Randomness, Computability, and Density

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Abstract

We study effectively given positive reals (more specifically, computably enumerable reals) under a measure of relative randomness introduced by Solovay [30] and studied by Calude, Hertling, Khoussainov, and Wang [6], Calude [2], Slaman [26], and Coles, Downey, and LaForte [12], among others. This measure is called *domination* or *Solovay reducibility*, and is defined by saying that α dominates β if there are a constant c and a partial computable function φ such that for all positive rationals $q < \alpha$ we have $\varphi(q)\downarrow< \beta$ and $\beta - \varphi(q) \leqslant c(\alpha - q)$. The intuition is that an approximating sequence for α generates one for β. It is not hard to show that if α dominates β then the initial segment complexity of α is at least that of β.

In this paper we are concerned with structural properties of the degree structure generated by Solovay reducibility. We answer a long-standing question in this area of investigation by proving the density of the Solovay degrees. We also provide a new characterization of the random c.e. reals in terms of splittings in the Solovay degrees. Specifically, we show that the Solovay degrees of computably enumerable reals are dense, that any incomplete Solovay degree splits over any lesser degree, and that the join of any two incomplete Solovay degrees is incomplete, so that the complete Solovay degree does not split at all. The methodology is of some technical interest, since it includes a priority argument in which the injuries are themselves controlled by randomness considerations.

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1 Introduction

In this paper we are concerned with effectively generated reals in the interval $(0,1]$ and their relative randomness. In what follows, *real* and *rational* will mean positive real and positive rational, respectively. It will be convenient to work modulo 1, that is, identifying $n+\alpha$ and α for any $n \in \omega$ and $\alpha \in (0,1]$, and we do this below without further comment.

Our basic objects are reals that are limits of computable increasing sequences of rationals. We call such reals *computably enumerable* (c.e.), though they have also been called *recursively enumerable*, *left computable* (by Ambos-Spies, Weihrauch, and Zheng [1]), and, together with the limits of computable decreasing sequences of rationals, *semicomputable.* If, in addition to the existence of a computable increasing sequence $q_0, q_1, \ldots$ of rationals with limit α, there is a total computable function f such that $\alpha - q_{f(n)} < 2^{-n}$ for all $n \in \omega$, then α is called *computable.* These and related concepts have been widely studied. In addition to the papers and books mentioned elsewhere in this introduction, we may cite, among others, early work of Rice [24], Lachlan [19], Soare [27], and Ceĭtin [8], and more recent papers by Ko [16, 17], Calude, Coles, Hertling, and Khoussainov [5], Ho [15], and Downey and LaForte [14].

A real is random if its dyadic expansion forms a random infinite binary sequence (in the sense of, for instance, Martin-Löf [23]). Chaitin's number Ω, the halting probability of a universal self-delimiting computer, is a standard random c.e. real. (We will define these concepts more formally below.)

Many authors have studied Ω and its properties, notably Chaitin [9, 10, 11] and Martin-Löf [23]. In the very long and widely circulated manuscript [30] (a fragment of which appeared in [31]), Solovay carefully investigated relationships between Martin-Löf-Chaitin prefix-free complexity, Kolmogorov complexity, and properties of random languages and reals. See Chaitin [9] for an account of some of the results in this manuscript.

Solovay discovered that several important properties of Ω (whose definition is model-dependent) are shared by another class of reals he called Ω-like, whose definition is model-independent. To define this class, he introduced the following reducibility relation among c.e. reals, called *domination* or *Solovay reducibility.*

1.1. Definition. Let α and β be c.e. reals. We say that α *dominates* β, and write $\beta \leqslant_{\mathrm{S}} \alpha$, if there are a constant c and a partial computable function $\varphi : \mathbb{Q} \to \mathbb{Q}$ such that for each rational $q < \alpha$ we have $\varphi(q)\downarrow < \beta$ and

$$\beta - \varphi(q) \leqslant c(\alpha - q).$$

We write $\alpha \equiv_{\mathrm{S}} \beta$ if $\alpha \leqslant_{\mathrm{S}} \beta$ and $\beta \leqslant_{\mathrm{S}} \alpha$.

The notation $\leqslant_{\text{dom}}$ has sometimes been used instead of $\leqslant_{\text{S}}$.

Solovay reducibility is naturally associated with randomness because of the following fact, whose proof we sketch for completeness. Let $H(\tau)$ denote the prefix-free complexity of τ and $K(\tau)$ its standard Kolmogorov complexity. (Most of the statements below also hold with $K(\tau)$ in place of $H(\tau)$. For the definitions and basic properties of $H(\tau)$ and $K(\tau)$, see Calude [3] and Li and Vitanyi [22]. Among the many works dealing with these and related topics, and in addition to those mentioned elsewhere in this paper, we may cite Solomonoff [28, 29], Kolmogorov [18], Levin [20, 21], Schnorr [25], and the expository article Calude and Chaitin [4].) We identify a real $\alpha \in (0, 1]$ with the infinite binary string σ such that $\alpha = 0.\sigma$. (The fact that certain reals have two different dyadic expansions need not concern us here, since all such reals are rational.)

1.2. Theorem (Solovay [30]). *Let $\beta \leqslant_S \alpha$ be c.e. reals. There is a constant $O(1)$ such that $H(\beta \upharpoonright n) \leqslant H(\alpha \upharpoonright n) + O(1)$ for all $n \in \omega$.*

Proof sketch. We first sketch the proof of the following lemma, implicit in [30] and noted by Calude, Hertling, Khoussainov, and Wang [6].

1.3. Lemma. *Let $c \in \omega$. There is a constant $O(1)$ such that, for all $n \geqslant 1$ and all binary strings σ, τ of length n with $|0.\sigma - 0.\tau| < c2^{-n}$, we have $|H(\tau) - H(\sigma)| \leqslant O(1)$.*

The proof of the lemma is relatively simple. We can easily write a program P that, for each sufficiently long σ, generates the $2c + 1$ binary strings τ' of length n with $|0.\sigma - 0.\tau'| < c2^{-n}$. For any binary strings σ, τ of length n with $|0.\sigma - 0.\tau| < c2^{-n}$, in order to compute τ it suffices to know a program for σ and the position of τ on the list generated by P on input σ.

Turning to the proof of the theorem, let φ and c be as in Definition 1.1. Let $\alpha_n = 0.(\alpha \upharpoonright n)$. Since α_n is rational and $\alpha - \alpha_n < 2^{-(n+1)}$, we have $\beta - \varphi(\alpha_n) < c2^{-(n+1)}$. Thus, by the lemma, $H(\beta \upharpoonright n) = H(\varphi(\alpha_n)) + O(1)$, and hence $H(\beta \upharpoonright n) \leqslant H(\alpha \upharpoonright n) + O(1)$. $\square$

Solovay observed that Ω dominates all c.e. reals, and Theorem 1.2 implies that if a c.e. real dominates all c.e. reals then it must be random. This led Solovay to define a c.e. real to be Ω-*like* if it dominates all c.e. reals. The point is that the definition of Ω-like seems quite model-independent, as opposed to the model-dependent definition of Ω. This circle of ideas was completed recently by Slaman [26], who proved the converse to the fact that Ω-like reals are random.

