Totally $< \omega^\omega$-computably enumerable degrees and m-topped degrees*

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1 Introduction

1.1 Degree classes

In this paper we will discuss recent work of the authors (Downey, Greenberg and Weber [8] and Downey and Greenberg [6, 7]) devoted to understanding some new naturally definable degree classes which capture the dynamics of various natural constructions arising from disparate areas of classical computability theory.

It is quite rare in computability theory to find a single class of degrees which capture precisely the underlying dynamics of a wide class of apparently similar constructions, demonstrating that they all give the same class of degrees. A good example of this phenomenon is work pioneered by Martin [22] who identified the high c.e. degrees as the ones arising from dense simple, maximal, hh-simple and other similar kinds of c.e. sets constructions. Another example would be the example of the promptly simple degrees by Ambos-Spies, Jockusch, Shore and Soare [2]. Another more recent example of current great interest is the class of K-trivial reals of Downey, Hirscheldt, Nies and Stephan [5], and Nies [23, 24].

We remark that in each case the clarification of the relevant degree class has lead to significant advances in our basic understanding of the c.e. degrees. We believe the results we mention in the present paper fall into this category. Our results were inspired by another such example, the array computable degrees introduced by Downey, Jockusch and Stob [10, 11]. This class was introduced by those authors to explain a number of natural "multiple permitting" arguments in computability theory. The reader should recall that a degree **a** is called array noncomputable iff for all functions $f \leq_{wtt} \emptyset'$ there is a a function g computable from **a** such that

$$\exists^\infty x \ (g(x) > f(x)).^1$$

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1 Of course, this was not the original definition of array noncomputability, but this version from [11] captures the domination property of the notion in a way that shows the way that it weakens the notion of non-low$_2$-ness, in that **a** would be non-low$_2$ using the same definition, but replacing $\leq_{wtt}$ by $\leq_T$.

1.2 Totally ω-c.e. degrees

Our two new main classes are what we call the *totally ω-c.e. degrees* and the *totally ω^ω-c.e. degrees.* These classes turn out to be completely natural and relate to natural definability in the c.e. degrees as we will discuss below. We begin with the ω case.

Definition 1 (Downey, Greenberg, Weber [8]). *We say that a c.e. degree* $\mathbf{a}$ *is* totally ω-c.e. *iff for all functions $g \leq_T \mathbf{a}$, g is ω-c.e.. That is, there is a computable approximation $g(x) = \lim_s g(x,s)$, and a computable function h, such that for all x,*

$$|\{s : g(x,s) \neq g(x,s+1)\}| < h(x).$$

The reader should keep in mind that array computability is a uniform version of this notion where h can be chosen independent of g. This class captures a number of natural constructions in computability theory.

As an illustration, recall that a c.e. prefix-free set of strings $A \in 2^{<\omega}$ *presents* a left c.e. real α if $\alpha = \sum_{\sigma \in A} 2^{-|\sigma|}$, that is, α is the measure of A. Now it is easy to use padding to show that every c.e. real has a presentation A which is computable (Downey [4]). On the other hand, bizarre things can happen. In [12], Downey and LaForte showed that there exist noncomputable left c.e. real α, all of whose c.e. presentations are computable. We have the following:

Theorem 1 (Downey and Greenberg [6]). *The following are equivalent.*

(i) $\mathbf{a}$ *is not totally ω-c.e..*
(ii) $\mathbf{a}$ *bounds a left c.e. real α and a c.e. set $B <_T \alpha$ such that if A presents α, then $A \leq_T B$.*

1.3 Natural definability

One of the really fascinating things is that this is all connected to *natural* definability issues within the computably enumerable Turing degrees. In terms of abstract results on definability, there has been significant success in recent years, culminating in Nies, Shore, Slaman [25], where the following is proven.

Theorem 2 (Nies, Shore, Slaman [25]). *Any relation on the c.e. degrees invariant under the double jump is definable in the c.e. degrees iff it is definable in first order arithmetic.*

The proof of Theorem 2 involves interpreting the standard model of arithmetic in the structure of the c.e. degrees without parameters, and a definable map from degrees to indices (in the model) which preserves the double jump. The beauty of this result is that it gives at one time a definition of a large class of relations on the c.e. degrees.

On the other hand, the result is somewhat unsatisfying in terms of seeking natural definitions of objects in computability theory as outlined in the paper

Shore [27]. Here we are thinking of results such as the following. (We refer the reader to Soare [28] for unexplained definitions below since they are mainly to provide background for the results of the current paper.)

