Questions in Computable algebra and Combinatorics

Rod Downey and J. B. Remmel

1. Introduction

Computable mathematics studies the effective content of theorems and methods in mathematics. There have been extensive studies of the effective content of theorems and methods of a wide variety of results from algebra, analysis, model theory and combinatorics. A good general introduction to the study of computable mathematics can be found in the two volume set "Handbook of Recursive Mathematics: vol. 1 and 2" edited by Yu. L. Ershov, S. S. Goncharov, A. Nerode, and J. B. Remmel [**42**].

In computable mathematics, one most often assumes that the underlying structures are computable, i.e. the universe of the structure is a computable set of natural numbers and the relations and functions of the structure are uniformly computable. One then uses the tools of modern computability theory to study the effective content of theorems and constructions on these computable structures. However, as we shall see, there are a number of interesting results in computable mathematics which are concerned with more general structures such as Σ_n^0 structures where the underlying universe, relations, and functions are Σ_n^0 as well as a number of interesting results concerning more restricted structures such as polynomial-time structures where there the underlying universe is a polynomial-time subset of $\{0,1\}^*$ and the relations and functions are restrictions of polynomial-time relations and functions.

In this article, we will focus on two areas of computable mathematics, namely computable algebra and combinatorics. The goal of this article is to present a number of open questions in both computable algebra and computable combinatorics and to give the reader a sense of the research activity in these fields. Our philosophy is to try to highlight questions, whose solutions we feel will *either* give insight into algebra or combinatorics, *or* will require new technology in the computability-theoretical techniques needed. A good historical example of the first phenomenom is the word problem for finitely presented groups which needed the development of a great deal of group theoretical machinery for its solution by Novikov [**110**] and Boone [**10**]. A good example of the latter phenomenon is the recent solution by Coles, Downey and Slaman [**17**] of the question of whether all rank one torsion free

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Abelian groups of finite type have first jump degree which relied on the discovery that every set has a least jump enumeration[1].

In view of this, in some instances we will try to comment on why we feel that the relevant question is interesting. Also, where possible we will try to give an attribution of the question. Any question without such an attribution is either ours or folklore. While we have done our best to get views of others, our article is necessarily a reflection of our own view of the mathematical universe, and hence we apologise in advance to those working in areas which we have not given due justice.

In a short article such as this we cannot hope to provide all the theoretical background which leads to our open questions, but we will give enough to frame the questions.

Our notation is standard and follows Soare [**145**].

2. Preliminaries

2.1. Presentations. Since we are concerned with questions of effectivity in algebraic and combinatorial structures, we need to be concerned with how we present the relevant structures.

Consider a model, or, more precisely, a structure $\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i\in S}, \{f_i^{\mathcal{A}}\}_{i\in T}, \{c_i^{\mathcal{A}}\}_{i\in U})$ where S, T, and U are initial segments of the natural numbers $\omega = \{0, 1, \dots\}$. We let $\phi_{e,n}$ denote the partial computable function of n-variables computed by the e-th Turing machine in some standard enumeration of Turing machines.

DEFINITION 2.1. $\mathcal{A}$ is *computable* if

(i) A is a computable subset of ω.
(ii) The set of relations $\{R_i^{\mathcal{A}}\}$ is uniformly computable, i.e. there exists a computable function G such that $G(i) = [e_i, n_i]$ where $R_i^{\mathcal{A}}$ is the n_i-ary relation whose characteristic function is computed by ϕ_{e_i,n_i}.
(iii) The set of functions $\{f_i^{\mathcal{A}}\}$ is uniformly computable, i.e. there exists a computable function F such that $F(i) = [e_i, n_i]$ where $f_i^{\mathcal{A}} = \phi_{e_i,n_i}|A^{n_i}$.
(iv) There is a computable function H such that $H(i) = c_i^{\mathcal{A}}$.

Thus $\mathcal{A}$ is computable iff its atomic diagram is computable. We say that a computable structure $\mathcal{A}$ is *decidable* if its complete diagram is computable and we say that $\mathcal{A}$ is Σ_n-*decidable* if its Σ_n-diagram is computable. We say that two computable structures $\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i\in S}, \{f_i^{\mathcal{A}}\}_{i\in T}, \{c_i^{\mathcal{A}}\}_{i\in U})$ and $\mathcal{B} = (B, \{R_i^{\mathcal{B}}\}_{i\in S}, \{f_i^{\mathcal{B}}\}_{i\in T}, \{c_i^{\mathcal{B}}\}_{i\in U})$ are *computably isomorphic* is there is an isomorphism $f : A \to B$ which is a partial computable function. It is quite easy to see that any infinite computable structure $\mathcal{A}$ is computably isomorphic to a computable structure $\mathcal{B}$ whose universe $B = \omega$. Since computable mathematics is only interested in results which are invariant under computable isomorphisms, there is no loss in generality in making the convention that all computable structures have universe ω. Thus, for example, a linear ordering is computable precisely if its ordering relation is a computable relation on ω.

As mentioned in the introduction, one often works with more general structures in computable algebra and combinatorics. For any structure $\mathcal{A}$ whose universe A is a subset of ω, one can assign a degree to $\mathcal{A}$, $deg(\mathcal{A})$, as the join of all the degrees

[1]That is, for all sets A, the collection $\{\deg(B)' : A \text{ is computably enumerable in } B\}$ always has a least element.

of $R_i^{\mathcal{A}} : i \in S, f_i^{\mathcal{A}} : i \in T, c_i^{\mathcal{A}} : i \in U$. One can also look at this classification of structures "descriptively". That is, we say that $\mathcal{A}$ is Σ_n^0 *presented* or, simply, Σ_n if its atomic diagram is Σ_n. Again, there is no loss in generality in assuming that a Σ_n structure has universe ω. Thus a linear ordering L is a linear ordering with domain ω is Σ_n iff its ordering relation $\preceq_L$ is a Σ_n^0 relation. We should note that a special case of Σ_1 structures arises naturally in many situations in algebra. Namely, one often considers the quotient structures, e.g. groups modulo a normal subgroup, rings or Boolean algebras modulo an ideal, vector spaces modulo a subspace. If the underlying structure is computable, but the normal subgroup, ideal, subspace etc. is only computably enumerable (c.e.), then one is naturally lead to consider structures $\mathcal{A} = (A, =^{\mathcal{A}}, \{R_i^{\mathcal{A}}\}_{i \in S}, \{f_i^{\mathcal{A}}\}_{i \in T},$
$\{c_i^{\mathcal{A}}\}_{i \in U})$ where A, $R_i^{\mathcal{A}}$, $f_i^{\mathcal{A}}$ and the underlying equality relation $=^{\mathcal{A}}$ are c.e. In such cases, we shall often refer to these structures as *c.e. presented* structures.

We should note that there is rich area of computable algebra and combinatorics where one considers more restricted resource bounded versions of computation such as polynomial-time, exponential time, polynomial-space, etc. For example, we say that a structure $\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i \in S}, \{f_i^{\mathcal{A}}\}_{i \in T}, \{c_i^{\mathcal{A}}\}_{i \in U})$ is **polynomial-time** if

1. A is a computable subset of $\{0,1\}^*$.
2. The set of relations $\{R_i^{\mathcal{A}}\}$ is uniformly polynomial-time computable, i.e. there exists a computable function G such that $G(i) = [n_i, e_i]$ where $R_i^{\mathcal{A}}$ is the n_i-ary relation whose characteristic function is computed by ϕ_{n_i,e_i} and there exists a computable function G' such that $G'(i) = m_i$ where for all $(x_1, \ldots, x_{n_i}) \in \{0,1\}^{n_i}$, it takes at most $(max\{2, |x_1|, \ldots, |x_{n_i}|\})^{m_i}$ steps to compute $\phi_{n_i,e_i}(x_1, \ldots, x_{n_i})$.
3. The set of functions $\{f_i^{\mathcal{A}}\}$ is uniformly polynomial-time computable, i.e. there exists a computable function F such that $F(i) = [n_i, e_i]$ where $f_i^{\mathcal{A}}$ is the n_i-ary function computed by ϕ_{n_i,e_i} and there exists a computable function F' such that $F'(i) = m_i$ where for all $(x_1, \ldots, x_{n_i}) \in \{0,1\}^{n_i}$, it takes at most $(\max\{2, |x_1|, \ldots, |x_{n_i}|\})^{m_i}$ steps to compute $\phi_{n_i,e_i}(x_1, \ldots, x_{n_i})$..
4. There is a polynomial-time computable function H such that $H(i) = c_i^{\mathcal{A}}$.

Exponential time, polynomial-space, and exponential space structures can be defined in a similar manner.

We note that if $\mathcal{A}$ is polynomial-time computable, it does not automatically follow that its atomic diagram is polynomial-time computable. Thus we say that $\mathcal{A}$ is *uniformly polynomial-time computable* iff its atomic diagram is polynomial-time computable. We say that two polynomial-time structures $\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i \in S}, \{f_i^{\mathcal{A}}\}_{i \in T},$ $\{c_i^{\mathcal{A}}\}_{i \in U})$ and $\mathcal{B} = (B, \{R_i^{\mathcal{B}}\}_{i \in S}, \{f_i^{\mathcal{B}}\}_{i \in T}, \{c_i^{\mathcal{B}}\}_{i \in U})$ are *polynomial-time isomorphic*, if there is are an isomorphism $f : A \to B$ such that both f and its inverse f^{-1} are the restrictions of polynomial-time functions. We note that it is not the case that any two polynomial-time sets are polynomial-time isomorphic. For example, if we let $Bin(\omega)$ denote the set of binary represention s of the natural numbers and $Tal(\omega) = \{0\}^*$ denote the tally representation of the natural numbers, then $Bin(\omega)$ and $Tal(\omega)$ are not polynomial-time isomorphic. Thus unlike the case of computable structures, we cannot just restrict ourselves to the consideration of structures whose underlying universe is $\{0,1\}^*$ or $Bin(\omega)$ or $Tal(\omega)$. We will look further at polynomial-time structures in Section 8.

