A NOTE ON REALIZATION OF INDEX SETS IN Π_1^0 CLASSES

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1. Introduction

We assume that the reader is familiar with Π_1^0 classes and index sets as per Cenzer and Jockusch [1], and Soare [4]. The Kriesel Basis Theorem says that each Π_1^0 class has a member of c.e. degree. In [2], Csima, Downey and Ng analysed the problem of determining which sets of c.e. degrees can be realised as members of Π_1^0 classes. Such sets of degrees can be considered as index sets. To wit, we say that e is realized in a Π_1^0 class $\mathcal{C}$ iff there a member P of $\mathcal{C}$ with $\deg_T(W_e) = \deg_T(P)$, and $\mathcal{C}$ is a Π_1^0 class, then $W[\mathcal{C}] = \{e : W_e \text{ is realisable in } \mathcal{C}\}$. Csima, Downey and Ng [2] have recently gaven a precise classification of the index sets which name precisely the c.e. degrees realised in some Π_1^0 class. This involved the following notion. A set S *represents* an index set I iff $I =_{\text{def}} G(S) = \{e : (\exists j \in S)\ W_e \equiv_T W_j\}$.

Theorem 1.1 (Csima, Downey and Ng [2]). *An index set I is realisable in a Π_1^0 class iff I has a Σ_3^0 representation iff I has a computable representation.*

Notice that a crude upper bound for the relevant index sets is Σ_4^0, while some Σ_4^0-complete index sets such as $\{e \mid W_e \text{ complete}\}$ have Σ_3^0 representations. (In this last case take the singleton consisting of any index for the halting problem.)

This led to Csima, Downey and Ng trying to ascertain precisely which index sets have Σ_3^0 representations. Classical index set results by Yates [5, 6] show that if A is low$_2$ then $\{e \mid W_e \leq_T A\}$ has a Σ_3^0 representation. Csima, Downey and Ng showed that the collection of superlow c.e. sets have Σ_3^0 representations, as do all upper cones. They asked the following question.

Question 1.2 (Csima, Downey and Ng [2]). *Is there some (Turing incomplete) non-low$_2$ c.e. set A such that the c.e. lower cone below A has a Σ_3^0 representation?*

In this note we solve this question verifying a conjecture from [2].

Theorem 1.3. *$\{e \mid W_e \leq_T A\}$ has a Σ_3^0 representation iff A is low$_2$ or Turing complete.*

2. The Proof

The proof is not difficult, but involves assembling a number of facts in a new way. First we consider the computable functions f and g defined by the uniform construction which, for each W_k, builds a splitting $W_k = W_k^1 \oplus W_k^2 \equiv_{\text{def}} W_{f(k)} \oplus W_{g(k)}$, and meets the requirements for $i = 1, 2$,

$$R_{\langle e,i\rangle} : \exists^\infty s(\Phi_e^{W_k^i}(e)\downarrow[s]) \to \Phi_e^{W_k^i}(e)\downarrow.$$

Downey and Melnikov thank the Marsden Fund of New Zealand.

We do this in the usual way: put x entering $W_k[s]$ into the set which does not injure the requirement of highest priority threatened. (This is the standard Sacks' method.)

We could assume that f and g are strictly increasing in their arguments, andnote that their domains do not overlap. In particular, without loss of generality we could assume that:

(1) $f(k) \neq f(j)$, $g(k) \neq g(j)$ whenever $j \neq k$;
(2) $f(k) \neq g(i)$ for any i and j;
(3) the sets $\{f(k) : k \in \omega\}$ and $\{g(k) : k \in \omega\}$ are both computable;

In particular, given $j \in \omega$ we can recognise whether $j = f(k)$ or $j = g(k)$ for some k, and thus compute this k. Also, note that the family of sets $\{W_{g(k)}, W_{f(k)}\}_{k\in\omega}$ is uniformly low.

Assuming $(1) - (3)$ above, the following lemma is immediate:

Lemma 2.1. *Let $S \subset \omega$ and let $\hat{S} = \{f(k) \mid k \in S\} \cup \{g(k) \mid k \in S\}$. Then $S \leq_1 \hat{S}$.*

Proof. Both f and g 1-reduce S to $\hat{S}$. □

Now suppose that A is non-low$_2$. Then $S = \{e \mid W_e \leq_T A\}$ is Σ^0_4 complete by Yates [5, 6]. Suppose that S has a Σ^0_3 representation R. Consider $\hat{S}$.

We claim that $e \in \hat{S}$ if, and only if, either $e = f(k)$ or $e = g(k)$ for some k, and if so then for this k we have

$$(\exists j, i)(W_{f(k)} \equiv_T W_j \,\&\, W_{g(k)} \equiv_T W_i \,\&\, R(j) \,\&\, R(i)).$$

If $e \in \hat{S}$ then e must be either $f(k)$ or $g(k)$ for some k, and since $W_{f(k)}$ and $W_{g(k)}$ split a set below A both halfs must be c.e. sets below A. In particular, their Turing degrees must be listed in the Σ^0_3 representation R of S. Conversely, if both $W_{f(k)}$ and $W_{g(k)}$ are listed in R, up to Turing equivalence, then they must be a split of a set Turing below A.

To produce the upper bound on the syntactical complexity of the definition above, recall that the sequence $\{W_{g(k)}, W_{f(k)}\}_{k\in\omega}$ is uniformly low. In particular, the $\Sigma_3^{W_{g(k)}}$ set

$$\{i : W_{g(k)} \equiv_T W_i\}$$

is Σ^0_3 uniformly in k, and similarly the $\Sigma_3^{W_{f(k)}}$ set

$$\{j : W_{f(k)} \equiv_T W_j\}$$

is Σ^0_3 uniformly in k.

This brings the complexity of the relation $e \in \hat{S}$ down to Σ^0_3. But $S \leq_1 \hat{S}$, contradicting the Σ^0_4-completeness of S. This concludes the proof.

3. Questions

There are a number of quite interesting questions which remain.

(1) For what intervals of c.e. degrees $[a, b]$ can we realize $\{\mathbf{c} \mid \mathbf{c} \in [a, b]\}$? We know that for any $\mathbf{a}$ and $\mathbf{b} = \mathbf{0}'$, and $\mathbf{a} = \mathbf{0}$ and $\mathbf{b}$ low$_2$. What else?
(2) (Csima, Downey, Ng) What is the situation for *separating classes*? If we insist that the host class is a separating class, what Index Sets can be realized. The only known singleton is $\mathbf{0}'$ as witnessed by, for example, the class of Martin-Löf random reals. It is known by using results of Downey,

Jockusch and Stob [3], no "array computable" singleton is possible. Is any incomplete (nonzero) singleton possible?

(3) What about strong reducibilities? For instance weak truth table reducibility? Again we know that $\emptyset'_{wtt}$ is possible using random reals, but it also seems that some singletons are *not* possible. Of course here the index sets will be Σ^0_3 as given.

References

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