Theorem 3 (Ambos-Spies, Jockusch, Shore, and Soare [2]). *A c.e. degree* $\mathbf{a}$ *is promptly simple iff it is not cappable.*

Theorem 4 (Downey and Lempp [13]). *A c.e. degree* $\mathbf{a}$ *is contiguous iff it is locally distributive, meaning that*

$$\forall \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}(\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a} \wedge \mathbf{b} \leq \mathbf{a} \rightarrow \\ \exists \mathbf{b}_1, \mathbf{b}_2(\mathbf{b}_1 \cup \mathbf{b}_2 = \mathbf{b} \wedge \mathbf{b}_1 \leq \mathbf{a}_1 \wedge \mathbf{b}_2 \leq \mathbf{a}_2))$$

holds in the c.e. degrees.

Theorem 5 (Ambos-Spies and Fejer [1]). *A c.e. degree* $\mathbf{a}$ *is contiguous iff it is not the top of the non-modular 5 element lattice in the c.e. degrees.*

Theorem 6 (Downey and Shore [14]). *A c.e. truth table degree is low*$_2$ *iff it has no minimal cover in the c.e. truth table degrees.*

At the present time, as articulated in Shore [27], there are very few such natural definability results.

In [6–8], we gave some new natural definability results for the c.e. degrees. Moreover, these definability results are related to the central topic of lattice embeddings into the c.e. degrees as analyzed by, for instance, Lempp and Lerman [19], Lempp, Lerman and Solomon [20], and Lerman [21].

A central notion for lattice embeddings into the c.e. degrees is the notion of a *weak critical triple.* The reader should recall from Downey [3] and Weinstein [30] that three incomparable elements $\mathbf{a}_0, \mathbf{a}_1$ and $\mathbf{b}$ in an upper semilattice form a weak critical triple if $\mathbf{a}_0 \cup \mathbf{b} = \mathbf{a}_1 \cup \mathbf{b}$ and there is no $\mathbf{c} \leq \mathbf{a}_0, \mathbf{a}_1$ with $\mathbf{a}_0 \leq \mathbf{b} \cup \mathbf{c}$. This notion captures the need for "continuous tracing" which is used in an embedding of the lattice M_5 into the c.e. degrees (first embedded by Lachlan [17]).[2]

The necessity of the "continuous tracing" process was demonstrated by Downey [3] and Weinstein [30] who showed that there are initial segments of the c.e. degrees where no lattice with a (weak) critical triple can be embedded. It was also noted in Downey [3] that the embedding of (weak) critical triples seemed to be tied up with multiple permitting in a way that was similar to non-low$_2$-ness. Indeed this intuition was verified by Downey and Shore [15] where it is shown that if $\mathbf{a}$ is non-low$_2$ then $\mathbf{a}$ bounds a copy of M_5.

The notion of non-low$_2$-ness seemed too strong to capture the class of degrees which bound M_5's but it was felt that something like that should suffice. On the other hand, Walk [29] constructed a array noncomputable c.e. degree bounding no weak critical triple, and hence it was already known that array non-computability was not enough for such embeddings. We proved the following definitive result:

[2] We recall that a lattice is not join semidistributive (also called principally indecomposable) iff it contains a copy of M_5 iff it contains a weak critical triple.

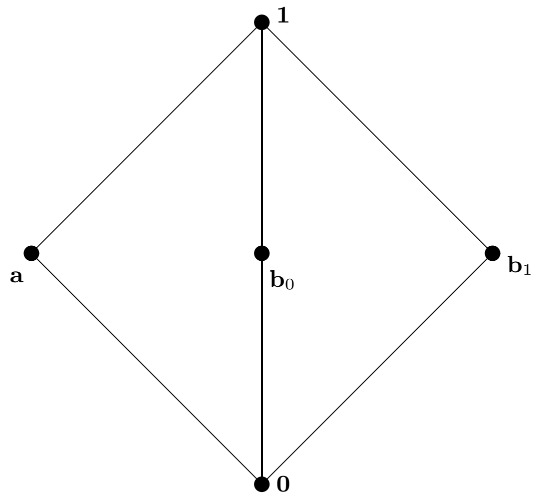

Fig. 1. The lattice M_5

Theorem 7 (Downey, Greenberg and Weber [8]). *A degree* $\mathbf{a}$ *is totally* ω*-c.e. iff it does not bound a weak critical triple in the c.e. degrees. Hence, the notion of being totally* ω*-c.e. is naturally definable in the c.e. degrees.*

Theorem 7 also allowed for the solution of certain technical problems from the literature.

Corollary 1 (Downey, Greenberg and Weber [8]). *The low degrees and the superlow degrees are not elementarily equivalent.*

Proof. As Schaeffer [26] and Walk [29] observe, all superlow degrees are array computable, and hence totally ω-c.e. Thus we cannot put a copy of M_5 below one. One the other hand there are indeed low copies of M_5.

Corollary 2 (Downey, Greenberg and Weber [8]). *There are c.e. degrees that are totally* ω*-c.e. and not array computable.*

Proof. Walk [29] constructed an array noncomputable degree $\mathbf{a}$ below which there was no weak critical triple. Such a degree must be totally ω-c.e.

The class of totally ω-c.e. degrees also captures other constructions.