Given any notion of effective structures, there is often a natural notion of effective categoricity associated with it. Below we give the definitions of computable and polynomial-time categoricity. We note that by our remarks above, the notion of polynomial-time categoricity naturally depends on the underlying universe.

DEFINITION 2.2. (i) We say a computable structure $\mathcal{A}$ is *computably categorical* if all computable structures $\mathcal{B}$ which are isomorphic to $\mathcal{A}$ are computably isomorphic to $\mathcal{A}$.
(ii) We say a p-time structure $\mathcal{A}$ with universe A is *p-time categorical over* A if all p-time structures $\mathcal{B}$ with universe A which are isomorphic to $\mathcal{A}$ are p-time isomorphic to $\mathcal{A}$.

Finally we should note that the Russian school of computable mathematics developed slightly different notions of computable structures. That is, in the Russian literature, one distinguishes the underlying abstract structure $\mathcal{A}$ from its various presentations. This point of view leads to the following definitions. Given a model,

$$\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i\in S}, \{f_i^{\mathcal{A}}\}_{i\in T}, \{c_i^{\mathcal{A}}\}_{i\in U})$$

where S,T, and U are initial segments of ω, a **constructivization (strong constructivization)** of $\mathcal{A}$ is a surjective mapping $\alpha : \omega \to A$ such that for any atomic formula (resp. *any* formula) ϕ, the relations

$$\{\overline{a} : \mathcal{A} \models \phi(\alpha(\overline{a}))\}$$

are computable uniformly in ϕ. It is easy to see that $\alpha(\mathcal{A})$ is a computable model if α is a constructivization of $\mathcal{A}$ and is a decidable model if α is a strong constructivization. Thus it is easy to translate results about computable and decidable models into the language of constructivizations and strong constructivizations and vice versa.

2.2. Basic Questions. In this paper, we will be concerned mainly with well-studied classical structures and theorems in algebra and combinatorics. Thus for computable algebra, we will be concerned with groups, rings, fields, orderings, etc. and for computable combinatorics we will consider classical results about graphs, colorings, matchings, etc. Our general concern will be loosely based around interpreting the following basic types of questions in the various arenas.

(i) *Study the presentations of a structure.*
What is the set of degrees of presentations of a structure? Compare and contrast c.e. presented vs. computable structures, computable vs. n-decidable structures, etc. Thus we try to understand the relationship between classical isomorphism-type and effective isomorphism-type. Richter's Thesis [**125**] was groundbreaking here.

An important category here consists of the *spectrum* problems, which look at the set of degrees within an isomorphism-type, or the degrees of images of a relation on a structure. Ash-Nerode [**5**] and Goncharov [**55**] are good examples of these types of considerations.

Related is the question of the *number* of computable, decidable, etc. models. For instance, is the structure, like the rationals as an ordering, computably categorical in the sense that it has only one computable model up to computable isomorphism? This is an example of a *computable dimension* problem. How many degrees can a relation on a computable model have? For instance, Downey and Moses [**35**] constructed a computably presented linear ordering for which the adjacency relation was *intrinsically complete*, whereas Hirschfeldt [**63**] has proven that a computable relation on a computably presented linear ordering with a spectrum with at least two elements has infinitely many elements.

Clearly, since we are considering *effective* structures, it is entirely natural to wish to understand how our classification tool, *effective* isomorphism, relates to the historical one. Aside from this intrinsic interest, *another* good reason for such analyses is that many of the remaining questions will need insight into the model theory of the relevant structures to solve them. For instance, it is unknown whether there is a low_n Boolean algebra with no computable copy, as we will see. This question would seem purely technical, but seems to need a complete understanding of a reasonable set of invariants for countable Boolean algebras. A related question was whether there was a Boolean algebra of rank 1 which was c.e. presented and not isomorphic to a computably presented one. This question was solved by Downey and Jockusch [**30**], by demonstrating a new algebraic construction for folding essentially any countable Boolean algebra into a rank 1 one in such a way that the original one can be recovered by two quantifiers.

(ii) *Study the effective content of classical theorems.* Two nice classical theorems are that every field has an algebraic closure, and that any two algebraic closures of the same field are isomorphic. Rabin [**117**] proved that any computably presented field has a computably presented algebraic closure, but Frölich and Shepherdson [**50**] proved that they are not necessarily computably unique. What's going on here?

In some sense the explanation was provided by Metakides and Nerode [**99, 100**] who showed that uniqueness happened iff there was an additional effectivity condition on the field, namely a *separable splitting algorithm.* If there was such an algorithm, then one could construct the algebraic closure by the usual sequential addition of roots, and this construction gives computable uniqueness. Therefore the Rabin construction actually constructs the algebraic closure *using a different method.*

Another nice example of this phenomenom stems from Kalantari's (and others (e.g. [**20**])) investigations of Stone's separation theorem in [**72**]. There one looked at a computable vector space over a computable ordered field with *both* a computable dependence algorithm and a computable algorithm to decide if a vector lay in the convex closure of a finite set of vectors. The question naturally arose as to whether both of these conditions were necessary. This question was answered by Nevins [**109**], who demonstrated that either implies the other.

Notice that this program has several subprograms.

(iia) *If a theorem is not effectively true, how can we modify the theorem to make it effectively true?* One example is the above. Another example would be the classical observation that every infinite linear ordering has either an infinite ascending chain or an infinite descending chain. Tennenbaum proved that this can fail if we ask that everything be computable. There are several analogues of this theorem which don't fail. For instance, Manaster (see [**23**]) proved that there is either an infinite Π_1 ascending or descending sequence. Rosenstein [**131**] proved that there must be a computable subsequence of type ω, ω^*, $\omega + \omega^*$

or $\omega + \zeta\eta + \omega^*$, and Lerman [**91**] proved that none from this list could be deleted.

(iib) *If a theorem is effectively false, can we measure how badly it fails?* A good instance is the result that every formally real field admits an ordering. Metakides and Nerode [**100**] proved that the cones of orderings of computable formally real fields are precisely the Π^0_1 classes.

(iic) *Study the set of "solutions" to a computable instance of a combinatorial problem*, e.g.: Can the set of k-colorings of a computable graph represent an arbitrary Π^1_0-class?

We remark that yet another good reason for this type of investigation is that there are usually reverse mathematical spinoffs. Here one uses proof theory in the form of subsystems of second order arithmetic to analyse and calibrate the theorems of classical mathematics. An immediate corollary to the proof of the quoted result of Metakides and Nerode is that the statement that every countable formally real field has an ordering is equivalent over RCA_0 to WKL_0 (See Friedman, Simpson, Smith [**47**]). Generally most results in reverse mathematics have, at their hearts, a computability theory construction. In many ways, the two subjects are simply reflections of the same basic philosophy, and enrich each other. In fact, many good questions in computable algebra and combinatorics are motivated by questions from reverse mathematics.

Ramsey's theorem for pairs and two colours, $RT(2,2)$, provides an excellent illustration of this phenomenom. It is unknown exactly what its proof theoretical strength is. In his Ph.D. Thesis, Hirst [**65**] used a computability-theoretical argument to show that it is not provable in WKL_0. Does it imply WKL_0? It was not known till the work of Seetapun (see Hummel [**67**] or Seetapun and Slaman [**140**]) that it was not as strong as ACA_0. Seetapun used a complicated forcing argument, later modified by Jockusch to a simpler forcing argument, to construct a computable coloring of pairs that avoided $\mathbf{0}'$, and concluded that $RT(2,2)$ did not imply ACA_0. We refer the reader to Simpson's article in this volume (Simpson, [**138**]) for background and an excellent source of questions of this ilk.

(iii) *Study the lattice of c.e. substructures of a computable structure, e.g. subspaces of a vector space, subfields of a field, ideals or filters of a Boolean algebra, subalgebras of a Boolean algebra, order ideals of a poset, etc.* There is a persistent intuition going back to Post that structural properties of the poset of c.e. sets have reflections in computability properties. For instance, the high degrees are exactly the degrees of maximal sets, and every double-jump class is an invariant class for the c.e. sets.

It seems reasonable to try to similarly understand computation via other lattices, to compare with the c.e. sets and to clarify their relationship with the degrees. Furthermore, since clasically the lattices of (e.g.) subgroups of groups give insight into groups, one would expect the same from the effective versions.

For instance, Metakides and Nerode [**99**] constructed a computable coinfinite dimensional subspace V of V_∞ such that every c.e. set independent of V is finite. The quotient space V_∞/V is then an infinite dimensional computable vector space, the only c.e. independent subsets of which are finite. Friedman, Simpson and Smith [**47**] constructed a computable infinite dimensional vector space, such that every infinite independent set computes the halting problem. Shore [**136**] proved that given any effective sequence of c.e. degrees $\mathbf{a}_1, \mathbf{a}_2, ...$ and a degree $\mathbf{a}$ above all the $\mathbf{a}_i$, there is a computable vector space V such that computing dependence for k-tuples (k-fixed) has degree $\mathbf{a}_k$, and computing dependence has degree $\mathbf{a}$. Thus, for instance, it is possible to have a vector space where for any fixed k we can compute dependence for k-tuples, but the complexity of computing dependence if k is allowed to vary is that of the Halting problem.

Another illustration is provided by the recent work of Cholak, Coles, Downey, and Herrmann [**15**], who proved that for the lattice of ideals of the free Boolean algebra, the analogous notion to maximal set provides an invariant class in the c.e. degrees consisting of the *array non-computable* degrees.