1.4. Theorem (Slaman). *A c.e. real is random if and only if it is Ω-like.*

It is natural to seek to understand the c.e. reals under Solovay reducibility. A useful characterization of this reducibility is given by the following lemma, which we prove in the next section.

1.5. Lemma. *Let α and β be c.e. reals. Then $\beta \leqslant_S \alpha$ if and only if for every computable sequence of rationals $a_0, a_1, \ldots$ such that*

$$\alpha = \sum_{n \in \omega} a_n$$

there are a constant c and a computable sequence of rationals $\varepsilon_0, \varepsilon_1, \ldots < c$ such that

$$\beta = \sum_{n \in \omega} \varepsilon_n a_n.$$

Phrased another way, Lemma 1.5 says that the c.e. reals dominated by a given c.e. real α essentially correspond to splittings of α under arithmetic addition.

1.6. Corollary. *Let $\beta \leqslant_S \alpha$ be c.e. reals. There is a c.e. real γ and a rational c such that $c\alpha = \beta + \gamma$.*

Proof. Let $a_0, a_1, \ldots$ be a computable sequence of rationals such that $\alpha = \sum_{n \in \omega} a_n$. Let $c \in \mathbb{Q}$ and $\varepsilon_0, \varepsilon_1, \ldots < c$ be as in Lemma 1.5. Define $\gamma = \sum_{n \in \omega} (c - \varepsilon_n) a_n$. Since each ε_n is less than c, the real γ is c.e., and of course $\beta + \gamma = \alpha$. □

Solovay reducibility has a number of other beautiful interactions with arithmetic, as we now discuss.

The relation $\leqslant_{\mathrm{S}}$ is symmetric and transitive, and hence $\equiv_{\mathrm{S}}$ is an equivalence relation on the c.e. reals. Thus we can define the *Solovay degree* $[\alpha]$ of a c.e. real α as its $\equiv_{\mathrm{S}}$ equivalence class. (When we mention Solovay degrees below, we always mean Solovay degrees of c.e. reals.) The Solovay degrees form an upper semilattice, with the join of $[\alpha]$ and $[\beta]$ being $[\alpha+\beta]$=$[\alpha\beta]$, a fact observed by Solovay and others ($\oplus$ is definitely not a join operation here). We note the following slight improvement of this result. Recall that an uppersemilattice U is *distributive* if for all $a_0, a_1, b \in U$ with $b \leqslant a_0 \vee a_1$ there exist $b_0, b_1 \in U$ such that $b_0 \vee b_1 = b$ and $b_i \leqslant a_i$ for $i = 0, 1$.

1.7. Lemma. *The Solovay degrees of c.e. reals form a distributive uppersemilattice with $[\alpha] \vee [\beta] = [\alpha + \beta] = [\alpha\beta]$.*

Proof. Suppose that $\beta \leqslant_{\mathrm{S}} \alpha_0 + \alpha_1$. Let $a^0_0, a^0_1, \ldots$ and $a^1_0, a^1_1, \ldots$ be computable sequences of rationals such that $\alpha_i = \sum_{n \in \omega} a^i_n$ for $i = 0, 1$. By Lemma 1.5, there are a constant c and a computable sequence of rationals $\varepsilon_0, \varepsilon_1, \ldots < c$ such that $\beta = \sum_{n \in \omega} \varepsilon_n (a^0_n + a^1_n)$.

Let $\beta_i = \sum_{n \in \omega} \varepsilon_n a_n^i$. Then $\beta = \beta_0 + \beta_1$ and, again by Lemma 1.5, $\beta_i \leqslant_S \alpha_i$ for $i = 0, 1$. This establishes distributivity.

To see that the join in the Solovay degrees is given by addition, we again apply Lemma 1.5. Certainly, for any c.e. reals β_0 and β_1 we have $\beta_i \leqslant_S \beta_0 + \beta_1$ for $i = 0, 1$, and hence $[\beta_0 + \beta_1] \geqslant_S [\beta_0], [\beta_1]$. Conversely, suppose that $\beta_0, \beta_1 \leqslant \alpha$. Let $a_0, a_1, \ldots$ be a computable sequence of rationals such that $\alpha = \sum_{n \in \omega} a_n$. For each $i = 0, 1$ there is a constant c_i and a computable sequence of rationals $\varepsilon_0^i, \varepsilon_1^i, \ldots < c_i$ such that $\beta_i = \sum_{n \in \omega} \varepsilon_n^i a_n$. Thus $\beta_0 + \beta_1 = \sum_{n \in \omega} (\varepsilon_n^0 + \varepsilon_n^1) a_n$. Since each $\varepsilon_n^0 + \varepsilon_n^1$ is less than $c_0 + c_1$, a final application of Lemma 1.5 show that $\beta_0 + \beta_1 \leqslant_S \alpha$.

The proof that the join in the Solovay degrees is also given by multiplication is a similar application of Lemma 1.5. $\square$

There is a least Solovay degree, the degree of the computable reals, as well as a greatest one, the degree of Ω. For proofs of these facts and more on c.e. reals and Solovay reducibility, see for instance Chaitin [9, 10, 11], Calude, Hertling, Khoussainov, and Wang [6], Calude and Nies [7], Calude [2], Slaman [26], and Coles, Downey, and LaForte [12].

Despite the many attractive features of the Solovay degrees, their structure is largely unknown. Coles, Downey, and LaForte [12] have shown that this structure is very complicated by proving that it has an undecidable first order theory.

One question addressed in the present paper, open since Solovay's original 1974 notes, is whether the structure of the Solovay degrees is dense. Indeed, up to now, it was not known even whether there is a *minimal* Solovay degree. That is, intuitively, if a c.e. real α is not computable, must there be a c.e. real that is also not computable, yet is strictly less random that α?

In this paper, we show that the Solovay degrees of c.e. reals are dense. To do this we divide the proof into two parts. We prove that if $\alpha <_S \Omega$ then there is a c.e. real γ with $\alpha <_S \gamma <_S \Omega$, and we also prove that every *incomplete* Solovay degree splits over each lesser degree.

The nonuniform nature of the argument is essential given the techniques we use, since, in the splitting case, we have a priority construction in which the control of the injuries is directly tied to the enumeration of Ω. The fact that if a c.e. real α is Solovay-incomplete then Ω must grow more slowly than α is what allows us to succeed. (We will discuss this more fully in Section 3.) This unusual technique is of some technical interest, and clearly cannot be applied to proving upwards density, since in that case the top degree is Ω itself. To prove upwards density, we use a different technique, taking advantage of the fact that, however we construct a c.e. real, it is automatically dominated by Ω.

In light of these results, and further motivated by the general question of how randomness can be produced, it is natural to ask whether the complete Solovay degree can be split, or in other words, whether there exist nonrandom c.e. reals α and β such that $\alpha + \beta$ is random. We give a negative answer to this question, thus characterizing the random c.e. reals as those c.e. reals that cannot be written as the sum of two c.e. reals of lesser Solovay degrees.