Theorem 8 (Downey, Greenberg and Weber [8]). *A c.e. degree* $\mathbf{a}$ *is totally* ω*-c.e. iff there are c.e. sets* A*,* B *and* C *of degree* $\leq_T \mathbf{a}$*, such that*

(i) $A \equiv_T B$
(ii) $A \not\leq_T C$
(iii) For all $D \leq_{wtt} A, B$*,* $D \leq_{wtt} C$*.*

1.4 Totally $<\omega^\omega$-c.e. degrees

The class of totally $<\omega^\omega$-c.e. degrees also arises quite naturally. Recall that if b is an ordinal notation in Kleene's $\mathcal{O}$, then a Δ_2^0 function g is *b-c.e.* if there is a computable approximation $g(x,s)$ for g such that the number of changes in the guessed value is bounded by some decreasing sequence of notations below b; that is, there is a function $o(x,s)$ such that for every x and s, $o(x,s) <_{\mathcal{O}} b$, $o(x,s+1) \leq_{\mathcal{O}} o(x,s)$ and if $g(x,s+1) \neq g(x,s)$ then $o(x,s+1) <_{\mathcal{O}} o(x,s)$. The definition of the class of totally $<\omega^\omega$-c.e. degrees involves *strong notations*, being notations for ordinals in Kleene's sense, except that we ask that below the given notation, Cantor normal form can be effectively computed. Exact formalization of this notion is straightforward for the ordinals below ϵ_0; such notations are computably unique, and so the corresponding class of functions is invariant under the chosen strong notation for a given ordinal; we thus call a function α-c.e. if it is b-c.e. for some (all) strong notations b for α. To make this definition explicit, we note how the lower levels correspond to functions that are given as increasing limits. Observe the following:

- A function g is ω-c.e. if there is a computable approximation $g(x,s)$ for g such that the number of changes in the guess for $g(x)$ is given in advance, in a computable fashion.
- A function g is $\omega \cdot 2$-c.e. if there is a computable approximation $g(x,s)$ for g such that the number of changes $n(x)$ in the guess for $g(x)$ has a computable approximation that changes at most once.
- Similarly, a function is $\omega \cdot n$-c.e. if we may change our mind at most $n-1$ times about the number of possible changes.
- A function is ω^2-c.e. if is has some computable approximation such that the number of changes $n(x)$ is ω-c.e., that is, the number of times we change our mind about $n(x)$ is computably bounded.
- Similarly, a function is ω^{n+1}-c.e. if it has a computable approximation for which the number $n(x)$ of changes in the guess for $g(x)$ is ω^n-c.e. (So for example, g is ω^3-c.e. if it has an approximation where there is a computable bound on the number of times we may change our mind about the number of times we may change our mind about the number of changes of our guess for $g(x)$.)

A degree $\mathbf{a}$ is totally $<\omega^\omega$-c.e. if every $g \leq_T \mathbf{a}$ is ω^n-c.e. for some n. In [6], Downey and Greenberg introduced this notion and showed that the collection of totally $<\omega^\omega$-c.e. degrees is naturally definable:

Theorem 9 (Downey and Greenberg [6]). *A c.e. degree is totally $<\omega^\omega$-c.e. iff it does not bound a copy of M_5.*

Again, Downey and Greenberg showed that a number of other constructions gave rise to the same class.

In the present paper, we will try to lead the reader to understanding how this class arises by showing how the class relates to the class of m-topped degrees of Downey and Jocksuch [9]. Whilst we cannot get the exact classification, the analysis is revealing, as we see in section 2.

2 m-topped degrees

Recall that a c.e. degrees **a** is called m-topped if it contains a c.e. set A such that for all c.e. $W \leq_T A$, $W \leq_m A$. Of course $\mathbf{0}'$ is m-topped as it contains the halting set, but there exist incomplete m-topped degrees (Downey and Jockusch [9]); by index-set considerations, all of these are low$_2$.

We look at the Downey-Jockusch construction to try to understand what is needed to make it work. We must meet the requirements

$$R_e : \quad \Phi_e^A = W_e \to W_e \leq_m A.$$

Additionally there will be a Friedberg strategy in the background making A noncomputable and some other one making sure that A is not complete.

To meet R_e we will have nodes $\tau = \tau(e)$ on a priority tree devoted to measuring

$$\ell(\tau, s) \;=\; \max\{\, x \,:\, \Phi_e^A \restriction x = W_e \restriction x \quad [s]\,\}.$$

The idea is crude and simple. For a given z, at the first suitable τ-expansionary stages s_z where $\ell(\tau, s_z) > z$, if z is not yet in W_{e,s_z}, we will take a fresh number $y > s_z$ and define $f(\tau, z) = y$. The promise is that if z enters W_e after stage s_z, then we will put $f(\tau, z)$ into A. Notice that A is controlling W_e and hence such a situation won't occur unless we change $A_{s_z} \restriction \varphi_e(z, s_z)$.