3. Linear Orderings and Boolean algebras

We begin with the spectrum problem. Recall the definition originally due to Harizanov for relations.

DEFINITION 3.1. (i) Given a structure $\mathcal{A}$,

$$Spec(\mathcal{A}) = \{deg(B) : B \text{ is isomorphic to } \mathcal{A}\}.$$

(ii) Given a relation R on a computable structure $\mathcal{A}$, the *spectrum* of R is the set

$$\{\deg(\widehat{R}) : \text{there is a computable } \widehat{\mathcal{A}} \cong \mathcal{A} \text{ with } \widehat{R} \text{ the image of } R\}.$$

The basic question is:

QUESTION 3.2. (i) What spectra can linear orderings have?
(ii) Can we classify the orderings that have computable copies? decidable copies? etc. Here we mean a classification by order-type.

We know that spectra are closed upwards (Knight [**86**]). We know that if $\mathbf{a}$ is in spec(L) then there is an ordering $\widehat{L} \cong L$ with degree forming a minimal pair with $\mathbf{a}$ and hence if L does not have a computable copy then there is no minimum in the spectrum. What else can be said? An important test case is the following question.

QUESTION 3.3 (Downey). Is there a linear ordering with no computable copy whose spectrum contains all non-zero degrees?

Of relevance here is the following theorem of Miller.

THEOREM 3.4 (Miller [**102**]). *There is a linear order $\mathcal{L}$ such that $Spec(\mathcal{L})$ contains all non-zero Δ^0_2 degrees but does not contain $\mathbf{0}$.*

The method used by Miller is a modification of an earlier technique of Jockusch and Soare [**69**]. It builds a certain ordering which only relies on "permissions" to work, and always gives the same order-type provided that we get such permisssions. This technique might be able to be pushed to hyperimmune degrees, but otherwise we seem to need a genuinely new idea.

Slaman [**139**] and Wehner [**151**] independently produced a structure $\mathcal{A}$ such that $Spec(\mathcal{A})$ consists of all non-zero degrees. Work of Hirschfeldt, Khoussainov, Shore and Slinko [**64**] shows that $\mathcal{A}$ can be a graph, a lattice, an integral domain, or an Abelian group. Downey and Jockusch [**29**] show that $\mathcal{A}$ cannot be a Boolean algebra. Miller also asked the following question.

QUESTION 3.5 (Miller). Can a spectrum of a linear ordering separate the high degrees H_1 from the low degrees L_1?

Notice that the refined notion of a spectrum considered by Miller gives rise to differing versions. For a class C of degrees the *C-spectrum* is $Spec(L) \cap C$. If C is the class of c.e. degrees (say) then the *strong* c.e. spectrum is the class of c.e. degrees realized by c.e. presented models isomorphic to $\mathcal{L}$. It is not too difficult to show that the strong c.e. spectrum and the c.e. spectrum coincide for *linear orderings.*

QUESTION 3.6. (i) What is the relationship between the c.e. spectrum and the strong c.e. spectrum of a c.e. structure?
(ii) What are the possible spectra of the adjacency relation for a computable linear ordering $\mathcal{L}$? We have seen it can have one element, and it can have all c.e. degrees except $\mathbf{0}$ (Downey [**24**]). But must it have either one or infinitely many degrees, etc?
(iii) (Hirschfeldt) In general, if a relation R on a linear ordering has a spectrum with more than one element, must it have an infinite spectrum? (In his thesis, Hirschfeldt proved that the answer is yes if the spectrum contains a computable copy of L with R computable.)

Related here is the question of whether there can be relations on orderings with one element spectra where that element is not $\mathbf{0}^{\alpha}$ for some α. Can, for instance, the adjacency relation have a one element spectrum and it not be a member of $\{\mathbf{0}, \mathbf{0}'\}$? An even more interesting question is whether one can have a nontrivial single element spectrum for a relation on a computable Boolean algebra.

Boolean algebras and linear orderings are related by the Stone construction. That is, each Boolean algebra is effectively isomorphic to $Intal(L)$, the algebra of left open right closed intevals of a linear ordering L.

QUESTION 3.7. (i) What is the relationship between the degree of B and the degree of L with $B \cong_{eff} Intal(L)$?
(ii) Given a computable Boolean algebra $A = Intal(L)$ for a computable linear ordering L and a computable $\widehat{B} \cong B$, is there a computable copy $\widehat{L}$ of L such that $\widehat{B} \cong_{comp} Intal(\widehat{L})$. The answer here is probably not. But what can be said?

Exact knowledge of the above would give insight into how to transfer some results from linear orderings to Boolean algebras. There is definite complexity here. For instance, Jockusch and Soare [**69**] constructed a low linear ordering not isomorphic to a computable one. However, Downey and Jockusch [**29**] proved that every low Boolean algebra is isomorphic to a computable one. Later Thurber [**149**] improved this to low_2 and then Knight and Stob [**88**] to low_4.

QUESTION 3.8. Is every low_n Boolean algebra isomorphic to a computable one?

This question would seem purely technical, but its solution either way would seem to need much greater understanding of countable Boolean algebras and their invariants than we currently possess.

Of course, there are analogous questions for Boolean algebras concerning spectra.

QUESTION 3.9. (i) What are the possible spectra of a Boolean algebra? By the work of Feiner [**43, 44**] we know that a Boolean algebra can be intrinsically non-low$_n$ for all $n \in \omega$ (that is, not isomorphic to any low$_n$ Boolean algebra). What else can be said?
(ii) What is the possible value for the spectrum of the atomicity relation for a computable Boolean algebra. In the Stone construction, adjacencies and atoms coincide, so the spectrum of the atomicity relation must be built as a union of spectra for adjacencies. Downey [**21**] proved that each spectrum must contain an incomplete degree, and Downey and Hirschfeldt [**28**] proved that such spectra are closed upwards.
(iii) **(Hirschfeldt)** If a relation on a Boolean algebra has a spectrum with at least 2 elements does it have to have infinitely many? What about if the relation has a computable copy?

Before we turn to questions intrinsic to Boolean algebras, we consider some questions more specific to orderings. We recall the following result. A self-embedding of an ordering $\mathcal{L}$ is an isotone injective map $f : \mathcal{L} \mapsto \mathcal{L}$.

THEOREM 3.10 (Dushnik and Miller [**41**]). *1) Every countable infinite linear order has a non-trivial self embedding.*
2) There is a linear order which is isomorphic to a dense subset of the reals which has no non-trivial self embedding.

The effective content of this theorem has been examined by Hay and Rosenstein (see Rosenstein [**131**]), Downey [**23**], and Downey and Lempp [**34**]. Let $f : \mathcal{Q} \mapsto \omega$. We say that an ordering L is *η-like* if $L = \Sigma_{q \in \mathcal{Q}} f(q)$. That is, L is obtained from the rationals by replacing points with finite blocks. We say that the ordering is *strongly* η-like if the range of f is bounded.

CONJECTURE 3.11 (Downey and Moses [**36**]). Every computable copy of a computable linear order $\mathcal{L}$ has a non-trivial computable self embedding iff $\mathcal{L}$ contains a closed interval which is strongly η-like.

We remark that by Downey [**23**] there is no *uniform* proof of this conjecture. There are a number of related questions concerning nontrivial automorphisms and categoricity. An automorphism $f : L \mapsto L$ is called strongly nontrivial if for all x, $[x, f(x)]$ has infinitely many points.

CONJECTURE 3.12 (Kierstead [**83**]). Every computable copy of an ordering L has a strongly nontrivial Π_1 automorphism iff L has an interval of type η.

There are a number of related more technical questions. There are basic questions also, which seem fairly unapproachable.

QUESTION 3.13. Classify, by order-type those computable orderings $\mathcal{L}$ such that
(i) Every computable copy of $\mathcal{L}$ has a Π_n nontrivial automorphism.
(ii) Every computable copy of $\mathcal{L}$ has a nontrivial Π_n self embedding.
(iii) $\mathcal{L}$ is Δ^0_n categorical. (For $n = 1$ the answer is that $\mathcal{L}$ has a finite number of adjacencies (Remmel [**118**], Goncharov [**54**]), and the answer is known for $n = 2$ by McCoy [**97**]).

In a slightly differing direction, Moses [**105**] proved the following.

THEOREM 3.14 (Moses [**105**]). *A relation R is intrinsically computable*[2] *in a computable linear ordering iff it can be defined by a quantifier free formula with finitely many parameters.*

This naturally leads to the question below.

QUESTION 3.15 (Moses). Can one characterize via some definability criterion the intrinsically computable relations in decidable (Σ_1 decidable, etc.) linear orderings?

Moses also asked some specific question related to spectra of adjacencies and decidable models.

QUESTION 3.16 (Moses). (i) Can one characterize the computable linear orderings with intrinsically complete successivities?
(ii) Can one characterize the computable linear orderings which have no computable copy with a computable set of successivities?
(iii) Can one characterize the computable linear orderings which have no decidable copy?

Naturally there are the related questions for atoms in Boolean algebras.

We finish this section with some questions which seem more specific to Boolean algebras. We recall the following definitions.
1) Let B be a Boolean algebra.
(i) $At(B) =$ set of atoms of B,
(ii) $Atl(B) =$ set of atomless elements of B,
(iii) $IAt(B) =$ ideal generated by the set of atoms of B,
(iv) $F_0(B) = \{0_B\}$
$F_1(B) = IAt(B)$
$F_{\alpha+1}(B) = \{x \in B : x/F_\alpha(B) \in IAt(B/F_\alpha(B)\}$
$F_\beta(B) = \bigcup_{\alpha<\beta} F_\beta(B)$ if β a limit ordinal.
(v) B is *atomic* if every element of B has an atom below it.
(vi) B is α-atomic if $B/F_\beta(B)$ is atomic for all $\beta < \alpha$.

