Downey [31].

**Proposition 2.16.** *Let  $C$  be a  $\Pi_1^0$  class. Then  $\text{deg}(C) = \mathbf{0}$  iff  $C$  is effectively compact.*

In this context, effective compactness of  $C$  is in fact equivalent to saying that  $C$  possesses a computable dense subset; this is because  $2^\omega$  is effectively compact. We will prove a more general result later. For now, we give a brute-force proof of the lemma.

*Proof.* Let  $C = [T]$  with  $\text{Ext}(T)$  a computable set. Given  $n$  we need to be able to compute a  $2^{-n}$  cover. Compute  $E_n = \{\sigma \mid \sigma \in \text{Ext}(C) \wedge |\sigma| = n + 1\}$ . Let  $E_n = \{\tau_1, \dots, \tau_k\}$ . For each  $i$  we can use the fact that  $\text{Ext}(C)$  is computable to calculate the leftmost extension  $\alpha_i$  of  $\tau_i$  in  $C$ , and then the balls of radius  $2^{-n}$  around the  $\alpha_i \cap C$  cover  $C$ .

We need to show that  $C$  has a computable dense set. We build  $D = \{\alpha_i \mid i \in \omega\}$  in stages. For each  $\alpha_i$  we need a Cauchy sequence for it. At stage  $n$  we will compute the nodes in  $T$  up to length  $n + 1$ . We begin with  $n = 0$ . Since we are assuming  $C$  is nonempty, one of  $\langle 0 \rangle, \langle 1 \rangle$  is in  $\text{Ext}(T)$ . If only one of these  $\langle i \rangle \in \text{Ext}(T)$ , then let  $\alpha_0$  have  $\langle i \rangle$  as the first element in its Cauchy sequence, and if both are extendible, let  $\alpha_0$  have  $\lambda$  (the empty string) as its first element. Suppose at step  $n - 1$  we have generated  $\alpha_0, \dots, \alpha_j$  with partial Cauchy sequences  $X_0, \dots, X_j$ . For each such Cauchy sequence,  $X_d$  look at the longest member  $\sigma_d$ . If  $\sigma_d$  has length  $n$  then proceed as we did with and if both are extendible in  $C$ ,  $X_d$  remains the same, and if not then add  $\sigma * \langle i \rangle$  to  $X_d$ . If  $\sigma_d$  is shorter than  $n$ , look at all the extensions of  $\sigma_d$  of length  $n + 1$ . By assumption (since  $\sigma_d$  has length below  $n$ ), all extensions of  $\sigma_d$  of length  $n$  are extendible in  $C$ . Working from right to left, let  $\tau_1, \dots, \tau_k$  list the the shortest extensions of  $\sigma_d$  of length  $\leq n$  where all the extensions of length  $n + 1$  are extendible. We look at these in left to right ordering. If  $\tau_1$  has length  $< n$  then we extend the Cauchy sequence for  $\alpha_d$  to be  $\tau_1$ . If  $\tau_1$  has length  $n$  then we let the extendible  $\tau_1 * \langle i \rangle$  be the new element of  $X_d$ . Then for all  $\tau_j$  for  $j \neq 1$  we start a Cauchy sequence for a new real  $\alpha_q$  with first entry  $X_q$  being  $\tau_j$  using the same method. Then the sequence  $\alpha_j$  define a computable collection of computable reals, forming a dense sequence.

Conversely, if  $C = [T]$  has a computable dense sequence, then  $\sigma \in T$  is in  $\text{Ext}(T)$  iff some member of the computable dense sequence extends it. This can be discovered once we know the computable dense sequence to precision  $2^{-(|\sigma|+1)}$ .  $\square$

Many other facts about  $\Pi_1^0$  classes can be found in Cenzer [22], Cenzer and Remmel [23] and Chapter 2 of Downey and Hirschfeldt [29].

### 3. THE DEFINITION OF EFFECTIVE COMPACTNESS IS ROBUST

**3.1. The other two standard definitions of effective compactness.** Our convention is that all of our spaces are nonempty Polish spaces. Definition of effective compactness says that for every  $n$  we can compute a cover of the space by basic  $2^{-n}$ -balls. This definition seems a weak form of compactness, as it seems that having a fixed cover for each  $2^{-n}$  does not seem quite as good as having access to all finite covers. The following definition would seem to give a stronger notion of effective compactness.

**Definition 3.1.** We define computably metrized space to be *\*-effectively compact* if the collection of *all* finite covers of  $M$  by basic open balls can be given as a c.e. collection of explicit finite sets.

We also note that Definition 3.1 is equivalent to:

**Definition 3.2.** We say that a computably metrized space is computably countably compact if there is a partial computable operator that on input any potential c.e. open basic cover outputs halts if it is a cover and some finite sub-cover.

It is easy to see that a space is effectively \*-compact iff it is computably countably compact. To see why computable countable compactness implies \*-effective compactness, we can enumerate all finite collections of basic open balls and apply the algorithm: if the procedure halts output its subcover. This enumerates a collection of finite covers, and to enumerate them all, we consider the union of this collection with the collection of all finite sets of balls. The other direction is also straightforward: if the space is \*-effectively compact, given a c.e. cover wait till we enumerate a finite subcover. These versions of countable compactness is essentially the approach used in reverse mathematics (e.g., Simpson [119]).

Interestingly, the two potential definitions suggested above (and a few more) turn out to be equivalent.

**Theorem 3.3 (Folklore).** *For a computably metrized (compact) Polish space  $M$ , TFAE:*

- (1)  $M$  is effectively compact;
- (2)  $M$  is \*-effectively compact.

*Proof.* The implication (2)  $\rightarrow$  (1) is obvious.

Assume (1), we prove (2). Take a finite collection  $(B_i)$  of basic open sets and assume it is a cover. We must argue that eventually we will be able to effectively recognise that it is indeed a cover. The idea is that there exists an  $\epsilon = 2^{-n}$  so small that every  $\epsilon$ -cover of  $M$  is formally contained in this given cover. (This will be the Lebesgue number of the cover, in particular.) This will also be true for the  $\epsilon$ -cover that will be given to us according to the definition of effective compactness. Since formal inclusion is c.e., we will be able to recognise that this formal inclusion has occurred. Noting that formal inclusion does imply set-theoretic inclusion, so if some  $\epsilon$  cover is formally included in some other finite collection of basic open balls, then this other collection must also be a cover. Thus, if we succeed, it will show that  $(B_i)$  is equivalent to saying that, for some  $n$ , every ball in the  $2^{-n}$ -cover given to us by the definition of effective compactness (and indeed, any other  $2^{-n}$ -cover) is formally included in one of the  $B_i$ . This is, of course, a  $\Sigma_1^0$ -property.

It remains to prove that such an  $\epsilon$  exists. We argue non-computably. Let  $c_i$  be the center of  $B_i$ , and  $r_i$  be its the radius. Define for every  $i$ , a function  $f_i(x) = r_i - d(x, c_i)$  if  $x$  is in the ball  $B_i$ , and 0 otherwise. This function is continuous. Now take the supremum of the finite family  $(f_i)$  to define a new continuous  $g$ ;  $g(x) = \sup_i f_i(x)$ . If  $(B_i)$  indeed was a cover, then the function  $g$  would be strictly positive, because each  $x$  is inside one of the  $B_i$ .

Let  $v$  be its infimum that is achieved somewhere, by compactness. Take a rational  $\epsilon = 2^{-m}$  less than  $v/2$ . Then for every point  $y$ , we have  $\epsilon < r_i - d(y, c_i)$ ; that is

$$d(y, c_i) + \epsilon < r_i,$$

equivalently,  $B(y, \epsilon) \subset_{form} B_i$ . This inclusion will still hold if we replace  $\epsilon$  with an even smaller  $\epsilon'$ . Thus, in particular, every basic open  $\epsilon'$ -ball is formally included in one of the  $B_i$ . Consequently, (1) implies (2).  $\square$

**Remark 3.4.** The proof of (1)  $\rightarrow$  (2) above additionally tells us that, for any given finite basic cover there is an  $\epsilon$  small enough so that any  $\epsilon$ -cover formally refines the given cover. Also note that to recognize formal inclusion in a c.e. way, we do not need the radii  $r_i$  to be rational numbers; (uniformly) computable  $r_i$  will suffice.

In view of Lemma 3.3, henceforth we use effectively compactness and  $*$ -effectively compactness interchangeably, and without further comment.

*Elementary properties of effectively compact spaces.* Examples of effectively compact spaces are the unit interval  $[0, 1]$ , the unit circle that can be viewed as the set of complex numbers having norm one:  $\{\xi \in \mathbb{C} : \|\xi\| = 1\}$ , the Hilbert cube, cantor space  $2^\omega$ , and also ‘natural’ (rational) geometric realisations of finite simplicial complexes that are central to algebraic topology. Simplicial complexes will be used as a tool later in the paper, and indeed will be discussed in the next subsection. We shall give much more intricate examples of effectively compact spaces in due course.

There are several properties of effectively compact spaces that are immediate from the definitions. These, for instance, include those summarised in the following:

**Proposition 3.5.** (1) *Let  $f : M \rightarrow \mathbb{R}$  be computable and  $M$  be effectively compact. Then  $\sup_{x \in M} f(x)$  and  $\inf_{x \in M} f(x)$  are computable real numbers. Furthermore, this is uniform.*

(2) *The class of (non-empty) effectively compact spaces is closed under taking (finite or computably infinite) direct products. More specifically, if  $(M_i)_{i \in I}$  is a uniformly computable sequence of spaces, where  $I \in \omega \cup \{\omega\}$ , then the direct product*

$$\prod_{i \in I} M_i$$

*under (say) the metric*

$$\sum_{i \in I} 2^{-i} \frac{d(x_i, y_i)}{1 + d(x_i, y_i)},$$

*where  $x_i$  denotes the  $i$ th projection of  $x \in \prod_{i \in I} M_i$ , is an effectively compact metric space.*

We omit the elementary proof. We remark that in (2), the choice of a dense computable sequence is not canonical. One way of choosing a dense computable sequence it is to fix *some* (e.g. the first found) sequence of special points  $\alpha$  in the product, and then using elements that are ‘eventually  $\alpha$ ’. That is the dense subset will be given by the collection of sequences of special points that are equal to  $\alpha$  for cofinitely many coordinates (projections). There are other potential metrics that we can use instead of the one suggested above, but the natural choices will be effectively equivalent (meaning that the identity map will be computable with respect to one and the other metric under consideration).

**3.2. Deciding the intersection.** The results in this section appear to be new as stated, however, similar arguments and some informal explanations closely related to what we do here can be found in the literature (e.g., in [54]). One standard way of using a (finite) cover of a compact space in dimension theory and algebraic topology is to use Alexandroff’s notion of a *nerve*.

**Definition 3.6** (Alexandroff [2]). *A nerve of a cover is a simplicial complex in which the faces are the collections of basic open sets that have a non-trivial intersection; i.e., each basic open set is a 0-dimensional simplex (a node), and balls  $\{B, C, D\}$  form a 2-dimensional face if  $B \cap C \cap D \neq \emptyset$ .*

From the computability-theoretic standpoint, the issue with this definition is that, for a fixed finite open cover, the non-emptiness of each specific intersection is merely  $\Sigma_1^0$ ; recall Example 2.9 in the preliminaries. However, if we choose our covers very carefully, we can use special covers where we can decide intersections. Our next result shows that in an effectively compact space we can find one such nice  $\epsilon$ -cover for every  $\epsilon \in \mathbb{Q}$ . To state the result formally, we push the notion of effective compactness to its limits.

**Definition 3.7.** A set of basic open balls is  $\cap$ -decidable if for every pair of balls  $B_0, \dots, B_i$  from the set, we can computably decide whether  $\bigcap_i B_i = \emptyset$ .

Before we proceed, we state and prove one elementary but important lemma. Recall that, for a basic open  $B$ , we write  $B^c$  for the basic closed ball with the same centre as  $B$ , and that the closure  $\overline{B}$  of  $B$  does not have to be equal to  $B^c$  in general.

**Lemma 3.8.** *Suppose  $M$  is effectively compact. Then, for basic closed balls  $B_i^c$  and  $B_j^c$ , the property  $B_i^c \cap B_j^c = \emptyset$  is c.e. uniformly in  $i, j$ . The same is true for any finite collection of basic closed balls.*

*Proof.* The open set  $M \setminus B_i^c$  is c.e. Indeed, we just list all the basic open balls that are formally disjoint from  $B_i^c$  via a standard argument<sup>3</sup>. Thus, the union of the complements, which is the complement of the intersection  $B_i^c \cap B_j^c$ , is also c.e. open. It covers the space if, and only if, the intersection is empty. By effective compactness of  $M$ , this is c.e. The case of finitely many balls is similar.  $\square$

**Definition 3.9.** A computably metrized (compact)  $M$  is *nerve-decidable*, or *\*\*-effectively compact*, if for every  $n > 0$  we can computably find a finite  $2^{-n}$ -cover  $K_n$  (represented as a finite tuple of basic open balls) of  $M$  so that  $K_n$  is  $\cap$ -decidable uniformly in  $n$ .

**Theorem 3.10.** *A computably metrized  $M$  is nerve-decidable (\*\*-effectively compact) if, and only if, it is effectively compact.*

*Proof.* Obviously, \*\*-effectively compactness implies effective compactness. To this end, we assume effective compactness of  $M$ . We will use the equivalence of effective compactness and \*-effective compactness throughout the rest of the proof without explicit reference.

We need to show that, for every  $\epsilon > 0$ , there exists a finite basic open  $\epsilon$ -cover  $K$  of the space. Fix a finite  $\epsilon/2$ -cover of the space by basic open balls, and replace each ball in the cover with a  $\epsilon$ -ball with the same centre. Let  $S$  be this new  $\epsilon$ -cover. Recall that  $B^c$  denotes the basic closed ball with the same centre as  $B$ . For each each basic open  $B_1, \dots, B_k \in S$ , (exactly) one of the possibilities is realized:

- (a)  $\bigcap_{i \leq k} B_i^c = \emptyset$ , or
- (b)  $\bigcap_{i \leq k} B_i \neq \emptyset$ , or
- (c)  $\bigcap_{i \leq k} B_i^c \neq \emptyset$  but  $\bigcap_{i \leq k} B_i = \emptyset$ .

Note that there are only finitely many conditions like that in total.

If we shrink the radii of all  $B \in S$  by a  $\delta < \epsilon/2$  (but keep the same centres), then the conditions of the form (a) will still hold, and the smaller balls will still cover the space because the  $\epsilon$ -balls do. If  $\delta$  is small enough, then the conditions of the form (b) will still be satisfied, since there are only finitely many conditions like that involved<sup>4</sup>. The third alternative (c) must be witnessed only by points  $y$  such that, for some  $B_i = B(c, r)$ ,  $d(y, c) = r$ . This means that, after the shrinking by  $\epsilon/2 > \delta > 0$  so small that the alternative (b') still holds for each tuple of balls, we completely exclude the third alternative for the new cover.

This argument shows that such a cover exists. Since the conditions are c.e. by Lemma 3.8, it remains to search for a cover such that each finite collection of basic balls in the cover satisfies either (a) or (b).  $\square$

By Remark 3.4, we can additionally assume that  $K_{n+1}$  formally refines  $K_n$ . Clearly, we get the following corollary which will be useful later:

**Corollary 3.11.** *Every effectively compact space admits uniformly computable sequence of  $2^{-n}$ -nerves (one nerve for each  $n$ ), where the latter are represented as a finite combinatorial simplices. Furthermore, the formal inclusion between covers  $K_{n+1}$  and  $K_n$  induces a simplicial map between the respective nerves; these maps are uniformly computable in  $n$ . (We cite Munkres [98] for the standard definitions from algebraic topology that we omit here.)*

<sup>3</sup>Every point in  $M \setminus B_i^c$  has the property  $d(\text{cntr}(B_i), y) > r(B_i) = r$ , and if we take  $B(y, q)$  where  $0 < q < \frac{d(\text{cntr}(B_i), y) - r(B_i)}{2}$  then  $d(\text{cntr}(B_i), y) > r + q$ .

<sup>4</sup>Each such intersection is witnessed by a (special) point which must be at distance strictly less than  $2\epsilon$  to all centres, say,  $\gamma$ -less. If we shrink the radii by a value less than the minimum of these  $\gamma$  (which is a positive value since there are only finitely many  $B_i$  involved) then the inequalities will still hold. Another way to think about it is as follows:  $B \cap D \neq \emptyset$  is an 'open property' of the parameters (radii and centres), and a finite conjunction of open properties is also open because open sets are closed under taking finite intersections.

*A stronger condition.* It will be convenient to have a system of covers  $(K_n)$  so that not only each  $K_n$  is  $\cap$ -decidable but the whole collection  $\bigcup_n K_n$  is  $\cap$ -decidable. (We strongly conjecture that there is an elementary counterexample.) For instance, we will see soon that having such a stronger system of covers will allow us to computably map  $2^\omega$  onto the space; this is (vii) of Theorem 1.1.

We are not sure whether such covers can be uniformly constructed for basic open balls with rational radii (represented as a pair of integers), but we can construct such a system for balls with centres in special points and uniformly computable radii<sup>5</sup>. We call such balls *basic computable open*.

**Definition 3.12.** A computably metrized  $M$  is strongly effectively compact if  $M$  admits a system of  $2^n$ -covers  $K_n$ ,  $n \in \omega$ , by basic computable open balls such that  $\bigcup_n K_n$  is  $\cap$ -decidable.

**Theorem 3.13.** *A computably metrized  $M$  is effectively compact if, and only if, it is strongly effectively compact.*

*Proof.* By slightly increasing the radii of all the balls in a cover, we can ensure their radii are rational. Thus, every strongly effectively compact space is effectively compact. To this end, we assume the space is effectively compact.

*The idea* behind the proof is as follows. We would like to argue that the idea from the proof of the previous Theorem 3.10 can be iterated. For example, suppose we have come up with a  $\cap$ -decidable  $K_0$  and need to find  $K_1$  so that  $K_0 \cup K_1$  is  $\cap$ -decidable. But to find such a cover, we might have to slightly shrink the radii of the balls that we have already put into  $K_0$ . This is because it could be that for some  $B \in K_0$  and  $C$  that we attempt to put into  $K_1$ , there is a point at distance  $r(B)$  from the centre of  $B$  that lies in  $C$  and is isolated, so there is nothing in  $B \cap C$ .

Suppose we iterate the strategy from the proof of Theorem 3.10 and allow the procedure to slightly shrink all the balls in  $K_0$ , thus updating the radii of balls in  $K_0$ . But note  $K_0$  must still satisfy the closed properties  $\bigcap_{i \leq k} B_i^c = \emptyset$  and finitely many open properties  $\bigcap_{i \leq k} B_i \neq \emptyset$ . The former is not an issue since the radii will decrease. The latter however needs to be maintained more carefully. When we first discover the finitely many open relations of the form of finitely many strict inequalities (when  $K_0$  is first introduced), we also *compute* a rational parameter  $\delta_0 > 0$  such that the relations will still hold if we decrease the radii of the balls by  $\delta_0$ . This is possible since the conditions build down to finitely many strict inequalities involving the radii and computable numbers:

$$d(c_i, x_j) < r_i,$$

where  $c_i$  are centres of the balls and  $x_i$  are special points witnessing that a certain intersection is not empty.

We then define  $\delta_{0,n} = 2^{-n-2}\delta_0$  and note that  $\sum_i \delta_{0,i} < \delta_0$ . We intend to shrink the radii of each ball in  $K_0$  by at most  $\delta_{0,s}$  at stage  $s$ . This will make the radii in the balls computable while maintaining the finitely many conditions that  $K_0$  needs to satisfy.

We also iterate this. When we define  $K_1$ , we will have more open conditions to maintain for  $K_0 \cup K_1$ . We compute a  $\delta_1 > 0$  and set  $\delta_{1,n} = 2^{-n-2}\delta_1$ . We also ensure that  $\delta_{1,n} \leq \delta_{0,n}$ , for every  $n$ . When we define (our first approximation to balls in)  $K_2$  at stage  $t$ , we will allow balls in  $K_0$  to shrink by at most  $\delta_{1,t} \leq \delta_{0,t}$  and balls in  $K_1$  by at most  $\delta_{1,t}$ . All the finitely many conditions will still be satisfied.

We iterate this process until, in the end of the construction, we finally get a collection of computable balls  $\bigcup_n K_n$ . At no stage we are stuck. By the choice of the parameters, all of the open conditions still hold, while the closed conditions will be satisfied vacuously.

Hopefully, the explanation above is sufficiently convincing, but we shall give a formal proof for completeness.

*Formal proof.*

**Lemma 3.14.** *For every  $\epsilon > 0$  and  $\delta > 0$  and any finite collection  $K'$  of basic open balls, there exists a finite basic open  $\epsilon$ -cover  $K$  of the space and a collection of basic open balls  $K''$  such that:*

- (i) *Every ball in  $K''$  has the same centre as some ball in  $K'$  but its radius is at most  $\delta$ -smaller;*
- (ii) *for each basic open  $B_1, \dots, B_k \in K'' \cup K$ , either  $\bigcap_{i \leq k} B_i^c = \emptyset$  or  $\bigcap_{i \leq k} B_i \neq \emptyset$  holds.*

Of course,  $\bigcap_{i \leq k} B_i^c = \emptyset$  implies  $\bigcap_{i \leq k} B_i = \emptyset$ , and thus  $K$  in the lemma has computable nerve, and for the same reason  $K \cup K'$  is  $\cap$ -decidable.

<sup>5</sup>The radii can likely be made rational if necessary, but they will be represented via Cauchy sequences, not as a fraction.

*Proof of Lemma 3.14.* Fix a finite  $\epsilon/2$ -cover of the space by basic open balls, and replace each ball in the cover with a  $\epsilon$ -ball with the same centre. Let  $S$  be this new  $\epsilon$ -cover. Recall that  $B^c$  denotes the basic closed ball with the same centre as  $B$ .

