A note on btt-degrees*

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Abstract

We prove that there are computable enumerable incomplete *btt*-degres with the anticupping property. In fact no simple set is *btt* cuppable in the (not necessarily computable enumerable) *btt*-degrees. This solves a question of Odifreddi and settles the situation for the one remaining reducibility where the answer was unknown.

1 Introduction

Reducibilities calibrate sets into equivalence classes of equicomputability. Turing reducibility is obtained by adding characteristic functions to Kleene

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schemes, or alternatively allowing arbitary oracle queries in the procedure. By restricting the types of oracle queries allowable, one obtains finer and finer calibrations of information content. For instance, m-reducibility is obtained by saying that $A \leq_m B$ iff there is a computable function f such that $x \in A$ iff $f(x) \in B$. m-degrees give a very fine classification of sets, and every T-degree splits into infinitely many m-degrees. (Jockusch [**?**])

While Turing reducibility is the most general type, strong reducibilities are also of considerable interest. Quite aside from the intrinsic interest which stems from asking what happens if we restrict the kinds of access mechanism one has to an oracle, if one considers applications one is naturally drawn to consider strong reducibilities. The point is that in applications of computability theory in classical mathematics, the kinds of reducibilies one usually obtains are, by and large, strong ones. For instance, there are word problems for finitely presented groups in each tt-degree but not in each m-degrees (nor even in each btt-degree). The natural reduction of the word problem to the conjugancy problem is an m-reduction. One gets an m-reduction in the solution to Hilbert's 10th problem, and many other undecidability results. For computably presented vector spaces one has bases in each wtt-degree below the wtt-degree of the dependence relation, but not each T-degree. There are many other examples.

In this paper, we will examine btt-reducibility. Recall that $A \leq_{btt} B$ if $A \leq_{tt} B$ via some truth table reduction with bounded norm. (That is, there is a computable function f such that, for all x, $x \in A$ iff $B \models \sigma_{f(x)}$ where σ_y denotes the y-th truth table (so $A \leq_{tt} B$), and furthermore the size of the truth table is bounded by some fixed n, called the norm.

btt-reducibility is of some interest since it throws considerable light upon both tt- and m- reductions, and points at limits of construction types. For instance, it is an old elementary result in computability theory that $A \equiv_T B$ iff $A' \equiv_m B'$ (e.g. Soare [**?**], Ch III 2.3). On the other hand, Mohrherr [**?**] and, independently, Bickford and Mills [**?**], constructed computably enumerable sets $A|_T B$ such that $A' \equiv_{tt} B'$. Finally Mohrherr [**?**] proved that the Mohrherr-Bickford-Mills result cannot be extended to btt reducibility byt proving that $A' \leq_{btt} B'$ implies $A \leq_T B$, and hence $A' \leq_{btt} B'$ iff $A' \leq_m B'$.

In the present paper we solve the remaining "anticupping" question of Odifreddi [?]. Results analogous to the following theorem are known for all other major reducibilities.

Theorem 1.1 *(i) There exists a computable enumerable btt-degree* $\mathbf{a}$ *strictly below* $\mathbf{0}'_{btt}$ *such that for all (not necessarily computably enumerable) btt degree* $\mathbf{b}$*,* $\mathbf{a} \cup \mathbf{b} \geq \mathbf{0}'$ *implies* $\mathbf{b} \geq \mathbf{a}$*.*
(ii) In fact, the btt-degree of a simple set is not btt-cuppable.

We remark that the proof technique is nonuniform and would seem applicable to other *btt*-questions. None of the other techniques such as Lachlan [?] and Downey [?] would seem applicable.

2 The proof of Theorem ??

Let A be a simple computably enumerable set. We will build an m-complete computable enumerable set B to meet the requirements

$$\mathcal{R}_e : (B \leq_{btt} A \oplus X \text{ via } \alpha_e) \rightarrow B \leq_{btt} X.$$

Here by abuse of notation, we regard the α_e'th truth table as having the norm in question. To make the argument clear, we first consider the easy case that α_e is simply the e-th possible m-reduction. Thus we will prove *no simple set is* m*-cuppable,* generalizing the well-known theorem of Post that no simple set is m-complete.