QUESTION 3.17. What can be said about the theory of Boolean algebra in extended logics?

For instance, we know:
a) **Goncharov** [**53**] Any ω-atomic computable Boolean algebra is isomorphic to a decidable Boolean algebra.

b) **Pinus** [**115**] Any ω-atomic computable Boolean algebra is isomorphic to an $L(Q)$-decidable Boolean algebra where $L(Q)$ is the extension of the first order language by the quantifier $Q =$ there exist infinitely many.

c) **Ershov (1973)** Any superatomic[3] computable Boolean algebra is isomorphic to an $L(I)$-decidable Boolean algebra where $L(I)$ is the restriction of the second order language whose unary predicate variables range over ideals.

While we are at it, we mention one long-standing question of Rosenstein.

[2]Recall that a relation on a structure $\mathcal{A}$ is *intrinsically computable* if it is computable in each computable copy of $\mathcal{A}$.

[3]That is, every sublgebra is atomic.

QUESTION 3.18 (Rosenstein). Suppose that T is a complete theory of linear orderings with a computable model and a prime model. Must T have a computable prime model?

4. Extensions of Partial Orderings to Linear Ones.

One of the basic theorems about partial orderings is the fact that they all have linear extensions. That is, given a poset $(P, \preceq)$ there is a linear ordering $(L, \leq)$ with the same field P such that for all x, y, $x \prec y$ implies $x < y$. This result is easily seen to be effectively true. This short section will look at several questions emanating from refinements of this theorem.

It is known that any well-founded partial ordering has a well-founded linear extension. The best way to think of this is that if we cannot embed ω^* into $(P, \preceq)$, then there is a linear extension also free of ω^*. Bonnet [**9**] has completely classified the linear order-types τ such that if $(P, \preceq)$ is τ-free, then there is an extension $(L, \leq)$ which is also τ-free. (Actually, this classifications uses higher cardinalities. Hence, one should look at Jullien's thesis [**71**] for a similar classification for countable orderings.)

QUESTION 4.1 (Downey). What is the effective content of the Bonnet-Jullien result? Is there an effective analogue?

Of particular interest is the result for $(P, \preceq)$ scattered. (That is, we cannot embed η into $(P, \preceq)$.) Slaman and Woodin [**141**] proved that those posets with dense extensions are essentially unclassifiable since the set is not Borel. Are there similar results for other constraints on extensions? We remark that the classical proofs of Bonnet's theorem that η can be avoided, part of the Bonnet-Jullien classification, are proof theoretically the same as Π^1_2-CA. It would be remarkable if the result was equivalent. Downey, Hirschfeldt, Lempp and Solomon [**27**] have results in this area.

One can also look at computable versions. Thus, a poset can be called *computably well-founded* if one cannot computably embed an ω^* sequence into it. Although every well-founded computable poset has a computable well-founded extension (Rosenstein, Kierstead [**132**]), there exist computable computably well-founded posets with no computably well-founded computable linear extension.

QUESTION 4.2 (Rosenstein). (i) What is the situation for other properties, e.g. it is known by Downey, Hirschfeldt, Lempp, and Solomon [**27**] that there is a $(P, \preceq)$ which is computably scattered yet no computable linear extension is computably scattered.
(ii) If $(P, \preceq)$ is computably well-founded (resp. computably scattered), then it has a Δ_2 computably well-founded (resp. computably scattered) linear extension. Can this be sharpened to, for instance, Π_1?

QUESTION 4.3 (Downey). Classify the computable partial orderings with precisely one computable linear extension.

There is a whole host of other questions of similar ilk we could examine and generalize up the arithmetical hierarchy, but we will confine ourselves to the above and return to posets when we look at computable combinatorics in Section 5.

5. Computable Combinatorics

We begin with some basic definitions. Let $G = (V, E)$ be a graph[4] and $x \in V$.
1) $N(x) = \{y \in V : (x, y) \in E\}$,
$\delta(x) = |N(x)|$,
$\Delta(G) = max\{\delta(x) : x \in V\}$.
2) G is *computable* if V and E are computable.
3) G is *highly computable* if V, E, and $N(x)$ are computable.
4) G is *A-computable* if V and E are computable and $N(x)$ is A-computable. (Gasarch and Lee [**51**])
5) $\omega(G) = max\{n : G \text{ contains an } n\text{-clique}\}$.
6) $\chi(G) = min\{n : G \text{ is } n\text{-colorable}\}$,
$\chi^c(G) = min\{n : G \text{ is computably } n\text{-colorable}\}$,
$\chi^p(G) = min\{n : G \text{ is polynomial-time } n\text{-colorable}\}$, etc.
7) A graph is *perfect* if for every induced subgraph H, $\omega(H) = \chi(H)$.

There have been a lot of theorems about computable analogues of theorems for graphs (and posets). To frame our question we recall some of those results.

THEOREM 5.1 (Bean [**7**]). (i) *Let a, b be such that $2 \leq a < b \leq \infty$. Let X be an infinite computable set. There exists a computable graph $G = (V, E)$ such that $\chi(G) = a$, $\chi^c(G) = b$, and $V \subseteq X$. If $a \leq 4$ then G can be taken to be planar.*

(ii) *There exists a computable graph G such that $\chi(G) = 2$, $\chi^c(G) = \infty$, and G is planar.*

THEOREM 5.2 (Gasarch and Lee [**51**]). *For any noncomputable c.e. set A, there exists an A-computable graph G such that $\chi(G) = 2$, $\chi^c(G) = \infty$, and G is planar.*

QUESTION 5.3. Gasarch asked if the theorem is true for all $A \leq_T 0'$. Jockusch observed that any such A has a non-computable c.e. set $B \leq_T A$. For graphs which are computable relative to A, Theorem 5.2 is true for degrees which contain a hyperimmune set.

What can be said about general $A \leq_T \emptyset'$?

THEOREM 5.4 (Remmel [**122**]). *For every computable tree T, there is a computable k-colorable graph G such that there is a degree preserving 1:1 correspondence between the infinite paths through T and the set of k-colorings of G (up to a permutation of the colors).*

QUESTION 5.5. Is the theorem true for restricted classes of graphs? E.g. planar graphs, comparability graphs, co-comparibility graphs, subgraphs of a given graph, etc.

THEOREM 5.6 (Schmerl [**133**]). *1) If G is highly computable and n-colorable, then G is computably $(2n-1)$-colorable. Moreover, given an index for G (as a highly computable graph), one can computably find an index for a $(2n-1)$-coloring of G.*
2) For every $n \geq 2$, there is a highly computable n-colorable graph H which is not computably $(2n-2)$-colorable.

[4]Here G is undirected, although similar questions and notions can be considered in the directed case.

THEOREM 5.7 (Kierstead [**80**]). *Every perfect, k-colorable, highly computable graph is computably $(k+1)$-colorable. Moreover there exist perfect, k-colorable, highly computable graphs which cannot be computably k-colored.*

THEOREM 5.8 (Brooks' Theorem). *For any graph G, $\chi(G) \leq \Delta(G)$ unless either $\Delta(G) \leq \omega(G)$, or both $\Delta(G) = 2$ and G contains an odd cycle.*

THEOREM 5.9 (Schmerl [**134**]). *For any highly computable graph G, $\chi^c(G) \leq \Delta(G)$ unless $\Delta(G) \leq \omega(G)$, or both $\Delta(G) = 2$ and G contains an odd cycle.*

QUESTION 5.10.
(i) **(Gasarch)** What happens to such results when we replace highly computable graphs by decidable graphs?
(ii) What happens to such results when we replace highly computable graphs by $L(Q)$-decidable graphs?
(iii) What is the complexity of such colorings if we assume the underlying graph is polynomial-time?

Remark: Bean [**7**] showed that all negative results that he obtained for highly computable graphs also hold for decidable graphs, e.g. for every $k \geq 2$ there exists a decidable graph G such that $\chi(G) = k$ and $\chi^c(G) = 2k - 1$.

A graph is called *k-edge colorable* if there is a coloring of the edges of G with k-colors such that any two edges of G which share a common vertex have different colors.

e-$\chi(G)$ = the minimum k such that G is k-edge colorable.

THEOREM 5.11 (Vizing's Theorem). *For any graph G, e-$\chi(G) \leq \Delta(G)$.*

THEOREM 5.12 (Kierstead [**80**]). *For any highly computable graph G, e-$\chi^c(G) \leq e$-$\chi(G) + 1 \leq \Delta(G) + 2$.*

THEOREM 5.13 (Manaster and Rosenstein [**96**]). *There exists a highly computable graph G such that e-$\chi(G) = \Delta(G)$, but e-$\chi(G) < e$-$\chi^c(G)$.*

QUESTION 5.14 (Kierstead). Is Vizing's Theorem effectively true?

THEOREM 5.15 (Kierstead). *If G is a computable directed graph that does not have an induced subgraph of the form*
(1) directed 3-cycle, or
(2) $\circ \longrightarrow \circ \longrightarrow \circ \longleftarrow \circ$, *or*
(3) $\circ \longleftarrow \circ \longrightarrow \circ \longrightarrow \circ$,
then $\chi^c(G) \leq 2^{\omega(G)}$ where $\omega(G)$ is the size of the largest clique.

We finish the section on graphs with one question which is apparently wide open.

QUESTION 5.16 (Kierstead). Lovasz proved that if $n_1, n_2 \in \omega$ have $n_1 + n_2 - 1 = \Delta(G)$, then G can be partitioned into V_1, V_2 with the maximum degree of the subgraph induced by $V_i \leq n_i$ for $i = 1, 2$. Kierstead asked if this result is effectively true for highly computable G.