For each tuple of basic open  $B_1, \dots, B_k \in S \cup K'$ , (exactly) one of the possibilities is realized:

- (a')  $\bigcap_{i \leq k} B_i^c = \emptyset$ , or
- (b')  $\bigcap_{i \leq k} B_i \neq \emptyset$ , or
- (c')  $\bigcap_{i \leq k} B_i^c \neq \emptyset$  but  $\bigcap_{i \leq k} B_i = \emptyset$ .

Note that there are only finitely many conditions in total.

If we shrink the radii of all  $B \in S$  by a  $\delta' < \min\{\delta, \epsilon/2\}$  (but keep the same centres), then the conditions of the form (a') will still hold, and the smaller balls will still cover the space because the  $\epsilon$ -balls do. If  $\delta'$  is small enough, then the conditions of the form (b') will also still be satisfied, since there are only finitely many conditions like that involved. Note that the third alternative must be witnessed only by points  $y$  such that, for some  $B_i = B(c_i, r)$ ,  $d(y, c_i) = r$ . This means that, after we shrink the radii by  $\epsilon/2 > \delta' > 0$ , (b') will still hold for each tuple of balls, but we completely exclude the third alternative. Define  $K''$  to be the balls in  $K'$  after the shrinking, and  $K$  is the shrunken balls from  $S$ .  $\square$

The rest of the proof proceeds by induction; we iteratively apply Lemma 3.14 to produce a system of covers that satisfies the properties required in the definition of a strongly effectively compact presentation. We produce a sequence  $(K_n)$  of covers, as follows.

At stage 0, search for a finite collection of basic open balls  $K$  satisfying conditions of Lemma 3.14 with  $\epsilon = \delta = 1$  and  $K' = \emptyset$ . Define  $\hat{K}_{0,0}$  equal to the first found such  $K$ . Also, for the finite collection of strict inequalities that witness non-emptiness of intersections in  $K_{0,0}$  calculate a parameter  $\delta' \in \mathbb{Q}$  such that the inequalities would still hold if we decrease the radii by  $\delta'$ . Set  $\delta_0 = \delta'$  and  $\delta_{0,i} = 2^{-i-2}\delta_0$ .

At stage  $s > 0$ , suppose  $K_{i,s-1}$  ( $i < s$ ) and  $\delta_{s-1,s-1}$  have already been defined. Search for a finite collection of covers  $K$  that satisfies the lemma with  $K' = \bigcup_{i < s} K_{i,s-1}$ ,  $\epsilon = 2^{-s}$ , and  $\delta = \delta_{s-1,s-1}$ . This will give a finite collection of balls  $K''$  having the same centres as balls in  $K'$  but perhaps having radii at most  $\delta$ -smaller than the radii of the respective balls in  $K'$ . For  $i < s$ , define  $K_{i,s}$  to be the collection of those balls in  $K''$  that have the same centre as some ball in  $K_{i,s-1}$ . Define  $K_{s,s} = K$ , where  $K$  is first found satisfying the conditions of the lemma. Compute a rational  $\delta' > 0$  so small that the finitely many strict inequalities that witness non-emptiness of finite collections of balls in  $\bigcup_{i \leq s} K_{i,s}$  will still hold if we decrease all the radii of all balls in  $\bigcup_{i \leq s} K_{i,s}$  by  $\delta'$ . Set  $\delta_s = \min\{\delta_s, \delta'\}$  and define  $\delta_{s,s} = 2^{-s-2}\delta_s$ . Proceed to the next stage.

*The verification* boils down to noting that at no stage of the construction we are stuck, so  $K_{i,s}$  and  $\delta_{s,s}$  are defined for every  $s$ . This is because of Lemma 3.8 implying that the conditions (a') and (b') are uniformly c.e., and because of Lemma 3.14 saying that balls and parameters with the needed properties exist. Thus, we just search for the first found balls and parameters.

At every stage  $s$ , the radii of the balls in  $K_{i,s}$  shrink by at most  $\delta_{s,s} \leq 2^{-s}$ , which makes each of the radii uniformly approach a computable real number as  $s$  goes to infinity. Set  $K_i$  equal to these balls that have their radii equal to the limit of the radii of the balls with the same centre in  $K_{i,s}$ . When compared to the radius of the ball in  $K_{s,s}$  when it was first introduced, the radius of the respective ball in  $K_s$  will be smaller by at most  $\sum_{n > s} \delta_{n,n} < \sum_{n > s} \delta_{s,n} \leq \delta_s$ , and  $\delta_s$  is not greater than the parameter  $\delta'$  that was calculated at stage  $s$  and that was sufficient to maintain the non-emptiness of finite intersections in  $K_{s,s}$ . Since the radii can only decrease, the conditions that say that the closed balls do not intersect will be preserved from a stage to a stage, and in the limit.

It follows therefore that  $\bigcup_n K_n$  consists of uniformly computable collection of basic computable balls and is  $\Pi$ -decidable.  $\square$

**Remark 3.15.** The reader should note that, instead of using basic open covers, we could just as well used basic closed covers in the proof of Theorem 3.13. For instance, for any finite collection of basic computable balls  $C_0, \dots, C_k$ , we have  $\bigcap_{i \leq k} C_i = \emptyset \iff \bigcap_{i \leq k} C_i^c = \emptyset$ , where  $C_i^c$  is the basic closed ball with the same centre and radius as  $C_i$ . Also, by Remark 3.4, we can always assume that  $K_{n+1}$  is formally contained in  $K_n$ , and this will still be true if we

choose working with closed covers. For that, define a new system of (closed or open) covers  $K_{f(n)}$  where  $f(n)$  is a computable monotone function that grows sufficiently fast so that  $K_{f(n+1)}$  formally refines  $K_{f(n)}$ .

However, even when we are working with closed basic computable balls, the intersection can always be witnessed by a special point, because the respective open balls intersect too.

Recall that  $\bar{C}$  denotes the closure of a basic computable open ball  $C$ , and recall that

$$C \subset \bar{C} \subset C^c,$$

and, in general, both inclusions can be strict. But these inclusions also guarantee that we can use closures of the open balls to form covers in Theorem 3.13 and still decide intersection. We can also come up with any combination of open, closed and closures (e.g., decide whether  $B_i \cap \bar{B}_j \cap B_k^c = \emptyset$ ) for any computable balls in  $K$  constructed in Theorem 3.13.

**Definition 3.16.** If a computable sequence  $(K_n)$  of finite  $2^{-n}$ -covers of computable balls satisfies the properties described above then we say that  $(K_n)$  is a *fully  $\cap$ -decidable system of covers* of the space. Such a  $K$  is given by a uniformly computable sequence of (finite sets  $K_n$  of) indices of radii and special points, and we can choose whether we want to consider open, closed, or closures of open balls that have these parameters.

For instance, when we say “ $B^c(r, q)$  is in  $K_n$ ” or “ $\overline{B(r, q)}$  in  $K_n$ ”, or the same for  $B(r, q)$ , we really mean that parameters  $\langle r, q \rangle$  are listed in  $K_n$  (where  $q$  is given as an index of a computable real).

The following lemma will be useful later.

**Lemma 3.17.** *Let  $K = \bigcup_n K_n$  be a fully  $\cap$ -decidable system of covers of a space  $M$ . Then, for every closed ball  $D^c$  in  $K$  we can enumerate all basic open  $B$  in  $M$  such that  $B \cap D^c \neq \emptyset$ .*

*Proof.* Suppose  $B \cap D^c \neq \emptyset$  and let  $x$  be any (not necessarily special) point in the intersection. Suppose the radius of  $B$  is  $\delta$ , and let  $c_1$  be the centre of  $B$ , and  $r_1$  its radius. For some positive  $\delta$ , we have

$$d(x, c_1) = r_1 - \delta.$$

Fix  $n$  so that  $2^{-n} < \delta/2$ , and consider the finite set  $K_n$ . Since  $K_n$  is a (closed or open) cover of the whole space, there must exist some  $C \in K_n$  such that  $x \in C$ . Since  $x \in D^c$ , it must be that

$$C \cap D^c \neq \emptyset,$$

and (by our assumption) this can be recognised computably. We claim that for this  $C$ , we have that  $C$  is formally included into  $B$ .

Indeed, if  $c_2$  is the centre of  $C$  and  $r_2$  is its radius, then we have that  $d(x, c_2) < r_2 \leq 2^{-n} < \delta/2$ , and therefore

$$d(c_1, c_2) + r_2 \leq d(x, c_1) + d(x, c_2) + \delta/2 < r_1 - \delta + \delta/2 + \delta/2 = r_1,$$

which is the same as to say that  $C$  is formally included into  $B$ .

It follows that  $B$  intersects  $D^c$  if, and only if, there is an  $n > 0$  and a ball  $C \in K_n$  such that  $C \cap D^c \neq \emptyset$  and  $C$  is formally included into  $B$ . This is a  $\Sigma_1^0$ -property.  $\square$

**3.3. Isometry-invariance of effective compactness.** Iljazovich [58] discovered that the notion of effective compactness admits another characterization that entails that it is isometry-invariant, i.e., every isometrically isomorphic computable metrization of the space must also be effectively compact. This property has recently been independently rediscovered in [107].

**Theorem 3.18.** *Suppose  $M$  is effectively compact and  $N$  is a computably metrized space isometrically isomorphic to  $M$ . Then  $N$  is effectively compact as well.*

*Proof.* We attempt to show that, in an effectively compact  $M$ , the following invariant  $D(M, 2^{-n})$  is uniformly computable in  $n$ :

$D(M, 2^{-n})$  is the maximal number of points of the space  $M$  that are  $> 2^{-n}$ -apart from one another.

We will perhaps fail to compute it; however, at this stage of the proof note that this is an isometry-invariant of  $M$ . We describe our attempt. Given  $\bar{x} \in M^m$ , we can calculate  $\inf_{i < j \leq m} d(x_i, x_j)$  and then

$$\sup_{\bar{x} \in M^m} \inf_{i < j \leq m} d(x_i, x_j).$$

If this supremum is  $< 2^{-n}$ , then  $m$  is too large, i.e.,  $m > D(M, 2^{-n})$ . Note that this is a c.e. event. On the other hand, by searching through all possible  $m$ -tuples we can bound the maximal number of such points from below. The issue is that the supremum could be exactly equal to  $2^{-n}$ , so we may end up with a pair of integers  $n_0, n_0 + 1$  each of which can potentially be equal to the invariant  $D(M, 2^{-n})$ ; here  $n_0 + 1$  corresponds to the situation when there are  $n_0 + 1$  points at distance exactly  $2^{-n}$  from one another.

In this case we shall wait long enough so that any  $(n_0 + 2)$ -tuple has at least one pair of points at distance  $> 2^{-n}$ , and for some  $\epsilon$  there exist  $n_0 + 1$  points at distance  $2^{-n} - \epsilon_n$  for  $\epsilon_n < 2^{-n-1}$ . Then

$$D(M, 2^{-n} - \epsilon_n) = n_0 + 1.$$

This allows us to compute a computable sequence of rationals

$$\xi_n = 2^{-n} - \epsilon_n,$$

where clearly  $\xi_n \leq 2^{-n}$ , such that  $D(M, \xi_n)$  is a computable sequence of natural numbers. Indeed, for this  $\xi_n$  there is a  $n_0 + 1$  tuple  $\bar{y}_n$  of points at distances strictly greater than  $\xi_n$  from each other. We can assume these points are special.

Note that, conversely, any such  $\bar{y}_n = \{y_i : i \leq n_0 + 1\}$  with the properties described above gives a  $2^{-n}$ -cover of the space:

$$B(y_i, 2^{-n}), \quad i \leq n_0 + 1.$$

In particular, we can search for one such tuple of special points in  $N$ . □

Note that the same argument also shows the following.

**Corollary 3.19.** *Suppose that  $M$  and  $N$  are isometrically isomorphic computable metrized spaces. Then both admit the same modulus of compactness up to Turing degree.*

**3.4. Calculus of effectively closed sets.** In this subsection we present some well-known basic results about effectively closed sets, and we also derive several pleasant consequences of these results that will be important in the sequel. The notion of an effectively closed set is a generalisation of a  $\Pi_1^0$  class, and it is especially useful if the ambient space is effective compact. We will need some basic facts of this generalised theory, but of course a lot more is known; see the very recent large survey [60].

**Definition 3.20.** A closed subset  $C$  of a computably metrized  $M$  is effectively closed if  $M \setminus C$  is c.e. open.

It should be clear that effectively closed sets are closed under finite unions and arbitrary computable intersections (meaning that the effective procedures listing the complements must be uniformly indexed). The following lemma is also an immediate consequence of the definition:

**Lemma 3.21.** *Suppose  $f : A \rightarrow B$  is a computable surjection, and assume  $C$  is effectively closed in  $B$ . Then  $f^{-1}(C)$  is effectively closed (in  $A$ ).*

*Proof.* This is because  $A \setminus f^{-1}(C) = f^{-1}(B \setminus C)$  is c.e. open. To see why, recall that  $f$  is computable if, and only if, it is effectively open. If a basic open  $D$  is enumerated into  $B \setminus C$ , then we will list all pairs  $(D', D'')$  in the continuous name of  $f$  such that  $D''$  is formally included in  $D$ . Since every  $x \in D$  is contained in such a  $D''$  (by surjectivity), this will give an enumeration of  $f^{-1}(D)$ . Putting these enumerations together for all such  $D$  in  $B \setminus C$ , we will list its preimage. □

Another observation is an easy generalization of a well-known fact about  $\Pi_1^0$  classes.

**Fact 3.22.** Suppose  $P = \{p\}$  is effectively closed singleton in an effectively compact space  $X$ . Then the only point  $p$  of  $P$  is (uniformly) computable.

(This can of course be pushed to show that isolated points can also be computed, though non-uniformly.)

*Proof.* Given  $n$ , wait for a basic open ball  $D$  of radius  $2^{-n}$  and finitely many basic open  $B_1, \dots, B_n \in X \setminus P$  such that  $D, B_1, \dots, B_n$  cover  $X$ . Then  $p \in D$ .  $\square$

More generally, effectively closed sets,  $\Pi_1^0$ -classes, computable functions, and effectively compact spaces are closely technically related. To make this relationship explicit, we need one more definition. As usual, we identify basic open balls with their indices.

**Definition 3.23.** A closed subset of a computably metrized  $M$  is c.e. if  $\{B : B \text{ basic open and } B \cap C \neq \emptyset\}$  is c.e.

The fact below is well-known; we are not sure who was the first to observe this.

**Lemma 3.24.** A closed subset  $C$  of a computably metrized space  $M$  is computably enumerable if, and only if,  $C$  possesses a uniformly computable (in  $M$ ) dense sequence of points.

Note that the dense sequence makes  $C$  a computable Polish space under the induced metric. Sets that possess a uniformly computable subsequence are sometimes called *computably overt* in the literature. That is, a closed subset of a computably metrized  $M$  is *computably overt* if it possesses a uniformly computable (in  $M$ ) dense sequence of points.

*Proof of Lemma 3.24.* Suppose  $C$  possesses such a computable sequence  $(\alpha_i)_{i \in \mathbb{N}}$ . Then the density of the sequence in  $C$  implies that  $B_i \cap C \neq \emptyset$  iff  $\exists j \alpha_j \in B_i$ , which is a uniformly  $\Sigma_1^0$  statement.

Now suppose  $C$  is a computably enumerable closed subset of  $M$ . Our goal is to construct a uniformly computable (finite or infinite) sequence of points  $(\alpha_i)_{i \in I}$  that is dense in  $C$ . The proof below does not have to be non-uniform, but for notational convenience we split it into two cases, namely, when  $C$  is finite or infinite.

If  $C$  is finite, then it clearly contains only computable points. To see why, assume it is not empty (in this case there is nothing to prove) and let  $x$  be any point in  $C$ . Take a ball small enough so that  $\{x\} \in B \cap C$ . To get an  $2^{-n}$ -approximation to  $x$ , wait for a basic open  $B'$  of radius  $< 2^{-n}$  so that  $B' \cap C \neq \emptyset$  and additionally  $B'$  is formally contained in  $B$ .

Without loss of generality, we assume  $C$  is infinite. We uniformly approximate a computable sequence by stages. Before we describe stage  $s$ , recall that two basic open balls  $U$  and  $W$  are formally  $s$ -disjoint if  $r(U) + r(W) < d(\text{cntr}(U), \text{cntr}(W))$  and this can be seen after calculating the radii and the distance with precision  $2^{-s}$ . Then  $U$  and  $W$  are formally disjoint if they are formally  $s$ -disjoint for some  $s$ .

At stage 0, search for a basic open ball  $B_{0,0}$  of radius  $< 1$  such that  $B_{0,0} \cap C \neq \emptyset$ . If such a ball is never found then do nothing. If it is every found, go to the next stage.

At stage  $s > 1$  first check whether there exists a basic open ball with index  $< s$  which is formally  $s$ -disjoint from  $B_{0,s-1}, \dots, B_{s-1,s-1}$ . If such a basic open  $B$  exists, then choose the first fund  $B_{s,s} \subseteq_{\text{form}} B$  and  $B_{i,s} \subseteq_{\text{form}} B_{i,s-1}$ ,  $i < s$  such that  $B_{j,s} \cap C \neq \emptyset$ , the  $B_{j,s}$  are pairwise formally disjoint and  $r(B_{j,s}) < 2^{-s}$ ,  $j = 0, \dots, s$ . Otherwise, if no such  $B$  exists, fix the first found pairwise formally disjoint  $B_{0,s}, \dots, B_{s,s}$  that intersect  $C$ , have radii  $< 2^{-s}$ , and such that  $B_{i,s} \subseteq_{\text{form}} B_{i,s-1}$  for  $i < s$  (note there is no further restriction on  $B_{s,s}$ ). This ends the construction.

Let  $\alpha_i$  be the unique point of the Polish space such that  $\{\alpha_i\} = \bigcap_{j \geq i} B_{i,j}$ . Since the construction is uniform and the radii of balls are rapidly shrinking, the points  $\alpha_i$  form a uniformly computable sequence. Since each of the  $B_{i,j}$  ( $j = i, i+1, \dots$ ) intersects  $C$  and  $C$  is closed, each  $\alpha_i \in C$ . It remains to check that  $(\alpha_i)_{i \in \mathbb{N}}$  is dense in  $C$ . Let  $\overline{(\alpha_i)_{i \in \mathbb{N}}}$  be the completion of  $(\alpha_i)_{i \in \mathbb{N}}$ .

Suppose  $c \in C$ . We claim that  $c \in \overline{(\alpha_i)_{i \in \mathbb{N}}}$ . Assume  $c \notin \overline{(\alpha_i)_{i \in \mathbb{N}}}$ , and there is a ball  $U$  centred in  $c$  which is outside  $\overline{(\alpha_i)_{i \in \mathbb{N}}}$ . There will be a basic open ball  $B' \ni c$  of radius at most  $2^{-n}$  and which is formally contained in  $U$  with precision  $2^{-n}$ :

$$d(\text{cntr}(U), \text{cntr}(B')) + r(B') < r(U) + 2^{-n}.$$

Then at every stage  $s > n + 4$  the balls  $B_{i,s-1}$ ,  $i = 0, \dots, s-1$  will be formally  $s$ -disjoint from  $B$ , as will be readily witnessed by the metric. At some late enough stage  $s'$  we will get a confirmation that  $B \cap C \neq \emptyset$ . There exist only finitely many basic balls that have their index smaller than the index of  $B$ . Therefore, eventually  $B$  will be used to define  $B_{t,t} \subseteq_{\text{form}} B$ , contradicting the assumption that  $U \cap \overline{(\alpha_i)_{i \in \mathbb{N}}} = \emptyset$ .  $\square$

**Definition 3.25.** A closed subset of a computably metrized  $M$  is *computable* if it c.e. and effectively closed.

As we mentioned immediately after the statement of Lemma 3.24, a c.e. closed subset of a computable metric space  $M$  can be viewed as a computably metrized space under the induced metric. It thus makes sense to ask when this subspace is effectively compact. If  $M$  is effectively compact, then it is both computable compact subset of itself and an effectively closed subset of itself. Interestingly, this trivial example is not misleading.

**Proposition 3.26.** *For a closed subset  $C$  of an effectively compact  $M$ , the following are equivalent:*

1.  $C$  is an effectively compact subspace of  $M$ ;
2.  $C$  is computable.

Before we proceed to the proof, the reader might well wonder what is wrong with the following analog of the classical argument that closed subsets of compact spaces are compact:

*Suppose that  $C$  is a closed subset of an effectively compact space  $P$ , then  $C$  is effectively compact.* Apply effective compactness we can compute a finite subcover and then attempt to ‘throw away’  $\overline{C}$  (that can be listed). One obvious problem with this idea is that we can never be sure whether a basic open ball  $B$  (in  $P$ ) intersects  $C$ , and thus we can never be sure whether we can keep  $B$  in our cover of  $C$ . By Lemma 3.24, this is equivalent to locating a computable dense sequence in  $C$ . This problem cannot be circumvented as  $C$  might not contain a computable dense sequence. Indeed, as Kleene showed in the 1950’s there are effectively closed subsets of  $\mathbb{R}$  containing no computable points at all. We discuss more about this issue in Section 4. We turn to the proof of Proposition 3.26.

*Proof.* Assume (1). It is clear that  $C$  is c.e. (by Lemma 3.24). To list its complement, fix  $x \in M \setminus C$ . Let  $\delta = \inf_{c \in C} d(x, c)$ . Then any  $\delta/4$ -cover of  $C$  must be formally disjoint from any ball centred in  $y$  with  $d(y, x) < \delta/4$ . For every  $n$ , fix a finite  $2^{-n}$ -cover  $\mathcal{K}$  of  $C$ . It follows that  $M \setminus C$  is equal to the union of the (uniformly) effectively open sets  $U_n$ , where

$$\{B : B \text{ basic open and } B \text{ is formally disjoint from every ball in } \mathcal{K}\}.$$

It follows that (2) holds.

Now assume (2).  $C$  is computably metrized by Lemma 3.24; let  $(y_j)$  be the computable dense subsequence. Fix  $\epsilon = 2^{-n}$ . We need to find an  $\epsilon$ -cover of  $C$  by basic open balls<sup>6</sup>. Regardless of whether the balls involved are basic or not, as long as their centres and radii are computable, the relation of formal containment remains c.e.

