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On co-simple isols and their intersection types

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Abstract

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We solve a question of McLaughlin by showing that if **A** is a regressive co-simple isol, there is a co-simple regressive isol **B** such that the intersection type of **A** and **B** is trivial. The proof is a nonuniform $0'''$ priority argument that can be viewed as the execution of a single strategy from a $0^{(4)}$-argument. We establish some limit on the properties of such pairs by showing that if $\mathbf{A}\times\mathbf{B}$ has low degree, then the intersection type of **A** and **B** cannot be trivial (solving negatively a stronger question of McLaughlin).

1. Introduction

If A and B are sets of integers, we say A is *recursively equivalent to* B, $A \simeq B$, if there is a partial recursive injective function f such that $A \subseteq \operatorname{dom} f$, $B \subseteq \operatorname{ra} f$ and $f(A) = B$. The recursive equivalence type (RET) of a set A is the set $[A] = \{B \mid B \simeq A\}$. We say an RET is an *isol* if one (or equivalently all) of its members is immune or finite. We will use boldface letters **A** and **B** to refer to isols. The isols are an effective version of the Dedekind finite ordinals.

The structure of the collection of the isols occurs naturally if one considers choice-free mathematics and appears in the work of Myhill, Dekker and others. This early work culminated in the monograph of Dekker and Myhill [4]. We remark that we now know the intuitive connection between choice-free mathe-

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matics and the theory of the isols can be made precise via Kleene realizability (see [8]).

Isols also have close connections with nonstandard models of arithmetic; see [5, 6, 11, 12].

In this paper, our interest is the set of co-simple isols, those with co-simple elements. In particular, we will discuss the collection of co-simple regressive isols. We denote this final class by $\Lambda_{\mathrm{ZR}}^{\infty}$. McLaughlin [10] is a good reference for background information on the isols.

We recall some nomenclature. A is *regressive* if either A is finite or there exists a 1-1 function $g: \omega \to \omega$ and a partial recursive function p such that $A \subseteq \operatorname{dom} p$, $\operatorname{ra} g = A$, $p(g(0)) = g(0)$ and $p(g(j+1)) = g(j)$. A is *retracable* if the elements of A recursively code its initial segments. Formally, A is retracable if there is a recursive function a such that if n is an element of A, then $a(n)$ is the restriction of A to $[0, n]$. If A is verified to be regressive by means of the function g and g is strictly increasing, then it is not hard to show that A is retracable.

Following [2], the *intersection type* $\mathbf{A} \tilde{\cap} \mathbf{B}$ of a pair of regressive isols $(\mathbf{A}, \mathbf{B})$ is defined as $\{[A \cap B]: A \in \mathbf{A}\ \&\ B \in \mathbf{B}\}$. If A and B are representatives of $\mathbf{A}$ and $\mathbf{B}$, let $\mathbf{A} + \mathbf{B} = [A \oplus B]$. As in [2], we can see that $\mathbf{A} \tilde{\cap} \mathbf{B} \subseteq \{Y: Y \leqslant A \oplus B\}$. On the other hand, if $\mathbf{A}$ and $\mathbf{B}$ are infinite, then $\mathbb{N} \subset \mathbf{A} \tilde{\cap} \mathbf{B}$ (where $\mathbb{N} = \{0, 1, 2, \ldots\}$). Furthermore, both limits are possible.

The behavior of intersection types is not well understood. In this paper we solve a question of McLaughlin [9, question 4(c)] and [10, §7 and appendix]. We show the following.

Theorem 1.1. *For all* $\mathbf{A} \in \Lambda_{\mathrm{ZR}}^{\infty}$ *there exists* $\mathbf{B} \in \Lambda_{\mathrm{ZR}}^{\infty}$ *such that* $\mathbf{A} \tilde{\cap} \mathbf{B} = \mathbb{N}$.

The proof of Theorem 1.1 is given in Section 2. We remark that the proof has an interesting nonuniformity. Given $\mathbf{A}$, we build an infinite set of candidates for $\mathbf{B}$. We show that *one* of these candidates satisfies the conclusion of the theorem. In Section 3 we show that this nonuniformity in the proof cannot be eliminated. An aspect of the argument that is of some technical interest is that it is a primitive $0^{(4)}$-argument. That is, it is a $0'''$-argument that involves the execution of one $0^{(4)}$-strategy in the same way as the Lachlan nondiamond theorem [7] is a finite injury argument that involves the execution of one $0''$-strategy.

In Section 3, we also examine a natural extension of Theorem 1.1 suggested by McLaughlin in [10, §7]. McLaughlin asked whether for each retracable co-simple A, there is a retracable co-simple set B such that $\mathbf{A} \tilde{\cap} \mathbf{B} = \mathbb{N}$ and $A \equiv_{\mathrm{T}} B$. This fails rather strongly, as can be seen from the next theorem.

Theorem 1.2. *If* A *and* B *are infinite* Π_1^0 *sets and* $A \times B$ *has low degree, then* $[A] \tilde{\cap} [B] \neq \mathbb{N}$.

Our notation follows [13]. W_e denotes the eth recursively enumerable set. In the context of a stage s in a recursive construction, we bound all computations and other parameters being approximated by s. We appeal to the existence of a recursive pairing function $\langle\cdot,\cdot\rangle$ that is monotone in both of its arguments. Soare [13, Chapter XIV] provides a good introduction to the tree method in priority constructions. We adopt that formalism in Section 2.

2. Positive solution to McLaughlin's question

Let A be a given infinite co-simple regressive set. As each regressive co-simple isol contains a retracable set, without loss of generality we shall take A to be retracable. Given that A is Π_1^0 and retracable, we can also assume that A is given by a *retracable construction*. That is, we may assume that A is represented as the intersection of a recursive sequence A_s: for each s, $\{a_{i,s}\colon i\in\omega\}$ enumerates the elements of A_s in increasing order; and for all $a_{i,s}$ less than s, if $a_{i,s}\in A_s-A_{s+1}$, then $i<s$ and for all j such that $i\leqslant j\leqslant s$, $a_{j,s}\in A_s-A_{s+1}$. Conversely, any retracable construction will produce a retracable set. Knowing that n is an element of A, we can compute the initial segment of A below n since the approximation to A does not change at any number below n after stage n.

Let $\{\varphi_e\colon e\in\omega\}$ denote an enumeration of all partial recursive *injective* functions. We shall produce retracable constructions of Π_1^0 sets X and Y_e for each $e\in\omega$. At stage s, we let X_s and $Y_{e,s}$ denote our approximations to X and Y_e. We let $\{x_{i,s}\colon i\in\omega\}$, $\{y_{e,i,s}\colon i\in\omega\}$ be increasing enumerations of X and Y_e. We satisfy the following Π_4 requirement R:

$$\Big[(\forall e)(|W_e|=\infty\Rightarrow W_e\not\subset X)\ \&\ (\forall e)\Big(\lim_s x_{e,s}=x_e \text{ exists}\Big)$$

$$\&\ (\forall j)(X\subset \operatorname{dom}\varphi_j\Rightarrow |\varphi_j(X)\cap A|<\infty)\Big]\quad\text{or,}$$

$$(\exists i)\Big[(\forall e)(W_e=\infty\Rightarrow W_e\not\subset Y_i)\ \&\ (\forall e)\Big(\lim_s y_{i,e,s}=y_{i,e} \text{ exists}\Big)$$

$$\&\ (\forall j)(\operatorname{dom}\varphi_j\supset Y_i\Rightarrow |\varphi_j(Y_i)\cap A|<\infty)\Big].$$