5.1. Dilworth's Theorem and Dimension. We turn now to combinatorial aspects of the theory of partial orderings. We recall some definitions.

Let $\mathcal{P} = (P, \leq_P)$ be a partially ordered set (poset).
1) $width(\mathcal{P})$ = maximal size of an antichain in $\mathcal{P}$.
2) $height(\mathcal{P})$ = maximal size of a chain in $\mathcal{P}$.

3) $dim(\mathcal{P})$ = the minimum number of linear orders $L_1 = (P, \leq_1), \ldots, L_n = (P, \leq_n)$ such that

$$x \leq_P y \iff x \leq_i y \text{ for } i = 1, \ldots, n$$

= the minimum n such that $\mathcal{P}$ can be embedded in Q^n (under the product ordering).

If $w \in N$, then a *w-cover* of $\mathcal{P} = (P, \leq_P)$ is a set of w disjoint chains such that every element of P is in some chain. The fundamental theorem of partial orderings is the following.

THEOREM 5.17 (Dilworth's Theorem). *Every finite poset of width w has an w-cover.*

While we will see that Dilworth's theorem is *not* effectively true, amazingly, there is an effective rescue.

THEOREM 5.18 (Kierstead [**79**]). *Every computable poset of width w can be partitioned into $\frac{5^w - 1}{4}$ computable chains.*

We let $w^c(\mathcal{P})$ denote the *computable* chain cover number. That is the minimum number of computable chains needed to cover the poset.

THEOREM 5.19 (Szemérédi, Trotter (see Kierstead [**82**])). *For every $w \geq 2$, there exists a computable partial order $\mathcal{P}$ of width w such that*

$$w^c(\mathcal{P}) = \binom{w+1}{2}.$$

THEOREM 5.20 (Kierstead [**82, 79**]). *There is a computable partial order of width 2 which cannot be partitioned into 4 computable chains.*

QUESTION 5.21 (Kierstead). Is the constant $\frac{5^w - 1}{4}$ the best we can do? The difference between the the quadratic lower bound and the exponential upper one is quite large.

Let $\mathcal{P} = (P, \leq_P)$ be a computable poset.

1) The *comparability graph* of $\mathcal{P}$, $C(\mathcal{P}) = (P, E_C)$ is the graph where $(x, y) \in E_C$ iff x and y are comparable in $\mathcal{P}$.
2) The *co-comparability graph* of $\mathcal{P}$, $I(\mathcal{P}) = (P, E_I)$ is the graph where $(x, y) \in E_I$ iff x and y are not comparable in $\mathcal{P}$.
3) Note that a chain in $\mathcal{P}$ is a clique in $C(\mathcal{P})$.
4) Note that an antichain in $\mathcal{P}$ is a clique in $I(\mathcal{P})$.
5) Dilworth's Theorem says that the co-comparability graph $C(\mathcal{P})$ of a poset of width w can be colored with w colors, i.e. $\chi(C(\mathcal{P})) = \omega(C(\mathcal{P}))$.

QUESTION 5.22 (Trotter, Schmerl). Does there exist a function f such that, for any computable co-comparability graph G, $\chi^c(G) \leq f(\omega(G))$? (Trotter in [**127**] asked the question also for Chvatal graphs.)

THEOREM 5.23 (Schmerl (see Kierstead [**82**]). *Every computable poset of height h can be covered by $\binom{h}{2}$ computable antichains. Moreover, for all positive integers h, there exists a computable poset of height h which cannot be covered by $\binom{h}{2} - 1$ computable antichains.*

THEOREM 5.24. *Every acyclic graph is the comparability graph of an ordered set of height two.*

Thus Bean's result shows that there exists a computable comparability graph G such that $\omega(G) = 2$ but $\chi^c(G) = \infty$.

1) An *interval order* is an ordered set whose points can be represented by intervals of the real line so that a point $x = (a, b)$ is less than a point $y = (c, d)$ iff $b \leq c$.
2) An *interval graph* is the comparability graph for an interval order.

THEOREM 5.25 (Kierstead and Trotter [**84**]). *For every computable interval graph G, $\chi^c(G) \leq 3\omega(G) - 2$. Moreover, for every w, there is a computable interval graph G such that $\omega(G) = w$ and $\chi^c(G) = 3w - 2$.*

Fact: The dimension of a partial order is $\leq$ its width.

THEOREM 5.26 (Kierstead, McNulty and Trotter [**85**]). *Let $\mathcal{P}$ be a computable poset. Then:*
(1) If $w^c(\mathcal{P}) \leq 2$, the computable dimension of $\mathcal{P}$ is ≤ 5.
(2) If $w^c(\mathcal{P}) \leq 3$, the computable dimension of $\mathcal{P}$ is ≤ 6 and moreover, there exists a computable partial order $\mathcal{Q}$ with $w^c(\mathcal{Q}) = 3$ and computable dimension 6).
(3) There is a computable partial order $\mathcal{Q}$ with $w^c(\mathcal{Q}) = 4$ which has no finite computable dimension ($\mathcal{Q}$ also has width 3).
(4) If the computable width of $\mathcal{P}$ is $\leq w$, then the computable dimension of $\mathcal{P}$ is $\leq 2w$.

THEOREM 5.27 (Hopkins [**66**]). *Let $\mathcal{P}$ be a computable interval order of width w. Then:*
(1) If $w = 2$, $\mathcal{P}$ has computable dimension ≤ 3.
(2) $\mathcal{P}$ has computable dimension $\leq 4w - 4$.
(3) For all $w \geq 2$, there exists a computable interval order $\mathcal{Q}$ of width w such that $\mathcal{Q}$ has computable dimension $\lceil \frac{4}{3}w \rceil$.

A *crown* is a partial order on $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ ($n \geq 3$) such that (1) for all $i \leq n$, $a_i < b_i$, (2) for all $i \leq n-1$, $a_{i+1} < b_i$, (3) $a_1 < b_n$, and (4) no other relation exists between the elements.
A partial order is *crown-free* if none of its induced suborders are crowns.

THEOREM 5.28 (Kierstead, McNulty, Trotter [**85**]). *(1) Every crown-free computable poset with computable width w has computable dimension $\leq w!$, and:*
(2) For $w \geq 3$ there is a computable crown-free poset with computable width w, but computable dimension at least $w\binom{w}{t}$ where $t = \lfloor \frac{(w-1)}{2} \rfloor$[5]*.*

COROLLARY 5.29 (Kierstead, McNulty, Trotter [**85**]). Every crown-free computable poset of width w has computable dimension $\leq (\frac{5^w-1}{4})!$.

QUESTION 5.30 (Kierstead et. al.). What are the best bounds in the theorems above?

[5] Here, as usual, $\lfloor z \rfloor$ denotes the greatest integer $\leq z$.

5.2. Ramsey-Type Theorems. Ramsey-type theorems assert that in large enough samples there is some well-behaved piece. Of course Ramsey's theorem itself asserts that if we colour the n element subsets of ω with k colours then there is an infinite homogeneous set: a subset A such that $[A]^n$ is monochromatic. But there are a host of Ramsey-type theorems also. For instance, Van der Waerden's theorem on arithmetical progressions, Szemérédi's theorem, the Hales-Jewett theorem, etc. Most of these have a constellation of computability-theoretic and associated "reverse mathematics" questions related to them. In fact, for most we don't know their proof theoretical strength, and thus don't know most recursion theoretical aspects.

Below are some assorted questions garnered from the literature. The reader is also directed to the articles by Simpson's [**138**] and Cenzer and Jockusch [**12**] in this volume, Hummel [**67**], and to the recent paper by Cholak, Jockusch and Slaman [**16**] where the connections between the computability-theoretical aspects and the reverse mathematical ones (in the sense of what is needed to get a reversal) are made quite explicit.

QUESTION 5.31 (Cholak, Jockusch, Slaman [**16**]). (i) For each noncomputable set C and each computable 2-colouring of $[\omega]^2$, is there an infinite low$_2$ homogeneous set X with $C \not\leq_T X$?
(ii) What degrees $\mathbf{d}$ have the property that every 2-colouring of $[\omega]^2$ has an infinite homogeneous set of degree at most $\mathbf{d}$?
(iii) For every 2-colouring C of $[\omega]^2$ not of PA degree, is there an infinite homogeneous H such that $C \oplus H$ does not have PA degree?

We refer the reader to Cholak, Jockusch and Slaman [**16**] for more on these questions. We remark that one of the questions there (For every infinite set $A \leq_T \emptyset'$, does one of A or $\overline{A}$ have an infinite low subset?) was solved negatively by Downey, Hirschfeldt, Lempp and Solomon [**25**].

Similar questions can be asked for the many well-quasi-ordering results in the literature. Kruskal's theorem (that finite trees are well-quasi-ordered under topological embedding as posets) has been investigated by Harvey Friedman, as have some variations. But what about the WQO results for graphs of bounded treewidth, posets under the chain minor ordering, etc? Moreover, how fast do obstruction sets grow as a function of the genus for Robertson and Seymour's Kuratowski Theorem for surfaces of genus $\leq g$?

5.3. Some Other Combinatorial Problems Which Have Not Been Studied.

The following is just a sample of other problems in combinatorics whose efective content has not been analysed.
1) Computable Genus
2) Spherical Partial Orders (see Felsner, Fishburn, and Trotter [**46**]).
Every partial order $\mathcal{P}$ has a representation of the form

$$x \leq_P y \iff F(x) \subseteq F(y)$$

Where F is some mapping of $\mathcal{P}$ to $\mathcal{R}^d$, for some d. Idea: restrict range of F to be sphere in R^d.
3) Computable Crossing Numbers
4) Stable Marriage Problems (But see Cenzer and Jockusch [**12**]).