Now suppose that we are trying to carry out this construction below a a given degree **b** represented by a c.e. set B. We look at this in the single requirement scenario. The action would occur as follows.

At some stage s_0 we would initiate something by enumerating some number p into A_{s_0}. By that stage, τ will have already defined $f(\tau, z)$ for all $z \leq n$ for some n. By the next τ-stage we see, s_1, perhaps some $z_1 \leq n$ entered W_{e,s_1}, causing us to now enumerate $f(\tau, z_1)$ into A_{s_1}. In turn, this number might be below the use $\varphi_e(z_i, s_1)$ of other z_i's at stage s_1, and hence this process could snowball so that it re-occurs many times before all pending coding actions are finished. It will finish since we won't define new $f(\tau, z')$ until we have a τ-expansionary stage s where there are no pending coding actions to be done.

The point is that *each* enumeration of some $f(\tau, z_i)$ really needs some B-permission. Thus the sequence we have began at stage s_0 could actually need a sequence of more or less (s_0) many B-permissions to be achieved.

Indeed, things are even worse when many requirements are considered. For example, if we consider two τ's, say τ_1, τ_2, each building their own $f(\tau_i, z)$'s, then assuming that τ_2 has weaker priority than τ_1, τ_1 could recover many times before we see any τ_2-expansionary stages. At each τ_1 expansionary stage, we would fulfill its obligation to enumerate $f(\tau_1, z)$ into A. Now, τ_1 cannot know if τ_2 will ever recover, so that before we did any enumeration of numbers like $f(\tau_2, z')$ we might have defined many *new* $f(\tau_1, \hat{z})$ where $\hat{z} > s_0$. Now the pending action at τ_2 of enumerating some $f(\tau_2, z')$ into A will likely cause new changes in W_{e_1} and hence yet further enumeration into A for the sake of τ_1. This process could repeat again more or less s_0 many times.

In summary, one R_e would seem to require $f(j)$ many permissions for some computable function f, for attack number j, and two requirements would seem to need $\sum_{i\leq f(j)} g(i,j)$ many permissions for some computable $g(i,j)$. Thus in some relatively natural way this construction would seem to need at least "ω^ω many permissions" to be carried out.

Now the construction of an m-topped degree *also* seems to need more in that once we have begun some action we must finish it. There is no way to allow us to "lose" on some $f(\tau, z)$. In the embeddin of M_5, we can think of the R_e's above as belonging to some gate of a pinball construction measuring some minimal pair requirement

$$\Phi^{A_k} = \Phi^{A_l} = h \to h \leq Q.$$

Here we will assume that the reader if familiar with the construction of a 1-3-1 using a pinball machine as in Downey and Shore [15].

The analogous action is that we have some sequence of numbers x, $T(x,s)$, $T^2(x,s)\ldots$ that have been realized and are traveling down the machine towards the pockets. They are a number x with its trace $T(x,s)$, etc. This can't pass the gate if they are a k,l sequence. For example, $k=1, l=2$ and x is targeted for A_2, $T(x,s)$ for A_1, $T^2(x,s)$ for A_2 etc. They must pass one at a time. We put the last one $p = T^n(x,s)$ (targeted, say, for A_1) out at the gate, and give it a trace $T^{n+1}(x,s)$ targeted for A_3 and so forth as a 1-3 sequence at the gate. When the gate opens at the next expansionary stage, we would drop the balls to the first unoccupied 1-3 gate and repeat.

To achieve this, we would need to repeat this n many times one per ball at gate G_e alone. For two gates, the situation is like the above, each ball from the first gate itself generates a long 1-3 entourage, and hence needs $g(i,j)$ many permissions for each descendent.

The critical difference between the situation for the M_5 lattice and the m-topped degree, is that if some set of balls is stuck forever at some gate then that causes no real grief. However, in the m-topped case, the failure of us fulfilling some $f(\tau, z)$ commitment is fatal. The issue seems to concern lowness; this is why we can't get a true reversal for the class of m-topped degrees:

Theorem 10. *There is a degree that is not totally $<\omega^\omega$-c.e., but does not bound any noncomputable m-topped degree.*

Proof. Downey and Jockusch [9] proved that no noncomputable m-topped c.e. degree is low. On the other hand, even Lachlan's original construction can be shown to produce a low degree that is the top of an embedding of M_5. By [7] mentioned above, such a degree cannot be totally $<\omega^\omega$-c.e. Of course, the low degrees form an initial segment of the c.e. degrees.