We write R as $R'{}^\frown R''$. We shall decompose R into infinitely many Π_3 requirements. It is probably easiest to think of trying to meet the pseudo-requirements:

$$P_e\colon\ |W_e|=\infty\Rightarrow W_e\not\subset X,$$

$$N_e\colon\ \lim_s x_{e,s}=x_e \text{ exists},$$

and the Π_4 requirement

$$R_e\colon\ (\operatorname{dom}\varphi_e\supset X\Rightarrow|\varphi_e(X)\cap A|<\infty)\ \text{or}\ R_e''.$$

Should we fail to meet one of the P_e or N_e or one of the first disjuncts of an R_e, then, for some fixed e, we must meet all the requirements in R_e'':

$$P_{e,i}\colon\ (\mathrm{dom}\ \varphi_e \supset X \ \&\ |\varphi_e(X)\cap A| = \infty)\ \ \&\ \ (|W_i| = \infty \Rightarrow W_i \not\subseteq Y_e),$$

$$N_{e,i}\colon\ (\mathrm{dom}\ \varphi_e \supset X \ \&\ |\varphi_e(X)\cap A| = \infty)\ \ \&\ \ \left(\lim_s y_{e,i,s} = y_{e,i} \text{ exists}\right),$$

$$R_{e,i}\colon\ (\mathrm{dom}\ \varphi_e \supset X \ \&\ |\varphi_e(X)\cap A| = \infty)\ \ \&\ \ \mathrm{dom}\ \varphi_e \supset Y_e \Rightarrow |\varphi_e(Y_e)\cap A| < \infty.$$

Note that we do not need to meet any of the $P_{e,i}$ and $N_{e,i}$ unless it *appears* that X is contained in the domain of φ_e and the set $\varphi_e(X)\cap A$ is infinite. The crucial requirements are, of course, the $R_{e,i}$, the others being met by standard techniques. Here our construction reduces to the Π_3 level since we will be able to obtain Y_e uniformly from e.

2.1. *The basic module*

Fix e and i. For a single $R_{e,i}$ we will have a primary strategy and a back up strategy.

The primary strategy is to try to force $|\varphi_e(X)\cap A| < \infty$ (i.e., should dom $\varphi_e \supset X$). We say that $x_{j,s}$ is *e-good* if $\varphi_{e,s}(x_{j,s})\downarrow \notin A_s$. Say that a set is e-good if all but finitely many of its elements are e-good. The idea is to make X e-good for every e such that X is contained in the domain of φ_e.

Remark. Dropping the requirement that X be retracable, Friedberg's construction of a maximal recursively enumerable set can be adapted to build a set X that is good. For a fixed e, if dom $\varphi_e \supset X$, we can wait for the appearance of an $x_{k,s}$ such that $x_{k,s}$ is e-good. Upon finding such an $x_{k,s}$, we can set $x_{0,s+1} = x_{k,s}$ by enumerating $x_{0,s}, \ldots, x_{k-1,s}$ into $\bar{X}_{s+1}$. Similarly, given that the first k elements of X are e-good, we can await an opportunity to make $x_{k+1,s}$ e-good. Infinitely many e-good elements would appear by the fact that A is co-simple. As Friedberg did in his maximal set construction, we can use an e-state construction to combine the strategies associated with different recursive functionals. We isolate the conclusion as follows.

Observation. *Given any co-simple isol A we can effectively find a co-simple isol B such that* $\mathbf{A} \tilde{\cap} \mathbf{B} = \mathbb{N}$.

The additional requirement that our sets be retracable prohibits our directly applying the Friedberg strategy. In particular, *we cannot enumerate* $x_{0,s}, x_{1,s}, \ldots, x_{k-1,s}$ into $\bar{X}_{s+1}$ *without also enumerating* $x_{k,s}$ into $\bar{X}_{s+1}$.

For the basic module, our primary goal is to ensure that Y_e is i-good. Failing this, we will ensure that X is e-good. Matters will be arranged so that if, for any i, we fail to make Y_e i-good, then we will make X e-good and end the effect of R_e.

R_e has a trivial outcome when X is not a subset of the domain of φ_e. Assume that this outcome is not realized.

R_e has two nontrivial outcomes. In its Π_3 outcome, R_e ensures that for every i either Y_e is i-good or Y_e is not contained in the domain of φ_i. In its Σ_3 outcome, there is an i such that Y_e is contained in the domain of φ_i yet R_e cannot ensure that Y_e is i-good. However, the condition that Y_e cannot be made i-good will imply that R_e can ensure that X is e-good. We have the following dichotomy. Either every R_e has the Σ_3 outcome and X is the good set or there is an e such that for every i, we ensure that Y_e is i-good. In the latter case, Y_e is the good set. Thus, our overall organization is to establish a disjunction of Π_4 and Σ_4 conditions, either of which implies the existence of a good set.

$R_{e,i}$ is the basic submodule of R_e, working on the pair e and i as described above. The program for $R_{e,i}$ is listed below. We define auxiliary functions f and g (equal to f_e and $g_{e,i}$) as *pointers* for this module. Also, the module need only act during e-expansionary stages, since we get a global win on R_e should X not be a subset of dom φ_e.

Step 0. Set $f(0)=0$, $g(0)=0$. (As usual $f(s+1)=f(s)$ etc. unless we explicitly change them.) Protect $y_{e,g(s),s}$ and $x_{f(s)}$.

Step 1. Wait for a stage s when $\varphi_{i,s}(y_{e,g(s),s})\downarrow$. Let y be equal to $y_{e,g(s),s}$. Note, if the strategy waits forever in Step 1, then dom $\varphi_i \not\supseteq Y_e$.

Case 1a. $\varphi_i(y) \notin A_s$.
Action. Declare y to be i-good. Set $g(s+1)=g(s)+1$ and return to Step 1.

Case 1b. $\varphi_i(y) \in A_s$.
Action. Protect y and go to Step 2.

Step 2. Wait for a stage t greater than or equal to s such that either $\varphi_{e,t}(x_{f(t),t})\downarrow$ or $\varphi_i(y) \notin A_t$.

Case 2a. $\varphi_i(y) \notin A_t$.
Action. Return to Case 1a (with t substituted for s).

Case 2b. $\varphi_e(x_{f(t),t}) \notin A_t$.
Action. Declare $x_{f(t),t}$ to be e-good, set $f(t+1)$ equal to $f(t)+1$ and return to Step 2.

Case 2c. Otherwise.
Action. Protect $x_{f(t),t}$ and go to Step 3.

Step 3. Enumerate $y_{e,g(t),t}, \ldots, y_{e,t,t}$ into $\bar{Y}_{e,t+1}$.

Step 4. Wait for a stage u greater than or equal to t such that $\varphi_{i,u}(y_{e,g(u),u})\downarrow$. Let $y=y_{e,y(u),u}$.

Case 4a. $\varphi_i(y) \notin A_u$.
Action. Declare y to be i-good, set $g(u+1)$ equal to $g(u)+1$ and return, to Step 4.

Case 4b. $\varphi_i(y) \in A_u$, but $\varphi_i(y) \leqslant \max\{\varphi_{i,u}(\hat{y})\colon \hat{y}<y\}$.
Action. Go to Step 3.

Case 4c. Otherwise.
Action. Protect y. See whether $\varphi_{e,u}(x_{f(u),u}) \notin A_u$. If this is the case, then declare $x_{f(u),u}$ as e-good and set $f(u+1)=f(u)+1$. Go to Step 5.