6. Groups

6.1. Groups in General. We begin with a question whose solution would seem to need real insight into group theory.

QUESTION 6.1 (Downey). (i) Is there a finitely presented group which is Σ_n-decidable but not Σ_{n+1}-decidable?
(ii) Is there an undecidable finitely presented group that is n-decidable for each n?

Actually the question is of interest in many cases, say for Abelian groups, even if the finitely presented hypothesis is removed.

There are also several questions that can be gleaned from Miller's survey paper [**101**].

Recall that a property P of groups G is called a *Markov Property* if

(i) There is some finitely presented group H such that for all finitely presented G, if H embeds in G then $G \notin P$.
(ii) There is a finitely presented X such that $X \in P$.

The Adian-Rabin theorem states that *for any Markov property P, there is no algorithm which, when given a finite presentation of a group G determines if G satisfies P.* This is sort of a Rice's theorem for groups. We remark that for many Markov properties the complexity of the index set is known. For instance, if the property is c.e. then the proof of the Adian-Rabin theorem (see Miller [**101**], Rabin [**116**]) shows that it is Σ_1-complete. Boone and Rogers [**11**] showed that having a solvable word problem is Σ_3-complete and Lempp [**90**] proved that being torsion free is Π_2-complete.

QUESTION 6.2 (Miller [**101**]). (i) Is the index set of finitely presented residually finite groups Π_2-complete?
(ii) Is the index set of finitely presented simple groups Π_2-complete?
(iii) Is the index set of solvable finitely presented groups Σ_3-complete?

It would be interesting to develop analogues of the Adian-Rabin theorem for higher index sets, such as an analogue of Markov properties that always gives $\mathbf{0}''$-hard problems.

6.2. Abelian Groups. To state some of the questions we will need some definitions leading to the classical invariants for Abelian groups.

Let G be a group, $G[p^n] = \{x \in G \mid p^n x = 0\}$. We let $\mathbf{Z}_{p^n}$ denote the cyclic group of order p^n and let $\mathbf{Z}_{p^\infty} = \overset{lim}{\leftarrow} \mathbf{Z}_{p^n}$ (The set of all rational numbers whose denominator is a power of p under addition mod 1.)

DEFINITION 6.3. 1) A group G is *divisible* iff for all $g \in G$ and $n \in \omega - \{0\}$, $n|g$ iff $nG = G$ for all $n > 0$.
2) A group G is *p-divisible* iff $p^n G = G$ for all $n > 0$ iff $pG = G$.
3) A group G is *reduced* if it has no divisible subgroups other than $\{0_G\}$.

It is a well-known classical theorem that: *Every Abelian group A is the direct sum of a divisible group D and a reduced group C, $A = D \oplus C$.* The effective content of this theorem has been investigated first by Khisamiev and Khisamiev [**76**] and later by Friedman-Simpson-Smith [**47**] who proved that its proof theoretical strength is Π^1_1-CA_0.

A subgroup G of an Abelian group A is called **pure** if for all $g \in G$, the equation $nx = g$ is solvable in G whenever it is solvable in A. A subgroup G of an Abelian group A is p-**pure** if for all $k > 0$,

$$p^k G = G \cap p^k A,$$

which happens iff $pG = G \cap pA$. We can now state the relevant invariants.

DEFINITION 6.4 (Szmielew invariants of G). 1) $G \models A_{pnk}$ iff G contains $\mathbf{Z}_{\mathbf{p^n}}^{\mathbf{k}}$ as a pure subgroup.
2) $G \models B_{pnk}$ iff $\mathbf{Z}_{\mathbf{p^n}}^{\mathbf{k}} \leq \mathbf{G}$.
3) $G \models C'_{pnk}(g_1, \ldots, g_k)$ iff the images $g'_1, \ldots, g'_k$ of $g_1, \ldots, g_k$ in the factor group $\overline{G} = G/G[p^n]$ are such that $g'_1 + p\overline{G}, \ldots, g'_k + p\overline{G}$ are linearly independent in $\overline{G}/p\overline{G}$.
4) $C_{pnk} = \exists x_1, \ldots, x_k C'_{pnk}(x_1, \ldots, x_k)$.

The following numbers are also called the Szmielew invariants of G.

$$\begin{aligned}
\alpha_{pn}(G) &= \sup\{k \in \omega \mid G \models A_{pnk}\}, \\
\beta_p(G) &= \inf\{\sup\{k \in \omega \mid G \models B_{pnk}\} \mid n \in \omega\}, \\
\gamma_p(G) &= \inf\{\sup\{k \in \omega \mid G \models C_{pnk}\} \mid n \in \omega\}.
\end{aligned}$$

THEOREM 6.5 (Szmielew [**147**]). *Two Abelian groups G and H are elementarily equivalent if and only if these invariants of G and H are the same.*

Let T be a complete theory of groups. We let

$$\begin{aligned}
A(T) &= \{(p^{n+1}, k) \mid A_{pnk} \in T\}, \\
B(T) &= \{(p^{n+1}, k) \mid B_{pnk} \in T\}, \\
C(T) &= \{(p, k) \mid \exists N_p[\forall n \geq N_p(\neg A_{pn1} \in T) \wedge C_{pN_pk} \in T]\}, \\
P(T) &= \{p \mid \exists N_p[\forall n \geq N_p(\neg A_{pn1} \in T)]\}.
\end{aligned}$$

A set C' of pairs (p, k), $k > 0$, is called a T*–extension* of the set $C(T)$ if
(i) $C(T) \subseteq C'$,
(ii) if $(p, k) \in C' \setminus C(T)$ then $p \notin P(T)$, and
(iii) if $(p, k) \in C'$ and $0 < s < k$ then $(p, s) \in C'$.

Khisamiev [**74**] was the first to really examine the computability aspects of Abelian groups. He proved the following.

THEOREM 6.6 (Khisamiev [**74**]). *A complete theory T of Abelian groups has an X-computable model if and only if the following hold.*
(i) $A(T) \in \Sigma_2^X$,
(ii) $B(T) \in \Sigma_1^X$,
(iii) The set $C(T)$ has a T–extension $C' \in \Sigma_2^X$.

This leads to several open questions.

QUESTION 6.7. 1) **(Khisamiev)** Find necessary and sufficient conditions, similar to the theorem above, for a theory of Abelian groups with a c.e. set of axioms to have a computable model.
2) **(Knight, Harizanov)** Find an algebraic characterization of the countable reduced Abelian p-groups which have computable copies.

There are several questions related to computable dimension.

THEOREM 6.8 (Goncharov [**55**], Smith [**148**]). *An Abelian p-group A is computably categorical iff either*
1) $A \cong \mathbf{Z}_{p^\infty}^{\omega} \oplus F$ *or*
2) $A \cong \mathbf{Z}_{p^\infty}^{m} \oplus \mathbf{Z}_{p^n}^{\omega} \oplus F$
where F is a finite p-group and $m, n < \omega$.

THEOREM 6.9 (Nurtazin). *A torsion-free Abelian group is computably categorical iff it has finite rank.*

The most basic question here is the following.

QUESTION 6.10 (Goncharov). Find algebraic criteria for when a computable Abelian group is computably categorical.

Naturally there are questions such as giving an algebraic characterization of when an Abelian group, p-group, or torsion free Abelian group is Δ_n^0 categorical.

Recently, the jump degrees of Abelian p-groups (Oates [**111**]) and torsion free Abelian groups were examined by Downey and Jockusch (in [**22**]). It was found that every rank 1 group has a 2nd jump degree, this being a kind of explanation of Baer's theorem about type sequences (see Coles, Downey, Slaman [**17**]). Is there a version for groups of finite rank not equal to 1?

6.3. Open Questions on the Groups of Computable Automorphisms of a Computable Model.

QUESTION 6.11. (i) **(Goncharov)** Find a description of the class of all groups isomorphic to a group of all computable automorphisms of a computable model.
(ii) **(Goncharov)** Is the class of all groups of computable automorphisms of a computable model closed under free products?
(iii) **(Nies)** Let G_δ be the group of all permutations computable in a Turing degree δ. Define G_δ^* as the quotient of this group modulo the group of all finitary permutations. Is it true that G_δ^* is embeddable into G_γ^* iff $\delta' \leq_T \gamma'$?

Morozov [**104**] proved that
1) G_δ is embeddable into G_γ iff $\delta \leq_T \gamma$.
2) G_δ^* is isomorphic to G_γ^* iff $\delta = \gamma$.
3) Similar results hold for G_δ modulo the set of all even permutations, i.e. permutations which are the product of an even number of two cycles.

6.4. Ordered Groups. The primary sources of work on computability aspects of the theory of ordered groups is Downey and Kurtz [**32**], Hatzikiriakou-Simpson [**61**], and above all, Solomon [**142**]. There, for instance, it is shown that the basic result that torsion free Abelian groups are orderable fails to hold effectively, and is equivalent to WKL_0. There are many open questions. Here is a selection.

QUESTION 6.12. (i) **(Downey and Kurtz)** Given a Π_1^0 class C, is there an orderable computable group whose cone of orderings (up to reversal) is in correspondence with the class C?
(ii) **(Downey and Kurtz)** Is every orderable computable group isomorphic to a computably orderable group?

(iii) **(Solomon)** Characterize the computably categorical ordered groups. Solomon remarks that since the groups are torsion free in the Abelian case, one would guess that the answer is "precisely when they have finite rank." In unpublished work, Goncharov, Lempp, and Solomon have shown that Archimedian ordered groups are computably categorical iff they have fine rank. But the case with infinitely many equivalence classes remains open. It is also not known if they can have finite computable dimension which is not 1.