But in the present paper we will prove that the analysis above works in one direction:

Theorem 11. *No totally $<\omega^\omega$-c.e. degree bounds a noncomputable, m-topped degree.*

On the other hand, it is possible to carry out the construction of an m-topped degree at a relatively low level. The reason for the interest in the next result is that the m-topped construction was a natural candidate for a construction that needed the "full power" of non-low$_2$ permitting. The reason for this is that Downey and Shore [14] proved that a c.e. degree $\mathbf{a}$ is low$_2$ iff it is bounded by an incomplete m-topped degree. The following theorem shows that Theorem 11 is optimal in the hierarchy of totally $< \alpha$-c.e. degrees; the next level above $< \omega^\omega$ is the class of totally ω^ω-c.e. degrees, the degrees that only compute ω^ω-c.e. functions.

Theorem 12. *There exists a m-topped c.e. degree that is totally ω^ω-c.e.*

3 Proof of Theorem 11

We sketch the proof of Theorem 11. As the class of totally $< \omega^\omega$-c.e. degrees is closed downwards, it is sufficient to show that no totally $< \omega^\omega$-c.e. degree is m-topped.

For a simplified start, suppose first that the given degree $\mathbf{a}$ is totally ω-c.e.; let $A \in \mathbf{a}$ be a candidate for having a maximal c.e. m-degree inside $\mathbf{a}$. Out goal is to build a c.e. set $V \leq_T A$ via $\Psi^A = V$ such that we meet the requirement

$$M_e : \ V \not\leq_m A \text{ via } \varphi_e.$$

That is, for some x, we would have $x \in V$ iff $\varphi_e(x) \notin A$ (or φ_e is not total.)

As with all these constructions, we will build an auxiliary function $\Delta^A = g$. Now suppose that we knew in advance the witness to the the fact that g is ω-c.e. That is we had in advance a computable function f so that g is ω-c.e. via some approximation $h(x, s)$, where the number of changes is bounded by $f(x)$.

We could then proceed as follows.

We choose some "permitting number" n, and a finite set X of size greater than $f(n)$, consisting of fresh potential diagonalisation witnesses. We wait until every $x \in X$ is *realised*, that is, $\varphi_e(x)\downarrow$; we then let $u = \max\{\varphi_e(x) : x \in X\}$, and define $\psi^A(x) = u$ for all $x \in X$ and $\delta^A(n) = u$ as well. [Strictly speaking, we need to define both $\delta(n)$ and $\psi(x)$ before the realisation, because the totality of Δ^A and Ψ^A cannot depend on φ_e being total; for this we use simple permitting.]

We are then ready to *attack* with some $x \in X$ (it doesn't matter which): we enumerate x into V. If we are unlucky, then at a later stage t_0 the attack *fails*: $\varphi_e(x)$ enters A. The way we defined $\delta(n)$ allows us to extract a price from A in exchange for this failure: since $\delta(n) \geq \varphi_e(x)$, we know that the failure of the attack allows us to redefine $\Delta^A(n)$ with new value that hasn't been guessed before as some $h(n, s)$. At a later stage s_0 we get a new guess $h(n, s_0) = \Delta^A(n)[t_0]$, and then we can attack with another $x' \in X$. Now note that we do not want to attack again before we get a change in $h(n, s)$, because the limit we have on the number of changes is used to show that some attack is eventually successful. Note that the reduction $\Psi^A = V$ is not damaged here: we defined $\psi(x') \geq \varphi_e(x)$, and so at stage t_0, $\psi(x') \uparrow$; at that stage we can define $\Psi(x') = 1$ with anticipation of stage s_0.

This plan succeeds because:

(i) h is indeed a correct approximation for g, and so every failure is followed by another attack; every stage t_0 as above is followed by some s_0. It follows also that the definition $\Psi(x') = 1$ made at stage t_0 is correct, and so indeed $V \leq_T A$.

(ii) Some attack must succeed, because $h(n, s)$ cannot change more than $f(n)$ many times. Hence M_e is met.

In the real construction, we don't know h and f in advance, so we list out all possibilities. We would use one V for each possible pair f, h. The point here is that *if* f is the real f, and h is the *real* witness for g, *then* the $V_{h,f}$ built for h and f will have $V_{h,f} \leq_T A$. *But* the key point is that g is total nevertheless - we never leave $\delta(n)$ undefined.