Step 5. Enumerate $x_{f(u+1),u+1}, \ldots, x_{u+1,u+1}$ into $\bar{X}_{u+2}$.

Step 6. Wait until either $\varphi_{e,q}(x_{f(q),q})\downarrow$ or $\varphi_i(y) \notin A_q$.

Case 6a. $\varphi_i(y) \notin A_q$.
Action. Go to Case 4a.

Case 6b. $\varphi_e(x_{f(q),q}) \notin A_q$.
Action. Declare $x_{f(q),q}$ as e-good, set $f(q+1)=f(q)+1$ and go to Step 6.

Case 6c. $\varphi_e(x_{f(q),q}) \in A_q$ but $\varphi_e(x_{f(q),q}) \leq \max\{\varphi_{e,q}(\hat{x})\colon \hat{x} < x\}$.
Action. Return to Step 5.

Case 6d. Otherwise.
Action. Protect $x_{f(q),q}$. Return to Step 3.

Remark. The basic idea is to ensure that during any stage either Y_e or X is *covering* A. Thus *any* A-change becomes helpful (at least in the basic module). Also note that the construction in some sense *favors* Y_e.

2.2. Verification of the basic module

X and Y_e are retracable as they have retracable constructions. For the basic module we see that

(i) $f(x) \to \infty$ implies $\varphi_e(x) \cap A = \emptyset$ (as f is incremented only when we achieve $x_{f(s),s}$ e-good);

(ii) $g(s) \to \infty$ implies $\varphi_i(Y_e) \cap A = \emptyset$ (similarly);

(iii) $\lim_s f(s) < \infty$ & $\lim_s g(s) < \infty$ iff one of dom $\varphi_e \not\supset X$ or dom $\varphi_i \not\supset Y_e$ holds.

To see that (iii) holds suppose otherwise. Let t be such that $f(t) = \lim_{s,f}(s)$, $g(t) = \lim_s g(s)$. Assume that dom $\varphi_e \supset X$ and dom $\varphi_i \supset Y_e$. We give a recursive procedure to compute A. Given a number z, we compute $A[z]$ as follows. We run our construction until a stage $u > t$ where $\varphi_{e,u}(x_{f(t),u})\downarrow > z$ and $\varphi_{i,u}(y_{e,g(u),u})\downarrow > z$. Recall that A is presented by a retracable construction. If A were to change below z during a stage v greater than u, then every number between z and v would enter the complement of A during stage v. Since this would cause a change in whichever of f or g was covering A during stage v, it follows that $A_u[z] = A[z]$.

2.2.1. The outcomes of the basic module (in order of priority)

(i, ∞) — almost all of Y_e is i-good.
(e, ∞) — almost all of X is e-good.
(i, w) — dom $\varphi_e \supset X$ but dom $\varphi_i \not\supset Y_e$.
(e, w) — dom $\varphi_e \not\supset X$.

2.2.2. Coherence of the strategies and the α-module

We first consider the coherence of the collection of $\{R_{e,i}\colon i \in \mathbb{N}\}$ amongst themselves.

The potential conflicts are handled by Π_2 strategies. The simplest case is for two requirements $R_{e,i}$ and $R_{e,j}$ where $i<j$. As usual there are essentially two important versions of $R_{e,i}$. One is guessing (i, w) and the other is to the left of this and is guessing (i, ∞). The other outcomes such as (e, ∞) of the $R_{e,i}$ module have no subsequent $R_{e,j}$ since this outcome wins all of $\{R_{e,k}\colon k \in \omega\}$. This will be taken care of in the definition of the priority tree.

The version of $R_{e,j}$ guessing (i, w) believes that dom $\varphi_e \supset X$ and dom $\varphi_i \not\supset Y_e$. Thus it appears correct at (e-expansionary) stages where we have set y equal to $y_{e,g(s),s}$ and we are waiting for $\varphi_{i,s}(y)$ to halt. Following the basic module, while we are waiting for $\varphi_{i,s}(y)$ to converge, there will be (unless $g(s)=0$) an x, equal to $x_{f(s),s}$, devoted to covering a portion of A_s.

The obvious strategy here for $R_{e,j}$ is to begin a new module working on $\hat{x}>x$ and $\hat{y}>y$. To do this we use new (e,j)-pointers $f_{e,j}(s)$ and $g_{e,j}(s)$. Note that if it later turns out that $\varphi_i(y)\downarrow$, then we can abandon these versions of $f_{e,j}$ and $g_{e,j}$. So this strategy does not cause any injury to either of $R_{e,i}$ or $R_{e,j}$ for a *single* pair.

The difference will be that now the rules for X have changed. Now we can get some element $x=x_{f_{e,i}(s)}$ permanently kept in X yet $\varphi_e(x) \in A$. Such an x draws attention to the fact that we win φ_i on Y_e but we must be very careful to argue that if there are infinitely many such x with $\varphi_e(x) \in A$ (due to the action of infinitely many $R_{e,i}$), then we ensure that $|\varphi_i(Y_e) \cap A| < \infty$ or dom $\varphi_i \not\supset Y_e$ for *all* i. This same problem will occur in the other version of $R_{e,j}$ as we shall see, and really is the crucial point of the whole argument.

The version of $R_{e,J}$ guessing (i, ∞) knows that $R_{e,i}$ will produce a stream of i-good numbers for Y_e. Then $R_{e,j}$ refines this stream and only works with i-good numbers. Furthermore, it seems reasonable to only let $R_{e,j}$ act on $y_{i,k,s}$ for $k>j$ and only when we have seen $g_i(s)$ increase. (Since it is guessing $g_i(s)\to\infty$). At any stage s, Y_e will appear as

$$q_0, \ldots, q_n, \quad z_0, \ldots, z_n, \quad p_0, \ldots, p_k, \ldots,$$

where the q_k are both j- and i-good, the z_k are only i-good and the p_k are neither.

Thus $R_{e,j}$ will be working on z_0 and $R_{e,i}$ on p_0. Again we see the same potential problem. While we await $\varphi_{j,s}(z_0)$ to halt, we will need a $\varphi_e(x)$ to cover it on A_s, yet when $\varphi_{i,t}(p_0)\downarrow$ we will wish to reset X_{t+1} by dumping all sufficiently large numbers less than t into the complement of X. Note that here $j>i$, so we are now getting injury from below (i.e., from lower priority subrequirements). Note that if we later see $\varphi_{j,u}(z_0)\downarrow$, then we can await a stage $u_1 \geqslant u$ where $\varphi_{i,u_1}(p_0)\downarrow$ and then dump from x onwards, so *if* $\varphi_j(z_0)\downarrow$, then Y_i can live with this.

The only problem is that (j, w) may be the *correct outcome* with witness z_0. Again this can force some $x \in X$ to have $\varphi_e(x) \in A$. The crucial point to realize though is that there can only be infinitely many such x only if (i, ∞) is the *true* outcome of $R_{e,i}$.

In general the setup is such that if we really do have infinitely many such bad x, it can only be that they are spread at infinitely many levels of the tree, and an inductive argument will allow us to argue that then for all i we meet $R_{e,i}$.

There is no problem with the coherence of the various $R_{e,i}$ and $R_{f,j}$ as if $f > e$, then f simply plays with e-numbers and the relevant outcomes will be that (e, w) or (e, ∞). For (e, w) the effect is finite. For (e, ∞) we know that almost all of X is e-good and there is only finitely much activity predicted on (i, ∞). So the Π_3 modules combine in a standard Π_2 way.

Now we turn to the details.