We remark that there are (non-c.e.) nonrandom reals whose sum is random; the following is an example of this phenomenon. Define the real α by letting $\alpha(n) = 0$ if n is even and $\alpha(n) = \Omega(n)$ otherwise. (Here we identify a real with its dyadic expansion as above.) Define the real β by letting $\beta(n) = 0$ if n is odd and $\beta(n) = \Omega(n)$ otherwise. Now α and β are clearly nonrandom, but $\alpha + \beta = \Omega$ is random.

Before turning to the details of the paper, we point out that there are other reducibilities one can study in this context. Coles, Downey, and LaForte [12, 13] introduced one such reducibility, called *sw-reducibility*; it is defined as follows. For sets of natural numbers A and B, we say that $A \leqslant_{\mathrm{sw}} B$ if there are a computable procedure Γ and a constant c such that $\Gamma^B = A$ and the use of Γ on argument x is bounded by $x + c$. For reals $\alpha, \beta \in (0, 1]$, we say that $\alpha \leqslant_{\mathrm{sw}} \beta$ if there are sets A and B such that $\alpha = 0.\chi_A$, $\beta = 0.\chi_B$, and $A \leqslant_{\mathrm{sw}} B$, where χ_S is the characteristic function of the set S.

As in the case of Solovay reducibility, it is not difficult to argue that if $\alpha \leqslant_{\mathrm{sw}} \beta$ then $H(\alpha \upharpoonright n) \leqslant H(\beta \upharpoonright n) + O(1)$ for all $n \in \omega$, and that Ω is sw-complete. Furthermore, Coles, Downey, and LaForte [12] proved the analog of Slaman's theorem above in the case of sw-reducibility, namely that if a c.e. real is random then it is sw-complete. They also showed that Solovay reducibility and sw-reducibility are different, since there are c.e. reals α, β, γ, and δ such that $\alpha \leqslant_{\mathrm{S}} \beta$ but $\alpha \nleqslant_{\mathrm{sw}} \beta$ and $\gamma \leqslant_{\mathrm{sw}} \delta$ but $\gamma \nleqslant_{\mathrm{S}} \delta$, and that there are no minimal sw-degrees of c.e. reals.

1.8. Question. Are the sw-degrees of c.e. reals dense?

Ultimately, the basic reducibility we seek to understand is H-reducibility, where $\sigma \leqslant_H \tau$ if there is a constant $O(1)$ such that $H(\sigma \upharpoonright n) \leqslant H(\tau \upharpoonright n) + O(1)$ for all $n \in \omega$. Little is known about this directly.

2 Preliminaries

Fix a self-delimiting universal computer M. (That is, for all binary strings σ, if $M(\sigma)\downarrow$ then $M(\sigma')\uparrow$ for all σ' properly extending σ.) Then one can define $\Omega = \Omega_M$ via

$$\Omega = \sum_{M(\sigma)\downarrow} 2^{-|\sigma|}.$$

(The properties of Ω relevant to this paper are independent of the choice of M.)

The c.e. real Ω is random in the canonical Martin-Löf sense. Recall that a *Martin-Löf test* is a uniformly c.e. sequence $\{V_e : e > 0\}$ of subsets of $\{0,1\}^*$ such that for all $e > 0$,

$$\mu(V_e\{0,1\}^\omega) \leqslant 2^{-e},$$

where μ denotes the usual product measure on $\{0,1\}^\omega$. The string $\sigma \in \{0,1\}^\omega$ and the real $0.\sigma$ are *random*, or more precisely, 1-random, if $\sigma \notin \bigcap_{e>0} V_e\{0,1\}^\omega$ for every Martin-Löf test $\{V_e : e > 0\}$.

An alternate characterization of the random reals can be given via the notion of a *Solovay test*. We give a somewhat nonstandard definition of this notion, which will be useful below. A Solovay test is a c.e. sequence $\{I_i : i \in \omega\}$ of intervals with rational endpoints such that $\sum_{i\in\omega} |I_i| < \infty$, where $|I|$ is the length of the interval I. As Solovay [30] showed, a real α is random if and only if $\{i \in \omega \mid \alpha \in I_i\}$ is finite for every Solovay test $\{I_i : i \in \omega\}$.

The following lemma, implicit in [30] and proved in [12], provides an alternate characterization of Solovay reducibility, which is the one that we will use below.

2.1. Lemma. *Let α and β be c.e. reals, and let $\alpha_0, \alpha_1, \dots$ and $\beta_0, \beta_1, \dots$ be computable increasing sequences of rationals converging to α and β, respectively. Then $\beta \leqslant_S \alpha$ if and only if there are a constant d and a total computable function f such that for all $n \in \omega$,*

$$\beta - \beta_{f(n)} < d(\alpha - \alpha_n).$$

Whenever we mention a c.e. real α, we assume that we have chosen a computable increasing sequence $\alpha_0, \alpha_1, \dots$ converging to α. The previous lemma guarantees that, in determining whether one c.e. real dominates another, the particular choice of such sequences is irrelevant. For convenience of notation, we adopt the convention that, for any c.e. real α mentioned below, the expression $\alpha_s - \alpha_{s-1}$ is equal to α_0 when $s = 0$.

We will also make use of two more lemmas, the first of which has Lemma 1.5 as a corollary.

2.2. Lemma. *Let $\beta \leqslant_S \alpha$ be c.e. reals and let $\alpha_0, \alpha_1, \ldots$ be a computable increasing sequence of rationals converging to α. There is a computable increasing sequence $\hat{\beta}_0, \hat{\beta}_1, \ldots$ of rationals converging to β such that for some constant c and all $s \in \omega$,*

$$\hat{\beta}_s - \hat{\beta}_{s-1} < c(\alpha_s - \alpha_{s-1}).$$

Proof. Fix a computable increasing sequence $\beta_0, \beta_1, \ldots$ of rationals converging to β, let d and f be as in Lemma 2.1, and let $c > d$ be such that $\beta_{f(0)} < c\alpha_0$. We may assume without loss of generality that f is increasing. Define $\hat{\beta}_0 = \beta_{f(0)}$.

There must be an $s_0 > 0$ for which $\beta_{f(s_0)} - \beta_{f(0)} < d(\alpha_{s_0} - \alpha_0)$, since otherwise we would have

$$\beta - \beta_{f(0)} = \lim_s \beta_{f(s)} - \beta_{f(0)} \geqslant \lim_s d(\alpha_s - \alpha_0) = \alpha - \alpha_0,$$

contradicting our choice of d and f. It is now easy to define $\hat{\beta}_1, \ldots, \hat{\beta}_{s_0}$ so that $\hat{\beta}_0 < \cdots < \hat{\beta}_{s_0}$, $\hat{\beta}_{s_0} = \beta_{f(s_0)}$, and $\hat{\beta}_s - \hat{\beta}_{s-1} \leqslant d(\alpha_s - \alpha_{s-1}) < c(\alpha_s - \alpha_{s-1})$ for all $s \leqslant s_0$.