If a finite collection  $\mathcal{K}$  of basic open (in  $C$ ) balls formally contains a cover  $\mathcal{K}'$  by basic open (in  $M$ ) balls, then clearly  $\mathcal{K}$  is a cover of  $C$ . We claim that this condition is also necessary (for  $\mathcal{K}$  to cover  $C$ ).

From the proof of Lemma 3.3 we know that, for a given cover  $\mathcal{K}$  of  $C$  by (basic or not) open balls there is a small enough  $\epsilon$  such that every  $\epsilon$ -cover of  $C$  will be formally contained in at least one ball of  $\mathcal{K}$ .

Take  $\delta = \epsilon/4$ . Fix a finite  $\delta$ -cover  $\mathcal{K}'$  of  $C$  by balls that are centred in special points of  $M$ , not  $C$ . Every  $B' \in \mathcal{K}'$  intersects  $C$  at some point  $x$ , and by the choice of  $\delta$ ,  $d(x, \text{cntr}(B')) + \delta < \epsilon$ , thus  $B(\epsilon, x) \supset_{\text{form}} B'$ . By transitivity of formal inclusion<sup>7</sup>, we have that  $B'$  must be formally contained in some ball in  $\mathcal{K}$ .

By effective compactness of  $M$  and computability of  $C$ , we can produce at least one  $\delta$ -cover  $\mathcal{K}'$  of  $C$  by basic open balls of  $M$ , uniformly in  $\delta$ . (To see why, replace every basic open ball in the c.e. open name of  $M \setminus C$  by the effective union of balls of radii at most  $\delta$  that are formally contained in it. This gives a new c.e. enumeration of the complement of  $C$  but with balls of radii at most  $\delta$ . Then take the c.e. collection of all basic  $\delta$ -balls that intersect  $C$ . Together these sets of balls cover  $M$ . Initiate the combined enumeration of these two c.e. sets and wait until at some finite stage we discover that we have a cover of  $M$ .)

Since formal inclusion is c.e., this gives a procedure of listing covers of  $C$  by basic balls (in  $C$ ). □

We see that effective compactness and computability of a closed set are very closely related notions. We have already mentioned above that an effectively closed set does not have to be computable, in general. However, suppose

<sup>6</sup>Note that  $y_j$  does not have to be special in  $M$ , and thus basic open balls in  $C$  do not have to be basic open in  $M$ . Nonetheless, an open ball of  $M$  centred in a computable point  $y$  and having a computable (more generally, left-c.e.) radius  $r$  is effectively open:

$$B(y, r) = \bigcup \{B(x, q) : d(x, y) + q < r\},$$

that is,  $B(y, r)$  is the union of basic open balls formally contained in it. Note effective openness of  $B(y, r)$  is uniform in  $y$  and  $r$ .

<sup>7</sup>This is because  $d(x, y) + r_2 < r_1$  and  $d(y, z) + r_3 < r_2$  (together with the triangle inequality) imply  $d(x, z) + r_3 < r_1$ .

$C$  is an effectively closed ( $\Pi_1^0$ ) subset of, say,  $\mathbb{R}^3$ , and suppose further that we know that it is a sphere or a ball. Is it computably closed? J. Miller [93] used algebraic topology to answer this question in the affirmative (in fact, in any dimension). The idea is that, roughly, we can non-uniformly localise it to a compact box in  $\mathbb{R}^n$  and then use that an effectively compact ball will eventually be contained in a simplex that “looks like the ball”; algebraic topology helps to make this formal. The results of Miller have been extended (e.g., to compact manifolds under some extra conditions) in [19, 57, 59, 61]. But of course, if we are interested in *presentations* of spaces and especially up to homeomorphism, then a sphere or a compact surface is clearly homeomorphic to an effectively compact space (e.g., given by a geometric realisation of its triangulation).

In general, there is no good reason why a basic closed ball (or the closure of a basic open ball) in an abstract Polish space needs to be computably closed; pathological examples similar to Ex. 2.9 can be constructed. Interestingly, it follows that there are always *enough* closed balls with computable radii that are computably closed as sets, and indeed uniformly so. More specifically, an immediate consequence of the proposition above is that, in Theorem 3.13, we can additionally state that the basic closed balls in the covers are computably closed sets:

**Proposition 3.27.** *Suppose  $K = \bigcup_n K_n$  is a fully  $\cap$ -decidable system of covers of a computably metrized  $M$ . Then each computable closed ball  $D^c$  in  $K$  is a computable closed set (thus, is an effectively compact subspace of  $M$ ), and this is uniform.*

*Proof.* Lemma 3.17 says that each such  $D^c$  is c.e. closed. If  $x \in M \setminus D^c$ , then  $x$  is inside an open ball  $C$  that is formally disjoint from  $D^c$ , and such balls can be computably enumerated<sup>8</sup>. Thus, the c.e. union of such open balls formally disjoint from  $D^c$  makes up the complement of  $D^c$ .  $\square$

The fact above will be useful when we talk about the universality of  $2^\omega$ . More generally, it seems to be useful in any iterated recursive argument in which a space is eventually replaced by its compact subset, and then a subset of this subset, etc. The lemma below will also be useful throughout the rest of the paper.

**Lemma 3.28.** *Suppose  $f : X \rightarrow Y$  is a computable map, and assume  $X$  is effectively compact. Then  $f(X)$  is c.e. closed (in  $Y$ ) and effectively compact.*

*Proof.* Let  $(x_i)$  be a computable dense sequence in  $X$ . Then  $(f(x_i))$  is dense in  $f(X)$ . (Every  $\alpha = \lim_j x_j$  for some subsequence  $(x_j)$  and, by continuity,  $f(\alpha) = \lim_j f(x_j)$ , so  $f(X) \subseteq cl(f(x_i))$ ). Suppose  $\xi \in cl(f(x_i))$ , say  $\xi = \lim_j f(x_j)$ . By compactness,  $(x_j)$  has a convergent subsequence  $(x_{j_k})$ , so let  $z = \lim_k x_{j_k} \in X$  be its limit. Then  $f(z) = \lim_k f(x_{j_k}) = \lim_j f(x_j) = \xi$ .

Given a cover  $B_j$  of  $f(X)$  by basic open (in  $f(X)$ ) balls of radius  $2^{-n}$  centred in  $f(x_j)$ , calculate the c.e. names of each  $B_j$  in  $Y$  and begin enumeration of  $f^{-1}(C)$  for each such open basic  $C$ ; note it could be that some of these  $f^{-1}(C)$  will be undefined. At some stage the preimages must cover the whole  $X$ . We can see which  $B_j$  included the basic open (in  $Y$ ) balls whose images were sufficient to cover  $X$ . This gives a way of producing at least one  $2^{-n}$ -cover of  $f(X)$  uniformly in  $n$ ; now apply Lemma 3.3.  $\square$

Combining Lemma 3.28 with Proposition 3.26, we get:

**Corollary 3.29.** *Suppose  $f : X \rightarrow Y$  is computable and  $X$  is effectively compact.*

- *If  $f$  is surjective then  $Y = f(X)$  must be effectively compact.*
- *If  $Y$  is effectively compact then  $f(X)$  is a computable closed subset of  $Y$ .*

In computable algebra, the inverse of a computable bijective map is clearly computable as well. In contrast, there is no reason why the inverse of a computable bijection between spaces has to be computable even if its inverse is continuous (we mention here that this is actually true for isometric maps). The theorem below is elementary, but it is rather important because it tells us that effectively continuous maps are the right morphisms in the category of effectively compact spaces.

**Theorem 3.30.** *Suppose  $f : X \rightarrow Y$  is a computable bijection between computably metrized spaces, and assume  $X$  is effectively compact. Then  $Y$  is also effectively compact, and  $f^{-1}$  is computable.*

<sup>8</sup>To see why, let  $c$  be the centre of  $D^c$  and  $r$  its radius, and assume  $d(c, x) = r + \delta$ . There must be a special  $x_i$  such that  $d(x_i, x) < \delta/2$ . Take the basic open ball  $C = B(x_i, \delta/2)$ . Then the distance between their centres is  $d(c, \alpha) > r + \delta - d(x_i, x) > r + \delta - \delta/2 = r + \delta/2$ , which is the sum of their radii. So the balls are formally disjoint.

It is easy to see that  $f$  is indeed a homeomorphism<sup>9</sup>. Our task is to produce a more subtle computable version of this observation.

*Proof.* Effective compactness of  $Y$  follows from the corollary above. Given a (not necessarily) special point  $y \in Y$ , act computably relative to  $y$ . The set  $Y \setminus \{y\}$  is effectively open relative to  $y$ . Indeed, for every  $z \neq y$  there must exist formally disjoint basic open  $B \ni y$  and  $D \ni z$ . Thus, it is sufficient to list, effectively in  $y$ , all basic open balls formally disjoint from some ball in the name of  $y$ .

Since  $f$  is computable,  $f^{-1}(Y \setminus \{y\})$  is effectively open relative to  $y$ . Since  $f$  is bijective, we have that

$$X \setminus f^{-1}(Y \setminus \{y\}) = \{f^{-1}(y)\}.$$

To list a basic open  $D$  into the name of  $x = f^{-1}(y)$ , wait for finitely many basic open balls  $B_1, \dots, B_k$  in  $X \setminus C_0 = f^{-1}(Y \setminus \{y\})$  such that  $D, B_1, \dots, B_k$  cover  $X$ . Note that, for each  $D \ni x$ , such a finite collection must exist by compactness. Since  $X$  is effectively compact, the process described above is uniformly computable in  $y$ , and thus  $f^{-1}$  is computable.  $\square$

One useful consequence says that partial inverses also exist under some conditions.

**Corollary 3.31.** *Suppose  $f : C \rightarrow M$  is a computable injective embedding of an effectively compact  $C$  into an effectively compact  $M$ . Then  $f^{-1}$  is computable on  $f(C)$  (when viewed as a map between the induced computable structure on  $f(C)$  and  $C$ ).*

*Proof.* Let  $N$  be the effectively compact induced computable metrization of  $f(C)$  that exists because of Lemma 3.28 and which is furthermore effectively compact by Corollary 3.29 and Proposition 3.26. The map  $f : C \rightarrow f(C)$  can be viewed as a computable map from  $C$  to the induced computable metrization on  $C$ , as follows. When  $(B, C)$  is enumerated in the name of  $f$ , find a basic open ball  $D$  in  $f(C)$  that formally contains  $C$ . The basic open balls in  $C$  are balls with centres that are special in  $f(C)$  but are computable in  $M$ , but formal containment is still c.e. So we enumerate  $(B, D)$  into the new name of  $f$ .

Another way to view this is to replace an  $\epsilon$ -approximation  $x_i \in M$  to  $f(y)$  by an  $2\epsilon$ -approximation  $c_i \in C$  to  $f(y)$ ; it must exist.

To compute  $f^{-1}$ , apply the previous theorem.  $\square$

We also include another nice fact connecting effective compactness with computable closed sets proved by Bratka [15]. The result will be used later (in Corollary 4.3) to clarify one of the well-known applications of effective compactness.

**Theorem 3.32.** *Let  $X$  and  $Y$  be effectively compact spaces and  $f : X \rightarrow Y$ . Then  $f$  is computable if and only if  $\text{graph}(f)$  is effectively closed and if and only if  $\text{graph}(f)$  is computable closed.*

*Proof.* We know that  $X \times Y$  is effectively compact. Suppose  $f$  is computable. Then  $f(x) = y$  is clearly a  $\Pi_1^0$  property.

Now assume  $\text{graph}(f)$  is effectively closed. The subspace  $\{x\} \times Y$  is effectively closed relative to  $x$ , thus  $\text{graph}(f) \cap \{x\} \times Y$  is an effectively closed singleton relative to  $x$ . By Fact 3.22 relativized to  $x$ , given  $x$  we can compute  $(x, f(x))$  and, thus  $f(x)$ .

It remains to note that  $\text{graph}(f)$  is actually c.e. closed for a computable  $f$ , because if a basic open  $B$  (in  $X \times Y$ ) intersects the graph then we will eventually recognise it.  $\square$

We did not really have to assume that  $Y$  is effectively compact; the proof would still work. But of course, by Theorem 3.30 the space  $f(X)$  has to be effectively compact, so we can always replace  $Y$  with  $f(X)$ .

<sup>9</sup>Fix any for every open  $Z$  in  $X$ , its complement is closed and thus is compact. Since the continuous image of a compact set is compact and therefore closed,  $f(X \setminus Z)$  is closed. Since  $f$  is a bijection,  $f(X \setminus Z) = f(X) \setminus f(Z) = Y \setminus f(Z)$ , which makes  $f(Z)$  open.

**3.5. Computable universality of the Hilbert cube.** We now discuss another way of looking at effectively compact spaces, using the *Hilbert cube*  $H$ . In  $H = [0, 1]^\omega$ , we define the distance as  $d((x_i), (y_i)) = \sum_i 2^{-i} d(x_i, y_i)$ . A canonical dense sequence is given by rational sequences that are eventually zero.

The Hilbert cube is a universal space for effectively compact spaces. To see this, recall that all our spaces are complete with respect to their metric. We embed a given compact computably metrized space  $M$  into the Hilbert cube, as follows. Assume the diameter of  $M$  is at most one<sup>10</sup>. Map the  $i$ th special point  $x_i$  into the sequence  $\xi_i = (2^{-k} d(x_i, x_k))_{k \in \omega}$ . We call this map *the canonical embedding of  $M$  into  $H$* .

It is easy to see that this embedding is computable, even though the inverse does not have to be computable. Its image is c.e. closed because it contains a dense computable sequence (more specifically,  $(\xi_i)_{i \in \omega}$ ). We will see that the image does not have to be effectively closed, and hence the image does not have to be computable by Proposition 3.26.

This embedding gives yet another characterisation of effective compactness, but this time up to homeomorphism:

**Theorem 3.33.** *For a computably metrized compact  $M$ , the following are equivalent.*

- (1)  $M$  is homeomorphic to an effectively compact space;
- (2)  $M$  is homeomorphic to a computable closed subset of  $H$ .

*Proof.* Note that  $H$  being a (computable) product of effectively compact spaces is itself effectively compact by Proposition 3.5.

If the space has an effectively compact presentation, then so will be its image under the canonical embedding (Proposition 3.26, Lemma 3.28), and thus the image will be computable closed by Corollary 3.28. On the other hand, if a closed subset of  $H$  homeomorphic to the space is a computable closed subset, then it gives an effectively compact homeomorphic presentation of the space by Proposition 3.26 because  $H$ , being a (computable) product of effectively compact spaces is itself effectively compact.  $\square$

**Remark 3.34.** We note that, by Theorem 3.30, if (the fixed computable complete metrization of)  $M$  is effectively compact, then it is computably homeomorphic to  $f(M)$ , meaning that both  $f$  and  $f^{-1}$  (when restricted to  $M$ ) have to be computable; see Corollary 3.31.

In other words, we can always effectively reduce the study of effectively compact spaces up to homeomorphism to the investigation of computable closed subsets of  $H$ . One pleasant and well-known characterization of computable closed sets in  $H$  is given below.

**Lemma 3.35.** *An closed subset  $C$  of  $H$  is computable if, and only if,  $D(x) = d(x, C)$  is a computable function.*

*Proof.* We can list the basic open balls  $B(x_i, r)$  for which  $D(x_i) > r$ , and they must cover  $M \setminus C$ . But we can also list the balls  $B(x_j, q)$  such that  $D(x_j) < r$ , and these are exactly the basic open balls that intersect  $C$ .  $\square$

Note that we can uniformly list  $r$  such that  $D(x_i) > r$ , i.e.  $D$  is left-c.e. (lower semi-computable) iff  $C$  is effectively closed, and we can uniformly list  $r$  such that  $D(x_i) < r$ , i.e.,  $D$  is right-c.e. (upper semi-computable) iff  $C$  is c.e. closed. We omit details. We also note that there is really nothing special about the choice of  $H$  in the lemma above; it could as well be some other effectively compact space.

The closed subsets of  $H$  give us yet another way to look at effectively compact spaces. Recall that compact spaces correspond to c.e. closed subsets of  $H$ , and effectively compact ones to computable closed subsets of  $H$ . Note that the space of all compact (or closed) subsets  $\mathcal{C}(H)$  of  $H$  is a Polish metric space in which the metric is given by the Hausdorff distance and the countable dense set is given by finite discrete subsets of special points of  $H$ .

**Fact 3.36** (Folklore). *For a c.e. closed subset  $C$  of  $H$ ,  $C$  is computable iff  $C$  is a computable point in  $\mathcal{C}(H)$ .*

*Proof.* If  $C$  is computable then it is effectively compact by Proposition 3.26. If  $(B_i)$  is a finite  $2^{-n}$  cover of  $C$  and  $x_i$  is a special point in  $B_i$ , then every point of  $C$  is at most  $2^{-n+1}$ -far from one of the  $x_i$ .

<sup>10</sup>If it is  $> 1$ , then replace the metric with the new metric  $\frac{1}{n}d(\cdot, \cdot)$ , where  $n$  is a large enough positive integer. You can also redefine the metric to be equal to one on a pair  $x$  and  $y$  if the original distance between  $x$  and  $y$  is greater than 1. The latter method is computably uniform and gives a metric computably compatible with the original one, i.e., the identity map is a computable homeomorphism between the old metrized space and the newly metrized one.

On the other hand, assume  $c \in B_i \cap C$ ; indeed  $c$  is contained in  $B_i$  together with an  $\epsilon$ -ball. Thus, there is  $\epsilon$  so small that any  $D$  that is  $\epsilon$ -close to  $C$  in  $\mathcal{C}(H)$  contains a special point in  $B_i$ . This will be eventually recognised, and thus such  $B_i$  can be computably enumerated, making  $C$  c.e. closed.  $\square$

Of course, there is again nothing really special about  $H$  is the fact above, and it can be replaced by some other effectively compact space if necessary. For instance, one could look at computability of graphs of functions  $f : [0, 1] \rightarrow [0, 1]$  and see that  $f$  is computable iff it can be approximated by piecewise linear functions iff the graph is a computable closed set (cf. Theorem 3.32). Generalizations of this fact can be found in [15].

**3.6. Computable universality of Cantor space.** We assume that our spaces are non-empty Polish ones, and so that our metrics are complete.

We identify  $2^\omega$  with its standard computable presentation of Cantor space by infinite strings under the usual ultrametric. It is well-known that every compact metric space is a homeomorphic image of  $2^\omega$ ; this is the classical Hausdorff-Alexandroff theorem. The computable version of this fact fully characterises effective compactness. As we have already discussed in the introduction, the result is new (as far as we know).

**Theorem 3.37.** *A computably metrized (non-empty) compact  $M$  is effectively compact if, and only if, there is a computable continuous surjective  $f : 2^\omega \rightarrow M$ .*

If  $M$  is a computable image of  $2^\omega$  then it has to be effectively compact by Theorem 3.30. To this end, we therefore assume  $M$  is effectively compact. We give two proofs of the harder direction of the theorem. The first proof exploits the strongest so far combinatorial characterisation of effective compactness given in Subsection 3.2 (specifically, Theorem 3.13) and basically follows the standard textbook argument pretty closely. The second proof is more indirect. It handles the combinatorics differently using a space-filling curve and a technical Lemma 3.38 that is of independent interest.

*The first proof.* Fix a fully  $\cap$ -decidable system of covers  $(K_n)$  of  $M$  that exists by Theorem 3.2. By Remark 3.15, we can use finite basic computable *closed* covers throughout.

We follow the standard classical topological proof very closely, and we use  $\cap$ -decidability of  $\bigcup_n K_n$  throughout. Suppose  $K_0 = \{D_1, \dots, D_k\}$ , and fix  $D_i \in K_0$ . Then  $K_1$  contains finitely many balls that cover  $D_i$ , denote them  $D_{i,j}$ . These can be computed and indeed, these  $D_{i,j}$  are exactly the closed balls in  $K_1$  that intersect  $D_i$ , because the rest are disjoint from it. Define  $\hat{D}_{(i,j)} = D_i \cap D_{i,j}$ .

We can computably proceed by recursion and define  $\hat{D}_\sigma$ , which is a finite non-empty intersection of basic closed balls, where  $\sigma$  ranges over a computably branching tree  $T$  with no dead ends (observe that  $\sigma \neq \tau$  does not necessarily imply that  $\hat{D}_\sigma \neq \hat{D}_\tau$ ). Equip the set of all infinite paths through  $[T]$  with the standard (longest common prefix) ultrametric; then  $[T]$  is computably metrized space in which the dense sequence is given by  $\sigma 1^\omega$ ,  $\sigma \in T$ . We identify  $\sigma$  with the basic clopen ball of  $[T]$  consisting of all strings with prefix  $\sigma$ .

Also recall that, by the construction of  $(K_n)$ , without loss of generality we can assume that basic open balls intersect whenever the respective closed balls intersect, and thus we can always calculate a special point  $x_\sigma$  in each  $\hat{D}_\sigma$  (see Remark 3.15). We could view  $x_\sigma$  to be an  $\epsilon$ -approximation to any path in  $[T]$  extending  $\sigma$ , where  $\epsilon = 2^{-n+1}$  for the largest  $n$  such that a ball from  $K_n$  is mentioned in  $\hat{D}_\sigma$ . So we define  $f(\sigma) = x_\sigma$  with precision  $2^{-n+1}$ .

For an infinite path  $\xi \in [T]$ ,  $\hat{D}_{\xi \upharpoonright n} \subseteq \hat{D}_{\xi \upharpoonright m}$  whenever  $m \leq n$  are prefixes of  $\xi$ , and since the diameter of  $\hat{D}_{\xi \upharpoonright n}$  is at most  $2^{-n+1}$  and it is non-empty, we conclude that

$$\bigcap_n \hat{D}_{\xi \upharpoonright n} = \{\alpha\}$$

for some  $\alpha \in M$ . So we set  $f(\xi) = \alpha \in M$ . It is routine to show that the procedure above defines a computable and surjective  $f : [T] \rightarrow M$ . It is not difficult to see that  $[T]$  is a computable image of  $2^\omega$ ; as we promised in the preliminaries, we include a proof of this well-known fact.

**Claim 1** (Folklore). (1) If  $T$  is a computable, computably branching tree with no dead ends then there is a computable surjective map from  $2^\omega$  onto  $[T]$ .