2.3. *The priority tree*

Generate the priority tree T as follows. Let λ denote the empty sequence. Assign λ to solving $(0, 0)$ (so that $e(\lambda) = i(\lambda) = 0$).

Assume α has been assigned to solving (e, i). Then α has 4 outcomes from left to right (i, ∞), (e, ∞), (i, w), (e, w) (so that $\alpha^\frown(i, \infty) \in T$). Then

$$\begin{aligned} &\alpha^\frown(i, \infty) \quad \text{is assigned to } (e, i+1), \\ &\alpha^\frown(e, \infty) \quad \text{is assigned to } (e+1, 0), \\ &\alpha^\frown(i, w) \quad \text{is assigned to } (e, i+1), \\ &\alpha^\frown(e, w) \quad \text{is assigned to } (e+1, 0). \end{aligned}$$

This gives the recursion to define T and the assignment of the $R_{e,i}$ requirements.

Remark. In our description of the construction, we write f_α and g_α for $f_{e(\alpha),i(\alpha)}$ and $g_{e(\alpha),i(\alpha)}$. We call a stage s *α-expansionary* if, for $e = e(\alpha)$, $l(e, s)$ is greater than $\max\{t, 0\colon t < s$ and t is α-expansionary$\}$. By initialization, we mean setting all parameters to zero and returning the module to zero. During a stage s, we let $M(\alpha, s)$ denote the current state of the α-module. $M(\alpha, s)$ will be a number $j \in \{0, 1, 2, 4, 6\}$ (corresponding to the indices of the steps in the basic module).

We make another notational convention. When a parameter assumes one value at some substage t which may be changed during a later substage, we indicate its value during substage t by appending the subscript t. In particular the functions $f(\alpha, s)$ and $g(\alpha, s)$ will be so indicated when the outcomes (e, ∞) and respectively (i, ∞) appear correct. This is because subsequent modules based on these guesses can cause enumeration into $\bar{X}$ (respectively $\bar{Y}_e$) and cause us to revise our belief as to how much of X (Y_e) we are prepared to believe is e-good (i-good).

Construction. The construction proceeds by stages. During stage 0, we set X and each Y_e equal to $\mathbb{N}$. During stage $s+1$, we proceed through the following sequence of substages.

0. Define $\sigma(0, s+1) = \lambda$.

$\boldsymbol{t+1}$. $(t \leqslant s)$.

Step 1. We are given $\alpha = \sigma(t, s+1)$. Let $e = e(\alpha)$ and $i = i(\alpha)$. Adopt the first case below to pertain.

Case 0. $M(\alpha, s) = 0$. Define $f(\alpha, s+1) = g(\alpha, s+1) = s+1$, $M(\alpha, s+1) = 1$, $r_1(\alpha, s+1) = s+1$ and set $\sigma_{s+1} = \alpha^\frown(i, \omega)$. Initialize all $\gamma \nleq_L \sigma_{s+1}$ and go to Step 4.

Case 1. $M(\alpha, s) = 1$. Adopt the first subcase to pertain.

1a. For some $\eta^\frown(j, \infty) \subseteq \alpha$, with $e(\eta) = e$, we have that $y_{e,g(\alpha,s),s}$ is not j-good.

Action. Set $\sigma = \alpha^\frown(i, w)$ and initialize all $\gamma \nleq_L \sigma_{s+1}$. Go to Step 4.

1b. For some $\eta^\frown(f, \infty) \subseteq \alpha$, $e(\eta) = f$, we have $x_{f(\alpha,s),s}$ is not f-good.

Action. As in subcase 1a.

1c. $\varphi_{i,s}(y)\uparrow$ for $y = y_{e,g(\alpha,s),s}$.

Action. Set $r_1(\alpha, s+1) = y$ and $r_2(\alpha, s+1) = |\alpha|$. Define $\sigma(t+1, s+1) = \alpha^\frown(i, w)$ and go to Step 2.

1d. $\varphi_i(y) \notin A_s$.

Action. Declare y as i-good. Set $g_{t+1}(\alpha, s+1) = g_{t+1}(\alpha, s) + 1$. Define $\sigma(t+1, s+1) = \alpha^\frown(i, \infty)$ and $M(\alpha, s+1) = 1$. Set $r_1(\alpha, s+1) = 0$, $r_2(\alpha, s+1) = |\alpha|$ and go to Step 2.

1e. $\varphi_i(y) \in A_s$.

Action. Set $r_1(\alpha, s+1) = y$ and define $\sigma(t+1, s+1) = \alpha^\frown(e, w)$. Set $M(\alpha, s+1) = 2$. Initialize all $\gamma \supset \alpha^\frown(i, w)$. Define $f(\alpha, s+1)$ to be the least z such that

$$\begin{aligned} &\text{(i)} \quad z > \max\{|\alpha|, r_2(\alpha, s)\colon\ \gamma \leqslant_L \alpha^\frown(i, w)\ \&\ r_2(\gamma, s) \neq 0\}, \\ &\text{(ii)} \quad x_{z,s} \text{ is not } e\text{-good at } s. \end{aligned} \tag{2.1}$$

Go to Step 2.

Case 2. $M(\alpha, s) = 2$

2a. for some $\eta^\frown(f, \infty) \subseteq \alpha$ with $e(\eta) = f$ we have $x_{f(\alpha,s),s}$ is not f-good.

Action. Set $\sigma_{s+1} = \alpha^\frown(e, w)$ and initialize all $\gamma \nleq_L \sigma_{s+1}$ with $\gamma \not\supseteq \sigma_{s+1}$. Go to Step 4.

2b. Neither $\varphi_{e,s}(x_{f(\alpha,s),s})\downarrow$ nor $\varphi_i(y) \notin A_s$.

Action. Define $\sigma(t+1, s+1) = \alpha^\frown(e, w)$ and $r_2(\alpha, s+1) = f(\alpha, s)$ and go to Step 2.

2c. $\varphi_i(y) \notin A_s$.

Action. As in subcase 1d.

2d. $\varphi_e(x_{f(\alpha,s),s}) \notin A_s$.

Action. Declare $x_{f(\alpha,s),s}$ as e-good. Set $f_{t+1}(\alpha, s+1) = f_{t+1}(\alpha, s) + 1$, $M(\alpha, s+1) = 2$ and $\sigma(t+1, s+1) = \alpha^\frown(e, \infty)$. Now set $r_{2,t}(\alpha, s+1) = |\alpha|$ (but keep $r_1(\alpha, s+1)$ the same). Go to Step 2.

2e. $\varphi_e(x_{f(\alpha,s),s}) \in A_s$, but $\varphi_e(x_{f(\alpha,s),s})$ is not greater than or equal to $\max\{\varphi_{e,s}(\hat{x})\colon\ \hat{x} \leqslant x_{f(\alpha,s),s}\}$.

Action. Set $M(\alpha, s+1)=2$. Initialize all $\gamma \supseteq \alpha^\frown(e, w)$. Enumerate $x_{f(\alpha,s),s}, \ldots, x_{s,s}$ into $\bar{X}_{s+1}$. For any $\gamma \subseteq \alpha$ with $f_t(\gamma, s+1) \geqslant f(\alpha, s)$, set $f_{t+1}(\gamma, s+1)=f(\alpha, s)$. Go to Step 2.

2f. Otherwise.