So here we can regard α_e as simply saying

$$x \in B \text{ iff } \phi_e(x) \in A \oplus X,$$

with ϕ_e the e-th partial computable function. Now B will have 0-th column $B^{(0)} = \{\langle x, 0\rangle : x \in K\}$. We will devote column $e+1$ to the satisfaction of $\mathcal{R}_e$. We will say that an element y is *bad* if $\phi(y) \in 2 \cdot \omega$. The idea is that we will try to code K into $B^{(e+1)}$ in such a way that X can recover this fact. We will do so via coding markers, $c(e,z)$ which will denote the coding location for $z \in K$? for requirement $\mathcal{R}_e$. At any stage s we will have a least as yet unassigned $z = z(e,s)$ which intially is 0.

All the requirements work independently. While $\mathcal{R}_e$ is not, as yet, declared met, we do the following.

Let $\ell(e,s)$ denote the length of convergence of α_e. That is, $\ell(e,s) = \max\{z : \forall z' \leq z(\phi_{e,s}(z') \downarrow\}$. We will also build an auxiliary computably enumerable set C_e for $\mathcal{R}_e$.

Let s be the least stage where $\ell(e,s) > \langle y, e+1\rangle$. First see if there is some $\langle m, e+1\rangle$ which is bad, and yet $\phi_e(\langle m, e+1\rangle) \in A_s \oplus \emptyset$. If this is the case, then declare $\mathcal{R}_e$ as met (since it will be the case that $\langle m, e+1\rangle \notin B$.

We need to figure out if $\langle y, e+1\rangle$ should be a coding marker or not.

Case 1. $\langle y, e+1\rangle$ is bad. There are two subcases.

Case 1a. For some $y' \leq y$, $\phi_e(\langle y', e+1\rangle) = \phi_e(\langle y, e+1\rangle)$. (Necessarily, $\langle y', e+1\rangle$ is bad.)
Action: Put $\langle y', e+1\rangle$ into B. Declare $\mathcal{R}_e$ as met.

Case 1b. Otherwise.
Action: Put n into C_e, where $2n = \phi_e(\langle y, e+1\rangle)$.

Case 2. $\langle y, e+1\rangle$ is good.
Action: Define $c(z) = \langle y, e+1\rangle$. For all stages $s' \geq s$, while $\mathcal{R}_e$ is not declared met, put $c(z)$ into $B_{s'}$ whenever $z \in K_{s'}$.

Verification for the m-degree case. First note that if there $\mathcal{R}_e$ is ever declared met then for some m, $\langle m, e+1\rangle \in B$ iff $\phi(\langle m, e+1\rangle \notin A$. This is because bad elements are never used for coding purposes, and are never put into B except for diagonalization . In case 1a, one of y or y' will take the role of m. Therefore if for some set X $A \oplus X \geq_m B$ via α_e, we can suppose that $\mathcal{R}_e$ is never declared satisfied.

As a consequence, we claim that Case 1b can pertain only finitely often.

Otherwise C_e would be an infinite computably enumerable set. Since A is simple, we see that for some n, $n \in C_e \cap A$. So, for some bad $\langle y, e+1\rangle$, $\phi_e(\langle y, e+1\rangle) = 2n \in A \oplus \emptyset$ so that $\mathcal{R}_e$ would be met, assuming that $\ell(e,s) \to \infty$.

Since Case 1b can never pertain, we see that for almost all y, $\langle y, e+1\rangle$ is good. To decide if $z \in K$, find the y with $c(e,z) = \langle y, e+1\rangle$. Then, assuming that $A \oplus X$ is m-above B via α_e, $z \in K$ iff $1/2(\phi_e(\langle y, e+1\rangle \in X$. $\square$

Turning to the full proof, not that the above can easily be adapted to the 1-tt case. In Case that the element is bad this time it will mean that the query is to the A-side. If the query is negative, rather than positive, as in the m-case, then we can put the element $\langle y, e+1\rangle$ into B and $2n$ into C_e^- a set built on the assumption that there are infinitely many reductions of the form $\langle y, e+1\rangle \in B$ iff $\phi(\langle y, e+1\rangle) \notin A \oplus \emptyset$.