We can repeat the procedure in the previous paragraph with s_0 in place of 0 to obtain an $s_1 > s_0$ and $\hat{\beta}_{s_0+1}, \ldots, \hat{\beta}_{s_1}$ such that $\hat{\beta}_{s_0} < \cdots < \hat{\beta}_{s_1}$, $\hat{\beta}_{s_1} = \beta_{f(s_1)}$, and $\hat{\beta}_s - \hat{\beta}_{s-1} < c(\alpha_s - \alpha_{s-1})$ for all $s_0 < s \leqslant s_1$.

Proceeding by recursion in this way, we define a computable increasing sequence $\hat{\beta}_0, \hat{\beta}_1, \ldots$ of rationals with the desired properties. □

We are now in a position to prove Lemma 1.5.

1.5. Lemma. *Let α and β be c.e. reals. Then $\beta \leqslant_S \alpha$ if and only if for every computable sequence of rationals $a_0, a_1, \ldots$ such that*

$$\alpha = \sum_{n \in \omega} a_n$$

there are a constant c and a computable sequence of rationals $\varepsilon_0, \varepsilon_1, \ldots < c$ such that

$$\beta = \sum_{n \in \omega} \varepsilon_n a_n.$$

Proof. The if direction is easy; we prove the only if direction.

Suppose that $\beta \leqslant_S \alpha$. Given a computable sequence of rationals $a_0, a_1, \ldots$ such that $\alpha = \sum_{n \in \omega} a_n$, let $\alpha_n = \sum_{i \leqslant n} a_i$ and apply Lemma 2.2 to obtain c and $\hat{\beta}_0, \hat{\beta}_1, \ldots$ as in that lemma. Define $\varepsilon_n = (\hat{\beta}_n - \hat{\beta}_{n-1})a_n^{-1}$. Now $\sum_{n \in \omega} \varepsilon_n a_n = \sum_{n \in \omega} \hat{\beta}_n - \hat{\beta}_{n-1} = \beta$, and for all $n \in \omega$,

$$\varepsilon_n = (\hat{\beta}_n - \hat{\beta}_{n-1})a_n^{-1} = (\hat{\beta}_n - \hat{\beta}_{n-1})(\alpha_n - \alpha_{n-1})^{-1} \leqslant c.$$

□

We finish this section with a simple lemma which will be quite useful below.

2.3. Lemma. *Let $\alpha \nleqslant_S \beta$ be c.e. reals. The following hold for all total computable functions f and all $k \in \omega$.*

1. *For each $n \in \omega$ there is an $s \in \omega$ such that either*

   (a) $\alpha_t - \alpha_{f(n)} < k(\beta_t - \beta_n)$ for all $t > s$ or

   (b) $\alpha_t - \alpha_{f(n)} > k(\beta_t - \beta_n)$ for all $t > s$.

2. *There are infinitely many $n \in \omega$ for which there is an $s \in \omega$ such that $\alpha_t - \alpha_{f(n)} > k(\beta_t - \beta_n)$ for all $t > s$.*

Proof. If there are infinitely many $t \in \omega$ such that $\alpha_t - \alpha_{f(n)} \leqslant k(\beta_t - \beta_n)$ and infinitely many $t \in \omega$ such that $\alpha_t - \alpha_{f(n)} \geqslant k(\beta_t - \beta_n)$ then

$$\alpha - \alpha_{f(n)} = \lim_t \alpha_t - \alpha_{f(n)} = \lim_t k(\beta_t - \beta_n) = k(\beta - \beta_n),$$

which implies that $\alpha \equiv_S \beta$.

If there are infinitely many $t \in \omega$ such that $\alpha_t - \alpha_{f(n)} \leqslant k(\beta_t - \beta_n)$ then

$$\alpha - \alpha_{f(n)} = \lim_t \alpha_t - \alpha_{f(n)} \leqslant \lim_t k(\beta_t - \beta_n) = k(\beta - \beta_n).$$

So if this happens for all but finitely many n then $\alpha \leqslant_S \beta$. (The finitely many n for which $\alpha - \alpha_{f(n)} > k(\beta - \beta_n)$ can be brought into line by increasing the constant k.) □

3 Main Results

We now proceed with the proofs of our main results. We begin by showing that every incomplete Solovay degree can be split over any lesser Solovay degree.

3.1. Theorem. *Let $\gamma <_S \alpha <_S \Omega$ be c.e. reals. There are c.e. reals β^0 and β^1 such that $\gamma <_S \beta^0, \beta^1 <_S \alpha$ and $\beta^0 + \beta^1 = \alpha$.*

Proof. We want to build β^0 and β^1 so that $\gamma \leqslant_S \beta^0, \beta^1 \leqslant_S \alpha$, $\beta^0 + \beta^1 = \alpha$, and the following requirement is satisfied for each $e, k \in \omega$ and $i < 2$:

$$R_{i,e,k} : \Phi_e \text{ total } \Rightarrow \exists n(\alpha - \alpha_{\Phi_e(n)} \geqslant k(\beta^i - \beta^i_n)).$$

By Lemma 2.2 and the fact that $\gamma/c \equiv_S \gamma$ for any rational c, we may assume without loss of generality that $2(\gamma_s - \gamma_{s-1}) \leqslant \alpha_s - \alpha_{s-1}$ for each $s \in \omega$. (Recall our convention that $\mu_0 - \mu_{-1} = \mu_0$ for any c.e. real μ.)

In the absence of requirements of the form $R_{1-i,e,k}$, it is easy to satisfy simultaneously all requirements of the form $R_{i,e,k}$: for each $s \in \omega$, simply let $\beta^i_s = \gamma_s$ and $\beta^{1-i}_s = \alpha_s - \gamma_s$. In the presence of requirements of the form $R_{1-i,e,k}$, however, we cannot afford to be quite so cavalier in our treatment of β^{1-i}; enough of α has to be kept out of β^{1-i} to guarantee that β^{1-i} does not dominate α.

Most of the essential features of our construction are already present in the case of two requirements $R_{i,e,k}$ and $R_{1-i,e',k'}$, which we now discuss. We assume that $R_{i,e,k}$ has priority over $R_{1-i,e',k'}$ and that both Φ_e and $\Phi_{e'}$ are total. We will think of the β^j as being built by adding amounts to them in stages. Thus β^j_s will be the total amount added to β^j by the end of stage s. At each stage s we begin by adding $\gamma_s - \gamma_{s-1}$ to the current value of each β^j; in the limit, this ensures that $\beta^j \geqslant_{\mathrm{S}} \gamma$.

We will say that $R_{i,e,k}$ is satisfied through n at stage s if $\Phi_e(n)[s]\downarrow$ and $\alpha_s - \alpha_{\Phi_e(n)} > k(\beta^i_s - \beta^i_n)$. The strategy for $R_{i,e,k}$ is to act whenever either it is not currently satisfied or the least number through which it is satisfied changes. Whenever this happens, $R_{i,e,k}$ initializes $R_{1-i,e',k'}$, which means that the amount of $\alpha - 2\gamma$ that $R_{1-i,e',k'}$ is allowed to funnel into β^i is reduced. More specifically, once $R_{1-i,e',k'}$ has been initialized for the mth time, the total amount that it is thenceforth allowed to put into β^i is reduced to 2^{-m}.