Action. Set $r_2(\alpha, s+1)=f(\alpha, s)$, $r_1(\alpha, s+1)=0$ and, enumerate $y_{e,g(\alpha,s),s}, \ldots, y_{e,s,s}$ into $\bar{Y}_{e,s+1}$. Set $\sigma(t+1, s+1)$ equal to $\alpha^\frown(i, w)$ and $M(\alpha, s+1)$ equal to 4. For any $\gamma \subseteq \alpha$ with $g_t(\gamma, s+1) \geqslant g(\alpha, s)$, set $g_{t+1}(\gamma, s+1)=g(\alpha, s)$. Go to Step 2.

Case 3. $M(\alpha, s+1)=4$.

3a. For some $\eta^\frown(j, \infty)$, with $e(\eta)=e$, we have $y_{e,g(\alpha,s),s}$ is not j-good.

Action. Set $\sigma_{s+1}=\alpha^\frown(i, w)$ and initialize all $\gamma \nleq_L \sigma_{s+1}$ such that $\gamma \not\supseteq \alpha^\frown(i, w)$. Go to Step 4.

3b. $\varphi_{i,s}(y)\uparrow$.

Action. Define $\sigma(t+1, s+1)=\alpha^\frown(i, w)$, $r_1(\alpha, s)=y$ and go to Step 2.

3c. $\varphi_i(y) \notin A_s$.

Action. Declare y as i-good and set $g_{t+1}(\alpha, s+1)$ equal to $g(\alpha, s)+1$, $\sigma(t+1, s+1)$ equal to $\alpha^\frown(i, \infty)$, $r_1(\alpha, s+1)$ equal to 0 and $r_2(\alpha, s+1)$ equal to $|\alpha|$. Set $M(\alpha, s+1)=1$ (note, '1' not '4') and go to Step 2.

3d. $\varphi_i(y) \in A_s$ but $\varphi_i(y) \ngtr \max\{\varphi_{i,s}(\hat{y})\colon \hat{y} \leqslant y\}$.

Action. As in subcase 2f except we also initialize all $\gamma \supseteq \alpha^\frown(i, w)$.

3e. Otherwise.

Action. Set $r_1(\alpha, s+1)=y$ and now see if $\varphi_e(x_{f(\alpha,s),s}) \notin A_s$. If so, declare $x_{f(\alpha,s),s}$ as e-good and set $f_{t+1}(\alpha, s+1)=f(\alpha, s)+1$, $r_{2,t}(\alpha, s+1)=|\alpha|$, $\sigma(t+1, s+1)=\alpha^\frown(e, \infty)$ and $M(\alpha, s+1)$ equal to 6. Go to Step 2.

If $\varphi_e(x_{f(\alpha,s)}) \in A_s$, set $\sigma(t+1, s+1)=\alpha^\frown(i, w)$ but initialize all $\gamma \supseteq \alpha^\frown(i, w)$, and set $r_2(\alpha, s+1)=|\alpha|$. Also enumerate $x_{f(\alpha,s),s}, \ldots, x_{s,s}$ into $\bar{X}_{s+1}$ and $M(\alpha, s+1)=6$. For any $\gamma \subseteq \alpha$ if $f_t(\gamma, s+1) \geqslant f(\alpha, s)$, then set $f_{t+1}(\gamma, s+1)=f(\alpha, s)$ and go to Step 2.

Case 4. $M(\alpha, s+1)=6$.

4a. $x_{f(\alpha,s),s}$ is not f-good for some $\eta^\frown(f, \infty) \subseteq \alpha$ with $e(\eta)=f$.

Action. Define $\sigma_{s+1}=\alpha^\frown(e, w)$ and initialize all $\gamma \nleq_L \sigma_{s+1}$ with $\gamma \supseteq \sigma_{s+1}$. Go to Step 4.

4b. $\varphi_i(y) \in A_s$ and $\varphi_{e,s}(x_{f(\alpha,s),s})\uparrow$.

Action. Define $\sigma(t+1, s+1)=\alpha^\frown(e, w)$. Set $r_2(\alpha, s+1)=x_{f(\alpha,s),s}$ and go to Step 2.

4c. $\varphi_i(y) \notin A_s$.

Action. As in subcase 3c.

4d. $\varphi_e(x_{f(\alpha, s),s}) \notin A_s$.

Action. Declare $x_{f(\alpha,s),s}$ as e-good. Set $f_{t+1}(\alpha, s+1)=f(\alpha, s)+1$, $\sigma(t+1, s+1)=\alpha^\frown(e, \infty)$, $r_{2,t}(\alpha, s+1)=|\alpha|$ and $M(\alpha, s+1)=6$. Go to Step 2.

4e. $\varphi_e(x_{f(\alpha,s),s}) \notin A_s$ but $\varphi_e(x_{f(\alpha,s),s})$ is not greater than or equal to $\max\{\varphi_e(\hat{x})\colon \hat{x} \leqslant x_{f(\alpha,s),s}\}$.

Action. As in subcase 3e.

4f. Otherwise.

Action. Set $r_2(\alpha, s+1) = x_{f(\alpha,s),s}$, $M(\alpha, s+1) = 4$ and $r_1(\alpha, s+1) = 0$. Define $\sigma(t+1, s+1) = \alpha^\frown(i, w)$. Initialize all $\gamma \supseteq \alpha^\frown(i, w)$ and enumerate $y_{e,g(\alpha,s),s}, \ldots, y_{e,s,s}$ into $\bar{Y}_{s+1}$. If $\gamma \subseteq \alpha$ and $g_t(\gamma, s+1) \geqslant g(\alpha, s)$ set $g_{t+1}(\alpha, s+1) = g(\alpha, s)$ and go to Step 2.

Step 2. Initialize all γ with $\sigma(t+1, s+1) \leqslant_L \gamma$ and $\sigma(t+1, s+1) \nsubseteq \gamma$. If $t = s$, define $\sigma_{s+1} = \sigma(t+1, s+1)$ and go to stage $s+2$. Otherwise go to Step 3.

Step 3. If $\sigma(t+1, s+1) = \alpha^\frown(e, \infty)$ or $\alpha^\frown(e, w)$, see if there exists numbers z and j such that $j < e$, $W_{j,s} \subseteq X_s$ and $x \in W_{j,s}$ with $z > x_{j,s}$ and $z > \max\{x_{r_2(\tau,s),s}\colon \tau \leqslant_L \alpha\}$. If so, then $z = x_{k,s}$ for some $k < s$. We enumerate $x_{k,s}, \ldots, x_{s,s}$ into $\bar{X}_{s+1}$ and initialize all $\gamma \supset \alpha^\frown(e, \infty)$ (respectively $\alpha^\frown(e, w)$) and go to Step 4 setting $\sigma_{s+1} = \alpha^\frown(e, \infty)$, $(\alpha^\frown(e, w))$. Otherwise go to substage $t+2$.

Similarly, if $\sigma(t+1, s+1) = \alpha^\frown(i, \infty)$ (respectively $\alpha^\frown(i, w)$) and there exist $j < i$ with $W_{j,s} \subseteq Y_{e,s}$, $z \in W_{j,s}$ with $z > \max\{r_1(\tau, s)\colon \tau \leqslant_L \alpha$ and $e(\tau) = e(\alpha)\}$ and $z > y_{e,j,s}$, we enumerate $y_{e,k,s}, \ldots, y_{e,s,s}$ into $\bar{Y}_{s+1}$ where $y_{e,k,s} = z$. We then go to Step 4, setting $\sigma_{s+1} = \alpha^\frown(i, \infty)$ $(\alpha^\frown(i, w))$. If not, we got to substage $t+1$.