(2) For every computable, non-empty  $\Pi_1^0$  class  $C$  there is a surjective computable map from  $2^\omega$  onto  $C$ .

*Proof.* We first reduce (2) to (1). Realise  $C$  as the set of (infinite) paths through a computably branching tree  $T$  without terminal nodes, as follows. Computably rearrange  $T$  into a new tree  $\Gamma$  such that  $[T]$  is computably homeomorphic to  $[\Gamma]$  and  $\Gamma$  is (at most) binary. To do so, split a node only if both basic clopen sets associated with the two successors of the node in  $T$  contain elements of  $C$ . (Note that, given a basic clopen set, we can *decide* whether it intersects  $C$ .) Neither  $\Gamma$  nor  $T$  has terminal nodes, and there is a computable homeomorphism between  $C$  and  $[\Gamma]$ .

For (1), we can also reduce the case of an arbitrary computably branching tree to the case when the tree is at most binary. If a node splits into  $n$  successors, where  $n > 2$ , replace it with a gadget in which every node has at most two successors. This gives a computable and (at most) binary tree  $\Gamma$  with no dead ends such that  $[\Gamma]$  is computably homeomorphic to  $[T]$ .

Thus, it remains to prove (1) for such a  $\Gamma$ . Define the map  $g$  from  $2^\omega$  onto  $[\Gamma]$  by recursion. We define the name of  $g$  by mapping clopen sets (on) to clopen sets. We also identify finite strings with the respective clopen sets in both trees.

At a stage  $s$ , assume  $g$  has already been declared on paths/basic clopen sets of length  $s-1$ . Suppose  $g(\sigma) = \tau \in \Gamma$  such that  $|\tau| = |\sigma| = n-1$ . If  $\tau 0$  and  $\tau 1$  both exist in  $\Gamma$ , then set  $g(\sigma 1) = \tau 1$  and  $g(\sigma 0) = \tau 0$ . Otherwise, without loss of generality, only  $\tau 0$  exists. In this case, set  $g(\sigma i) = \tau 1$  for  $i = 0, 1$ . Do that for every string of length  $n$ , and then go to the next stage. The map is clearly computable and surjective, and thus induces a computable surjective map of  $2^\omega$  onto  $C$ .  $\square$

In combination with  $f$  defined above, this gives a computable surjection of  $2^\omega$  onto  $M$ .  $\square$

*The second proof.* Recall that we assumed that  $M$  is effectively compact. Without loss of generality, we can assume it is a computable closed subset of  $H$ ; see Remark 3.34. Recall that there is a computable map from  $2^\omega$  onto  $[0, 1]$ ; for instance, map every infinite sequence  $\sigma$  in  $2^\omega$  into the binary expansion  $\sum_{i \in \omega} 2^{-i-1} \sigma(i)$ . Also, the famous Hilbert's curve computably maps  $[0, 1]$  onto  $H = [0, 1]^\omega$ ; see, e.g., [114] for a *primitive recursive* version due to Schoenberg. (See [5] for a detailed explanation of computability of this particular construction. We also cite [26] for more about computability of space-filling curves.) This gives a computable surjective  $f : 2^\omega \rightarrow H \supseteq M$ , and we know from Lemma 3.21 that  $f^{-1}(M)$  is a  $\Pi_1^0$ -class which, unfortunately, does not have to be computable in general. Recall that, in the beginning of the subsection, we assumed that  $M$  (thus,  $P$ ) is non-empty. If it *were* a computable  $\Pi_1^0$  class, then we would be able to computably subjectively map  $2^\omega$  onto it by Claim 1. But first, we must be in the position to apply the claim. The lemma below helps.

**Lemma 3.38.** *Suppose  $f : 2^\omega \rightarrow K$  is computable, surjective, and  $K$  is effectively compact. If  $P \subseteq K$  is (non-empty) computable closed, then there is a computable  $\tilde{f} : 2^\omega \rightarrow K$  and a computable  $\Pi_1^0$ -class  $C \subseteq 2^\omega$  such that  $\tilde{f}(C) = P$ .*

Furthermore,  $\tilde{f}$  can be chosen so that  $\tilde{f}$  agrees with  $f$  on  $f^{-1}(P)$  and  $\tilde{f}(2^\omega \setminus C) \subseteq K \setminus P$ ; we will also include the verification of these properties in our proof below. Also, we will see that the simultaneous construction of  $\tilde{f}$  and  $C$  is uniform. The proof below is somewhat informal: after all, we have already proven the theorem. We hope that the elementary formal details that are missing should be easy to reconstruct.

*Proof.* This is done as follows. We use a c.e. formal open name of  $f$  and effective compactness of  $K$  and computability of  $P$  throughout. In particular, we will use that  $f$  is uniformly computably continuous. This is done by listing  $2^{-n}$ -covers  $K_n$  of  $K$  and using the formal name of  $f$  to find a cover  $S$  of  $2^\omega$  such that, for every  $\sigma \in S$  there is a  $D \in K_n$  so that  $(\sigma, D)$  is in the name.

Since  $P$  is computable and thus is an effectively compact subspace (see Proposition 3.26), we can assume that we know which open sets in  $K_n$  intersect  $P$  and which do not. To see why, create two lists: one is the list of all finite covers of  $P$ , and the second list includes finite collections of basic open balls in  $K$  that are formally disjoint from *some* cover from the first list. For every  $\epsilon$ , there is a finite  $\epsilon$ -cover  $K' \cup K''$  of  $K$ , where  $K'$  is from the first list and  $K''$  is from the second list<sup>11</sup>. The finite set of balls  $K''$  can be empty, but  $K'$  is never empty. If  $\epsilon = 2^{-n}$  then we can set  $K_n = K' \cup K''$ .

<sup>11</sup>Every point in  $K \setminus P$  is contained in a basic ball formally disjoint from some finite cover of  $P$  by basic open balls. If we take the collection of all such balls around all such  $x \in K \setminus P$  of radius at most  $\epsilon$ , then, together with any (finite) cover of  $P$ , they must cover the whole space  $K$ . Thus, there is a finite subcover in which we can keep all the finitely many  $\epsilon$ -balls that cover  $P$  and intersect  $P$ . By

The construction of  $C$  and  $\tilde{f}$  proceeds as follows.

Suppose at a stage we see that, for some basic open set  $\sigma$  in  $2^\omega$ ,  $f(\sigma) \subseteq B$  for some  $\epsilon$ -ball  $B$  such that  $B \cap P = \emptyset$ . Then declare  $\sigma$  outside of the closed set  $C$  that we build, and let (the name of)  $\tilde{f}$  copy (the name of)  $f$ .

Now suppose, using the same notation and premises,  $B \cap P \neq \emptyset$ . It still possible that  $f(\sigma) \cap P = \emptyset$  (because all we know is that  $f(\sigma) \subseteq B$ , so it can miss  $P$ ), but we do not know that yet. In this case declare that  $\sigma$  intersects  $C$  and proceed to the next stage. (As we made sure above, for any such  $B$  and  $\sigma$  that we use at the stage we can decide whether  $B$  intersects  $P$  or not.)

At the next stage we will find a refined cover of  $K$  and  $2^\omega$  and, in particular, of  $\sigma$ . If at least one such  $\sigma'$  refining (extending)  $\sigma$  has the property that  $f(\sigma') \subset B'$ , where  $B' \cap P \neq \emptyset$  (here  $B'$  is taken from the more refined cover of  $K$  and  $B'$ ) then we let (the name of)  $\tilde{f}$  copy (the name of)  $f$ .

Of course, it is entirely possible that we discover that for *all* of the finitely many extensions  $\sigma'$  of  $\sigma$ , the respective ball  $B'$  (such that  $f(\sigma') \subseteq B'$ ) does *not* intersect  $P$ . However, we have already declared that  $\sigma \cap C \neq \emptyset$ .

In this case, go to the previous stage and find a computable point  $x \in B \cap P$  (that can be uniformly extracted from the proof of Lemma 3.24) and declare

$$\tilde{f}(\xi) = x$$

for every  $\xi \in B$ . In this case we also say that the clopen ball  $B$  is declared *artificially in  $C$* . (This is also the only case when  $\tilde{f}$  does not agree with  $f$ ; we set  $f$  and  $\tilde{f}$  equal in all other cases.)

By construction,  $\tilde{f}$  is well-defined on all of  $2^\omega$  and is computable. We need to argue that it additionally has the properties claimed in the statement of the lemma.

If  $\xi \in B$  then clearly  $\tilde{f}(\xi) \in P$ . If  $\xi \in f^{-1}(P)$  then at no stage it can be declared in a ball which is out or artificially in  $C$ , so it must be that  $\tilde{f}$  and  $f$  agree on  $\xi$ .

If  $\xi$  is such that it is never in any ball that is artificially in or out, it means that for every  $2^{-n}$  there must be a point  $\alpha_n \in P$  that is  $2^{-n}$ -close to  $\tilde{f}(\xi)$ , and we see that  $\tilde{\xi} = \lim_n \alpha_n \in P$  because  $P$  is compact and thus closed.

Finally,  $f(2^\omega \setminus C) \subseteq K \setminus P$  follows from the fact that if a basic clopen  $\sigma$  is declared out of  $C$ , we let  $\tilde{f}$  follow  $f$  in its definition within  $\sigma$ .  $\square$

It follows that  $M$  is a computable surjective image of a computable  $\Pi_1^0$ -class. It remains to map  $2^\omega$  to such class surjectively using Claim 1. This finishes the second proof.  $\square$

Note that, in the second proof, the combinatorics is handled using Lemma 3.38 rather than using  $\cap$ -decidable covers; we did not need them here.

## 4. APPLICATIONS

**4.1. A few elementary applications of  $\Pi_1^0$ -classes.** In this subsection we briefly discuss several applications of effectively closed sets and  $\Pi_1^0$  classes in classical computable analysis. Most of these applications are well-known and are not difficult. However, they serve as a good illustration of the convenience of the techniques we have described in the previous sections. Indeed, in each case we can come up with a brute-force direct proof which would however be much less pleasant and often would give a less general result too.

Consider a computable function  $f : [0, 1] \rightarrow \mathbb{R}$ . Then the set of zeroes  $Z_f = \{x \in [0, 1] : f(x) = 0\}$  is effectively closed in  $[0, 1]$ . Following result shows that any  $\Pi_1^0$  class can be ‘realized’ as the set of zeros of a computable function. This allows for simple applications of  $\Pi_1^0$  classes in real analysis.

**Theorem 4.1** (Nerode and Hwang [100]). *Given a  $\Pi_1^0$  class  $C$  ( $C$  is thought of a subset of the Cantor set) there is a computable function  $f : [0, 1] \rightarrow \mathbb{R}$  whose zeroes  $Z_f = \{x : f(x) = 0\}$  exactly the members of  $C$ .*

*Sketch.* Define a computable function by stages on the Cantor set (linear elsewhere) so that, while an interval is not yet declared out of  $C$ ,  $f$  keeps getting closer to 0, say, from below. If the interval is declared out, freeze the function at this interval and, thus, keep it away from zero. It shall approach zero but only at points that correspond to paths in  $C$ .  $\square$

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our assumption,  $K$  is effectively compact / computable closed, and thus all these conditions are c.e.; we just wait until such a finite  $\epsilon$ -cover of  $K$  is found.

Ignoring the  $\Pi_1^0$  coding, this method of diagonalization sketched above goes back to Specker [122] and is a mainstay of Aberth [1], and arguably has roots in the work of Bishop [9]. Here are well-known easy applications:

**Corollary 4.2.** *For a function  $f : [0, 1] \rightarrow \mathbb{R}$ , let  $Z_f = \{x : f(x) = 0\}$  denote the set of its zeros.*

- (1) *There is a computable function  $f : [0, 1] \rightarrow \mathbb{R}$  with uncountably many zeros and no computable zeros.*
- (2) *If  $f : [0, 1] \rightarrow \mathbb{R}$  is computable and  $Z_f \neq \emptyset$ , then  $Z_f$  contains a low point.*
- (3) *If  $Z_f \neq \emptyset$  is finite then all of its members are computable.*
- (4) *If  $Z_f$  is infinite and countable, then it contains infinitely many computable points.*

*Proof.* (1) Fix an uncountable  $\Pi_1^0$ -class  $P$  without computable points and then apply Theorem 4.1; for example, some  $\Pi_1^0$  class consisting of only Martin-Löf random reals (e.g., Downey-Hirschfeldt [30]).

(2) By Theorem 3.37, there is a computable surjection  $g : 2^\omega \rightarrow [0, 1]$ , and by Lemma 3.38 the pre-image of the effectively closed  $X = f^{-1}Z_f$  is also effectively closed in  $2^\omega$ , that is, it is a  $\Pi_1^0$  class. Using this argument, we easily see that if  $Z_f$  is nonempty then it has a low point, by the Low Basis Theorem.

(3) This is because isolated members of effectively closed sets are computable.

(4) If  $Z_f$  is infinite and countable, and thus it has no perfect kernel. In particular, there must be infinitely many isolated points; apply (3).  $\square$

The elementary corollary below entails that the intersection of two computable closed sets is not computable in general. This unfortunate property of computable closed sets is well-known. The issue is that, although we can list balls that intersect *both* sets, some balls like that can have no points from the intersection of the closed sets.

To see why the corollary implies this counter-intuitive property of closed sets, note that every non-empty computable closed set has at least one computable point (since it has to be effectively overt).

**Corollary 4.3.** *There exist two computable compact subsets of the unit square  $[0, 1]^2$  that intersect but have no computable points in the intersection.*

*Proof.* In view of Theorem 3.32, it is sufficient to take  $C_1 = \text{graph}(f)$  for  $f$  in (2) of the theorem above, and  $C_2 = [0, 1]$  (the “ $x$ -axis”) which is clearly computable too.  $\square$

Another application of  $\Pi_1^0$ -classes, similar technically to (1) of Theorem 4.2 above, it is concerned with Markov computability. As mentioned in the introduction, the original definition of computable function used by Turing in [127], was that  $f$  is computable if it is an effective operator taking (indices of) computable reals to (indices of) computable reals.

**Theorem 4.4** (Specker [122]). *There is a continuous, Markov computable  $f : [0, 1] \rightarrow \mathbb{R}$  such that  $\sup_{x \in [0, 1]} f(x)$  is not computable and, thus,  $f$  is not computable (by, e.g., (1) of Proposition 3.5).*

*Proof.* Similar to the proof of (1) of Theorem 4.2. Fix a  $\Pi_1^0$  class without computable members and define  $f$  on the Cantor set (and linearly elsewhere), but this time make its value approach a left-c.e. non-computable real  $r$ . Note that, if a point  $x$  is computable, then it has to be either on a segment of the Cantor set that was declared out at some stage, or it is on a linear segment connecting two such points that have been declared out. Observe also that each linear segment connects computable points. In either case, we can wait for the point to be listed in the effectively open complement of the homeomorphic image of the  $\Pi_1^0$  class, go to the stage of the construction where that happened, and compute the index of the image.  $\square$

Another more recent application of  $\Pi_1^0$  classes in computable analysis is by Barrett, Downey and Greenberg [6], and concerns Cousin’s Lemma. This is a core lemma in the theory of the Denjoy integral. Recall that a *gauge* is a function  $\delta : [0, 1] \rightarrow \mathbb{R}^+$ . A *tagged partition* is a partition  $0 = a_1 < a_2 < \dots < a_n = 1$  together with a sequence  $z_i \in (a_i, a_{i+1})$ . For a gauge  $\delta$  we say that a tagged partition is  $\delta$ -*fine* iff for all  $i$ ,  $[a_i, a_{i+1}] \subset (z_i - \delta(z_i), z_i + \delta(z_i))$ . Cousin’s Lemma states that for any gauge  $\delta$ , there is a  $\delta$ -fine partition. Using similar codings to those of Theorem 4.2, we have the following theorem.

**Theorem 4.5** (Barrett, Downey, and Greenberg [6]). *There is a computable gauge  $\delta$  with no computable  $\delta$ -fine partition.*

Actually this was stated in [6] as “RCA<sub>0</sub> proves that “Cousin’s Lemma for continuous functions” is equivalent to WKL<sub>0</sub>. Many results in Reverse Mathematics concerning WKL<sub>0</sub> correlate to coding  $\Pi_1^0$  classes. (See Simpson [119].) In passing we remark that in [6], the authors looked also at Borel gauges  $\delta$ . We say that  $f$  is effectively Baire 0 if it is computable, and effectively Baire  $n + 1$ , if it is the pointwise limit of a computable collection of effectively Baire  $n$  functions. In [6] it is shown that if  $\alpha$  is a computable ordinal, then there is a effective Baire 2 function  $f$  such that any  $\delta$ -fine partition computes  $\emptyset^{(\alpha)}$ .

**4.2. The space of isometries.** An isometry is a metric-preserving map. It is clearly continuous. Note that an isometry is always injective, and if  $f$  is surjective then we say that it is an isometric isomorphism. Using a brute-force search, we can easily show that the inverse of a computable isometric isomorphism is always a computable map even if the spaces are not effectively compact. In particular, we do not need to refer to Theorem 3.30 to compute the inverse of an isometric isomorphism.

**Remark 4.6.** However, we can argue that its proof can be used to find a more satisfying way to compute it. For instance, we might be able to argue that a primitive recursive procedure might be possible under the right choice of definitions. The subject of primitive recursive or “punctual” analysis (see [7, 33]) is largely unexplored. Moreover, connections between polynomial time analysis (Ko and Friedman [74], Ko [73]) and compactness also remain to be analysed.

The following result was stated in [89] without proof.

**Theorem 4.7** (Melnikov and Nies). *Suppose  $\mathcal{X}$  is effectively compact and  $\mathcal{Y}$  is computably metrized and is isometrically isomorphic to  $\mathcal{X}$ . Then the collection of all isometric isomorphisms  $\text{Iso}(\mathcal{X}, \mathcal{Y})$  can be viewed as a  $\Pi_1^0$  class.*

*Proof.* We first note that, by Theorem 3.18,  $\mathcal{Y}$  has to be effectively compact as well. Let  $h$  be a computable compactness modulus of  $\mathcal{Y}$  as defined in Definition 2.7. The idea is that we have at most  $h(n)$ -many essential extension of a given partial  $2^{-n+1}$ -isometry from  $\mathcal{X}$  to  $\mathcal{Y}$  since any other choice will be within a  $2^{-n}$ -error. We will also use the fact that every isometric embedding  $\mathcal{X}$  in  $\mathcal{Y}$  has to be onto.

Suppose the special points of  $\mathcal{Y}$  are given by the sequence  $(q_i)_{i \in \mathbb{N}}$ , and let  $(p_i)_{i \in \mathbb{N}}$  be the dense computable sequence in  $\mathcal{X}$ .

We define a tree  $B \subseteq \omega^{<\omega}$ . The  $n$ -th level of  $B$  is given by Gödel numbers of (some) tuples from  $\{q_0, \dots, q_{h(n)}\}^n$ . We view these tuples  $\bar{r} = \langle r_0, \dots, r_{n-1} \rangle$  as possible isometric images of  $\langle p_0, \dots, p_{n-1} \rangle$ , up to an error of  $2^{-n+1}$ . Thus, we require the  $\Pi_1^0$  condition that  $|d_Y(r_i, r_k) - d_X(p_i, p_k)| \leq 2^{-n+1}$  for each  $i < k < n$ . For a tuple  $\bar{u}$  at level  $n$  and a tuple  $\bar{v}$  at level  $n + 1$ , we posit as a further  $\Pi_1^0$  condition that  $v$  is a child of  $u$  if  $d(u_i, v_i) \leq 2^{-n}$  for each  $i < n$ . We let  $B$  consist of all strings  $\sigma$  such that for each  $n < |\sigma|$ ,  $\sigma(n)$  is on level  $n$ , and if  $n > 0$  then  $\sigma(n)$  is a child of  $\sigma(n - 1)$ . Then  $B$  is  $\Pi_1^0$ ; furthermore, clearly there is a function  $\hat{h} \leq_T h$  that bounds any  $f \in [B]$ .

**Claim 2.** Suppose there is an isometric embedding  $\Theta : \mathcal{X} \rightarrow \mathcal{Y}$ . Then  $[B] \neq \emptyset$ .

To see this, let  $f(n)$  be a tuple of special points on level  $n$  that is at distance less than  $2^{-n}$  from  $\langle \Theta(p_0), \dots, \Theta(p_{n-1}) \rangle$ . Then  $f \in [B]$ .

**Claim 3.** Let  $f \in [B]$ . Then  $f$  uniformly computes an isometric embedding  $\Theta_f : X \rightarrow Y$ .

We prove the claim. Note that for each  $i$ , we have a Cauchy name  $s_n^i = f(n)_i$  ( $n > i$ ), namely,  $d(s_n, s_{n+1}) \leq 2^{-n}$ . Thus  $f$  uniformly computes the function  $\Theta_f$  given by  $\Theta_f(i) = \lim_{n > i} f(n)_i$ . For each  $i < k < n$  we have

$$|d_Y(s_n^i, s_n^k) - d_X(p_i, p_k)| \leq 2^{-n+1}.$$

Thus,  $\Theta_f$  is an isometric embedding. □

Again we can now appeal to facts about  $\Pi_1^0$  classes.

**Corollary 4.8.** *For an effectively compact space  $X$ , if  $Y \cong_{iso} X$  then there is a low isometric isomorphism witnessing this.*

Using different methods, Il’jazovich [58] proved a special case of the corollary for the case when  $\text{Aut}_{iso}(M)$  is finite.

**Corollary 4.9.** *Suppose an effectively compact  $X$  has only at most countably many self-isometries. If  $Y \cong_{iso} X$  then there is a computable isometric isomorphism witnessing this.*

*Proof.* This follows from the fact that  $Iso(X, Y)$  must contain an isolated point. To see why, fix an arbitrary  $\psi : X \rightarrow Y$ . Then every isometry  $\phi$  from  $X$  to  $Y$  gives an automorphism  $\psi^{-1}\phi\psi$  of  $X$ , and since there are only countably many isometric isomorphisms, there could be only countably many members in  $Iso(X, Y)$ . A  $\Pi_1^0$ -class that has countably many members must have an isolated member, for otherwise the class would be perfect and thus uncountable. An isolated member of a  $\Pi_1^0$  class is computable.  $\square$

It is natural to ask whether the isomorphism in the corollary above can be reconstructed uniformly from a given pair of compact presentations of the space. The answer to this question is (perhaps, not surprisingly) negative; an intricate example can be found in [46]. The cited paper also contains a subtle definability-theoretic analysis of computably unique metric spaces, i.e., up to computable isometry. It is also known that every compact Polish space admits a Scott sentence of very low complexity; see [89].