The above strategy guarantees that if $R_{1-i,e',k'}$ is initialized infinitely often then the amount put into β^i by $R_{1-i,e',k'}$ (which in this case is all that is put into β^i except for the coding of γ) adds up to a computable real. In other words, $\beta^i \equiv_{\mathrm{S}} \gamma <_{\mathrm{S}} \alpha$. But it is not hard to argue, with the help of Lemma 2.3, that this means that there is a stage s after which $R_{i,e,k}$ is always satisfied and the least number through which it is satisfied does not change. So we conclude that $R_{1-i,e',k'}$ is initialized only finitely often, and that $R_{i,e,k}$ is eventually permanently satisfied.

This leaves us with the problem of designing a strategy for $R_{1-i,e',k'}$ that respects the strategy for $R_{i,e,k}$. The problem is one of timing. To simplify notation, let $\hat{\alpha} = \alpha - 2\gamma$ and $\hat{\alpha}_s = \alpha_s - 2\gamma_s$. Since $R_{1-i,e',k'}$ is initialized only finitely often, there is a certain amount 2^{-m} that it is allowed to put into β^i after the last time it is initialized. Thus if $R_{1-i,e',k'}$ waits until a stage s such that $\hat{\alpha} - \hat{\alpha}_s < 2^{-m}$, adding nothing to β^i until such a stage is reached, then from that point on it can put all of $\hat{\alpha} - \hat{\alpha}_s$ into β^i, which of course guarantees its success. The problem is that, in the general construction, a strategy working with a quota 2^{-m} cannot effectively find an s such that $\hat{\alpha} - \hat{\alpha}_s < 2^{-m}$. If it uses up its quota too soon, it may find itself unsatisfied and unable to do anything about it.

The key to solving this problem (and the reason for the hypothesis that $\alpha <_{\mathrm{S}} \Omega$) is the observation that, since the sequence $\Omega_0, \Omega_1, \ldots$ converges much more slowly than the sequence $\hat{\alpha}_0, \hat{\alpha}_1, \ldots$, Ω can be used to modulate the amount that $R_{1-i,e',k'}$ puts into β^i. More specifically, at a stage s, if $R_{1-i,e',k'}$'s current quota is 2^{-m} then it puts into β^i as much of $\hat{\alpha}_s - \hat{\alpha}_{s-1}$ as possible, subject to the constraint that the total amount put into β^i by $R_{1-i,e',k'}$ since the last stage before stage s at which $R_{1-i,e',k'}$ was initialized must not exceed $2^{-m}\Omega_s$. As we will see below, the fact that $\Omega >_{\mathrm{S}} \alpha$ implies that there is a stage v after which $R_{1-i,e',k'}$ is allowed to put in all of $\hat{\alpha} - \hat{\alpha}_v$ into β^i.

In general, at a given stage s there will be several requirements, each with a certain amount that it wants (and is allowed) to direct into one of the β^j. We will work backwards, starting with the weakest priority requirement that we are currently considering. This requirement will be allowed to direct as much of $\hat{\alpha}_s - \hat{\alpha}_{s-1}$ as it wants (subject to its current quota, of course). If any of $\hat{\alpha}_s - \hat{\alpha}_{s-1}$ is left then the next weakest priority strategy will be allowed to act, and so on up the line.

We now proceed with the full construction. We say that $R_{i,e,k}$ has stronger priority than $R_{i',e',k'}$ if $2\langle e,k\rangle + i < 2\langle e',k'\rangle + i'$.

We say that $R_{i,e,k}$ is satisfied through n at stage s if

$$\Phi_e(n)[s]\!\downarrow \wedge\ \alpha_s - \alpha_{\Phi_e(n)} > k(\beta^i_s - \beta^i_n).$$

Let $n^{i,e,k}_s$ be the least n through which $R_{i,e,k}$ is satisfied at stage s, if such an n exists, and let $n^{i,e,k}_s = \infty$ otherwise.

A stage s is e-expansionary if

$$\max\{n \mid \forall m \leqslant n(\Phi_e(m)[s]\!\downarrow)\} > \max\{n \mid \forall m \leqslant n(\Phi_e(n)[s-1]\!\downarrow)\}.$$

Let q be the last e-expansionary stage before stage s (or let $q = 0$ if there have been none). We say that $R_{i,e,k}$ requires attention at stage s if s is an e-expansionary stage and there is an $r \in [q, s)$ such that either $n^{i,e,k}_r = \infty$ or $n^{i,e,k}_r \neq n^{i,e,k}_{r-1}$.

If $R_{i,e,k}$ requires attention at stage s then we say that each requirement of weaker priority than $R_{i,e,k}$ is initialized at stage s.

Each requirement $R_{i,e,k}$ has associated with it a c.e. real $\tau^{i,e,k}$, which records the amount put into β^{1-i} for the sake of $R_{i,e,k}$.

We decide how to distribute $\delta = \alpha_s - \alpha_{s-1}$ between β^0 and β^1 at stage s as follows.

1. Let $j = s$ and $\varepsilon = 2(\gamma_s - \gamma_{s-1})$, and add $\gamma_s - \gamma_{s-1}$ to the current value of each β^i.

2. Let $i < 2$ and $e, k \in \omega$ be such that $2\langle e,k\rangle + i = j$. Let m be the number of times $R_{i,e,k}$ has been initialized and let t be the last stage at which $R_{i,e,k}$ was initialized. Let

$$\zeta = \min(\delta - \varepsilon, 2^{-(j+m)}\Omega_s - (\tau^{i,e,k}_{s-1} - \tau^{i,e,k}_t)).$$

(It is not hard to check that ζ is non-negative.) Add ζ to ε and to the current values of $\tau^{i,e,k}$ and β^{1-i}.

3. If $\varepsilon = \delta$ or $j = 0$ then add $\delta - \varepsilon$ to the current value of β^0 and end the stage. Otherwise, decrease j by one and go to step 2.

This completes the construction. Clearly, $\gamma \leqslant_{\mathrm{S}} \beta^0, \beta^1 \leqslant_{\mathrm{S}} \alpha$ and $\beta^0 + \beta^1 = \alpha$.

We now show by induction that each requirement initializes requirements of weaker priority only finitely often and is eventually satisfied. Assume by induction that $R_{i,e,k}$ is initialized only finitely often. Let $j = 2\langle e, k\rangle + i$, let m be the number of times $R_{i,e,k}$ is initialized, and let t be the last stage at which $R_{i,e,k}$ is initialized. If Φ_e is not total then $R_{i,e,k}$ is vacuously satisfied and eventually stops initializing requirements of weaker priority, so we may assume that Φ_e is total. Now the following are clearly equivalent:

1. $R_{i,e,k}$ is satisfied,

2. $\lim_s n_s^{i,e,k}$ exists and is finite, and

3. $R_{i,e,k}$ eventually stops requiring attention.