Step 4. For each $\alpha \subset \sigma_{s+1}$ with $\alpha^\frown(e, \infty) \subset \sigma_{s+1}$ set $r_{1,s+1}(\alpha, s+1) = f_{s+1}(\alpha, s+1)$. (End of Construction)

It is clear that X and Y_e are retracable for all $e \in \omega$, as they are built by retracable constructions. Let β denote the true path. Note that any P_e or $P_{e,j}$ can act at most once. They will therefore be met with finite effect, provided we argue that along the true path the lim inf of the restraints is finite. The following is the crucial lemma.

Lemma 2.1. *Let $\alpha \subset \beta$ and $(e, i) = (e(\alpha), i(\alpha))$.*

(i) *If $\alpha^\frown(e, w) \subset \beta$, then* dom $\varphi_e \nsubseteq X$. $\lim_s f(\alpha, s) = f(\alpha)$ *exists and* $f(\alpha) > |\alpha|$, $\lim_s r_1(\alpha, s)$ *exists and* $\lim_s(r_2(\alpha, s))$ *exists and* $\lim_s x_{j,s}$ *exists for* $j \leqslant f(\alpha)$.

(ii) *If $\alpha^\frown(i, w) \subset \beta$ then* dom $\varphi_i \not\supseteq Y_e$; $\liminf f_s g(\alpha, s) = g(\alpha)$ *exists and* $g(\alpha) > |\alpha|$, $\liminf_s r_1(\alpha, s)$ *and* $\lim_s r_2(\alpha, s)$ *exist and* $\liminf_s y_{e,j,s}$ *exists for* $j \leqslant g(\alpha)$ *and furthermore* $(\exists t)(\forall s > t)(\alpha = \sigma(u, s) \Rightarrow \alpha^\frown(i, w) = \sigma(u+1, s))$.

(iii) *If $\alpha^\frown(e, \infty) \subset \beta$, then* $(\exists s_0)(\forall t \geqslant s_0)(f(\alpha, t) \geqslant |\alpha|)$, $\liminf_s x_{j,s} = x_j$ *exists for $j \leqslant |\alpha|$, and there exists a stage $s_1 \geqslant s_0$ such that for all $k \ngtr \liminf f(\alpha, t)$, $x_{k,s_1} \in X$ implies that x_{k,s_1} is e-good (and hence almost all of X is e-good),* $\lim_s r_1(\alpha, s)$ *exists and* $\liminf r_2(\alpha, s) = |\alpha|$ *(and drops down when $\sigma(t, s) = \alpha^\frown(e, \infty)$).*

(iv) *If $\alpha^\frown(i, \infty) \subset \beta$, then* $(\exists s_0)(\forall t \geqslant s_0)(g(\alpha, t) \geqslant |\alpha|)$, $\liminf_s y_{e,j,s} = y_{e,s}$ *exists for $j \leqslant |\alpha|$, for all $k \geqslant \liminf g(\alpha, s)$, $y_{e,k,s_0} \in Y_e$ implies that y_{e,k,s_0} is i-good (and* *hence almost all of Y_e is i-good), and* $\liminf \min\{r_1(\alpha, s), r_2(\alpha, s)\}$ *exists* $(=|\alpha|)$ *and both drop down when* $\sigma(t, s) = \alpha^\frown(i, \infty)$.

Once we have the lemma, we finish the proof by observing that one of (a) or (b) below must occur:

(a) $(\forall e)(\exists \alpha)(e = e(\alpha) \,\&\, (\alpha^\frown(e, \infty) \subset \beta \vee \alpha^\frown(e, w) \subset \beta))$;

(b) $(\exists e)(\forall \alpha)(e = e(\alpha) \rightarrow (\alpha^\frown(i, \infty) \subset \beta \vee \alpha^\frown(i, w) \subset \beta))$.

In case (a), then (i) and (iii) imply that $|X| = \infty$, and for all e either $\operatorname{dom} \varphi_e \not\supset X$ or $|\varphi_e(X) \cap A| < \infty$. Also the restraints having finite lim inf allows us to meet P_e. Case (b) similarly implies that Y_e does the job.

We prove Lemma 2.1 by induction. Let $\alpha \subseteq \beta$ and let s_0 be a stage where for all $\tau \leq_L \alpha$ with $\tau \neq \alpha$, τ has ceased acting, for all j with P_j $(P_{\hat{e},j})$ having higher priority than α, P_j $(P_{\hat{e},j})$ have ceased acting, and for all $\gamma \subsetneq \alpha$ the hypotheses of the lemma hold for γ at stage s_0 (that is, for example, if $\gamma^\frown(f, w) \leq \alpha$ and $f = e(\gamma)$, then $f(\gamma, s_0) = f(\gamma)$, $r_1(\gamma, s_0) = r_1(\gamma)$, etc.).

Now suppose $\alpha^\frown(e, p) \subset \beta$. Additionally we may assume at s_0 that all $\eta \supsetneq \alpha$ with $\eta \lneq_L \alpha^\frown(e, p)$ have ceased acting. Let $p = w$. Note by initialization we can suppose that if $\rho \supset \alpha^\frown(e, w)$, then ρ is initialized at s_0 (or has not yet been visited) so that $M(\rho, s_0) = 0$.

The construction allows for $\alpha^\frown(e, w) \subset \beta$ only via Case 2 or Case 4. Note that when either of these are visited, we will set $r_2(\alpha, s) = x_{f(\alpha,s),s}$. Also note that after s_0, $f(\alpha, s)$ is only reset when we leave the relevant case and hence $\lim_s f(\alpha, s) = f(\alpha)$ exists. To finish the proof of (i) it will suffice to argue that $\lim_s x_{j,s} = x_j$ exists for $j \leq f(\alpha)$ (and from this it will follow that $x_{f(\alpha),s}$ will witness the fact that $\operatorname{dom} \varphi_e \not\subset X$ as subcase 4c cannot pertain more than finitely after to any particular $f(\alpha, s)$ (as φ_e is 1-1)).

For $j < f(\alpha)$ how can $x_{j,s}$ change? By initialization $f(\rho, s_0) > f(\alpha)$ for $\rho \supset \alpha^\frown(e, w)$ (this is set at a very large number in Case 0 and (2.1)). Therefore $\rho \supset \alpha^\frown(e, w)$ can change $x_{j,s}$ for $j < f(\alpha)$. α cannot change such $x_{j,s}$ either and by choice of s_0 the positive requirements will not change $x_{j,s}$ after s_0. Thus we see that $x_{j,s}$ for $j < f(\alpha)$ will only change due to the action of $\eta \subsetneq \alpha$. By choice of s_0 the only nodes η that will act cofinally with the construction and $\eta \subsetneq \alpha$ are of the form $\eta^\frown(f, \infty) \subseteq \alpha$ and $\eta^\frown(i, \infty) \subseteq \alpha$. Let η be the shortest note whose action causes $x_{j,s}$ not to reach its limit (j least).

Case 1. $\eta^\frown(f, \infty) \subset \alpha$. Now by construction we see that $x_{j-1,s}$ is f-good for all $s \geq t$, some $t \geq s_0$ (minimality of j). Without loss of generality, we may take $t = s_0$. Since η causes $x_{j,s}$ not to reach its limit, we must have $j \geq f(\eta, s)$ for infinitely many $s \geq s_0$. As $x_{j,s}$ is protected until we see $\varphi_{f,s}(x_{j,s})\downarrow$ and $\eta^\frown(f, \infty) \subseteq \alpha$, it follows that subcase 2d or 3e must pertain infinitely often. When each of these subcases pertains (say at substage t), we set $f_t(\eta, s+1) = f(\eta, s) + 1 = k + 1$. Now by η we finish with $x_{j,s}$ (as it is now f-good) unless some γ with $\eta^\frown(f, \infty) \subset \gamma \subset \alpha$ causes $x_{j,s}$ to enter (and hence cut the sequence back to $x_{j,s}$ again). It cannot be that all such γ are of the form $\gamma^\frown(h, \infty) \subset \alpha$ where $h = e(\gamma)$

since, like an e-state construction $x_{j,s}$ would be h-good for all such h. Thus this case reduces to the case that for some γ we have $\gamma^\frown(i, \infty) \subseteq \alpha$ causing $x_{j,s}$ to not have a limit. We deal with this below in Case 2.