Fix an effective listing  $(M_i)$  of all (partial) computable Polish spaces. Each such  $M_i$  is given by a dense sequence that can be identified with  $\omega$  and a (partial) computable metric on it. (The space  $M_i$  represents is the completion of  $M_i$ .) We could list all partial effectively compact spaces in a similar way, but this approach is not standard and has never been used in the literature.

**Corollary 4.10** (Melnikov and Nies [89]). *The following index sets are arithmetical:*

- (1) *The characterisation problem  $\{i : M_i \text{ is compact}\}$ .*
- (2) *The isometric isomorphism problem  $\{\langle i, j \rangle : M_i \cong_{isom} M_j \ \& \ M_i, M_j \text{ are compact}\}$ .*

*Sketch.* For (1), say that the metric is total, is indeed a metric, and for every  $n$  there is a  $2^{-n}$ -cover of the space by closed basic balls.

To see why (2) holds, note that (by Corollary 4.8) it is sufficient to state that there exists a  $0'$ -computable isometry. All conditions that express that it ‘works’ are arithmetical.  $\square$

In contrast with compact spaces, the characterisation problem for locally compact Polish spaces is  $\Pi_1^1$ -complete as conjectured in [89] and then formally clarified in [101]. We will return to index sets later when we discuss computable Stone duality.

We now discuss potential converses to Theorem 4.7. J. Miller (personal communication) suggested the following example.

**Proposition 4.11.** *Let  $A, B$  be disjoint c.e. sets. There are isometric effectively compact computable metric spaces  $L, R$  such that any representation of an isometry computes a set  $S$  such that  $A \subseteq S$  and  $B \cap S = \emptyset$ .*

If we choose  $A, B$  effectively inseparable, then this shows that the  $\Pi_1^0$  class of isometries from  $\mathcal{X}$  to  $\mathcal{Y}$  is Medvedev complete within the  $\Pi_1^0$  classes, meaning that whenever there is a member of some other class of degree  $\mathbf{d}$  then this Medvedev-complete class also has a member of that degree. Thus, the isometries are as complicated as allowed by Theorem 4.7. In particular, there exist isometric effectively compact spaces that are not computably isometric.

*Proof.* We describe the metric spaces by giving a connected graph of special points. If there is an edge the distance is defined directly. All the other distances between pairs of special points  $p, q$  will be given indirectly as the path distance. We ensure in the construction that this distance function is consistent and computable. There will be only one limit point in the space. Thus the space has Cantor-Bendixson rank 2.

Let  $\delta_n = 4^{-n}$ . For each  $n$ , there are special points  $a_n, u_n, v_n, a_{n+1}$  in  $L$  that have pairwise distance  $\delta_n$ . We call this basic configuration the  $n$ -th diamond. The space  $R$  looks similar with special points  $a'_n, u'_n, v'_n, a'_{n+1}$  sharing the same properties.

At any stage of the construction, the procedure  $\text{mark}(x, \gamma)$ , where  $x$  is a special point already introduced, adds a new special point  $y$  with  $d(x, y) = \gamma$ .

*Construction of  $L, R$ .* We may assume at most one number enters  $A \cup B$  at any stage.

If  $n$  enters  $A$  at stage  $s$ , call  $\text{mark}(u_n, 3^{-s})$  and  $\text{mark}(u'_n, 3^{-s})$ .

If  $n$  enters  $B$  at stage  $s$ , call  $\text{mark}(u_n, 3^{-s})$  and  $\text{mark}(v'_n, 3^{-s})$ .

It is clear that  $L, R$  are isometric. Each space is effectively compact: let  $h(n)$  be so large that the special points up to  $h(n)$  include the first  $n + 1$  diamonds, and the points used for marking up to stage  $n$ . Then in either space the special points up to  $h(n)$  form a  $2^{-n}$ -net. Now suppose that  $\Theta: L \rightarrow R$  is an isometry. The special points  $a_n$  and  $a'_n$  ( $n > 0$ ) are singled out by having three points at distance  $\delta_n$ , and three points at distance  $\delta_{n+1}$ . Thus,  $\Theta(a_n) = a'_n$ . Therefore  $\Theta(u_n) \in \{u'_n, v'_n\}$ . Let  $S = \{n: \Theta(u_n) = u'_n\}$ . Then  $A \subseteq S$  and  $B \cap S = \emptyset$  as required.

Since  $d(u'_n, v'_n) = \delta_n$ , using a term sufficiently far out in the Cauchy name for  $\Theta(u_n)$ , we can decide whether  $n \in S$  using the representation as an oracle.  $\square$

We conjecture the following.

**Conjecture 4.12.** Every  $\Pi_1^0$ -class can be realised as  $Iso(M, N)$  for two isometric effectively compact spaces.

Conjecture 4.12 holds for computable discrete algebraic structures in the place of effective compact spaces [40], and we conjecture that it should be possible to turn such structures into compact spaces. As correctly noted in [40] the space of isometries between any computable Polish spaces can be viewed as a  $\Pi_1^0$ -class in Baire space, but since we are only interested in compact spaces and classes we shall not explain this. The details can be found in [39].

**4.3. Basic sequences in Banach spaces.** Some of the most used Polish Spaces are Banach spaces, and there is a reasonably well-developed theory of effective Banach spaces beginning with Pour-El and Richards [110]. We note that Banach spaces are usually viewed under isometric isomorphism, and it is well-known that every isometric isomorphism has to be affine (this is Mazur-Ulam Theorem).

**Definition 4.13.** A *computable Banach space* is a computably metrized Banach space in which the Banach space operations are computable.

This definition means that any computable Banach space needs to be separable, since it needs a computable dense set. We regard this as presented with a computable norm  $\|\cdot\|$ , and for simplicity will consider the space as a complete normed vector space  $B$  over the reals, although the results also work if the field is the complex numbers.

Some consequences of effective compactness, such as an effective version of the open mapping theorem, can be found in [13]. We will give a couple of recent applications of effective compactness to the theory of computable Banach spaces. A simple application of effective compactness is Pour-El Richards [111] result that linear independence in a Banach space is  $\Sigma_1^0$ . To see this, we note that  $\{x_1, \dots, x_n\}$  is independent iff

$$\min_{\lambda_i \in \mathbb{R} \setminus \{0\}} \left\| \sum_{i=1}^n \lambda_i x_i \right\| > 0.$$

But by normalizing we can consider the ball

$$S = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \setminus \{0\} \mid \sum_{i=1}^n |\lambda_i| = 1\},$$

for the quantification, and this is effectively compact, meaning that the minimum  $\min_{(\lambda_1, \dots, \lambda_n) \in S} \left\| \sum_{i=1}^n \lambda_i x_i \right\|$  is  $\Sigma_1^0$ .

Almost all elementary linear algebra works via the fact that vector spaces have bases. However, the analog for Banach spaces is not so easy. The most accepted candidate of a basis for a Banach space is called a *Schauder Basis*.

**Definition 4.14.** A *sequence*  $X = \{x_i \mid i \in \omega\}$  is called a Schauder Basis for  $B$  iff for each  $z \in B$  there is a *unique* sequence  $\{\lambda_i \mid i \in \omega\}$  such that

$$\sum_{i=1}^{\infty} \lambda_i x_i = z.$$

Brattka and Dillhage [16] showed that the theory computable Banach spaces with well-behaved computable Schauder bases<sup>12</sup> is relatively smooth. In particular, many theorems using duality lift quite smoothly to have computable versions.

However, one of the unfortunate aspects of Banach space theory is that not only don't (computable) Banach spaces necessarily have computable Schauder bases, but in fact some don't have Schauder bases *at all*. This is

<sup>12</sup>Specifically, those with "monotone" or "shrinking" bases.

a remarkable result of Enflo [36] who solved Banach's question 45 year old of whether a separable Banach space necessarily has a basis negatively: There is a separable Banach space with no Schauder basis. Bosserhoff [11] proved that Enflo's construction could be made computable to give a computable Banach space.

The fundamental fact about Schauder bases is the following characterization by Banach.

**Lemma 4.15** (Banach). *Let  $X = x_1, x_2, \dots$  be a sequence of elements of  $B$ . Then this sequence forms a Schauder basis iff*

- (1)  $x_i \neq 0$  for all  $i$
- (2) The finite span of  $\{x_i \mid i \in \omega\}$  is dense in  $B$ .
- (3) There is a  $K \in \mathbb{R}$  such that for all  $m < n$ , and all sequences of scalars  $\lambda_i$ ,

$$\left\| \sum_{i=1}^m \lambda_i x_i \right\| \leq K \left\| \sum_{i=1}^n \lambda_i x_i \right\|.$$

$K$  in (3) above is called the *Basis Constant*  $bc(X)$  of the Schauder basis. The hard direction of this lemma is to suppose that  $X$  is a basis and consider the projections  $S_k(\sum_{i=1}^{\infty} \lambda_i x_i) = \sum_{i=1}^k \lambda_i x_i$ . We need to prove that  $\sum_j \|S_j\|$  is finite, and this is achieved by considering the equivalent norm  $\|\cdot\|'$  defined by  $\|\sum_{i=1}^{\infty} \lambda_i x_i\|' = \sup_m \|\sum_{i=1}^m \lambda_i x_i\|$ . This is bounded by the Open Mapping Theorem.

Note that we can also define the basis constant of a *space* as the infimum of the basis constants of Schauder bases for the space. We remark that there are even finite dimensional spaces without a basis with  $K = 1$ . But in the finite dimensional case, we can at least get a computable basis constant for the whole space. That is, a simple application of effective compactness yields the following:

**Lemma 4.16.** (1) (Bossorhoff [11]) *Let  $X$  be a computable Banach space and  $x_1, \dots, x_n$  an independent sequence. Then  $bc(x_1, \dots, x_n)$  is a computable real.*  
 (2) (Downey, Long and Greenberg [81]) *For a finite dimensional  $X$ ,  $bc(X)$  is a computable real.*

Notice that Lemma 4.16 means that each finite dimensional projection in the proof of the existence of  $K$  must have a computable basis constant. Therefore the sup  $K$  is a computable sup of computable reals. That is  $K$  is a left c.e. real for a computable Schauder basis  $X$ . In fact, in Long's MSc thesis it is shown that any left c.e. real can be the basis constant of a computable Banach space. It is presently unknown what can be said about the basis constants of computable *spaces*.

The one theorem we will look at is the following. It provides a counterpoint to the theorem of Metakides and Nerode [90] where they constructed a computable vector space over  $\mathbb{Q}$  where every c.e. independent set is finite.

**Theorem 4.17** (Downey, Long and Greenberg [81]). *If  $B$  is an infinite computable Banach space, then  $B$  has an infinite dimensional subspace with a computable Schauder basis  $Z = z_1, z_2, \dots$ .*

*Sketch.* Let  $E = \{e_i \mid i \in \omega\}$  be an effective dense set for  $B$ . We begin with a lemma of Mazur: If  $B$  is an infinite dimensional Banach space and  $Y$  a finite dimensional subspace,  $\epsilon > 0$  then there is  $x \in B$  with  $\|x\| = 1$  and

$$\|y\| \leq (1 + \epsilon)\|y + \lambda x\|,$$

for all  $y \in B$  and  $\lambda \in \mathbb{R}$ . Since  $E$  is dense, it is not hard to show that we may choose  $x = e_i$  for some  $i$ , by playing with the triangle inequality, choosing one close to  $x$ . Now we can follow the classical argument of Banach. Choose a sequence of reals  $\epsilon_i$  with  $\prod_{i=1}^{\infty} (1 + \epsilon_i) < \infty$ . Then construct the basic sequence in stages. Having constructed  $z_1, \dots, z_n$ , find an  $e_i$  in the effectively dense sequence  $E$  with  $bc(x_1, \dots, x_n, e_i) \leq \prod_{i=1}^{n+1} (1 + \epsilon_i)$ , and we know by effective compactness, that this procedure is computable.  $\square$

We will return to Banach spaces a bit later, when we discuss the effective content of Banach-Stone duality that establishes a 1-1 correspondence between computable presentability of Banach spaces in a broad class with effectively compact presentability of totally disconnected compact spaces.

**4.4. Computable Stone duality with applications.** Recall that two basic open balls  $B(c_1, r_1)$  and  $B(c_2, r_2)$  are formally disjoint if  $r_1 + r_2 < d(c_1, c_2)$ . Two sets of basic open balls are formally disjoint if any pair of basic open balls, one coming from the first set and the other from the second, are formally disjoint. A *clopen split* of  $M$  is a pair of (cl)open sets  $X, Y$  such that  $X \sqcup Y = M$ .

*Computable Stone duality.* Various versions of the elementary lemma below can be found in [17, 51, 54, 85], but in some of the cited papers the proof contains minor but misleading errors, thus we give a proof.

**Lemma 4.18.** *Suppose  $X$  can list all basic finite covers of  $M$ . Then  $X$  can list all clopen splits of  $M$ .*

*Proof.* Suppose  $M = X \sqcup Y$  is a split, and let  $\delta$  be the infimum-distance between these compact open sets

$$\delta = \inf_{(x,y) \in X \times Y} d(x,y).$$

(Since  $X \times Y$  is compact and  $d$  is continuous, it attains its infimum at some pair  $(x_0, y_0)$ . In particular,  $\delta > 0$ .)

Suppose  $0 < \epsilon < \delta/4$ . Then every finite  $\epsilon$ -cover will consist of two formally disjoint subsets of basic open balls. Indeed, every ball covering a point in  $X$  cannot contain a point in  $Y$ , and every ball covering a point in  $Y$  cannot contain a point in  $X$ . If a basic open  $B$  has its centre in  $X$  and  $D$  has its centre in  $Y$ , then the distance between their centres is at least  $\delta$ , while the sum of their radii is at most  $\delta/2 < \delta$ , making them formally disjoint.

On the other hand, if a finite open cover of  $M$  consists of two formally disjoint subcovers, then these subcovers induce a split of  $M$  into clopen components. Since the property of being formally disjoint is a c.e. property,  $X$  is able to list all such covers.  $\square$

Another way to state the lemma above is that any modulus of compactness of  $M$  can computably enumerate the clopen splits of  $M$ . Also note that we could have used  $\cap$ -decidable covers in the proof of the lemma above, and this way we can additionally assume that, for the clopen sets that we list, we can additionally decide whether they intersect or not. This will be very convenient in the proof of the next result.

**Theorem 4.19** ([54]). *Let  $M$  be an effectively compact Stone space (a totally disconnected compact Polish space). Then the Boolean algebra of its clopen subsets admits a computable presentation.*

*Proof.* Fix a  $\cap$ -decidable system of covers  $K = \bigcup_{n \in \omega} K_n$ . Using the previous Lemma 4.18, effectively list all clopen splits of  $M$  into (open, formally disjoint names of) pairs of clopen sets. Let  $(X_i, Y_i)$  be the enumeration of these clopen splits. Note that we can also wait and see whether both  $X_i$  and  $Y_i$  are non-empty; just wait for a special point to appear in one of the two. In this case we say that the split is *proper*. This is a c.e. event because  $X_i$  and  $Y_i$  are both given by their open as well as their closed covers, whichever is more convenient. Thus, without loss of generality we can assume that we list only the proper splits. Write  $X_i^{-1}$  for the corresponding  $Y_i$  in a proper split; and let  $X_i^0$  be another notation for  $X_i$ .

The Boolean algebra is generated by the empty set and arbitrary finite *non-empty* conjunctions of the form  $\bigwedge_i X_i^{\epsilon_i}$ , where  $\epsilon_i \in \{0, 1\}$ . Since the system of covers  $K$  is  $\cap$ -decidable, we can indeed decide whether such a finite intersection is empty. In other words, if  $F$  is the free Boolean algebra generated by the  $X_i$ , then the Boolean algebra of clopen sets of  $M$  is isomorphic to  $F/I$ , where  $I$  is a computable ideal. This makes the Boolean algebra computable.  $\square$

**Remark 4.20.** For an effectively compact Stone space, we can *uniformly* produce the dual Boolean algebra of its clopen sets. (As usual, we assume our spaces are non-empty. Alternatively, we can stretch our terminology a bit and view the 1-element lattice as a Boolean algebra.) It is also not difficult to see that the construction is also *locally uniform* in the following sense. Using clopen components we can produce a computably branching, computable tree  $T$  without dead ends such that the space is homeomorphic to  $[T]$ . If we take the usual ultrametric on the infinite paths through  $T$ , then this metric will be computably compatible with the original metric in the sense that  $Id : M \rightarrow [T]$  is a computable map.

This result allows us to establish the following representation theorem.

**Theorem 4.21** ([51]). *Let  $M$  be a computably metrized Stone space. Then the Boolean algebra of its clopen subsets admits a computable presentation.*

*Proof.* Let  $M$  be the computable space. Recall that  $0'$  can compute a modulus of continuity of the space. Relativize the previous theorem to get  $0'$ -computable presentation of the Boolean algebra of clopen sets. Given an element of the Boolean algebra, we can use its representation via a finite union of basic computable balls and ask whether there exist two unequal special points  $x, y$  that are contained in this clopen set; this is a  $\Sigma_1^0$ . This element is an atom if, and only if, no such pair of points exists. Thus, the atom relation is also  $0'$ -computable. It is well-known that every

$0'$ -computable Boolean algebra in which the atom relation is also  $0'$ -computable has a computable presentation; see [72] for the explicit statement and [32] for the first implicit use of this property. It follows that the Boolean algebra of clopen sets has a computable copy.  $\square$

Given a computable Boolean algebra  $B$ , it is not difficult to represent its dual Stone space  $\widehat{B}$  as the collection of infinite paths  $[T]$  through a computably branching, computable tree  $T$  without dead ends; see [43] for the details. Thus, we have:

**Theorem 4.22** ([51, 54]). *For a countable Boolean algebra  $B$ ,  $B$  has a computable presentation iff its dual Stone space  $\widehat{B}$  can be computably metrized iff  $\widehat{B}$  has an effectively compact metrization.*

**Corollary 4.23** ([51]). *Every computably metrized Stone space is homeomorphic to an effectively compact Stone space.*

*Computable topological vs. computable Polish spaces.* Working under the supervision of Nerode, Tran Ying-Ying recently proved:

**Theorem 4.24** ([126]). *For a countable Boolean algebra  $B$ , the following are equivalent:*

- (1)  $B$  admits a c.e. presentation.
- (2) The dual Boolean algebra  $\widehat{B}$  is homeomorphic to  $\Pi_1^0$ -class.

*Idea.* To see why (1)  $\Rightarrow$  (2) should hold, think of a  $B = A/I$ , where  $A$  is atomless and  $I$  is a c.e. ideal. One can view  $A$  as a full binary tree in which some nodes can be declared equal. Instead of making them equal, we can declare them to be *outside* of a  $\Pi_1^0$  tree  $T$  such that the Stone space of  $B$  is  $[T]$ . To understand why (2)  $\Rightarrow$  (1) should hold, define a c.e. ideal by setting a node  $x$  to be equal to the left-most node  $y$  at the same level that still remains in the tree (representing the  $\Pi_1^0$  class).  $\square$

Working independently, Bazhenov, Melnikov and Harrison-Trainer [8] proved another closely related computable version of Stone duality. To state it, note that the basic notion of effective compactness (Definition 3.1) can be defined without the assumption that the metric is computable<sup>13</sup>. An example of such a right-c.e. “effectively compact” space is any non-empty  $\Pi_1^0$  class. (This requires a relatively straightforward inductive argument that can be found in, e.g., [8].)

**Theorem 4.25** ([8]). *For a countable Boolean algebra  $B$ , the following are equivalent:*

- (1)  $B$  admits a c.e. presentation.
- (2) the dual Boolean algebra  $\widehat{B}$  is homeomorphic to an effectively compact right-c.e. completely metrized space.

*Idea.* The proof of (1)  $\Rightarrow$  (2) is similar to the proof of (1)  $\Rightarrow$  (2) the previous theorem. (A non-empty  $\Pi_1^0$  class can be viewed as a right-c.e. metrized space.)

To prove (2)  $\Rightarrow$  (1) use a theorem of Odintsov and Selivanov [104] who showed that a  $\Pi_2^0$ -presented Boolean algebra admits a c.e. presentation. It is therefore sufficient to show that the Boolean algebra of clopen sets has the form  $\beta/I$ , where  $\beta$  is the atomless algebra and  $I$  is its  $\Pi_2^0$  ideal. This can be done using methods similar to the techniques described in this section; we omit the details.  $\square$

Recall that every right-c.e. space is a computable topological space (Proposition 2.4). Recall also that Feiner [38] constructed an example of a c.e. presented Boolean algebra that does not admit a computable presentation. Thus, Theorems 4.24 and 4.25 imply:

**Corollary 4.26** ([8]). *There exists a computable topological Polish space that is not homeomorphic to any computably metrized topological space.*

In spite of appearing as a standard classical result, Corollary 4.26 is very recent. We really have a stronger consequence. The corollary above follows from the one below because every right-c.e. completely metrized space is a computable topological space (Proposition 2.4).

<sup>13</sup>Definition 3.1 also works for computable topological spaces: just say that we can effectively enumerate all finite covers of the space by basic open sets.

**Corollary 4.27** ([8]). *There exists a right-c.e. metrized space not homeomorphic to any computably metrized Polish space.*

*Sketch.* We do not need the full strength of right-c.e. Stone duality to see why the corollary holds. The Stone space of a c.e. presented Boolean algebra is isomorphic to a  $\Pi_1^0$  class by Theorem 4.24. Then argue that a non-empty  $\Pi_1^0$  class admits a right-c.e. complete ultrametric. It remains to take Feiner's Boolean algebra and apply Theorem 4.24 and Theorem 4.22.  $\square$

In fact, the right-c.e. space in the corollary above is effectively compact in the more general sense (as discussed above). In view of Theorem 4.24, it follows also that effective compactness for computably metrized and right-c.e. metrized spaces differ up to homeomorphism.