Assume for a contradiction that $R_{i,e,k}$ requires attention infinitely often. Since $\Omega \nleqslant_{\mathrm{S}} \alpha$, part 2 of Lemma 2.3 implies that there are $v > u > t$ such that for all $w > v$ we have $2^{-(j+m)}(\Omega_w - \Omega_u) > \alpha_w - \alpha_u$. Furthermore, by the way the amount ζ added to $\tau^{i,e,k}$ at a given stage is defined in step 2 of the construction, $\tau_u^{i,e,k} - \tau_t^{i,e,k} \leqslant 2^{-(j+m)}\Omega_u$ and $\tau_{w-1}^{i,e,k} - \tau_u^{i,e,k} \leqslant \alpha_{w-1} - \alpha_u$. Thus for all $w > v$,

$$\begin{aligned}
&\alpha_w - \alpha_{w-1} = \alpha_w - \alpha_u - (\alpha_{w-1} - \alpha_u) < \\
&\quad 2^{-(j+m)}(\Omega_w - \Omega_u) - (\alpha_{w-1} - \alpha_u) = 2^{-(j+m)}\Omega_w - (2^{-(j+m)}\Omega_u + \alpha_{w-1} - \alpha_u) \leqslant \\
&\qquad 2^{-(j+m)}\Omega_w - (\tau_u^{i,e,k} - \tau_t^{i,e,k} + \tau_{w-1}^{i,e,k} - \tau_u^{i,e,k}) = 2^{-(j+m)}\Omega_w - (\tau_{w-1}^{i,e,k} - \tau_t^{i,e,k}).
\end{aligned}$$

From this we conclude that, after stage v, the reverse recursion performed at each stage never gets past j, and hence everything put into β^i after stage v is put in either to code γ or for the sake of requirements of weaker priority than $R_{i,e,k}$.

Let τ be the sum of all $\tau^{1-i,e',k'}$ such that $R_{1-i,e',k'}$ has weaker priority than $R_{i,e,k}$. Let $s_l > t$ be the lth stage at which $R_{i,e,k}$ requires attention. If $R_{1-i,e',k'}$ is the pth requirement on the priority list and $p > j$ then $\tau^{i',e',k'} - \tau_{s_l}^{i',e',k'} \leqslant 2^{-(p+l)}\Omega$. Thus

$$\tau - \tau_{s_l} \leqslant \sum_{p \geqslant 1} 2^{-(p+l)}\Omega = 2^{-l}\Omega \leqslant 2^{-l},$$

and hence τ is computable.

Putting together the results of the previous two paragraphs, we see that $\beta^i \leqslant_S \gamma$. Since $\alpha \nleqslant_S \gamma$, this means that $\alpha \nleqslant_S \beta^i$. It now follows from Lemma 2.3 that there is an $n \in \omega$ such that $R_{i,e,k}$ is eventually permanently satisfied through n, and such that $R_{i,e,k}$ is eventually never satisfied through any $n' < n$. Thus $\lim_s n_s^{i,e,k}$ exists and is finite, and hence $R_{i,e,k}$ is satisfied and eventually stops requiring attention. $\square$

We now show that the Solovay degrees are upwards dense, which together with the previous result implies that they are dense.

3.2. Theorem. *Let $\gamma <_S \Omega$ be a c.e. real. There is a c.e. real β such that $\gamma <_S \beta <_S \Omega$.*

Proof. We want to build $\beta \geqslant_S \gamma$ to satisfy the following requirements for each $e, k \in \omega$:

$$R_{e,k} : \Phi_e \text{ total } \Rightarrow \exists n(\beta - \beta_{\Phi_e(n)} \geqslant k(\gamma - \gamma_n))$$

and

$$S_{e,k} : \Phi_e \text{ total } \Rightarrow \exists n(\Omega - \Omega_{\Phi_e(n)} \geqslant k(\beta - \beta_n)).$$

As in the previous proof, the analysis of an appropriate two-strategy case will be enough to outline the essentials of the full construction. Let us consider the strategies $S_{e,k}$ and $R_{e',k'}$, the former having priority over the latter. We assume that both Φ_e and $\Phi_{e'}$ are total.

The strategy for $S_{e,k}$ is basically to make β look like γ. At each point of the construction, $R_{e',k'}$ has a certain fraction of Ω that it is allowed to put into β. (This is in addition to the coding of γ into β, of course.) We will say that $S_{e,k}$ is satisfied through n at stage s if $\Phi_e(n)\downarrow$ and $\Omega_s - \Omega_{\Phi_e(n)} > k(\beta_s - \beta_n)$. Whenever either it is not currently satisfied or the least number through which it is satisfied changes, $S_{e,k}$ initializes $R_{e',k'}$, which means that the fraction of Ω that $R_{e',k'}$ is allowed to put into β is reduced.

As in the previous proof, if $S_{e,k}$ is not eventually permanently satisfied through some n then the amount put into β by $R_{e',k'}$ is computable, and hence $\beta \equiv_S \gamma$. But, as before, this implies that there is a stage after which $S_{e,k}$ is permanently satisfied through some n and never again satisfied through any $n' < n$. Once this stage has been reached, $R_{e',k'}$ is free to code a fixed fraction of Ω into β, and hence it too succeeds.

We now proceed with the full construction. We say that a requirement $X_{e,k}$ has stronger priority than a requirement $Y_{e',k'}$ if either $\langle e, k\rangle < \langle e', k'\rangle$ or $\langle e, k\rangle = \langle e', k'\rangle$, $X = R$, and $Y = S$.

We say that $R_{e,k}$ is satisfied through n at stage s if $\Phi_e(n)\downarrow$ and

$$\beta_s - \beta_{\Phi_e(n)} > k(\gamma_s - \gamma_n).$$

We say that $S_{e,k}$ is satisfied through n at stage s if $\Phi_e(n)\downarrow$ and

$$\Omega_s - \Omega_{\Phi_e(n)} > k(\beta_s - \beta_n).$$

For a requirement $X_{e,k}$, let $n_s^{X_{e,k}}$ be the least n through which $X_{e,k}$ is satisfied at stage s, if such an n exists, and let $n_s^{X_{e,k}} = \infty$ otherwise.

As before, a stage s is e-expansionary if

$$\max\{n \mid \forall m \leqslant n(\Phi_e(m)[s]\downarrow)\} > \max\{n \mid \forall m \leqslant n(\Phi_e(n)[s-1]\downarrow)\}.$$

Let $X_{e,k}$ be a requirement and let q be the last e-expansionary stage before stage s (or let $q = 0$ if there have been none). We say that $X_{e,k}$ requires attention at stage s if s is an e-expansionary stage and there is an $r \in [q, s)$ such that either $n_r^{X_{e,k}} = \infty$ or $n_r^{X_{e,k}} \neq n_{r-1}^{X_{e,k}}$.

At stage s, proceed as follows. First add $\gamma_s - \gamma_{s-1}$ to the current value of β. If no requirement requires attention at stage s then end the stage. Otherwise, let $X_{e,k}$ be the strongest priority requirement requiring attention at stage s. We say that $X_{e,k}$ acts at stage s. If $X = S$ then initialize all weaker priority requirements and end the stage. If $X = R$ then let $j = \langle e, k\rangle$ and let m be the number of times that $R_{e,k}$ has been initialized. If s is the first stage at which $R_{e,k}$ acts after the last time it was initialized then let t be the last stage at which $R_{e,k}$ was initialized, and otherwise let t be the last stage at which $R_{e,k}$ acted. Add $2^{-(j+m)}(\Omega_s - \Omega_t)$ to the current value of β and end the stage.