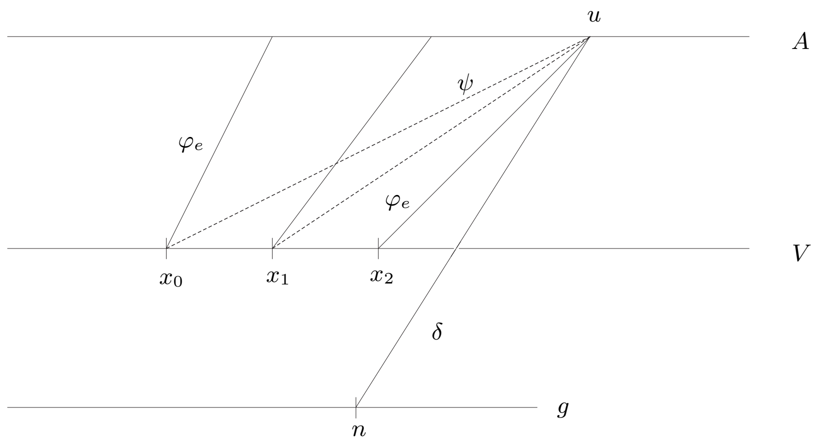

Fig. 2. The ω-c.e. construction

Now consider the case that A is totally ω^2-c.e. To continue the analogy above with the gates of a pinball machine construction, we see that the ω-c.e. case corresponds exactly to the failure of a single node τ to meet R_e. A set that has totally ω^2-c.e. degree may be able to win on all single gates alone, but fails to meet the combined requirements R_{e_0} and R_{e_1}. The analogy suggests that we need to build *two* sets V_{e_0} and V_{e_1} and succeed on one of them. We now describe how the necessities of the construction lead us to require these two sets.

Again we construct an auxiliary function $g = \Delta^A$ and guess at an approximation $h(x, s)$ for g, which is accompanied by a bounding function $o(x, s)$ which gives, for every x, a non-increasing sequence of ordinals below ω^2; every change in the guess for $g(x)$ is matched by a change in $o(x, s)$. Consider the following naïve plan. At first, we get $o(x, 0) = \omega \cdot k_0 + k_1$; we set up a collection of potential witnesses X of size greater than k_1 and repeatedly attack with these witnesses as before. Since each attack is related to a decrease of $o(x, s)$, before we run out, we have a new value $\omega \cdot l_0 + l_1$ where $l_0 < l_1$, so we'd like to have at least l_1 more new witnesses to throw into X. Of course this cannot work as it would translate to an argument that no totally b-c.e. degree (for any notation b) can be m-topped, contradicting Theorem 12. [For the naïve plan to "work" we need to work with some b, otherwise we may repeat the process infinitely many times and $\Delta^A(n)$ would be undefined.] The problem is one of timing: before X runs out, we can appoint l_1 new witnesses; but at some point we need to wait for them to get realised. This has to happen before some stage as s_0 above where we can redefine $\delta(n)$ to be at least $\varphi_e(x)$ for all *new* witnesses x. This means that before, say, we make the last attack with an old witness, we first need to wait for realisation of the new witnesses. But if φ_e is not total then this may never happen, and this spoils the reduction $\Psi^A = V$. Here the nonuniformity, familiar from these constructions, creeps in: the solution is to build a backup set V_e that is only needed if we fail to meet M_e for the main set V. All work regarding V_e (including the definition of a reduction $\Psi_e^A = V_e$) is based on the assumption that φ_e is total. Thus, when we run out of old witnesses, we appoint new witnesses, wait for realisation, and then attack with a V_e-witness; when this attack fails, $\delta(n)$ frees up and we can redefine it as larger as is required to start working with the new witnesses.

Here's the construction for the ω^2-case, assuming that we guessed h and o correctly. We show how to meet the requirement

$M_{e,j}$: Either M_e holds, or V_e is not 1-1 reducible to A via φ_j.

The algorithm is as follows:

1. Appoint a permitting number n. Let $o(n, 0) = \omega \cdot k_0 + k_1$. Appoint a set of witnesses Y, targeted for V_e, of size greater than k_0. Wait for realisation, i.e. for $\varphi_j(y)\downarrow$ for all $y \in Y$.
2. Let $u_Y = \max\{\varphi_j(y) : y \in Y\}$; let $\psi_e(y) = u_Y$ for all $y \in Y$ and let $\delta(n) = u_Y$ as well. Appoint a set of witnesses X, targeted for V, of size greater than k_1. Wait for realisation, i.e. for $\varphi_e(x)\downarrow$ for all $x \in X$. [In the meanwhile we can define $\psi(x) = u_Y$ for all $x \in X$.]
3. Attack with some $y \in Y$: enumerate it into V_e. Wait for the failure of the attack, i.e. for $\varphi_j(y)$ to enter A.
4. Let $u_X = \max\{\varphi_e(x) : x \in X\}$ and $u = \max\{u_X, u_Y\}$. Redefine $\delta(n) = u$ and $\psi(x) = u$ for $x \in X$ and $\psi_e(y) = u$ for $y \in Y$. However, reserve one $x \in X$ for attack, and wait for a new guess $h(n, s) = \Delta^A(n)$.
5. Attack with x: enumerate it into V. Wait for the failure of this attack, i.e. for $\varphi_e(x)$ to enter A; repeat as in the ω-case, until X runs out.
6. Upon the failure of the attack of the last $x \in X$, we have a new number l_1 as above; we appoint a new X with more than l_1 many fresh witnesses; we let $\delta(n) = u_Y = \psi(x)$ for $x \in X$. We wait for realisation of all $x \in X$.