*Applications to classification problems.* The next corollary measures the classification problem for compact computable Polish spaces up to homeomorphism. Recall that we fixed an effective listing  $(M_i)_{i \in \omega}$  of all (partial) computable Polish spaces.

As was pointed out by Selivanov (in personal communication with the second author), it seems that the corollary below has actually never been explicitly stated in the literature. The corollary is, of course, not really new. Compare the corollary below with Corollary 4.10.

**Corollary 4.28.** *The homeomorphism problem  $\{(i, j) : M_i \cong_{\text{hom}} M_j \ \& \ M_i, M_j \text{ are compact}\}$  for computable compact Polish spaces is  $\Sigma_1^1$ -complete.*

*Proof.* The  $\Sigma_1^1$ -hardness follows from the  $\Sigma_1^1$ -completeness of the isomorphism problem for computable Boolean algebras [44] and the fact that Stone duality is computably uniform.

We need to argue that the index set is  $\Sigma_1^1$ . We give only a sketch since this fact seems to be well-known. A rather similar fact is folklore in descriptive set theory [42]. A detailed proof in the lightface case (and in the harder context of topological groups) can be found in [85]. A very closely related (and, in some sense, stronger) argument is Lemma 3.6 of [50].

It is arithmetical to say that  $M_i$  is a (presentation of a) compact Polish space; see (1) of Corollary 4.10. To say that there is a homeomorphism  $f : M_i \rightarrow M_j$ , it is sufficient to state that there exist continuous surjective  $f_1 : M_i \rightarrow M_j$  and  $f_2 : M_j \rightarrow M_i$  such that  $f_1 \circ f_2 = \text{Id}_{M_i}$ . Every  $g : X \rightarrow Y$  between compact  $X$  and  $Y$  can be represented by, e.g., a pair  $(\tilde{g}, m)$  where  $\tilde{g} : \omega^2 \rightarrow \omega$  and  $m : \omega \rightarrow \omega$ , where the function  $\tilde{g}(n, k)$  is interpreted as the image of the  $n$ th special point with precision  $2^{-m}$ , and  $m$  as the modulus of uniform continuity.

It is arithmetical to say that  $(\tilde{g}, m)$  represents a continuous function  $\lim_k g(\cdot, k) : X \rightarrow Y$ . This is because totality is arithmetical, and also one can express that  $m$  is a modulus of continuity that works for  $\tilde{g}$  as a closed property. Thus, as before, if it fails then it must fail for some special points. Since the continuous image of a compact space is closed, it is arithmetical to say that  $(\tilde{g}, m)$  represents a surjective function. (If it does not, then again there is a special point in the complement witnessing this.)

This allows to state the existence of  $f_1$  and  $f_2$  in a  $\Sigma_1^1$  way. Finally, to say that  $f_1 \circ f_2 = \text{Id}_{M_i}$ , it is sufficient to say that it is true for special points because the property is (again) closed. This can be expressed arithmetically (in the presentations of)  $f_1$  and  $f_2$ .  $\square$

Of course, the result above can be relativized to any oracle, thus implying the classical descriptive-theoretic result saying that the homeomorphism problem for compact Polish spaces is analytic complete.

**Remark 4.29.** The  $\Sigma_1^1$ -completeness in the proof above is witnessed by effectively compact Stone spaces. It is also not difficult to see that the index set of Stone spaces is arithmetical, and thus the homeomorphism problem is actually complete within the class of Stone spaces (that can further be assumed effectively compact). To see why saying that  $M_i$  is a Stone space is an arithmetical property, iterate the process of splitting the space and search for a non-trivial connected component using Lemma 3.6. Every connected component can be expressed as a finite union of basic open balls, and thus the existence of a non-trivial connected component can be expressed as a first-order, arithmetical property.

A *substantially different proof* of Corollary 4.28 can be extracted from a closely related result for topological groups established in [85]. A computably metrized topological group is a computable Polish space with computable

group operations defined on the space. We say that it is effectively compact if the space is furthermore effectively compact.

**Theorem 4.30** ([85]). *The topological isomorphism problem for computably metrized connected compact abelian groups is  $\Sigma_1^1$ -complete.*

The proof of the result above uses a computable version of *Pontryagin duality* that was proved in [85] and has recently been extended in [82]; we omit the definitions. In fact, it follows from the version of computable duality established in [82] that the  $\Sigma_1^1$ -completeness is witnessed by effectively compact groups. We now explain how the theorem above gives a different proof of Corollary 4.28. As we shall discuss later in the paper (see Subsection 4.7, the second proof of Theorem 4.38), using algebraic topology it is possible to show that two compact connected abelian Polish groups are homeomorphic (as spaces) if, and only if, they are isomorphic as topological groups. Assuming this result, Corollary 4.28 follows from the theorem above, but this time the corollary is witnessed by connected effectively compact spaces, not totally disconnected effectively compact spaces.

To conclude the subsection, we mention one more application of effectively compact Stone spaces, this time to Banach spaces. Recall that *computable Banach space* is a computably metrized Banach space in which the Banach space operations are computable. Recall that Banach-Stone duality states that  $C(K_0; \mathbb{R})$  and  $C(K_1; \mathbb{R})$  are linearly isometrically isomorphic if, and only if, the respective compact spaces  $K_0$  and  $K_1$  are homeomorphic. In [8], a computable version of Banach-Stone duality has been established:

**Theorem 4.31.** *For a countable Boolean algebra  $B$ ,  $B$  has a computable copy iff the space  $C(\widehat{B}; \mathbb{R})$  is a computable Banach space.*

In view of Theorems 4.24 and 4.25, this correspondence gives an explicit application of effective compactness to computable Banach space theory. In particular, we obtain the following corollary. Fix a computable list  $(B_i)_{i \in \omega}$  of all (partial) computable linear spaces over  $\mathbb{Q}$  with a computable norm, and write  $\overline{B_i}$  for the completion of  $B_i$  with respect to its norm.

**Corollary 4.32.** *The linear isometric isomorphism problem  $\{(i, j) : \overline{B_i} \cong_{iso} \overline{B_j}\}$  for computable separable Banach spaces is  $\Sigma_1^1$ -complete.*

*Proof.* We have that  $C(\widehat{B_0}; \mathbb{R}) \cong_{iso} C(\widehat{B_1}; \mathbb{R})$  iff  $\widehat{B_0} \cong_{hom} \widehat{B_1}$  (iff  $B_1 \cong B_2$ ). The proof of aforementioned result from [8] is uniform when passing from effectively compact Stone spaces to the respective Banach spaces. So  $\Sigma_1^1$ -hardness follows from the previous corollary.

It remains to note that the upper bound is also  $\Sigma_1^1$ . It is sufficient to state that there is an isometry that works for special points, maps zero to zero, and is, furthermore, surjective (these properties are closed). The well-known Mazur-Ulam theorem asserts that every isometry with these properties has to be linear.  $\square$

Further results and open questions related to index sets in analysis, and to compact spaces and compact groups more specifically, can be found in [34].

**4.5. Profinite groups.** Profinite groups are the Galois groups. They are inverse limits of finite groups. The study of effective profinite groups began with Metakides and Nerode [91], La Roche [79, 80] and Smith [120, 121], where they defined the group to have a *co-r.e. presentation* if it was isomorphic to a computably bounded  $\Pi_1^0$  class  $[T]$  (in Baire space) where the group operations were computable. La Roche and Smith defined the group to be recursively presentable (computable) if the set of extendible nodes in  $T$  forms a computable set. If  $F$  is a computable (countable) field, and  $K$  is a c.e. subfield of  $F$ , then the Galois group  $G(F \setminus K)$  is a co-r.e. profinite group. As Smith [121] observed, Waterhouse's result [129] can be effectivized to show that each co-r.e. profinite group is effectively isomorphic to  $G(F \setminus K)$  for some computable  $F$  and  $K$ .

A profinite group is *recursive* if it can be represented as the projective limit of computable linear sequence of finite groups  $(F_i)$  given by their strong indices and computable *surjective*  $f_i : F_{i+1} \rightarrow_{onto} F_i$ . Smith [121] showed that a profinite group is recursive if, and only if, it is isomorphic to a decidable computably bounded  $\Pi_1^0$  class  $[T]$  where the group operations were computable. Since every infinite profinite group is homeomorphic to  $2^\omega$ , and any two effectively compact presentations of  $2^\omega$  are computably homeomorphic (e.g., [8]), without loss of generality we can assume that  $[T] = 2^\omega$ .

A natural question arises what happens when the metric on the group is not necessarily an ultrametric. For example, it is sometimes convenient to think of abelian profinite groups as subgroups of an infinite direct power of the unit circle group.

We give a classification of profinite groups with an arbitrary compatible metric that have a recursive presentation. The result below is new. A similar result that establishes an arithmetical bound without the assumption of effective compactness can be found in [85]. Recall that the notion of an effectively compact group was defined before Theorem 4.30.

**Theorem 4.33.** *For a profinite group  $G$ , the following are equivalent:*

- (1)  $G$  has a recursive presentation;
- (2)  $G$  has an effectively compact presentation.

*Proof.* Clearly, every recursive presentation is effectively compact. Now assume we are given an effectively compact presentation. Using Lemma 4.18, computably list all clopen components of the group. At this stage there are two ways we can proceed to prove the theorem.

*The first possibility* is to use the materials of the previous section to construct a computably branching tree  $T$  with no dead ends such that the domain of  $G$  is  $[T]$ , and note that the natural shortest-prefix ultrametric inherited from the tree is computably compatible with the original metric (in the sense that the identity map  $G \rightarrow [T]$  is computable; see Remark 4.20). Then we can use the aforementioned result of Smith and conclude that  $G$  admits a recursive presentation (i.e., via a surjective linear computable inverse system). We will not give any further details.

*The second possibility* is to directly calculate the recursive presentation without any reference to effective compatibility and the result of Smith. To make the paper self-contained, we give the details below.

To say that a clopen component is a normal subgroup, use the fact that every clopen component is a computable subspace of the group, and thus is effectively compact, by Proposition 3.26. To see if a clopen  $C$  is a subgroup, search for a pair of finite covers, say  $(B_i)$  and  $(D_j)$ , of  $C$  such that for every  $i, j$  there is a  $k$  with the property

$$B_i \cdot B_j' \subseteq D_k$$

and for every  $i$  there is a  $k$  such that

$$B_i^{-1} \subseteq D_k.$$

We also search for a finite cover  $(U_n)$  of  $G$  such that for all  $n, m$  and  $i$  there is a  $k$  with

$$U_n^{-1} \cdot B_i \cdot U_m \subseteq D_k.$$

We argue that such a cover exists, and this will imply that we can computably list all clopen subgroups of  $G$ . Then we explain how to use these subgroups to build a recursive presentation of the group.

Since the clopen component  $C$  can be expressed as a (finite) union of open balls, the preimage of the clopen component under the computable maps  $x, y \rightarrow xy$ ,  $x \rightarrow x^{-1}$  and  $z, x \rightarrow z^{-1}xz$  in the respective product space (respectively,  $C \times C$ ,  $C$ , and  $G \times C \times G$ ) can be uniformly listed. If  $C$  were not a normal subgroup then there will be special points witnessing this, and these would be witnessed together with sufficiently small basic open balls containing them. On the other hand, if  $C$  is a normal subgroup then every equation of the form, say,

$$z^{-1}xz = y,$$

where  $x, y \in C$  and  $z \in G$ , would be witnessed by small enough basic open balls containing these points, i.e.,

$$U^{-1} \cdot B \cdot U \subseteq D,$$

where  $z \in U$ ,  $x \in B$ , and  $y \in D$ . These products of these balls would give a cover of the respective compact product space (in the case of conjugation and in the notation above,  $B \times U \times D$  cover  $G \times C \times G$ .) It follows that we can find a finite subcover.

We conclude that we can list all clopen normal subgroups of  $G$ . Note that, by the uniform effective compactness of each such clopen  $C$ , we can compute the diameter of  $C$ , which is  $\sup_{x, y \in C} d(x, y)$ . Using the techniques of Lemma 4.18 and Theorem 4.19 – that basically can be summarised by saying that we take the next cover by very

small balls – we can furthermore produce a nested sequence of (finite open names of) clopen normal subgroups  $\{C_i : i \in \omega\}$  such that:

- (1)  $C_{i+1} \subseteq C_i$  is formal<sup>14</sup>,
- (2)  $\text{diam } C_i < 2^{-i}$ ,
- (3) for every  $i$  there exists a computable finite tuple  $(x_{i,j})$  of special points (given by its strong index) such that  $(x_{i,j}C_i)$  is a cover of  $G$ .
- (4) For every  $i, j, n$ , if  $x_{i,j}C_{i+1} \subseteq x_{i+1,n}C_{i+1}$  then this inclusion is formal.
- (5) When  $j \neq j'$ ,  $x_{i,j}C_{i+1} \cap x_{i,j'}C_{i+1} = \emptyset$ .

If we succeed, then  $\bigcap_{i \in \omega} C_i = \{0\}$ , so it is a uniformly computable basis of clopen normal subgroups of  $G$ . We will then use the cosets to calculate the finite  $G/C_i$  and the homomorphisms from  $G/C_{i+1}$  onto  $G/C_i$ .

More formally, we proceed by recursion. Assume  $C_{i-1}$  has been defined. We search for a  $C_i$  that satisfies all these four conditions. If we drop ‘formal’ in all these conditions, then it should be clear that such a  $C_i$  and  $x_{i,j}$  must exist. Then fix such a  $C_i$ .

By Lemma 4.18 and the analysis of normality above, a normal clopen  $C_i$  will eventually be found, and furthermore both  $C_i$  and the finitely many cosets mod  $C_i$  will be represented as a finite collections of balls. Our task it to show that we can effectively recognise that these finite parameters describing the cosets define what we need. For that, we might need to adjust the finite covers by refining them so that, for instance, the inclusion is witnessed by formal inclusion of covers. This is done as follows.

We satisfy (1) by choosing the radii of a finite cover describing  $C_i$  small (see Remark 3.4), and we satisfy (2) by evaluating the computable diameter of the clopen set (this is again essentially done by further refining the cover). Here we use that  $C_i$  is indeed a computable closed set because of Lemma 4.18, so we can apply Proposition 3.26.

We elaborate why we will eventually find special points  $(x_{i,j})$  and will eventually recognize that they satisfy (3). For that, note that each coset of  $C_i$  is open, and thus in particular contains a special point, say  $x$ . In particular, every coset mod  $C_i$  has the form  $xC_i$ . Since for every special  $x$  its coset  $xC_i$  is the image of  $C_i$  under the computable map  $y \rightarrow xy$  and  $C_i$  is effectively compact with all possible uniformity, by Lemma 3.28 we conclude that  $xC_i$  is also effectively compact, and with all possible uniformity. By refining the cover of  $xC_i$  (see Remark 3.4), we can ensure that all set-theoretical inclusions of  $xC_i$  into the clopen sets seen so far in the construction hold formally. We can also ensure that if two cosets do not intersect then this is also witnessed formally<sup>15</sup>. This gives a way of computably recognising condition (5). We can also wait for finitely many such special points  $x_{i,j}$  so that the respective cosets  $x_{i,j}C_i$  cover the whole space.

To reconstruct the computable operation on  $G/C_i$ , calculate the product and the inverse on the special points  $x_{i,j}$  with a sufficient precision until you see that the result is in one of the cosets modulo  $C_i$ . This is all computable because the cosets  $x_{i,j}C_i$  are (uniformly) given by their finite open covers, and the operations on  $G$  are computable.

Finally, use effectiveness of condition (4) to calculate the surjective group-homomorphism  $\phi_i : G/C_{i+1} \rightarrow G/C_i$  that maps every  $x_{i+1,j}C_{i+1}$  to the unique coset  $x_{i,j'}C_i$  that contains it. This gives a computable surjective inverse system

$$(G/C_i, \phi_i)_{i \in \omega}$$

the (inverse, projective) limit of which is topologically isomorphic to  $G$ . Since the system is uniformly computable (in the sense of strong indices of finite sets), this gives a recursive presentation of  $G$ .  $\square$

**Remark 4.34.** In view of the results in the previous subsection that connect c.e.-presented Boolean algebras with  $\Pi_1^0$ -classes, we (strongly) conjecture that co-c.e. presented profinite groups should correspond to effectively compact right-c.e. metrized groups. Melnikov [85] gives the first example of a profinite computably metrized group that does not admit a recursive presentation. In view of our theorem above and the results of Smith, the notion of a recursive profinite group seems to be the ‘‘right’’ notion of computability for profinite groups. See [85] for a complete

<sup>14</sup>Meaning that each ball from the open cover of  $C_{i+1}$  is formally included into some basic open ball in the fixed cover of  $C_i$ . Similar for condition (4) below.

<sup>15</sup>Just take the radii of open balls much smaller than the pairwise distances between the finitely many clopen sets to see that it can be done. It is crucial here that the sets are clopen, see, e.g. Corollary 4.3 for a potential issue in general.

description of computably categorical profinite abelian groups and an effective version of Pontryagin duality that works for such groups.

**4.6. Computability of Čech cohomology.** The earliest application of simplicial homology in computable analysis we are aware of can be found in J. Miller's thesis [93] (this application has already been discussed above). Simplicial (co)homology is computable in its nature, and this can be made formal. For example, Chapter 1 (§11) of [98] contains a careful verification of the computability of the homology groups for finite simplicial complexes. This of course entails computability of cohomology groups as well. More specifically, given a (strong index of a) simplicial complex, we can uniformly compute its  $i$ th homology group represented as  $\bigoplus_{i \leq k} \langle a_i \rangle$ , where  $a_0, \dots, a_k$  are the generators of the group such that the orders of the cyclic  $\langle a_i \rangle$  are also uniformly computable. Since  $A^i = \text{Hom}(A_i, \mathbb{Z})$ , we can easily observe that that respective cohomology groups are also computable in this strong sense.

In this section we extend these results to arbitrary effectively compact spaces and their Čech cohomology groups that will be defined shortly. For a finite simplicial complex, its Čech cohomology is isomorphic to its simplicial cohomology; see that last chapter of [98]. One of the convenient features of Čech cohomology is that it does not rely on triangulation and works for an arbitrary compact metric space.

*Background from algebraic topology.* Given a compact  $M$ , let  $\mathcal{N}$  be the directed set of all its finite open covers (under refinement). Since the covers by basic  $\epsilon$ -balls, where  $\epsilon$  ranges over positive rationals, are cofinal among all covers, without loss of generality we can restrict ourselves only to covers by basic open balls with rational radii. For instance,  $\mathcal{N}$  could be the  $\cap$ -decidable system of covers nested under formal refinement instead of the usual refinement (at this stage, computability of these conditions is not important).

For each member  $C$  of  $\mathcal{N}$ , recall that its *nerve*  $N(C)$  is the collection of all sets in the cover that intersect non-trivially. One can view  $N(C)$  as a (finite) simplicial complex in which the  $n$ -dimensional faces are exactly the  $n$ -element subsets  $X$  of  $N(C)$  such that  $\bigcap\{Y : Y \in X\}$  is a non-empty set. For these finite simplicial complexes we can define their *cohomology groups*  $H^*(N(C))$  (with coefficients in  $\mathbb{Z}$ ) as follows.

We follow §73 of [98] and define the Čech cohomology group of a compact metrized space as follows. For a fixed finite set of basic open balls  $C \in \mathcal{N}$  and the respective simplex  $N(C)$ , define the simplicial chain complex as usual:

$$\dots \rightarrow_{\delta_3} A_2 \rightarrow_{\delta_2} A_1 \rightarrow_{\delta_1} A_0$$

where  $A_i$  are finitely generated free abelian groups and  $\delta_i$  are boundary homomorphisms, and then define the associated cochain complex  $A^i = \text{Hom}(A_i, \mathbb{Z})$  and define  $d_i : A^i \rightarrow A^{i-1}$  to be the dual homomorphism of  $\delta_{i+1}$ . Then  $H^i(N(C)) = \text{Ker}(d_i)/\text{Im}(d_{i-1})$  is the  $i$ th cohomology group<sup>16</sup> of the simplex  $N(C)$  which is a finitely generated abelian group which can be thought of as given by finitely many generators and relations. Let  $H^*(M)$  be the direct limit of  $H^*(N(C))$  induced by the inverse system  $\mathcal{N}$  under the refinement maps.

*The result.* Recall that a (discrete, countable) group is c.e.-presented if it is isomorphic to a factor of a computable free group by its computably enumerable subgroup. In other words, the operations of the group are computable by equality is c.e., thus the name.

**Theorem 4.35** ([82]). *For an effectively compact  $M$ , its  $i$ th Čech cohomology group admits a c.e. presentation uniformly in  $i$ .*

A version of this proof for effectively compact spaces can be found in [82], and similar result for computable Polish spaces (and with a simpler proof, but giving merely  $0'$ -computable nerves) is contained in [84]. The proof in [82] relies on a new constructive version of Čech cohomology that was designed to circumvent the following obvious obstacle: for a given cover, we cannot (in general) compute its nerve. However, Theorem 3.13 tells us that this difficulty can be circumvented even if we use the standard notion of a nerve. Thus, we do not need to out-source to the notationally heavy apparatus of algebraic topology compared to which the somewhat tedious proof of Theorem 3.13 looks rather tame.

*Proof of Theorem 4.35.* As we noted above, we can assume that we are given a system of  $2^{-n}$  covers  $\mathcal{N}$  that is linearly nested under formal inclusion and is  $\cap$ -decidable; by Theorem 3.13 and Remark 3.15 this can be done computably. We say that a sequence of finitely generated uniformly computable abelian groups  $(A_j)$  is *strongly*

<sup>16</sup>We are not really interested in the case when  $i = 0$  when  $d_{-1}$  is not really defined; just assume  $\text{Im}(d_{-1}) = \{0\}$ .

*completely decomposable* if each  $A_i$  uniformly splits into a direct sum of its cyclic subgroups, and furthermore the sets of generators of the cyclic summands are given by their strong indices.

Fix a  $\cap$ -decidable finite cover  $C$ .