This completes the construction. Since β is bounded by $\gamma + \sum_{i\geqslant 0} 2^{-i}\Omega = \gamma + 2\Omega$, it is a well-defined c.e. real. Furthermore, $\gamma \leqslant_{\mathrm{S}} \beta$.

We now show by induction that each requirement initializes requirements of weaker priority only finitely often and is eventually satisfied. Assume by induction that there is a stage u such that no requirement of stronger priority than $X_{e,k}$ requires attention after stage u. If Φ_e is not total then $X_{e,k}$ is vacuously satisfied and eventually stops requiring attention, so we may assume that Φ_e is total. Now the following are clearly equivalent:

1. $X_{e,k}$ is satisfied,
2. $\lim_s n_s^{X_{e,k}}$ exists and is finite,
3. $X_{e,k}$ eventually stops requiring attention, and
4. $X_{e,k}$ acts only finitely often.

First suppose that $X = R$. Let $j = \langle e, k\rangle$ and let m be the number of times that $R_{e,k}$ is initialized. (Since $R_{e,k}$ is not initialized at any stage after stage u, this number is finite.) Suppose that $R_{e,k}$ acts infinitely often. Then the total amount added to β for the sake of $R_{e,k}$ is $2^{-(j+m)}\Omega$, and hence $\beta \equiv_{\mathrm{S}} 2^{-(j+m)}\Omega \equiv_{\mathrm{S}} \Omega \nleqslant_{\mathrm{S}} \gamma$. It now follows from Lemma 2.3 that there is an $n \in \omega$ such that $R_{e,k}$ is eventually permanently satisfied through n, and such that $R_{e,k}$ is eventually never satisfied through $n' < n$. Thus $\lim_s n_s^{R_{e,k}}$ exists and is finite, and hence $R_{e,k}$ is satisfied and eventually stops requiring attention.

Now suppose that $X = S$ and $S_{e,k}$ acts infinitely often. If $v > u$ is the mth stage at which $S_{e,k}$ acts then the total amount added to β after stage v for purposes other than coding γ is bounded by $\sum_{i\geqslant 0} 2^{-(i+m)}\Omega < 2^{-m+1}$. This means that $\beta \equiv_{\mathrm{S}} \gamma \ngeqslant_{\mathrm{S}} \Omega$. It now follows from Lemma 2.3 that there is an $n \in \omega$ such that $S_{e,k}$ is eventually permanently satisfied through n, and such that $S_{e,k}$ is eventually never satisfied through $n' < n$. Thus $\lim_s n_s^{S_{e,k}}$ exists and is finite, and hence $S_{e,k}$ is satisfied and eventually stops requiring attention. □

Combining Theorems 3.1 and 3.2, we have the following result.

3.3. Theorem. *The Solovay degrees of c.e. reals are dense.*

We finish by showing that the hypothesis that $\alpha <_{\mathrm{S}} \Omega$ in the statement of Theorem 3.1 is necessary. This fact will follow easily from a stronger result which shows that, despite the upwards density of the Solovay degrees, there is a sense in which the complete Solovay degree is very much above all other Solovay degrees. We begin with a lemma giving a sufficient condition for domination.

3.4. Lemma. *Let f be an increasing total computable function and let $k > 0$ be a natural number. Let α and β be c.e. reals for which there are infinitely many $s \in \omega$ such that $k(\alpha - \alpha_s) > \beta - \beta_{f(s)}$, but only finitely many $s \in \omega$ such that $k(\alpha_t - \alpha_s) > \beta_{f(t)} - \beta_{f(s)}$ for all $t > s$. Then $\beta \leqslant_S \alpha$.*

Proof. By taking $\beta_{f(0)}, \beta_{f(1)}, \ldots$ instead of $\beta_0, \beta_1, \ldots$ as an approximating sequence for β, we may assume that f is the identity.

By hypothesis, there is an $r \in \omega$ such that for all $s > r$ there is a $t > s$ with $k(\alpha_t - \alpha_s) \leqslant \beta_t - \beta_s$. Furthermore, there is an $s_0 > r$ such that $k(\alpha - \alpha_{s_0}) > \beta - \beta_{s_0}$. Given s_i, let s_{i+1} be the least number greater than s_i such that $k(\alpha_{s_{i+1}} - \alpha_{s_i}) \leqslant \beta_{s_{i+1}} - \beta_{s_i}$.

Assuming by induction that $k(\alpha - \alpha_{s_i}) > \beta - \beta_{s_i}$, we have

$$k(\alpha - \alpha_{s_{i+1}}) = k(\alpha - \alpha_{s_i}) - k(\alpha_{s_{i+1}} - \alpha_{s_i}) > \beta - \beta_{s_i} - (\beta_{s_{i+1}} - \beta_{s_i}) = \beta - \beta_{s_{i+1}}.$$

Thus $s_0 < s_1 < \cdots$ is a computable sequence such that $k(\alpha - \alpha_{s_i}) > \beta - \beta_{s_i}$ for all $i \in \omega$.

Now define the computable function g by letting $g(n)$ be the least s_i that is greater than or equal to n. Then $\beta - \beta_{g(n)} < k(\alpha - \alpha_{g(n)}) \leqslant k(\alpha - \alpha_n)$ for all $n \in \omega$, and hence $\beta \leqslant_{\mathrm{S}} \alpha$. $\square$

3.5. Theorem. *Let α and β be c.e. reals, let f be an increasing total computable function, and let $k > 0$ be a natural number. If β is random and there are infinitely many $s \in \omega$ such that $k(\alpha - \alpha_s) > \beta - \beta_{f(s)}$ then α is random.*

Proof. As in Lemma 3.4, we may assume that f is the identity. If α is rational then we can replace it with a nonrational computable real α' such that $\alpha' - \alpha'_s \geqslant \alpha - \alpha_s$ for all $s \in \omega$, so we may assume that α is not rational.

We assume that α is nonrandom and there are infinitely many $s \in \omega$ such that $k(\alpha - \alpha_s) > \beta - \beta_s$, and show that β is nonrandom. The idea is to take a Solovay test $A = \{I_i : i \in \omega\}$ such that $\alpha \in I_i$ for infinitely many $i \in \omega$ and use it to build a Solovay test $B = \{J_i : i \in \omega\}$ such that $\beta \in J_i$ for infinitely many $i \in \omega$.

Let

$$U = \{s \in \omega \mid k(\alpha - \alpha_s) > \beta - \beta_s\}.$$

Except in the trivial case in which $\beta \equiv_{\mathrm{S}} \alpha$, Lemma 2.3 guarantees that U is Δ^0_2. Thus a first attempt at building B could be to run the following procedure for all $i \in \omega$ in parallel. Look for the least t such that there is an $s < t$ with $s \in U[t]$ and $\alpha_s \in I_i$. If there is more than one number s with this property then choose the least among such numbers. Begin to add the intervals

$$[\beta_s, \beta_s + k(\alpha_{s+1} - \alpha_s)], [\beta_s + k(\alpha_{s+1} - \alpha_s), \beta_s + k(\alpha_{s+2} - \alpha_s)], \ldots \qquad (*)$$

to B, continuing to do so as long as s remains in U and the approximation of α remains in I_i. If the approximation of α leaves I_i then end the procedure. If s leaves U, say at stage u, then repeat the procedure (only considering $t \geqslant u$, of course).

