THE GEOMETRY OF COMPUTABLE BANACH SPACES

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Abstract. We investigate the complexity of a computable Banach space having a Schauder basis, and of related properties, such as the approximation property, and having a local basis structure.

1. Introduction

Vladimir V'yugin made extensive contributions to computability theory, an area concerned with the algorithmic content of mathematics. It asks what processes and theorems can be performed on an idealised machine, and if they cannot, how complex are they? This paper lies under the umbrella of *computable analysis*, which applies this critique to continuous processes. Computable analysis goes back to the dawn of modern studies in computability via Turing [Tur36, Tur37] and is an area of significant current interest. V'yugin contributed to the computability of analytic structures (for example, [V'y97, V'y98]), so we feel that the paper is appropriate for this special issue.

Our focus in this paper is the geometry of computable Banach spaces, in particular, questions concerning bases. Answering a long-standing question of Banach himself, Enflo [Enf73] constructed a separable Banach space without a basis. In his Ph.D. thesis, Bosserhof ([Bos08], see also [Bos09]) showed that Enflo's example has a computable copy. This raises the natural question: how hard is it to tell whether a given Banach space has a basis? A closely related question is: assuming a given Banach space has a basis, what does it take to build one? Can it be done inductively, step by step, as in the familiar construction of bases of vector spaces? Similar questions were answered in the setting of discrete structures, for example for countable vector spaces [MN77a], countable free abelian groups [DM13], and uncountable such groups [GTW18].

The question is much harder in the context of Banach spaces. Indeed, we will present evidence that standard techniques in Banach space theory are not sufficient for answering this question. This is because the common constructions of Banach spaces with no bases produce spaces in which properties weaker than having a basis fail, for example, the bounded approximation property, or having a local basis structure. These properties are known to be simpler than the expected complexity of having a basis. Thus, new techniques need to be developed.

This paper is a sequel of the paper [DGQ24], which was the basis for Downey's invited lecture for his S.B. Cooper Prize. That paper gives a overview of current knowledge, and lists some open questions. In the current paper we will give some full proofs as well as new results. As we shall see below, the theory of Banach spaces provides a number of genuine challenges to someone wishing to study the effective

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content of mathematical structures. We hope that together, the current paper and [DGQ24] can serve as the basis for future studies. For more on computable analysis, the classic texts are [PER89] and [Wei00]. For more recent treatments, that concentrate on computability of metric structures, see, for example, [DM25], and the upcoming [FGM].

1.1. **Computable Banach spaces.** Recall that a normed vector space has associated with it a distance $d(x, y) = \|x - y\|$. If this is a complete metric space, it is called a Banach space. For simplicity, we will consider Banach spaces over $\mathbb{F} = \mathbb{R}$, but most results work just as well for complex spaces. Banach spaces are fundamental to the field of functional analysis, and have extensive applications particularly in physics. The study of the pure theory of Banach spaces in their own right began with the work of Banach, Haar, Schauder, and others in the early 20th century (See e.g. Banach [Ban32]). The classical theory has proven very challenging with many open questions apparently far from resolution, some of which are pertinent to the present paper (see e.g. Megginson [Meg98]). Banach spaces provide a rich resource for workers in descriptive set theory (e.g. Ferenczi, Louveau and Rosendal [FLR09]).

We wish to understand and classify algorithmic properties of Banach spaces, and hence will be using the lens of computability theory for our studies. The modern theory of computable Banach spaces, the effective content the theory, likely began with the work of Pour-El and Richards [PER83], who showed how the effective theory gave insight into issues from classical physics. They motivated their studies as follows:

> "Which processes in analysis and physics preserve computability, and which do not?... Among the processes of physics we expect to include are those associated with the wave equation, the heat equation, Laplace's equation and many others. Among the processes of analysis we expect to consider are Fourier series, Fourier transform and others... Hence we will be concerned with linear operators on Banach spaces of functions. They will be our 'processes'." [PER83, p.77]

Metakides-Nerode-Shore [MNS85] and others studied the computable content of the Hahn-Banach theorem. For example, they showed that the Theorem is computably true for finite-dimensional Banach spaces, but not uniformly so. The infinite dimensional Hahn-Banach theorem fails to be effective in general. Brattka [Bra16] further studied the Hahn-Banach theorem, as well as the open mapping theorem, the closed graph theorem, and the Banach-Steinhaus theorem.