(iv) (Downey) Does every low ordered computable group have a computable copy? That is, we fix the low order and ask for a copy where both the orderings and the group have are computable. Solomon remarks that the answer seems open even in the Abelian case.

(v) (Downey) Same as (iv) for low ordered fields.

(vi) (Downey) Does every computable formally real computable field have a presentation which admits a computable ordering?

(vii) (Solomon) Does every computable ordered Abelian group have a presentation which admits a computable basis. Dobritsa [**19**] proved this for computable Abelian groups, but maybe in the ordered case, it is not possible to make *both* the basis and the ordering computable at the same time.

We remark that (i) above has a negative answer if we ask that the group be Abelian (Solomon [**142**]).

7. Fields, Rings and the Like

A longstanding question of Friedman, Simpson and Smith [**48**] is the following.

Question 7.1. Let P be a Π_1^0 class. Is there a computable commutative ring R with 1 such that the ideals of R are in 1-1 correspondence with the members of P?

For many types of algebraic structures the possible computable dimension is known.

Question 7.2 (Goncharov). Is there a field with finite computable dimension $\neq 1$?

If the answer is no then it will be interesting to look at skew fields (Shore). For integral domains, Kudinov (personal communication) has shown the answer to be yes.

There are other more general questions one might ask about rings. Many classical theorems were proven using noneffective means and await analysis. For instance, while the effective content of Stone duality is fairly well-understood, the more general Zariski topology is not. Also radical properties occupy a central topic in the general structure of rings, but have not been analysed effectively.

8. Polynomial-Time and Other Special Structures

There is a lot of work on various time and space bounded models. We look at some of this material and the particular sorts of questions one can ask below. Let $\mathcal{A}$ be a structure as before. We recall from the introductory section the basic definitions for polynomial-time structures.

DEFINITION 8.1. $\mathcal{A}$ is **polynomial-time** (abbrev. **p-time** if
1) A is a polynomial-time subset of $\{0,1\}^*$.
2) The set of relations $\{R_i^{\mathcal{A}}\}$ is uniformly polynomial-time computable, i.e. there exists a computable function G such that $G(i) = [n_i, e_i]$ where $R_i^{\mathcal{A}}$ is the n_i-ary relation whose characteristic function is computed by ϕ_{n_i,e_i} and there exists a computable function G' such that $G'(i) = m_i$ where for all $(x_1, \ldots, x_{n_i}) \in \{0,1\}^{n_i}$, it takes at most $(max\{2, |x_1|, \ldots, |x_{n_i}|\})^{m_i}$ steps to compute $\phi_{n_i,e_i}(x_1, \ldots, x_{n_i})$.
3) The set of functions $\{f_i^{\mathcal{A}}\}$ are uniformly polynomial-time computable.
4) There is a polynomial-time computable function H such that $H(i) = c_i^{\mathcal{A}}$.

Note: The fact that $\mathcal{A}$ is polynomial-time computable does not necessarily imply that its atomic diagram is polynomial-time computable. Thus we say that $\mathcal{A}$ is **uniformly polynomial-time computable** iff its atomic diagram is polynomial-time computable.

Similarly we can define $\mathcal{A}$ being **p-time decidable**, **p-time Σ_n-decidable**, etc. A related idea is that of automatic structures. Let $\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i\in S}, \{f_i^{\mathcal{A}}\}_{i\in T}, \{c_i^{\mathcal{A}}\}_{i\in U})$ where $A \subseteq \Sigma^*$ for some finite alphabet Σ.

DEFINITION 8.2. Let $\nu : D \to A$ be a surjective mapping where D is an automaton recognizable subset of Σ^*. The mapping ν is an **automatic (strongly automatic, asynchronous automatic)** presentation of $\mathcal{A}$ if ν satisfies:
1) There exists a 2-variable automaton (strong automaton, asynchronous automaton) which for any two words $\alpha, \beta \in D$ decides whether $\nu(\alpha) = \nu(\beta)$.
2) For each $j \in S$, there exists an n_j-variable automaton (strong automaton, asynchronous automaton) which, for all $(\alpha_1, \ldots, \alpha_{n_j})$, decides whether $\mathcal{A} \models R_j^{\mathcal{A}}(\nu(\alpha_1), \ldots, \nu(\alpha_{n_j})$.
3) For each $j \in t$, there exists an $n_j + 1$-variable automaton (strong automaton, asynchronous automaton) which, for all $(\alpha_1, \ldots, \alpha_{n_j}, \alpha_{n_j+1})$, decides whether

$$\mathcal{A} \models f_j^{\mathcal{A}}(\nu(\alpha_1), \ldots, \nu(\alpha_{n_j}) = \nu(\alpha_{n_j+1}).$$

There are several variations depending upon how we let the automata act on n-tuples:

$\alpha_1 = \sigma_{01} \ldots \sigma_{k_1 1}$
$\vdots$
$\alpha_n = \sigma_{0n} \ldots \sigma_{k_n n}$

n-automata: Let an ordinary automaton act on the words individually but have the final or accepting states be n-tuples.

strong n-automata: Pad the words with a new symbol so that all words have the same length and then have the underlying letters of the automaton alphabet be n-tuples.

asynchronous automata: Allow transitions where we make a move on some subset of the n-tuples.

8.1. Basic Questions for Polynomial-Time Models. Let $\mathcal{C}$ be a class of computable models, e.g. linear orderings, groups, etc.
1) Is every $\mathcal{A}$ in $\mathcal{C}$ isomorphic to a p-time model?

2) Is every $\mathcal{A}$ in $\mathcal{C}$ computably isomorphic to a p-time model?
3) Is every $\mathcal{A}$ in $\mathcal{C}$ isomorphic to a p-time model over a standard universe such as $Bin(\omega)$ or $Tal(\omega)$?
4) Is every $\mathcal{A}$ in $\mathcal{C}$ computably isomorphic to a p-time model over a standard universe such as $Bin(\omega)$ or $Tal(\omega)$?
5) Uniqueness? $A \cong_p B$ iff there exist p-time functions f and g such that $f \mid A : A \to B$ and $g \mid B : B \to A$.

Analogously to the computable version we have the following definition.

DEFINITION 8.3. (i) We say a p-time structure $\mathcal{A}$ is **strongly p-time categorical** if all p-time structures $\mathcal{B}$ which are isomorphic to $\mathcal{A}$ are p-time isomorphic to $\mathcal{A}$.
(ii) We say a p-time structure $\mathcal{A}$ with universe A is **p-time categorical over** A if all p-time structures $\mathcal{B}$ with universe A which are isomorphic to $\mathcal{A}$ are p-time isomorphic to $\mathcal{A}$.

To clarify the situation we give some examples below.
Example 1. $\mathcal{A} = (Tal(N), S, A)$ where $S(x) = x+1$ and $A \subseteq N$. $\mathcal{A}$ is isomorphic to a p-time model $\mathcal{B}$ implies that A is double exponential time.

Example 2. $\mathcal{A} = (Tal(N), f)$ where $f : Tal(N) \to Tal(N)$. The complexity of possible presentations of $\mathcal{A}$ is characterized by the orbits of f. Consider the extremes given by $f(x) = 2x$ versus $f(x)$ equals the Ackermann function.
Example 3. (Cenzer-Remmel [**13**]) $(N, S, +, -, \times, 2^x, \leq)$ has a polynomial-time model.

QUESTION 8.4. Does $(N, S, +, -, \times, 2^x, 3^x \leq)$ have a polynomial-time model?

THEOREM 8.5 (Grigorieff [**56**]). *Any computable relational structure*
$$\mathcal{A} = (A, \{R_i^{\mathcal{A}}\}_{i \in S}, \{c_i^{\mathcal{A}}\}_{i \in U})$$
is computably isomorphic to a p-time structure.

THEOREM 8.6 (Cenzer-Remmel [**13**]). *Let $A \subseteq \{0,1\}^*$ be any p-time set. There exists a computable linear ordering $\mathcal{L}$ isomorphic to $\omega + \omega^*$ which is not computably isomorphic to any p-time linear ordering whose universe is A.*

THEOREM 8.7 (Grigorieff [**56**]). *Every computable linear ordering $\mathcal{L}$ is isomorphic to a real time linear ordering $\mathcal{L}' = (Bin(\omega), <_{\mathcal{L}'})$.*

The proof of Theorem 8.7 boils down to two cases as follows.
Case A:
$\mathcal{L}$ has a computable increasing sequence S
$$S = (s_0 <_{\mathcal{L}} s_1 <_{\mathcal{L}} s_2 <_{\mathcal{L}} \dots)$$
such that S is cofinal, or S has a limit, or $\mathcal{L}$ has a computable decreasing sequence D
$$D = (d_0 >_{\mathcal{L}} d_1 >_{\mathcal{L}} d_2 >_{\mathcal{L}} \dots)$$
such that D is cofinal or D has a limit.

Case B: Not case A.

Then $\mathcal{L} \cong \omega + Z \cdot \lambda + \omega^*$.

THEOREM 8.8 (Remmel). *Let $A \subseteq \{0,1\}^*$ be any infinite p-time set and let $\mathcal{L}$ be a computable linear ordering which is isomorphic to $\omega + Z \cdot \lambda + \omega^*$ for some linear ordering λ. Then there exists a computable linear ordering $\mathcal{K}$ which is isomorphic to $\mathcal{L}$ but which is not computably isomorphic to any p-time linear ordering whose universe is A.*

Similar results hold for Boolean algebras and graphs.

QUESTION 8.9. Are there similar results for other structures?

COROLLARY 8.10 (Cenzer-Remmel [**13**]). Every computable Boolean algebra is computably isomorphic to a computable Boolean algebra over B where $B = Tal(\omega)$ or $B = Bin(\omega)$.