**Claim 4.** The groups  $H^i(N(C))$  are strongly completely decomposable (uniformly in  $C$  and  $i$ ).

*Proof.* The finite complex  $N(C)$  is computable because the cover  $C$  is  $\cap$ -decidable. A close examination of the definitions shows that, given  $C$  (as a finite set of parameters) and  $i$ , we can compute the generators of  $A^i = \text{Hom}(A_i, \mathbb{Z})$  and compute  $d_i$ . We will need the fact below which is well-known; see [41] for a proof.

**Fact 4.36.** Let  $G \leq F$  be free abelian groups. There exist generating sets  $g_1, \dots, g_k$  and  $f_1, \dots, f_m$  ( $k \leq m$ ) of  $G$  and  $F$ , respectively, and integers  $n_1, \dots, n_k$  such that for each  $i \leq k$ , we have  $g_i = n_i f_i$ .

We can computably find the set of generators  $(a_j)$  of  $\text{Ker}(d_i)$  and a set of generators  $(b_s)$  of  $\text{Im}(d_{i-1})$  such that for each  $s$  there is an integer  $m$  and an index  $i$  such that  $ma_i = b_s$ ; we know that such generators exist so we just search for the first found ones. It follows that the factor  $H^i(N(C)) = \text{Ker}(d_i)/\text{Im}(d_{i-1})$  is strongly completely decomposable with all possible uniformity.  $\square$

Recall that a group admits a  $\Sigma_1^0$  presentation if it is isomorphic to a factor of a computable group by a  $\Sigma_1^0$  subgroup.

**Claim 5.** The direct limit  $\lim_{C \in \mathcal{N}} H^i(N(C))$  admits a  $\Sigma_1^0$  presentation.

*Proof.* We can list the  $\epsilon$ -covers and decide whether two given basic open balls intersect in the listed covers. The refinement relation between two covers  $C \subseteq_{\text{form}} C'$  in  $\mathcal{N}$  induces a simplicial map between the respective nerves  $N(C)$  and  $N(C')$ , and this induces a homomorphism between the respective cohomology groups  $H^i(N(C)) \rightarrow H^i(N(C'))$ . By Claim 4, these finitely generated abelian groups are effectively completely decomposable uniformly in  $C$  and  $i$ . Note that  $\text{Im} \phi$  is generated in  $H^i(N(C'))$  by the images of the generators of  $H^i(N(C))$ . Similarly to the proof of Claim 4, choose new generators of  $H^i(N(C'))$  and  $\text{Im} \phi$  so that the latter are integer multiples of the former. In particular, it is easy to see that  $\text{Im} \phi$  is a computable subgroup of  $H^i(N(C'))$ . This means that we can augment  $\text{Im} \phi$  with extra generators in a computable way to expand it to  $H^i(N(C'))$ . It follows that

$$\lim_{C \in \mathcal{N}} H^i(N(C)) = H^i(G)$$

can be consistently defined as the “union” of the  $H^i(N(C))$ ,  $C \in \mathcal{N}$ , to obtain a group in which the operations are computable and the equality is  $\Sigma_1^0$ . (The equality is merely  $\Sigma_1^0$  because an element  $a \in H^i(N(C))$  can be mapped to 0 in some  $H^i(N(C'))$  which appears arbitrarily late in the directed system.)  $\square$

This finishes the proof of the theorem.  $\square$

We will also see that in Theorem 4.35 “uniformly c.e. presented” cannot be improved to “uniformly computably presented”; this is Corollary 4.41. (In fact, in the example given in the next subsection each individual cohomology group will actually have a computable presentation, just not uniformly so.)

*Applications.* Perhaps the most significant application to date is computability of Pontryagin duality for effectively compact connected groups; see [82]; we omit the definitions. Further applications of computability of cohomology include various index set results in topology; see [82]. One sample result is:

**Corollary 4.37** ([82]). *The index set of solenoid spaces<sup>17</sup> is arithmetical among all compact Polish spaces.*

It follows that in any class of compact manifolds in which non-homeomorphic members have non-isomorphic cohomology groups (for some  $i$ ), the homeomorphism problem is arithmetical provided that the respective cohomology groups have arithmetical isomorphism problem. The significance of this fact is that, in general, even if a computably metrized manifold admits a triangulation, it is not known whether it always admits an arithmetical

<sup>17</sup>A solenoid (space) is a compact connected topological space which the inverse limit of a system  $(S_i, f_i)$  with  $f_i : S_{i+1} \rightarrow S_i$ , where each  $S_i$  is a circle and  $f_i$  is the map that uniformly wraps  $S_{i+1}$   $n_i \geq 2$  times around  $S_i$ . These constructions are important in the area of hyperbolic dynamical systems.

triangulation<sup>18</sup>. Indeed, even in the seemingly trivial case of compact surfaces, producing an arithmetical triangulation based entirely on the given metric takes some 18 Turing jumps [50]. It is believed that complexity is likely close to being optimal. In contrast, calculating Čech cohomology groups allows to completely avoid triangulation (that does not even have to exist, let alone an arithmetical triangulation). Some further discussion can be found in [50].

In the next subsection we discuss how Čech cohomology can be used to find a computably metrized space not homeomorphic to an effectively compact one.

**4.7. Computably metrized spaces not homeomorphic to effectively compact ones.** As we noted in the introduction, finding an example of a computably metrized compact space not isometric to an effectively compact one is easy: just take  $[0, \alpha]$  for a left-c.e. non-computable real  $\alpha$ . The situation is significantly more difficult if we want to work up to homeomorphism. For instance, we have seen that every computably metrized Stone space is homeomorphic to an effectively compact one, so no such example can be found among totally disconnected compact Polish spaces.

It seems that constructing such an example necessarily requires some relatively advanced techniques. A few years ago, a closely related result was established by Bosserhoff and Hertling [12]: For any  $n \geq 2$  there exists a c.e. compact subset  $C \subseteq \mathbb{R}^n$  such that  $\phi(C)$  is not computable compact for any self-homeomorphism  $\phi$  of  $\mathbb{R}^n$ . However, the result and the techniques that were used to establish it are restricted to  $\mathbb{R}^n$ . Hoyrup, Kihara and Selivanov [54] were the first to announce a general construction of a computably metrized space that is not homeomorphic to any effectively compact space. Using completely different techniques, a connected example has recently been suggested in [82].

We outline two constructions of a compact computable space not homeomorphic to any effectively compact space. The first proof is more similar to what Hoyrup, Kihara and Selivanov [54] announced; it will be given in almost complete detail. The second proof can be found in [82]. It produces a relatively natural example using Pontryagin- van Kampen duality; because too much background is necessary to fully explain the proof, we will only briefly sketch it here. Both proofs rely heavily on Čech cohomology. Finally, we will briefly discuss whether we can completely avoid cohomology to prove the theorem below.

**Theorem 4.38.** *There exists a computably (completely) metrized compact Polish space not homeomorphic to any effectively compact space.*

*First proof of Theorem 4.38.* The proof is very similar to the one given in Hoyrup, Kihara and Selivanov [54]. The proof that we give here replaces the most complex definability part of their proof with an argument that involves computability of Čech cohomology first established in [82] and then improved in the proceeding subsection.

The definition below “encodes” a set into a space. We view an isolated point as a 0-sphere  $S^0$ .

**Definition 4.39.** For a set  $X \subseteq \omega$ , let  $CP(X)$  be the one-point compactification of the disjoint union of spheres  $S^k$ , with infinitely many copies for each  $k \in X$ .

One way to think about  $CP(X)$  is as follows. Fix a fishbone (a 1-atom) in the Cantor space  $2^\omega$  in which every isolated path (an atom) is replaced with a copy of  $S^k$  for some  $k \geq 0$ .

**Proposition 4.40.**

- (1)  $X$  is  $\Sigma_3^0$  if, and only if,  $CP(X)$  is computably (completely) metrizable.
- (2)  $X$  is  $\Sigma_2^0$  if, and only if,  $CP(X)$  admits an effectively compact presentation.

*Proof.* (1) Recall that using a modulus of compactness we can find a split of a computable Polish space into two clopen subspaces, as explained in Subsection 4.4. Recall also that  $0'$  can compute a modulus of compactness. Also, a space is connected if it does not have a non-trivial clopen split. Thus,  $0''$  can produce a uniform list of computable indices of the clopen connected subspaces of  $CP(X)$ . Each such index is a finite collection of (open or closed) open balls that contain (only) the component. For a (finite) simplicial complex, Čech cohomology groups are isomorphic with the respective simplicial cohomology groups, see the final chapter of [98]. For the  $n$ -sphere we have:

$$H^p(S^n) \cong \mathbb{Z} \text{ iff } p = 0, n,$$

<sup>18</sup>Classically, classification of manifolds works via triangulations.

and it vanishes otherwise.

Suppose that we know that a computably metrized  $M$  is homeomorphic to  $S^k$ ,  $k > 0$ , but we do not necessarily know what this  $k$  is. The modulus of compactness of each connected component of  $M$  is  $0'$ -computable uniformly in the finite set of parameters that isolates this component. Since the Čech cohomology groups are uniformly  $\Sigma_2^0$ -presentable (by Theorem 4.35 relativized to  $0'$ ), in this case  $M \cong S^k$  is equivalent to saying that  $\check{H}^k(M)$  contains at least one non-zero element, which is a  $\Sigma_3^0$  property (the equality in the group is  $\Sigma_2^0$ ). It follows that  $0''$  can list the components for which the  $k$ th cohomology group is non-trivial, and it can also list such  $k > 0$ . The set of these  $k$  is equal to  $X$ .

Now we prove that, given  $X \in \Sigma_3^0$ , we can produce a computable metric on  $CP(X)$ . Represent  $\Sigma_3^0$  as the set of all  $k$  such that  $\exists x \exists^\infty y R(x, y, k)$ , for some computable predicate  $R \subseteq \omega^3$ ; see [113]. Say we are testing whether  $k \in X$ ,  $k > 0$ . For each existential witness  $x$  corresponding to  $k$ , create a new component and do more steps in making it look like  $S^k$ . This is done by enumerating more points into the components when more  $\exists^\infty$ -witnesses are found for the given existential witness  $x$ ; abandon the finitely many points until the next expansionary stage.

This gives a uniformly computable sequence of spaces, each space is either finite discrete or is equal to  $S^k$ . We can make sure there are infinitely many copies  $S^k$  for each  $k \in X$ . Put them together (uniformly shrink the  $i$ th component by  $2^{-i}$  and use an ultra-metric of the Cantor space to define the distance between different components).

(2) It follows from the material in Subsection 4.4 that, using effective compactness we can list indices (names) clopen connected components of a (compact) space using  $0'$ . It also follows from Proposition 3.27 that each component can be viewed as a computable closed subset of  $M$  (being a finite union of computable sets), and thus it is also effectively compact uniformly in its name.

Effective compactness makes the Čech cohomology groups uniformly c.e.-presented, and saying that it is not trivial is now merely  $\Sigma_2^0$ . This makes  $X$  a  $\Sigma_2^0$ -set.

For the other direction, given a  $\Sigma_2^0$  infinite  $X$ , represent it via  $\exists \forall$ . Assume we are guessing whether  $k \in X$ . For each  $\exists$ -witness keep building an effectively compact copy of  $S^k$  unless a counterexample to the universal quantifier is found (in which case abandon the component). Then put the spaces together as before, but this time observe the space is effectively compact since each component can be easily made uniformly effectively compact.  $\square$

It remains to fix a  $\Sigma_3^0$ -complete set  $X$ .  $\square$

**Corollary 4.41.** *Theorem 4.35 cannot be improved to state that the Čech cohomology groups are uniformly computably presented.*

*Proof.* Let  $X$  be  $\Sigma_2^0$ -complete and consider  $CP(X)$  defined above. Then  $CP(X)$  admits an effectively compact presentation. In fact, the construction of  $CP(X)$  is based on the uniform construction of a sequence of effectively compact disjoint components, each being either  $S^k$  or a finite union of isolated points.

Assume that the Čech cohomology groups were uniformly computably presented in general (and for these components in particular). It is well-known that the cohomology of a finite disjoint union is the direct sum of the cohomologies of the components; this follows from Mayer–Vietoris sequence calculations [98]. Thus, the components that are a finite union of isolated points will have trivial cohomology groups for  $i > 0$ . Also, by assumption, saying that the  $i$ th computable cohomology group has a non-zero element is now  $\Sigma_1^0$ . Thus, to decide if  $i \in X$  ( $i > 0$ ) it is sufficient to ask whether there is a component whose  $i$ th cohomology group is non-trivial, and this is also  $\Sigma_1^0$  contradicting the choice of  $X$ .  $\square$

As we already mentioned above, Khisamiev [70] showed that every c.e. presented torsion-free abelian group has a computable presentation. All known proofs of this result are non-uniform, but the only non-uniformity comes from deciding whether there is a non-zero element on the group. As a byproduct of the proof of the corollary above, it follows that this obstacle cannot be circumvented.

The basic idea of the first proof above was to code information into connected components of the space. Producing a connected example seems to necessarily require some advanced techniques that are outside the scope of this paper. We outline an argument that relies on the recent results from [82, 84] and which gives a connected example of covering dimension 1. We leave many terms and notions undefined, see [82, 84] for the definitions and more explanation.

*Second proof (sketch).* For a discrete torsion-free abelian group  $G$ , the 1st Čech cohomology group of the space of its compact *connected* Pontryagin - van Kampen dual  $\widehat{G}$  is isomorphic to  $G$ ; see, e.g., Part 5 of Chapter 8 of [53]. Using the aforementioned result of Khisamiev and two new computable versions of Pontryagin - van Kampen duality from [82] and [84] we can conclude that, for some broad enough class of groups, namely  $q$ -divisible groups,  $G$  has a  $\Delta_2^0$ -presentation iff  $\widehat{G}$  is computably metrizable, and  $G$  has a computable presentation iff  $\widehat{G}$  has an effectively compact presentation. As there are plenty of  $\Delta_2^0$   $q$ -divisible groups that have no computable presentation (including examples having  $X$ -computable copies iff  $X$  is non-low [83, 84]), the result follows. Indeed, we can find a subgroup of the rationals with this property; this will give a connected example of a solenoid space that satisfies the theorem. In particular, there exist examples like that having covering dimension 1.  $\square$

We *conjecture* that one can completely avoid using homological algebra to prove Theorem 4.38. We suspect that one way to do this would rely on a combinatorially involved construction similar to one that can be found in [51]. We outline a plan of this argument; a detailed verification would take too much space (if it works).

*Cohomology-free proof idea (for Theorem 4.38).* An  $n$ -star is a the Wedge sum of  $n$ -copies of the unit interval (identify the left most-points of  $n$  copies of  $[0, 1]$ ). The basic idea here is to replace  $n$ -spheres with  $n$ -stars in the previously discussed proof of Theorem 4.38. A 0-star is just an isolated point. One can use a fairly basic technique of  $\epsilon$ -chains to produce a  $\Sigma_4^0$ -enumeration of the set of  $n \in \mathbb{N}$  such that the space has an  $n$ -star component. As has been suggested by Ng (personal communication with the second author), under the assumption of effective compactness, this definition should become  $\Sigma_3^0$ . It remains to prove that, given a  $\Sigma_4^0$ -set, we can produce a computably metrized (compact) space that codes the set into its  $n$ -star components. This requires a relatively involved priority construction that can be viewed as a  $0'''$ -argument; see [51]. In [51], we can find a construction of this sort that produces a locally compact space. As explained in [51], we can use the 1-point compactification of this space to produce a compact space.  $\square$

It is not clear at all whether this approach (if it works) is any simpler than the approach that uses cohomology since it relies on a  $0'''$ -argument.

**4.8. Computable universality of  $C[0,1]$ .** Fix the standard computable presentation of  $C[0,1]$  under the supremum metric given by piecewise linear functions with finitely many rational breaking points. We should note that there are also “non-standard” computable presentations of this space that are isometric but not computably isometric to the standard one. We also cite [8, 88] for further results about the computability-theoretic aspects of this space. The theorem below has recently been established in [5] using a direct combinatorial argument. We give a new proof that uses effective compactness to sort out the combinatorics.

**Theorem 4.42.** *Every computably (completely) metrized Polish space can be computably isometrically embedded into  $C[0,1]$ .*

*Proof.* We computably embed the Urysohn space  $\mathbb{U}$ ; it is known (and is not hard to show) that it is computably universal in the sense that every computable Polish space can be computably isometrically embedded into the Urysohn space; see [5, 69]. (There is no ambiguity here because the Urysohn space admits a unique computable presentation, up to computable isometry [87].)

Recall that  $\mathbb{Q}\mathbb{U} = (p_i)_i$ , the rational Urysohn space, is dense in  $\mathbb{U}$ . It is also known that the distances between special points  $p_i$  and  $p_j$  are rational numbers uniformly computable from  $i, j$  (as fractions).

In the Hilbert cube, basic open box is a product of intervals only finitely many of which are open rational sub-intervals of  $[0, 1]$  and the rest are  $[0, 1]$ . It is clear that basic open boxes are effectively open. We computably adjust the metric in the Hilbert cube and view it as  $H = [-d(p_0, p_n), d(p_0, p_n)]^\omega$ , and thus adjust the notion of a basic open box accordingly.

Observe that

$$-d(p_0, p_n) \leq d(p, p_n) - d(p, p_0) \leq d(p_0, p_n),$$

for any  $p_n$ . Say that a point  $\xi$  in  $H = [-d(p_0, p_n), d(p_0, p_n)]^\omega$  corresponds to  $p_j$  if the projections  $\pi_n(\xi)$  of the point to the edges of the cube are exactly the

$$\gamma_n(p_j) = d(p_j, p_n) - d(p_j, p_0),$$

and let  $U$  be the collection of all such points. We can effectively enumerate  $U$  as a sequence of computable points.

**Lemma 4.43.**  $P = cl(U)$  is computable closed.

*Proof.* Since we can list  $U$  which is dense in  $cl(U)$ , by Lemma 3.24 it is sufficient to show that  $cl(U)$  is effectively closed. One way to do this is as follows.

Say that reals  $d_1, \dots, d_k$  are *legit* if there is a real  $d$  such that

$$\{d(p_i, p_j), d, d_1 + d, \dots, d_k + d, i < j \leq k\}$$

is a diagram of a metric space (on points  $p_i, p$ ), where  $d_i = d(p, p_i) - d(p, p_0)$  and  $d = d(p, p_0)$ . Recall also that  $\mathbb{U}$  has the extension property, in other words if there is a finite metric space extending  $p_0, \dots, p_k$  then it is isometrically embeddable to  $\mathbb{U}$  over  $p_0, \dots, p_k$ . The existence of a 1-point extension  $p, p_0, \dots, p_k$  is equivalent to saying that  $d_0, \dots, d_k$  are legit, where  $d_i = d(p, p_i) - d(p, p_0)$ , as witnessed by  $d = d(p, p_0)$ .

**Claim 6.** We can decide whether a given basic open box in  $H$  contains a legit tuple.

*Proof.* There is a first-order formula in the language of  $(R, +, \times, 0)$  that says that, for some real  $d$  and reals  $d_1, \dots, d_k$  that range between some *fixed* rational parameters (describing the intervals in a given open box),  $\{d(p_i, p_j), d, d_1 + d, \dots, d_k + d : i, j < k\}$  is a diagram of a metric space. Recall that  $d(p_i, p_j)$  are rational. Using Tarski's elimination of quantifiers, we can computably find an equivalent quantifier-free formula with rational parameters. It follows that the property is decidable because equality and order are decidable for rational numbers.  $\square$

Recall that  $\mathbb{QU} = (p_i)_i$ , the rational Urysohn space, is dense in  $\mathbb{U}$ . Since  $\mathbb{QU}$  is dense in the Urysohn space, a basic open box of  $H$  (determined by the projection onto the first  $k$  coordinates) contains a tuple of legit reals if, and only if, it contains  $d(p_j, p_i) - d(p_j, p_0)$ , where  $p_j$  is sufficiently close to  $p$ , where  $p \in \mathbb{U}$  are such that  $d_i = d(p, p_i) - d(p, p_0)$  and  $d = d(p, p_0)$  witness that the tuple is legit.

Since we can decide whether a basic open box is free of legit tuples, it follows that we can decide which basic open box contains no  $(\gamma_i(p_j))_i$  for any  $j$ . Since the collection of such sequences  $U$  is dense in  $cl(U)$ , and since basic open boxes are uniformly effectively open and form a basis of topology in  $H$ , we conclude that we can effectively enumerate the complement of  $cl(U)$ .  $\square$

We effectively identify  $2^\omega$  with the ternary Cantor set  $\mathcal{C}$  in  $[0, 1]$  (in the sense that we do not distinguish between these computably homeomorphic spaces; see also Theorem 3.30). Using Theorem 3.37 and Proposition 3.26, fix a computable surjective  $g : \mathcal{C} \rightarrow P$ , and let  $g_i = \pi_i g$ , where  $\pi_i$  is the  $i$ th projection in the cube. Define a computable  $f_i$  to be equal to  $g_i$  on the Cantor set, and to be linear otherwise (this standard technique was discussed in Subsection 4.1). We define a computable embedding of  $\mathbb{U}$  into  $C[0, 1]$  by mapping  $p_i$  into  $f_i$ .