Case 2. $\eta^\frown(i, \infty) \subseteq \alpha$. At the same stage s after stage s_0, when we visit α, we will have defined $r_2(\alpha, s)$ to protect $x_{f(\alpha,s),s}$. Now thereafter when we visit η, η can only cause enumeration into $\bar{X}_{t+1}$ only for those $k > f(\alpha, s)$ because of the definition of $f(\eta, s)$ in (2.1) and the fact that we *delay* enumerating $x_{f(h,s)}$ until all those $x_{q,s}$ for $q \leq f(h, s)$ are h-good for all h with $\gamma^\frown(h, \infty) \subset \eta$ and $h = e(\gamma)$ (as in Case 1). Therefore η must respect (in particular) $x_{j,s}$ until we have seen $\varphi_{e,s}(x_{f(\alpha,s),s})\downarrow$. This establishes (i).

To see (ii) holds, again the crucial part is to show that $\lim_s y_{e,g(\alpha),s} = y_{e,g(\alpha)}$ exists. (Also here we must argue the existence of the stage t.) Here the argument is much easier. If $\alpha^\frown(i, w) \subset \beta$, then by construction of the priority tree, for all η, if $e = e(\eta)$, then $\eta^\frown(e, \infty) \nsubseteq \alpha$. Thus the only nodes which can cause infinitely much enumeration into $\bar{Y}_e$ of higher priority than α are ones with $\eta^\frown(j, \infty) \subset \alpha$. But now the argument mimics the one for (i) since such nodes work like e-states together.

We must also argue for (ii) that $(\exists t)(\forall s > t)(\alpha = \sigma(u, s) \Rightarrow \alpha^\frown(i, w) = \sigma(u + 1, s))$. (This is really needed to get dom $\varphi_i \not\supseteq Y_e$.) Thus we must argue that we cannot switch infinitely often from $\alpha^\frown(i, w)$ to $\alpha^\frown(e, w)$ and must get struck awaiting the relevant i-computation to halt. For this we argue exactly as in the basic module. When we switch from (e, w) to (i, w) we will protect y. This y is f-good for all $\gamma^\frown(i, \infty) \subset \alpha$ and is therefore immune from enumeration. Similarly when we have the switch from (i, w) to (e, w), we do so to some $x_{f(\alpha,s),s}$ which is h-good for all $\gamma^\frown(h, \infty)$ for $k = e(\gamma)$ (and be protected from such $\gamma^\frown(h, \infty)$) and furthermore this $x_{f(\alpha,s),s}$ is protected from enumeration by $\gamma^\frown(i, \infty)$ by the clause (2.1)(i). Therefore at any stage one side or the other covers A_s. But then as A is retracable and nonrecursive, we must see some outcome $\alpha^\frown(f, \infty)$ or $\alpha(i, \infty)$ after stage s_0, this being a contradiction and giving (ii).

To establish (iii) suppose that $\alpha^\frown(e, \infty) \subset \beta$. This implies that there are only finitely many restraints α must respect that are permanently generated by $\rho \supseteq \eta^\frown(i, \infty)$ for $\eta \subseteq \alpha$ but $\eta^\frown(i, \infty) \not\supseteq \alpha^\frown(e, \infty)$ (via (2.1)). Also after stage s_0, $r_2(\alpha, s) \geq |\alpha|$ henceforth. It follows that any $\eta^\frown(i, \infty) \subset \alpha$ must respect $r_2(\alpha, s)$ (by (2.1)) and hence using the argument we used for (i) we see that $\lim_s x_{j,s} = x_j$ exists for $j \leq |\alpha|$. To complete the proof of (iii) we need to argue that for all $k \nleq \liminf f(\alpha, t)$, $x_{k,s} \in X$ implies that $x_{k,s}$ becomes e-good at some stage $s > s_1$ and some s_1. By construction, we know that there will be a stage s_1 such that $s_1 > s_0$, $x_{f(\alpha,s_1),s_1}$ is h-good for all $h \leq e$ with $\eta^\frown(h, \infty) \subset \alpha$ and $h = e(\eta)$. When we play the outcome $\alpha^\frown(e, \infty)$, we will allow $r_2(\alpha, s_1)$ to be $|\alpha|$. This may allow some enumeration into X by nodes $\gamma \supset \alpha^\frown(e, \infty)$, but the virtue of $f(\alpha, s)$ at the end of substage s will be preserved by r_2 (by Step 4) and hence by any $\eta^\frown(i, \infty) \subseteq \alpha$. It follows that at the end of stage s, $x_{f(\alpha,s)-1,s}$ is e-good, and for some $t > s$ with $f(\alpha, t) = f(\alpha, s)$, we will have $x_{f(\alpha,t),t}$ is e-good (i.e., at the next $\alpha^\frown(e, \infty)$ stage).

There are now two cases. Either $\liminf f(\alpha, t) \to \infty$ in which case almost all of X is e-good or $\liminf f(\alpha, t) < \infty$ in which case X is finite. We now get (iii).

To see (iv) is even easier. There is no injury from below for (iv), and we argue as in (ii) that $\lim y_{e,j,s} = y_{e,j}$ exists for $j \leqslant |\alpha|$, and similarly almost all of Y_e is i-good. Finally, whenever we have an $\alpha^\frown(i, \infty)$-stage, we always set $r_2 = |\alpha|$ and $r_1 = 0$.

3. Related results

In this section we shall prove two related results. First we will show that in some sense the complexity of the argument of Section 2 cannot be avoided in the sense there is no *uniform* solution to McLaughlin's question.

Theorem 3.1. *There is no recursive function f such that $\bar{W}_{f(e)}$ is co-simple regressive and whenever $\bar{W}_e$ is co-simple regressive, then $[\bar{W}_e] \bar{\cap} [\bar{W}_{f(e)}] = \mathbb{N}$.*

Proof. We shall prove that we have the ability to build A satisfy the requirements, for all $e \in \omega$ and any f,

$$P_e: \quad |W_e| = \infty \Rightarrow W_e \not\subseteq A,$$

$$N_e: \quad \lim_s a_{e,s} = a_e \text{ exists}$$

(where $\{a_{e,s}\colon e \in \omega\}$ lists A_s), and

$$R_{f,i}: \quad \text{if } |\bar{V}_f| = \infty \text{ and } \gamma_f \text{ retraces } \bar{V}_f, \text{ then } D_f \simeq \bar{V}_f \text{ and } |D_f \cap \bar{A}| \geqslant i.$$

Here $\langle V_f, \gamma_f \rangle_{e \in w}$ denotes a list of all pairs consisting of a recursively enumerable set and a partial recursive retracing function (D_f is built by us as the range of a partial recursive function). That is, by withholding the enumeration of V_f and γ_f we ask that V_f and γ_f satisfy the following conditions: γ_f is injective; if $\{v^f_{j,s}\colon j \in \omega\}$ lists $\bar{V}_{f,s}$, we ask that if $\gamma_{e,s}(v^f_{j,s})\downarrow$, then $(\forall k < j)[\gamma_{f,s}(v^f_{j,s})\downarrow]$; if $j = 0$, then $\gamma_{f,s}(v^f_{0,s}) = v^f_{0,s}$: and if $j \geqslant 1$, $\gamma_{f,s}(v^f_{j,s}) = \gamma_{f,s}(v^f_{f-1,s})$. Finally if $l(f, s) = \max\{x\colon (\forall y \leqslant x)[\gamma_{f,s}(v^f_{y,s})\downarrow]\}$, we ask that $V_{f,s+1} = V_{f,s}$ implies that $l(f, s+1) > l(f, s)$. Note that this means that if $|\bar{V}_f| = \infty$, then if γ_f does not retrace $\bar{V}_f$, (V_f, γ_f) will be frozen at some stage.