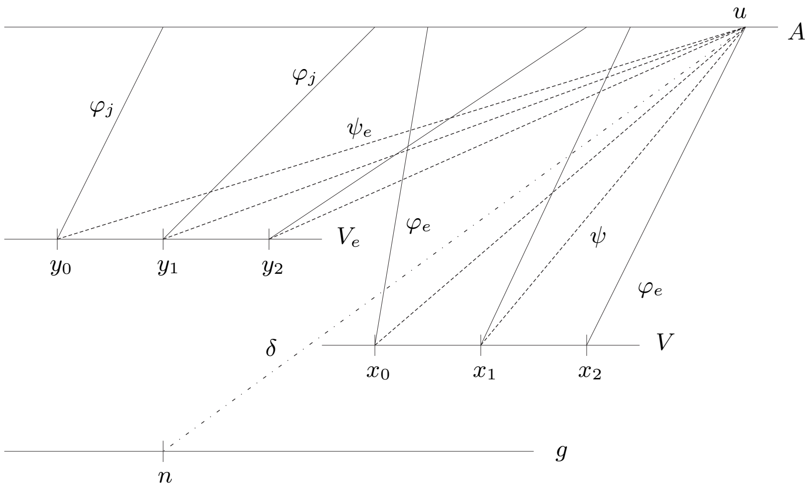

Fig. 3. The ω^2-c.e. construction

7. We then attack with another $y \in Y$; repeat as in step (3) and onwards.

For ω^n, we need n sets $V, V_{e_0}, V_{e_0,e_1}, \ldots$, nested by layers of nonuniformity; the idea is the same. In the more complex case of $\Delta^B = g$ being only $< \omega^\omega$-c.e., we must guess for which n it is ω^n-c.e., and the witnesses for this. Again this is typical.

4 Proof of Theorem 12

In this last section we sketch the proof of Theorem 12. In fact this comes from analyzing the complexity of the natural construction of Downey and Jockusch [9], carefully controlling where numbers are assigned as followers.

We enumerate a set A. We meet the following requirements:

P_e**:** φ_e is not the characteristic function of A.
R_e**:** If $\Phi_e(A) = W_e$ then $W_e \leq_1 A$.
Q_e**:** If $\Psi_e(A)$ is total then it is ω^ω-c.e.

Here $\langle \Psi_e \rangle$ is a listing of all Turing functionals, and $\langle \varphi_e \rangle$ is a listing of all partial computable functions. $\langle (\Phi_e, W_e) \rangle$ is a listing of all pairs of Turing functionals and c.e. sets. Note that the fact that A is totally ω^ω-c.e. guarantees its incompleteness; in fact, it guarantees that it is low$_2$. Recall that the key is that (in the setting of R_e) $\Phi_e(A)$ controls W_e in the sense that if at some stage t we have $\Phi_e(A, x) = 0 = W_e(x)$ $[t]$ and $s > t$, then if A did not change below the use $\phi_e(A, x)[t]$ between t and s then $x \notin W_e$ $[s]$.

The construction is done on a tree of strategies, with every level working for a single requirement. The tree of strategies is $2^{<\omega}$ (and as usual, the priority ordering is the lexicographic one); we identify 0 with the infinite outcome for a node working for R_e or Q_e, and with the positive satisfaction for P_e. Recall that a node σ that works for R_e builds a recursive function f_σ that attempts to be a one-one reduction of $\Phi_e(A)$ to A.

At stage s, we construct (by induction) the path of accessible nodes; for an accessible node σ, we describe σ's action and which successor (if any) is next accessible.

σ works for P_e: If σ has no follower, appoint a fresh follower x; let $\sigma^\frown 1$ be accessible. If σ has a follower x then there are three cases:

1. If we already have $x \in A$ (so P_e is satisfied) we let $\sigma^\frown 0$ be accessible.
2. If it is not the case that $\varphi_e(x) \downarrow= 0$, then let $\sigma^\frown 1$ be accessible.
3. If $\varphi_e(x) \downarrow= 0$, but x has not yet been enumerated into A, then we enumerate x into A and let $\sigma^\frown 0$ be accessible.

σ works for R_e: First, we need to correct potential inconsistencies between $W_e = \Phi_e(A)$ and $f_\sigma^{-1}A$. For every x such that $f_\sigma(x)$ is defined, $x \in W_e\,[s]$ and $f_\sigma(x) \notin A\,[s]$, enumerate $f_\sigma(x)$ into A.

If some numbers were enumerated into A, then we end the stage; we initialise all nodes that lie to the right of $\sigma^\frown 0$, but not nodes that extend $\sigma^\frown 0$.

Assuming that we did not end the stage, we define $\ell(\sigma)[s]$ be the length of agreement between $\Phi_e(A)$ and W_e. Let $t < s$ be the last stage at which $\sigma^\frown 0$ was accessible ($t = 0$ if no such stage exists); if $\ell(\sigma)[s] > t$ then let $\sigma^\frown 0$ be accessible, otherwise let $\sigma^\frown 1$ be accessible. In the first case, for every $x < \ell(\sigma)[s]$ for which $f_\sigma(x)$ is not yet defined, define it (with fresh value).