In the case that the norm is bigger than one one proceeds similarly, except that the number of cases grows. The 2-tt case is enough for the problems of a general norm. So suppose now that $\alpha_e(x)$ is a 2-tt reduction concerning two elements x_1, x_2. We say that x *projects* to x_1 and x_2. Again we devote $B^{(e+1)}$ to $\mathcal{R}_e$. This time the coding of K into X is nonuniform depending on the outcome. We assume in the following that α_e is total, and the module acts at α_e expansionary stages. We will have coding markers $c_(x)$ (dropping the "e"). for the good x. If we fail for some x to define this, then we will try for back up markers $c'(x,p)$ for $p \in \omega$. Finally if we fail for some x and as well for some p, then we will have additional back-up markers $c''(x,p,d)$.

As before we look at a new y. A good y is one where $\langle y, e+1\rangle$ projects to two (or fewer) elements both of which are odd. In that case, for the leat unassigned c-coding marker z we will assign $c(z) = \langle y, e+1\rangle$ *and initialize all $c'(z,p)$ and $c'(z,p,d)$ which have been defined up to this point, in the sense that they will be no longer coding markers. Also initialize all 1-reasonable, 1-active, 2-reasonable and 2-active elements. (See below for definitions)* We declare that $c(z+1,0)$ is active.

Analysis. *Notice that, exactly as before, if $c(x)$ is assigned for every x, then if $A \oplus X = B$ via α_e then we can use the $c(x)$ and the fact that each $c(x)$*

projects purely into $\emptyset \oplus X$ to construct a 2-tt reduction computing B from X.

If y is not good then perhaps it is 1-*reasonable.* We say that y is 1-reasonable, in the case that $\langle y, e+1\rangle$ projects to two elements, one y_1 of which is even and one odd. In this case we put $1/2 \cdot y_1$ into the auxiliary set $C'_{e,z}$.

Having done this we see if there is some 1-reasonable element $y' \leq y$, such that y''s even projection is in $A \oplus \emptyset$. and y''s even projection is bigger than $r(1,s)$. For the least such y' declare that y' is $1-$ active, and for the least $c'(z,p)$ not yet assigned, assign $c'(z,p)$ to $\langle y', e+1\rangle$. Declare that *while $c(z)$ is not assigned, $p \in K$* iff $c'(z,p) \in B$. Reset $r(1, s+1) = s+1$.

Initialise all $c''(p,q,r)$ and 2-active and 2-reasonable elements.

Analysis. *Notice that if the first case does not pertain, then there must be some least z such that $c(z)$ is never defined. If we find infinitely many 1-reasonable elements, then, by simplicity of A, infinitely many of them must become 1-active, since $C'_{e,z}$ will be infinite because $r(1,s) \mapsto \infty$. This means that $c'(z,p)$ will be defined for all p. Notice that $c'(z,p)$ is defined only if its even projection lies already in $A \oplus \emptyset$. Therefore, we can conclude that $K \leq_{1-tt} X$ via the 1-good elements.*

The next case is that perhaps y is 2-reasonable. This means that, although $\langle y, e+1\rangle$ has an even projection, this projection is below $r(1,s)$. For the least unassigned $c''(z,p,d)$ assign $c''(z,p,d)$ to $\langle y, e+1\rangle$. The promise is that *provided that $c'(z,p)$ is not defined, $d \in K$ iff $c''(z,p,d) \in B$.*

Analysis. *Now if this is the eventual outcome we are stuck on some pair z, p. For this pair, we will define $c''(z,p,d)$ for all d, and* furthermore *$r(1,s)$ has a fixed limit r_1. Let s be any stage where $A_s \upharpoonright r_1 = A \upharpoonright r_1$. The for all stages t, the answers to the even projection of $c''(z,p,d)$ will be known, and hence we again conclude that $B \leq_{1-tt} X$.*

The final case is that all of the above cease to pertain. This means that $\langle y, e+1\rangle$ projects purely into A. Now we can diagonalize as before, in the sense that either the total projection is finite so we can have repition as in Case 1a of the m-case, or it is infinite, and hence for some element we can force A to include both of its projections by simplicity. For instance, if the projection is purely positive, then we wait for them to enter, and keep $\langle y, e+1 \rangle$ out of B, if they are both negative, we put the $\langle y, e+1 \rangle$ into B first.

The genral case follows the same pattern with many details, but no greater insight. $\square$

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