8.2. Questions for Automatic Models.

THEOREM 8.11 (Khoussainov-Nerode [**77**]). *1) For each n, ω^n has an automatic presentation. The rationals have an automatic presentation. The ordinal ω^ω has an asynchronous automatic presentation.*
2) For each n, the Boolean algebra $\mathcal{B}_{n\times\omega}$ has an asynchronous automatic presentation.
3) Automatic (asynchronous) orderings are closed under $+$ and $\times$.

QUESTION 8.12 (Khoussainov-Nerode). Find the first ordinal α which does not have an automatic (asynchronous) presentation?

9. Lattices of C.E. Substructures

9.1. Subspaces and the Like. Let V_∞ be an infinite dimensional computable vector space over a computable field F with a dependence algorithm. Let $L(V_\infty)$ denote the lattice of c.e. subspaces of V_∞ and $L^*(V_\infty)$ denote the lattice of c.e. subspaces of V_∞ modulo finite dimensional subspaces.

QUESTION 9.1. 1) Does every r-maximal subset of a computable basis for V generate an r-maximal subspace?
2) Does every atomless subset of a computable basis for V generate an atomless subspace?
3) What are the principal filters of $L^*(V_\infty)$? (Conjecture: **(Downey)** Every Σ^0_3 bounded modular lattice is a filter in $L^*(V_\infty)$.)
4) Is the $\forall\exists$ theory of $L^*(V_\infty)$-decidable?

We remark that this is all related to the classical question below. For $X \subset V_\infty$ let $sp(X)$ denote the subspace spaned by X.

QUESTION 9.2 (Downey). Suppose that $V_1 \oplus V_2 = R$ for a vector space R, and B is a basis of R. Are there $B_1 \sqcup B_2 = B$ such that $V_1 \oplus sp(B_1) = V_2 \oplus sp(B_2) = R$? (Note that this means that we require $V_i \cap sp(B_i) = \{0\}$.) What if $R = V_\infty$? If both V_i are decidable, do there exist computable B_i and is there an algorithm for effectively computing them?

We remark that a positive solution to this question for $R = V_\infty$ implies that r-maximal subsets of bases generate r-maximal spaces.

We remark that V_∞ is unlikely to have any interesting orbits (although this is an interesting question) since every automorphism is induced by a computable semilinear invertible transformation (Guichard [**57**]). Hence all automorphic spaces have, in particular the same 1-degree.

9.2. Lattice of C.E. Subalgebras and Ideals (Filters) of a Computable Boolean algebra. Let $\mathcal{N}$ denote a decidable presentation of the Boolean algebra of finite and cofinite subsets of N, $\mathcal{Q}$ denote a decidable presentation of the countable atomless Boolean algebra, $\mathcal{C}$ denote a decidable presentation of the Boolean algebra generated by the closed intervals of the rationals Q.

QUESTION 9.3. Is there an analogue of supermaximal subspaces in $\mathcal{N}$? (Recall that c.e. $V \subseteq V_\infty$ is called supermaximal iff for all superspaces W of V, either $W = V_\infty$ or $W =^* V$.)

We can look at the lattice of ideals of $\mathcal{Q}$. This is dually the lattice of Π^0_1 classes. It is distributive and has infinitely many orbits. A c.e. ideal I with infinitely many extensions is called *maximal* if for all c.e. extensions J of I, there is an x such that $J = \langle I \cup \{x\}\rangle$. I is called perfect if $\mathcal{Q}/I \cong \mathcal{Q}$. There is a general programme of examining the lattice of ideals of $\mathcal{Q}$.

QUESTION 9.4. What degrees contain maximal ideals?

It is known by the work of Cholak, Coles, Downey, and Herrmann [**15**] that the degrees containing perfect maximal ideals are exactly the array noncomputable degrees of Downey, Jockusch and Stob [**31**].

QUESTION 9.5. What is the relationship between the Boolean algebra $\mathcal{Q}/I$ and the possible degrees for I, where I is maximal?

QUESTION 9.6 (Cholak, Coles, Downey and Herrmann). It was proved by Cholak, Coles, Downey and Herrmann [**15**] that any two perfect maximal ideals are automorphic. Is this true for any other $\mathcal{Q}/I$? For instance, does the result still hold if we ask that $\mathcal{Q}/I$ be a copy of the Boolean algebra of finite and cofinite sets with the ideal generated by the atoms immune?

There are clearly a large number of interesting questions concerning the automorphism group of the lattice of ideals of $\mathcal{Q}$. We state only one more.

QUESTION 9.7 (Herrmann). Given any two nontrivial ideals I and J, are the lattices of c.e. subideals isomorphic? It is known that the number of isomorphism-types is at most 2.

10. Subalgebras

QUESTION 10.1 (Remmel). What is the relationship between the degrees of splittings and c.e. splittings of c.e. subalgebras of a computable Boolean algebra?

The background to the question is provided by the following set of results.

THEOREM 10.2 (Downey and Remmel [**39**]). *Let $V \in L(V_\infty)$. Suppose that A is a c.e. set. Then V has a basis B such that $B \equiv_{wtt} A$ iff $A \leq_{wtt} V$.*

That is, the correct reduction here is exactly weak truth-table reduction.

THEOREM 10.3 (Downey, Remmel, Welch [**40**]). *1) Let V be a c.e. subspace of V_∞. Then δ is the degree of a c.e. basis of V iff there exist c.e. subspaces W_1 and W_2 such such that $W_1 \oplus W_2 = V$ and $deg(W_1) = depdeg(W_1) = \delta$, where $depdeg(Z)$ denotes the dependence degree of Z which is the degree of computing independence over Z.*
2) There exist $Q, V, W \in L(V_\infty)$ such that $Q \oplus V = W$ and $\emptyset <_T Q <_T W$, but such that for all $V', Q' \in L(V_\infty)$ if

$$V' \oplus Q' = W \text{ and } Q \equiv_T Q',$$

then $deg(Q') \vee deg(V') = deg(W)$. (And same for depdeg.)

For Boolean algebras we have the following.

THEOREM 10.4 (Remmel, J. Yang (1997) see [**152**]). *Let δ_1 and δ_2 be any two c.e. degrees.*
1) There exist c.e. subalgebras W_1, W_2 of $\mathcal{N}$ such that $W_1 \oplus W_2 = \mathcal{N}$ and $deg(W_i) = \delta_i$ for $i = 1, 2$.
2) Similar results hold for $\mathcal{Q} \times B$, $\tilde{N} \times B$, and $\mathcal{C}$ for any computable Boolean algebra B.

3) Every non-computable c.e. subalgebra of $\mathcal{N}^k$ has a non-computable c.e. generating sequence.

10.1. NP and Similar Subspaces. Let $st(V_\infty)$ denote the standard representation of V_∞ (i.e. over $\{0,1\}^*$ and and $tal(V_\infty)$ the tally represetation.

A c.e. subspace S of V_∞ is *simple* if $dim(V_\infty/S) = \infty$ and for any infinite dimensional c.e. subspace W of V_∞, $W \cap S \neq \{\vec{0}\}$. By analogy, we define the following.

Let A be an oracle. An NP^A subspace S of $st(V_\infty)$ ($tal(V_\infty)$) is *NP^A-simple* if the dimension of $st(V_\infty)/S$ ($tal(V_\infty)/S$) is infinite and for any infinite NP^A subspace W of $st(V_\infty)$ ($tal(V_\infty)$), $W \cap S \neq \{bin(\vec{0})\}$ ($W \cap S \neq \{tal(\vec{0})\}$). An NP^A subspace M is *NP^A-maximal* if the dimension of $st(V_\infty)/M$ ($tal(V_\infty)/M$) is infinite and for any NP^A subspace W of $st(V_\infty)$ ($tal(V_\infty)$), either the dimension of $st(V_\infty)/W$ ($tal(V_\infty)/W$) or the dimension of W/M is finite. An NP^A subspace M is *NP^A-supermaximal* if the dimension of $st(V_\infty)/M$ ($tal(V_\infty)/M$) is infinite and for any NP^A subspace W of $st(V_\infty)$ ($tal(V_\infty)$), either $st(V_\infty) = W$ ($tal(V_\infty) = W$) or the dimension of W/M is finite. A NP^A subspace S of $st(V_\infty)$ ($tal(V_\infty)$) is *P^A-simple* if the dimension of $st(V_\infty)/S$ ($tal(V_\infty)/S$) is infinite and for any infinite dimensional P^A subspace W of $st(V_\infty)$ ($tal(V_\infty)$), $W \cap S \neq \{bin(\vec{0})\}$ ($W \cap S \neq \{tal(\vec{0})\}$).

It follows by Nerode and Remmel [**108**] that there exist oracles A such that there exists an infinite dimensional NP^A subspace V of $tal(V_\infty)$ such that V has no infinite dimensional subspace $W \in P^A$. Thus while a subspace W which is NP^A-simple is certainly P^A-simple, it is not clear that every P^A-simple subspace of $tal(V_\infty)$ is NP^A-simple.

There are a lot of variations to explore. The main theme is to understand the possible structure of NP^A spaces for various oracles A. Recently Downey and Nies [**37**] proved that, for subsets of ω (that is the lattice of NP^A *sets*), the structure can be very complicated indeed.

Assume the underlying field F of V_∞ is finite.

QUESTION 10.5 (Remmel). Does there exist a c.e. oracle D such that there exists an NP^D-supermaximal NP^D-simple subspace in $tal(V_\infty)$?

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School of Mathematical and Computing Sciences, Victoria University, P.O. Box 600, Wellington, New Zealand
E-mail address: rod.downey@vuw.ac.nz

Mathematics Department, University of California at San Diego, La Jolla, CA 92093-0012, U.S.A.
E-mail address: jremmel@ucsd.edu