σ works for Q_e: Let $t < s$ be the last stage at which $\sigma^\frown 0$ was accessible. If $\operatorname{dom} \Psi_e(A)[s] > t$ then let $\sigma^\frown 0$ be accessible; otherwise let $\sigma^\frown 1$ be accessible.

If the stage was not halted by the time we got to a node of length s, we end the stage and initialise all nodes that are weaker than the last accessible node. [This means that all followers are cancelled, and all functions are restarted from scratch.]

Verification. The existence of a true path is standard. On the true path, each node is eventually never initialised. The point here is that if σ works for R_e and $\sigma^\frown 1$ is on the true path then only finitely many values $f_\sigma(x)$ are defined, so after some stage, σ does not halt the stage (and initialise $\sigma^\frown 1$) because it enumerates some $f_\sigma(x)$ into A. It follows that every P_e requirement is met. It is also easy to see that each R_e requirement is met: if σ on the true path works for R_e (and the hypothesis $\Phi_e(A) = W_e$ holds), then for every x, $f_\sigma(x)$ is eventually defined, and enumerated into A iff x enters W_e; this is because $\sigma^\frown 0$ is accessible infinitely many times.

It thus remains to show that each Q_e is met; fix $e < \omega$, assume that $Z = \Psi_e(A)$ is total, and let σ be the node on the true path that works for Q_e; we know that the next node on the true path must be $\sigma^\frown 0$. Let r^* be a stage after which σ is never initialised.

Let $d < \omega$; we describe how to approximate $Z(d)$ in an ω^ω-c.e. fashion. The approximation itself is simple: at a stage $s > r^*$ at which $\sigma^\frown 0$ is accessible and $d < \operatorname{dom} \Psi_e(A)\,[s]$, we guess that $Z(d) = \Psi_e(A, d)\,[s]$. The point is of course to find the ordinal bound on the number of possible injuries to these computations. Of course such a computation can only be injured by nodes that are compatible with $\sigma^\frown 0$.

Recall that the key to this construction (as is in Fejer's branching degree construction, Slaman's density and other constructions) is the establishment of natural barriers and the preservation of the sets below these barriers by a concert of all agents involved. Let s_0 be a stage at which $\sigma^\frown 0$ is accessible and such that $d < \operatorname{dom} \Psi_e(A)\,[s_0]$. Suppose, for example, that at stage s_0, there is only one node τ compatible with $\sigma^\frown 0$ which works for some $R_{e'}$ and has any value $f_\sigma(x)$ defined (say $\sigma^\frown 0 \subseteq \tau$) and that there is only one node $\rho \supseteq \tau^\frown 0$ that works for some $P_{e''}$ and has a follower y defined at s_0. Until the computation $\Psi_e(A, d)[s_0]$ is injured (possibly at stage s_0, but possibly later), all new values $f_\tau(x)$ and followers y appointed for any node are greater than the use $\psi_e(d)[s_0]$, and so the injury has to result from action by either τ or ρ. To begin with, some such injuries can happen by τ enumerating values $f_\tau(x)$ into A; the number of such injuries is bounded by s_0. After each such injury, nodes to the right of $\tau^\frown 0$ are initialised, and nodes extending τ are not accessible, so the next injury still must come from τ or ρ. Eventually, new values $f_\tau(x)$ are defined at a stage s_1 at which $\tau^\frown 0$ is accessible. The barrier mechanism now comes into place: these values are defined only for $x < \ell(\tau)[s_1]$, and the only node that has a number smaller than the $\Phi_{e'}$-use is ρ. By $\Phi_{e'}(A)$'s controlling of $W_{e'}$, no $x < \ell(\tau)[s_1]$ will enter $W_{e'}$ (and no $f_\tau(x)$ will enter A) unless ρ acts at some $s_2 \geq s_1$. At that stage some new cascading by τ may begin, yielding at most s_2-many new injuries for $\Psi_e(A, d)$. Thus the approximation for $Z(d)$ is $\omega \cdot 2$-c.e. If there are further P-nodes ρ then we get $\omega \cdot n$-c.e.

However, if we have two R-nodes τ_0 and τ_1 (say $\tau_0 \subset \tau_1$), then the effect is multiplicative (in a reverse fashion). After each time τ_1 enumerates a number into A, the total τ_0, τ_1-equilibrium is damaged and a barrage of new numbers can be enumerated into A by τ_0. The result (together with several P-nodes weaker than τ_1) is an $\omega^2 \cdot n$-c.e. approximation. As d increases, more and more nodes τ have values $f_\tau(x)$ defined when $\Psi_e(A, d)$ is first encountered, which means that we get ω^k-c.e. approximations where $k \to \infty$. Overall, we get an ω^ω-approximation for Z.

Here is the formal argument.

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