The reader should note that if we can meet the P_e, N_e for all $e \in \omega$ and $R_{f,i}$ for all $i \in \omega$ any arbitrary f, the recursion theorem will give Theorem 3.1.

We will now describe the strategies, omitting the formal details as they are routine. We build A by a retracable construction. We meet P_e as usual. At any stage s if we see an unrestrained $x \in W_{e,s}$ with $W_{e,s} \subset A_s$, we enumerate x into $\bar{A}_{s+1}$. Note that if $x = a_{i,s}$, then we ask that $i > e$, so we automatically meet the N_j. The hard part is to get things into A.

Let us drop the f. We must meet the overall requirement

$$R: \quad |\bar{V}| < \infty \text{ or } |V| < \infty \text{ or } (\forall i)(R_i).$$

In particular R_i attempts to either ensure that $\bar{V} = \{v_{0,s}, \ldots, v_{i,s}\}$ or $(\exists t)(\forall s > t)(V_s = V_t)$ or we get a new witness for R_i. For the sake of R_i, we define a restraint $r(i, s)$. Then the basic strategy is this.

Suppose we have met all the higher priority P_e and R_j for $j < i$. For those $j < i$ we can suppose that R_j has a *stable assignment* (as becomes clear below). Now R_i will require attentions when we see a stage s where $l(s) > k$ for some least k not yet assigned. We then set $i(s) = k$ and assign $v_{k,s}$ to $a_{s,s}$. (That is, we extend our partial recursive function δ we are building to include $v_{k,s}$ and define $\delta(v_{k,s}) = a_{s,s}$.)

We then raise $r(i, s) = a_{s,s}$ and restrain A from losing $a_{s,s}$ with priority i. This temporarily satisfies R_i and wins with finite effect unless the assignment is *not stable*: that is, there exists a stage $t > s$ such that $v_{k,t+1} \neq v_{k,t}$. Note that by our assumptions on the enumeration of V this means that all assignments of $\hat{k}$ for $\hat{k} \geqslant k$ are unstable.

Should this case occur, we must seek a new assignment for $v_{k,u}$, some $u \geqslant t + 1$. However, if we assigned $v_{k,t+1}$ to $a_{t+1,t+1}$ *immediately*, we would potentially fail, since then perhaps $\limsup l(s) \to \infty$ yet $\liminf f(s) = k$. The obvious solution is to use a standard infinite injury strategy to give all the P_j requirements a *window* at stage $t + 1$. That is, drop the restraint to zero for one state and *then* give the new assignment. In this way we get to meet all the P_l no matter what the outcome of R. □

The other result we shall examine is generated by the stronger question of McLaughlin as to whether the main result would hold in a given degree. That is, given any regressive co-simple A, is there a regressive co-simple $B \equiv_{\mathrm{T}} A$ such that $\mathbf{A} \tilde{\cap} \mathbf{B} = \mathbb{N}$? The answer is no in a very strong sense.

Theorem 3.2. *Suppose that A and B are infinite Π_1 with $A \times B$ of low degree. Then* $[A] \tilde{\cap} [B] \neq \mathbb{N}$.

Proof. The argument we now give uses a very well-known technique called the *Robinson trick* (cf. [13, Chapter XI]), so we again sketch the details.

Let $A = \bigcap_s A_s$ and $B = \bigcap_s B_s$ with $A_0 = B_0 = \omega$ and $A_s = \{a_{i,s}\colon i \in \omega\}$, $B_s = \{b_{i,s}\colon i \in \omega\}$. We shall define a partial recursive function f so that $|f(A) \cap B| = \infty$. To do this we meet the requirements

$$R_e: \quad |f(A) \cap B| \geqslant e$$

by a finite injury argument. Note that as $A \times B$ is low (where $A \times B = \{\langle x, y\rangle\colon x \in A \;\&\; y \in B\}$) and Π_1^0, it follows that (cf. [13, Chapter XI])

$$\{e\colon W_e \cap (A \times B) \neq \emptyset\} \leqslant_{\mathrm{T}} \emptyset'.$$

By the limit lemma, there is a recursive {0,1}-valued function g such that $\lim_s g(e, s) = g(e)$ exists and

$$g(e) = 1 \quad \text{iff} \quad e \in \{e\colon W_e \cap (A \times B) \neq \emptyset\}.$$

For the same of R_e we define numbers $x(e, s)$ and $y(e, s)$. We say that R_e requires attention if e is least such that if $x(e, s)$ and $y(e, s)$ are undefined or $x(e, s) \in \bar{A}_s$ or $y(e, s) \in \bar{B}_s$.

Construction. During stage $s + 1$, the construction proceeds as follows:

Find the least e such that R_e requires attention and $g(h(e), s) = 0$, where for the sake of R_e we have a test set V_e whose index $h(e)$ is given by the recursion theorem. Find the least i and j such that $a_{i,s} \notin \operatorname{dom} f_s$ and $b_{j,s} \notin \operatorname{dom} g_s$. Enumerate $\langle a_{i,s}, b_{r,s}\rangle$ into $V_{e,s+1}$. Compute $t \geqslant s + 1$ such that either $g(h(e), t) = 1$ or $a_{i,s} \notin A_t$ or $b_{j,s} \notin B_t$. (One must occur.) If $g(h(e), t) = 1$ occurs, define $f_{s+1}(a_{i,s}) = b_{j,s}$ and $x(e, s+1) = a_{i,s}$ and $y(e, s+1) = b_{j,s}$. If one of the other cases occurs (say $a_{i,s} \notin A_t$, the other is dual), find the least i' such that $a_{i',t} \notin \operatorname{dom} f_s$ and enumerate $\langle a_{i',s}, b_{j,s}\rangle$ into $V_{e,t+1}$ and await a stage $u \geqslant t + 1$ such that either $g(h(e), u) = 1$ or $a_{i',t} \in A_u$ or $b_{j,s} \in B_u$, etc. We continue in this way until the $g(h(e), \cdot) = 1$ option occurs as it must since $|A| = |B| = \infty$. □

(End of Construction)

To see that the construction works we need only observe that if we map $a_{i,s}$ to $b_{j,s}$ at stage s, it is only because $g(h(e), t) = 1$. If this is a false assignment, then $\langle a_{i,s}, b_{j,s}\rangle \notin A \times B$ and hence at some stage $u > t$ we must see $g(h(e), w) = 0$. Now as $\lim_s g(z, s) = g(z)$ exists, we can only so switch finitely often. □

It seems reasonable the stronger conjecture might possibly have an affirmative solution if A has degree $0'$. One would need to check that coding combined with the argument of Section 2. It also seems that one might even prove the stronger conjecture for any A of high recursively enumerable degree.

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