To show that the map  $p_n \rightarrow f_n$  is isometric, first note that

$$d(p_i, p_k) = (d(p_k, p_i) - d(p_k, p_0)) - (d(p_k, p_k) - d(p_k, p_0)) = \gamma_i(p_k) - \gamma_k(p_k) = f_i(t_k) - f_k(t_k),$$

where  $t_k$  is any pre-image (under  $g$ ) of the point in  $H$  corresponding to  $p_k$ . It thus follows that

$$d(p_i, p_k) \leq d_{sup}(f_i, f_k).$$

On the other hand, for any  $i, k \in \omega$  and any  $t \in \mathcal{C}$  with  $g(t) = \lim_j(\gamma_s(p_j))_s \in cl(U)$ , we have

$$f_i(t) - f_k(t) = \pi_i(\lim_j(\gamma_s(p_j))_s) - \pi_k(\lim_j(\gamma_s(p_j))_s) = \lim_j \gamma_i(p_j) - \lim_j \gamma_k(p_j) = \lim_j (d(p_j, p_i) - d(p_j, p_k)).$$

By the triangle inequality we get  $|d(p_j, p_i) - d(p_j, p_k)| \leq d(p_i, p_k)$  for every  $j$ , and therefore

$$|f_i(t) - f_k(t)| \leq d(p_i, p_k),$$

for any  $t \in \mathcal{C}$ . By the definition of  $f_n$ , any maximum difference must be attained on  $\mathcal{C}$ . Thus

$$d_{sup}(f_i, f_k) \leq d(p_i, p_k).$$

Since the maps  $p_i \rightarrow f_i$  are uniformly computable in  $i$ , it follows that the isometry defined above for  $(p_i)_{i \in \omega}$  induces a computable isometric embedding  $\mathbb{U} \rightarrow C[0, 1]$ .  $\square$

#### 4.9. Covering dimension and embeddings into $\mathbb{R}^n$ .

**Definition 4.44.** The covering dimension of  $M$  the least  $n \in \mathbb{N} \cup \{\infty\}$  such that every open cover of  $M$  has a refinement of order  $n + 1$ , i.e., each point belongs to at most  $n + 1$  sets.

We know that every compact space is a subspace of the Hilbert cube; it is also well-known that a compact space of covering dimension  $n$  can be homeomorphically embedded into  $\mathbb{R}^{2n+1}$ . Is this also computably true? One pleasant application of  $\cap$ -decidable covers is the following theorem that answers the question in the affirmative:

**Theorem 4.45.** *Let  $M$  be a  $n$  effectively compact Polish space of covering dimension  $n$ . Then there is a computable homeomorphic embedding of  $M$  into  $\mathbb{R}^{2n+1}$ .*

The proof is an improved version of a result of Melnikov and Harrison-Trainor [50] that states that every computably metrized compact Polish space of covering dimension  $n$  can be  $0'$ -computably homomorphically embedded into  $\mathbb{R}^{2n+1}$ . The proof that we give below is more subtle and relies heavily on a classical argument from [108] but with some modifications. It uses computability of nerves, so strictly speaking  $**$ -effective compactness would be enough to run the proof.

*Proof.* We say that a continuous  $f : M \rightarrow \mathbb{R}^{2n+1}$  is an  $\epsilon$ -homeomorphism if  $f^{-1}(x)$  has diameter at most  $\epsilon$  for every  $x$  in the range. We will need to prove a computable version of the following well-known fact:

**Fact 4.46.** The set of  $\epsilon$ -homeomorphisms form a dense open set in  $C[M, \mathbb{R}^{2n+1}]$ .

Let's first explain how at least one  $\epsilon$ -homomorphism can be found. The proof below is an adaptation of the argument that can be found in [108] (see Theorems 4 and 5), however, our definition of an  $\epsilon$ -homomorphism is a bit different.

Fix  $\epsilon = 2^{-m}$  for some  $m$ . Construct a computable  $\epsilon$ -homeomorphism of  $M$  to  $\mathbb{R}^{2n+1}$  as follows. Use Theorem 3.13 and fix a strongly  $\cap$ -decidable basis of computable balls  $K$  in  $M$ , recall that this means that the non-emptiness of intersection (of finite families) is decidable. (By Remark 3.15, we could alternatively use computable closed balls with the same centres and same radii.)

- (1) Find an open  $\epsilon$ -cover  $C_1, \dots, C_k$  of  $M$  having order  $n + 1$ , where each  $C_i$  is a finite union of open computable balls from  $K$ .
- (2) Compute the nerve  $N$  of  $C_1, \dots, C_k$ .
- (3) Find special points  $c_1, \dots, c_k$  in  $\mathbb{R}^{2n+1}$  and a (geometrical) simplicial complex on vertices  $c_1, \dots, c_k$  isomorphic to  $N$  via a simplicial map which maps vertices to vertices.
- (4) Define  $d_i(x)$  as follows. First, assume  $C_i = \cup_{j \in J_i} B(k_{i,j}, r_{i,j})$  and set  $d_{i,j}(x) = \sup\{r_i - d(x, k_{i,j}), 0\}$ . (In Theorem 4 of [108] Pontryagin uses the distance from  $x$  to the complement of  $C_i$ .) Then define

$$d_i(x) = \sup_{j \in J_i} d_{i,j}(x),$$

let  $u(x) = d_1(x) + \dots + d_k(x)$ . Noting that  $u$  is strictly positive (because  $C_1, \dots, C_k$  cover the space), set  $\theta_i(x) = d_i(x)/u(x)$ , and finally define

$$f(x) = \sum_i \theta_i(x) c_i.$$

We will argue that  $f$  is an  $\epsilon$ -homomorphism of  $M$  to  $\mathbb{R}^{2n+1}$  with some additional properties. But first, we argue that the steps above can be performed computably. The first step is possible because *some*  $\epsilon$ -cover  $C'_1, \dots, C'_r$  having order  $n + 1$  exists. By compactness, each  $C'_i$  can be replaced with a finite union  $C_i$  of (closed or open) basic computable balls from  $K$  of radii at most  $\epsilon$  and that are contained in  $C'_i$ , and so that together  $C_1, \dots, C_r$  cover the space. The new cover  $C_1, \dots, C_r$  has order at most the order of  $C'_1, \dots, C'_r$  because  $C_i \subseteq C'_i$ . We conclude that, in (1), such a cover *exists* among finite subsets of  $K$ . We can decide intersection for computable balls in  $K$ , and therefore we can also decide which finite families (representing  $C_i$ ) intersect. The diameter of each  $C_i$  can be easily computably estimated from above<sup>19</sup>. Together with  $\cap$ -decidability of  $K$  this implies that, given  $\epsilon = 2^{-m}$ , we can

<sup>19</sup>This can be done using, e.g., the distances between the centres and the radii. Alternatively, replace all computable balls in  $C_i$  with the respective closed balls that are computable closed sets by Proposition 3.27. Then use that the union of finitely many computable closed sets is computable, and that the diameter is a supremum of a computable function defined on the effectively compact space; see Proposition 3.5. This gives an arbitrary tight upper estimate on the diameter of  $C_i$ , with all possible uniformity.

computably search for such an  $\epsilon$ -cover satisfying (1). This also implies that the nerve formed in the second step is uniformly computable (as a finite object, i.e., is uniformly given by its strong index).

The third step is possible because in topology we prove that finite simplicial complexes are realizable in Euclidean spaces, this is done using points in a general position, and being in a general position is an open property<sup>20</sup>. If the (combinatorial) dimension of the simplex is  $n$  then this can be done in  $\mathbb{R}^{2n+1}$  using points that are in general position, which is a c.e. property. See, e.g., Theorem 3 of [108]. There is much freedom in the choice of points in general position, in particular, those can be found in any collection of open balls  $V_1, \dots, V_k$  in  $\mathbb{R}^{2n+1}$ ; this is Theorem 2 of [108]. Theorem 1 of [108] also implies that  $c_1, \dots, c_k$  can be chosen special. Searching for a simplicial map is a finitistic task and can be done in finite time.

The third step uses elementary properties of computable functions to define a computable  $f$ .

The proof that  $f$  is an  $\epsilon$ -homeomorphism is essentially literally the same as the proof of the analogous property of the  $\epsilon$ -homeomorphism constructed in the proof of Theorem 4 of [108], because our function satisfies the same properties (sufficient to prove that it is an  $\epsilon$ -homeomorphism) as the function built in the proof of the aforementioned Theorem 4 of [108]. More specifically,  $\theta_i$  is continuous and has support  $C_i$ , and also for every  $x$  we have that  $\sum_i \theta_i(x) = 1$ . For instance, it follows that a face in the nerve of the cover is mapped to the corresponding face in the geometric complex on  $c_1, \dots, c_k$ . We refer the reader to Theorem 4 of [108] for further details.

It is rather important that in step (3) of the definition of  $f$  we only needed that  $c_1, \dots, c_k$  were in general position. By the aforementioned Theorem 2 of [108], such points can be found in any collection of open neighbourhoods of  $\mathbb{R}^{2n+1}$ . In particular, this is exploited in the proof of Theorem 5 of [108] to show that the set of  $\epsilon$ -homeomorphisms form a dense open set in  $C[M, \mathbb{R}^{2n+1}]$ . Although our definition of  $f$  is different from that in [108], it shares all the properties needed from an  $\epsilon$ -isomorphism in the proof of Theorem 5 of [108]; in particular, all that is needed is that it maps a face of the nerve to the respective face of the geometric complex.

In other words, we have:

**Fact 4.47.** For every  $g \in C[M, \mathbb{R}^{2n+1}]$  and every  $m, k > 0$ , there exists a computable  $2^{-k}$ -homeomorphism  $f$  such that

$$\sup_{x \in M} \|f(x) - g(x)\| < 2^{-m}.$$

Such an  $f$  from the fact above will have a rather clear definition given by the construction described above, and for some specific choice of special points  $c_1, \dots, c_k$ .

We thus iterate this process. Given a  $2^{-n}$ -homeomorphism  $f_n$ , search for an  $2^{-n-1}$ -homeomorphism  $f_{n+1}$  (according to (1)-(4) above), such that  $\sup_{x \in M} \|f_n(x) - f_{n+1}(x)\| < 2^{-n}$ . The limit of the process exists and gives a computable injective continuous embedding of  $M$  into  $\mathbb{R}^{2n+1}$ , thus, it is a homeomorphic embedding (as injective continuous functions on compacta are homeomorphisms).  $\square$

**4.10. Probability spaces and Haar measure.** For a computable compact space  $X$ , the space of all probability measures  $\mathcal{P}(X)$  is a computably metrized space under the Wasserstein metric defined to be

$$d_w(\mu, \nu) = \sup \left| \int f d\mu - \int f d\nu \right|,$$

where the supremum is taken over all 1-Lipschitz functions upon  $X$ ; that is,  $|f(x) - f(y)| \leq d(x, y)$  for every  $x, y \in X$ . The dense set is given by Dirac measures which are the probability measures concentrated at finitely many special points of  $X$ . (We refer the reader to [56] for further background on computability of measure spaces.)

Perhaps the first known construction of a surjective computable  $\Phi : 2^\omega \rightarrow \mathcal{P}(2^\omega)$  can be found in Day and Miller [27], but a very closely related argument can be found in the earlier paper [56]. Effective compactness of  $\mathcal{P}(2^\omega)$  was later used in [112]. Interestingly, this is a special case of Theorem 3.37 and the (seemingly well-known) lemma below. We also note that in the special case of  $\mathcal{P}(2^\omega)$  the lemma essentially becomes a triviality; its two-line proof can be extracted from [27].

**Lemma 4.48.** *If  $X$  is effectively compact then so is  $\mathcal{P}(X)$ .*

<sup>20</sup>Points  $c_1, \dots, c_k \in \mathbb{R}^d$  are in general position if any subset of at most  $d$ -many points of  $\{c_1, \dots, c_k\}$  is linearly independent.

*Proof.* To cover  $\mathcal{P}(X)$ , take a finite  $2^{-n}$ -cover of  $X$  and let  $(x_i)$  be the finitely many centres of the open balls forming the cover. Say, there are  $N$  such balls. Take the finite collection of Dirac measures concentrated in the points  $(x_i)$  and taking the values of the form  $k\frac{2^{-n}}{N^2}$ , where  $k \in \omega$ . There are only finitely many such measures, let  $D$  be the set of these measures. We claim that balls of radius  $2^{-n+2}$  centred at these points cover the whole space.

Take any other Dirac measure  $\mu$  concentrated at finitely many points  $(y_j)$ . We can find, for each  $j$ , the least index  $i = c(j)$  such that the ball centred at  $x_i$  contains  $y_j$ . For each  $i$ , let

$$C_i = \{y_j : c(j) = i\},$$

and note that  $d(x_i, y_j) < 2^{-n}$  for every  $y_j \in C_i$ . Define

$$\nu(x_i) = \sum_{j \in C_i} \mu(y_j)$$

and let  $\rho \in D$  be a measure (from the fixed above finite set) that differs by at most  $\frac{2^{-n}}{N}$  from  $\nu$  at every  $x_i$ . Fix any 1-Lipschitz function  $f$  and assume it takes value 0 at  $x_0$  (recall this means  $|f(x) - f(y)| \leq d(x, y)$ ) and assume the diameter of the space  $X$  is 1, which makes the absolute value of  $f$  also bounded by 1. Then

$$\left| \int f d\nu - \int f d\rho \right| = \left| \sum_{i < N} f(x_i)(\mu(x_i) - \rho(x_i)) \right| \leq \left| \sum_{i < N} (\mu(x_i) - \rho(x_i)) \right| \leq N \frac{2^{-n}}{N} = 2^{-n}.$$

On the other hand,

$$\left| \int f d\mu - \int f d\nu \right| = \left| \sum_{i < N} \left( \sum_{j \in C_i} f(y_j)\mu(y_j) - f(x_i)\nu(x_i) \right) \right| = \left| \sum_{i < N} \sum_{j \in C_i} (f(y_j) - f(x_i))\mu(y_j) \right|,$$

and noting that  $d(x_i, y_j) < 2^{-n}$  for every  $y_j \in C_i$ , this is bounded from above by  $2^{-n} \sum_j \mu(y_j) = 2^{-n}$ . It follows that the distance between  $\rho$  and  $\mu$  is at most  $2^{-n+1}$ . To finish the proof, recall that such Dirac measures are dense in the space, and the choice of  $\mu$  was arbitrary.  $\square$

A few years ago Jason Rute suggested the proof below in a personal communication with Melnikov and Nies (and shortly before he left academia). A proof sketch similar to the one that we give below can be found on the logic blog edited by Nies; see [35, Section 17]. Recently this result has been rediscovered in [107]; see [107] for a complete and detailed proof.

**Theorem 4.49.** *For a compact computable group  $G$ , the Haar measure is computable iff  $G$  is effectively compact.*

*Sketch.* Suppose  $G$  is effectively compact. The property of being translation invariant is a  $\Pi_1^0$  property. So the Haar measure is contained in an effectively closed singleton of the effectively compact (by Lemma 4.48) space  $\mathcal{P}(G)$ . Therefore, the Haar measure is computable by Fact 3.22.

Now take a computable compact group  $G$  that has a computable Haar measure. We want to show it is effectively compact. Replace  $d(x, y)$  with the (integral) average of  $d(gx, gy)$ , where the average is taken in the Haar measure as  $g$  varies across the group. This gives a computable  $G$ -invariant metric compatible with the original metric.

Now, to show that  $G$  is effectively compact (in this new metric), it is enough for each rational  $k$ , to effectively find a finite set of points  $a_0^k, \dots, a_{n-1}^k$  for which every point in  $G$  is within distance  $2^{-k}$  of one of these points. Fix  $k$ . Using our Haar measure find the measure of a ball of radius  $2^{-(k+1)}$ . Call this measure  $\delta$ . (Since the new distance is  $G$ -invariant, all balls of the same radius have the same measure.) Find a collection of balls  $B_0, \dots, B_{n-1}$  with radius  $2^{-(k+1)}$  whose union  $C = B_0 \cup \dots \cup B_{n-1}$  has measure  $> 1 - \delta$ ; recall that the measure is left-c.e. Now, consider any point  $x$  not in this union  $C$ . It has to be distance  $< 2^{-(k+1)}$  from the union. Otherwise, there would be a ball centered at  $x$  with radius  $2^{-(k+1)}$ , and hence measure  $\delta$ , which is disjoint from the union  $C$ . But the union  $C$  has measure  $> 1 - \delta$ , so this cannot happen. Therefore all points of  $G$  are within distance  $2^{-k}$  of the centers of  $B_0, \dots, B_{n-1}$ . This algorithm shows that the space is effectively compact in the new metric. To show it is effectively compact in the original metric, for any finite list of rational balls in original metric, convert it to a list of balls in the new metric. Now, if this list of balls covers the space  $G$ , by effective compactness, we will eventually find this out.  $\square$

It follows from Theorem 4.33 that a profinite group admits a recursive presentation iff it has a computably metrization with computable Haar measure iff it admits an effectively compact presentation. (In the latter two cases we need to also assume the operations are computable.) It was established in [85] that recursive profinite groups are exactly the Pontryagin duals of computable torsion groups, and it is not difficult to construct an example of a procyclic, computably metrized group whose dual has no computable copy [85]. (A similar result has recently been established in [107].) In the connected case, one of the proofs of Theorem 4.38 actually builds a computably metrized group with computable operations whose space is not homeomorphic to any effectively compact space. We summarise this below:

**Corollary 4.50** ([82, 85]). *In the classes of connected compact abelian and profinite abelian groups there exist examples of computably metrizable groups no computable metrization of which can compute Haar measure.*

**4.11. Some further open questions.** In Subsection 4.4 we explained why the characterisation problem and the isometric isomorphism problem for compact sets are arithmetical, and also why the homeomorphism problem for compact Polish spaces is  $\Sigma_1^1$ -complete. Similar results for compact Polish groups can be found in [85]. The following related questions are left open:

**Question 4.51.**

- (1) (Melnikov and Harrison-Trainor<sup>21</sup>) *What is the complexity of the isomorphism problem for (not necessarily compact) Polish spaces? (Is the naive upper bound optimal?)*
- (2) (Melnikov and Harrison-Trainor) *What is the complexity of the characterisation problem  $\{i : M_i \cong_{\text{hom}} S^3\}$  for the 3-sphere  $S^3$ ? What about the 2-ball? More generally, is it true that the (topological) characterisation problem  $\{i : M_i \cong_{\text{hom}} S\}$  for any compact manifold  $S$  is arithmetical?*

It is known that the characterisation problem for every compact 2-surface (including the 2-sphere, obviously) is arithmetical [50]. Their key step in their proof produces an arithmetical atlas of a given computable surface.

**Question 4.52** (Melnikov and Harrison-Trainor [50]). *Suppose a compact computable manifold  $S$ . Does  $S$  admit an arithmetical atlas?*

We have already mentioned above that the index set approach has not yet been applied to the effective enumeration of all (partial) effectively compact spaces. This seems reasonable assuming the thesis of the article (that effective compactness is a natural approach to computability in the compact case). Also, there are not many arithmetical *completeness* index set results in the literature (some can be found in [85]). The approach via effective compactness seems rather natural for such potential completeness results in the compact case (cf. Remark 4.29).

**Question 4.53.**

- (1) *Develop the index set approach to classification using the enumeration of all (partial) effectively compact spaces.*
- (2) *Prove completeness results for the arithmetical index set estimates stated above and also for other results that can be found in, e.g., [34, 50, 82].*

In Subsection 4.4 we also explained how to construct a compact computable Polish space not homeomorphic to any effectively compact space; this is Theorem 4.38. We also conjectured that a  $0'''$  proof can be used to replace the use of algebraic topology, but this approach is by no means elementary (if it works). It is rather natural to ask whether there is a less involved proof of Theorem 4.38. The question below is, of course, loosely stated.

**Question 4.54.** *Find an elementary (elegant?) proof of Theorem 4.38.*

In Subsection 4.3 we discussed an application of effective compactness to constructing basic sequences in computable Banach spaces. Bosserhoff [11] constructed a computable Banach space with a Schauder basis and no computable Schauder basis, and Downey, Long and Greenberg [81] showed that the index set of computable Banach spaces with computable bases is  $\Sigma_3^0$  complete. Using the characterisation of Schauder bases together with effective compactness, it is possible to show [81] that having a basis is a  $\Sigma_1^1$  property. The following question seems rather challenging.

<sup>21</sup>This question and question (2) below have been posed in [50] in a slightly different form.

**Question 4.55.** *Is the complexity of the index set  $\{i \mid W_i \text{ is a computable Banach space with a Schauder basis}\} \Sigma_1^1$  complete?*

There are many interesting open questions in the area of computable Banach spaces; see, e.g., Long [81]. It is highly plausible that effective compactness can be used to attack some of these questions.

Algebraic topology has played a considerable role in many proofs throughout the paper. We believe that there is much more to be said about the algorithmic content of algebraic topology, so we state:

**Problem 4.56.** Develop a general theory of computable algebraic topology and computable homological algebra.

Classically, the class of locally compact spaces is perhaps the narrowest natural class that contains both compact and discrete spaces. Even though there have been many attempts to define effective local compactness in the literature (e.g., [94]), it seems that there is no commonly accepted and robust notion that would be considered ‘standard’.

**Problem 4.57.** Suggest a robust (and useful) notion of effectively locally compact Polish space.

We of course do not exclude the possibility that some of the known definitions will already be good enough, but certainly we need to accumulate more results to draw any conclusions.

Our last problem is concerned with primitive recursive analysis. Historically a lot of elementary computable analysis was in fact developed primitively recursively; see, e.g., book [45]. However, gradually, primitive recursiveness was abandoned, perhaps because of technical difficulties that arise while dealing with primitive recursive procedures. Beginning with the 1980-s pretty much all computable analysis has been done using general Turing computability, see [111], [130]. On the other hand, there has been an increasing interest in polynomial-time analysis of continuous functions; see book [73]. Recently in [33] and [115, 116], it has been proposed to revive primitive recursive analysis using modern methods; this program could potentially serve as a link between abstract computable analysis and the more practical polynomial time and computational analysis.

For instance, say that a Polish space is *punctual* if the distances between special points are uniformly primitive recursive *nonzero* reals (to avoid dealing with equality). Some recent results about punctual spaces can be found in [5]. The role of compactness in primitive recursive analysis is very poorly understood. For example, using effective compactness of  $[0, 1]$ , it is easy to show that every computable continuous function on  $[0, 1]$  is effectively uniformly continuous, i.e., has a computable modulus of uniform continuity. The last section of [33] outlines a primitive recursive version of this elementary fact. The proof also uses compactness, but it uses it rather differently from the usual proof. On the other hand, one of the main results in [5] relies on the *standard* effective compactness (in the sense of this paper) to establish a primitive recursive result. So perhaps more insight is needed to attack the following:

**Problem 4.58.** Give a robust (and useful) definition of a punctually compact space.

We suspect that there are several potentially useful definitions of a punctually compact space that are not equivalent. For instance, we would like to know whether the results discussed in this paper, especially the Main Theorem, hold primitively recursively.

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