If $\alpha \in I_i$ then the variable s in the above procedure eventually assumes a value in U. For this value, $k(\alpha - \alpha_s) > \beta - \beta_s$, from which it follows that $k(\alpha_u - \alpha_s) > \beta - \beta_s$ for some $u > s$, and hence that $\beta \in [\beta_s, \beta_s + k(\alpha_u - \alpha_s)]$. So β must be in one of the intervals $(*)$ added to B by the above procedure.

Since α is in infinitely many of the I_i, running the above procedure for all $i \in \omega$ guarantees that β is in infinitely many of the intervals in B. The problem is that we also need the sum of the lengths of the intervals in B to be finite, and the above procedure gives no control over this sum, since it could easily be the case that we start working with some s, see it leave U at some stage t (at which point we have already added to B intervals whose lengths add up to $\alpha_{t-1} - \alpha_s$), and then find that the next s with which we have to work is much smaller than t. Since this could happen many times for each $i \in \omega$, we would have no bound on the sum of the lengths of the intervals in B.

This problem would be solved if we had an infinite computable subset T of U. For each I_i, we could look for an $s \in T$ such that $\alpha_s \in I_i$, and then begin to add the intervals $(*)$ to B, continuing to do so as long as the approximation of α remained in I_i. (Of course, in this easy setting, we could also simply add the single interval $[\beta_s, \beta_s + k\,|I|]$ to B.) It is not hard to check that this would guarantee that if $\alpha \in I_i$ then β is in one of the intervals added to B, while also ensuring that the sum of the lengths of these intervals is less than or equal to $k\,|I_i|$. Following this procedure for all $i \in \omega$ would give us the desired Solovay test B. Unless $\beta \leqslant_{\mathrm{S}} \alpha$, however, there is no infinite computable $T \subseteq U$, so we use Lemma 3.4 to obtain the next best thing.

Let

$$S = \{s \in \omega \mid \forall t > s(k(\alpha_t - \alpha_s) > \beta_t - \beta_s)\}.$$

If $\beta \leqslant_{\mathrm{S}} \alpha$ then β is nonrandom, so, by Lemma 3.4, we may assume that S is infinite. Note that $k(\alpha - \alpha_s) \geqslant \beta - \beta_s$ for all $s \in S$. In fact, we may assume that $k(\alpha - \alpha_s) > \beta - \beta_s$ for all $s \in S$, since if $k(\alpha - \alpha_s) = \beta - \beta_s$ then $k\alpha$ and β differ by a rational amount, and hence β is nonrandom.

The set S is co-c.e. by definition, but it has an additional useful property. Let

$$S[t] = \{s \in \omega \mid \forall u \in (s,t](k(\alpha_u - \alpha_s) > \beta_u - \beta_s)\}.$$

If $s \in S[t-1] - S[t]$ then no $u \in (s,t)$ is in S, since for any such u we have

$$k(\alpha_t - \alpha_u) = k(\alpha_t - \alpha_s) - k(\alpha_u - \alpha_s) \leqslant \beta_t - \beta_s - (\beta_u - \beta_s) = \beta_t - \beta_u.$$

In other words, if s leaves S at stage t then so do all numbers in (s,t).

To construct B, we run the following procedure P_i for all $i \in \omega$ in parallel. Note that B is a sequence rather than a set, so we are allowed to add more than one copy of a given interval to B.

1. Look for an $s \in \omega$ such that $\alpha_s \in I_i$.

2. Let $t = s + 1$. If $\alpha_t \notin I_i$ then terminate the procedure.

3. If $s \notin S[t]$ then let $s = t$ and go to step 2. Otherwise, add the interval

   $$[\beta_s + k(\alpha_{t-1} - \alpha_s), \beta_s + k(\alpha_t - \alpha_s)]$$

   to B, increase t by one, and repeat step 3.

This concludes the construction of B. We now show that the sum of the lengths of the intervals in B is finite and that β is in infinitely many of the intervals in B.

For each $i \in \omega$, let B_i be the set of intervals added to B by P_i and let l_i be the sum of the lengths of the intervals in B_i. If P_i never leaves step 1 then $B_i = \emptyset$. If P_i eventually terminates then $l_i \leqslant k(\alpha_t - \alpha_s)$ for some $s, t \in \omega$ such that $\alpha_s, \alpha_t \in I_i$, and hence $l_i \leqslant k\,|I_i|$. If P_i reaches step 3 and never terminates then $\alpha \in I_i$ and $l_i \leqslant k(\alpha - \alpha_s)$ for some $s \in \omega$ such that $\alpha_s \in I_i$, and hence again $l_i \leqslant k\,|I_i|$. Thus the sum of the lengths of the intervals in B is less than or equal to $k \sum_{i \in \omega} |I_i| < \infty$.

To show that β is in infinitely many of the intervals in B, it is enough to show that, for each $i \in \omega$, if $\alpha \in I_i$ then β is in one of the intervals in B_i.

Fix $i \in \omega$ such that $\alpha \in I_i$. Since α is not rational, $\alpha_u \in I_i$ for all sufficiently large $u \in \omega$, so P_i must eventually reach step 3. By the properties of S discussed above, the variable s in the procedure P_i eventually assumes a value in S. For this value, $k(\alpha - \alpha_s) > \beta - \beta_s$, from which it follows that $k(\alpha_u - \alpha_s) > \beta - \beta_s$ for some $u > s$, and hence that $\beta \in [\beta_s, \beta_s + k(\alpha_u - \alpha_s)]$. So β must be in one of the intervals $(*)$, all of which are in B_i. □

3.6. Corollary. *If α^0 and α^1 are c.e. reals such that $\alpha^0 + \alpha^1$ is random then at least one of α^0 and α^1 is random.*

Proof. Let $\beta = \alpha^0 + \alpha^1$. For each $s \in \omega$, either $3(\alpha^0 - \alpha^0_s) > \beta - \beta_s$ or $3(\alpha^1 - \alpha^1_s) > \beta - \beta_s$, so for some $i < 2$ there are infinitely many $s \in \omega$ such that $3(\alpha^i - \alpha^i_s) > \beta - \beta_s$. By Theorem 3.5, α^i is random. □

Combining Theorem 3.1 and Corollary 3.6, we have the following results, the second of which also depends on Theorem 1.4.

3.7. Theorem. *A c.e. real γ is random if and only if it cannot be written as $\alpha + \beta$ for c.e. reals $\alpha, \beta <_S \gamma$.*

3.8. Theorem. *Let $\mathbf{d}$ be a Solovay degree. The following are equivalent:*

1. $\mathbf{d}$ *is incomplete.*
2. $\mathbf{d}$ *splits.*
3. $\mathbf{d}$ *splits over any lesser Solovay degree.*

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