In this paper we will highlight some recent work concerning the algorithmic content of work around the geometry and topology of Banach spaces, specifically those associated with bases, and decompositions.

We remark that the questions provide a fascinating "logician's eye view" of classical constructions, in that it seems that *all* of the classical constructions are insufficient to answer some of the basic questions such as the complexity of finding a Schauder basis.

Basic definitions. For the definition, recall that a *computable metric space* is a complete metric space (M, d) equipped with a dense sequence $\bar{x} = (x_i)_{i \in \mathbb{N}}$ of points, on which the metric is uniformly computable. This uses Turing's definition of a computable real number r, as one that is the limit of a computable *fast Cauchy sequence* — a computable sequence (q_n) of rationals satisfying $|q_n - r| \leqslant 2^{-n}$. Such a sequence is also called a *Cauchy name* of r (this naming scheme is called the *Cauchy representation* of $\mathbb{R}$). This notion of representation can be extended to any computable metric space, replacing rational numbers by the dense sequence $\bar{x}$ specifying the computable structure of the space (the points x_i are sometimes called "special points"). Thus, a name of a point y in the space $(M, d, \bar{x})$ is a sequence $(i_n) \in \mathbb{N}^{\mathbb{N}}$ such that for all n, $d(y, x_{i_n}) \leqslant 2^{-n}$. A point is computable if it has a computable name.

For computable *functions* between computable spaces, we use the "type 2" definition, that requires computability (rather than constructivist approaches that focus on functions only defined on the computable points). A subset of a computable metric space $(M, d, \bar{x})$ is *effectively open* if it is the union of a c.e. collection of open balls whose centre is a special point and whose radius is rational. That is, a set of the form $\bigcup\{B(x_i, q) : (i, q) \in W\}$ for some c.e. set $W \subseteq \mathbb{N} \times \mathbb{Q}^+$. A function $f\colon X \to Y$ between computable metric spaces is computable if the pull-backs $f^{-1}[W]$ of effectively open subsets $W \subseteq Y$ are effectively open in X, uniformly. An equivalent characterisation of computable functions is: uniformly, given a name $\bar{q}$ of a point $x \in X$, we can compute a name $\bar{p}$ of $f(x)$. Since the effectively open sets form a basis for the topology of a computable metric space, every computable function is continuous. Indeed, a function between computable metric spaces is continuous if and only if it is computable relative to some oracle.

With the notion of computable functions on computable metric spaces, we can endow computable metric spaces with extra structure, making them *computable metric structures*. For our purpose:

Definition 1.1. A *computable Banach space* is a Banach space $X = (X, \|\cdot\|, +, \cdot)$ endowed with a dense sequence of points $\bar{x}$ which makes X a computable metric space using the metric determined by the norm, for which the norm, vector addition, and scalar multiplication, are all computable functions.

Note that we are only considering complete spaces. Some authors have extended the theory to incomplete computable normed spaces; we will not discuss there here. For basic properties and equivalent characterisation of computable Banach spaces see [Bra16] and [DM25, §2.4.3].

Remark 1.2. In [DGQ24, §1.3] we mention the notion of a *generalised computable Banach space*, due to Brattka [Bra16]. The issue is that a computable Banach space is necessarily separable, however many interesting Banach spaces, such as ℓ_∞ and many other dual spaces, are not separable, and so their effective content remains to be explored. In the classical literature on Banach spaces, duality is the principal tool. As a consequence of non-seperability, this technique is not available when we study computable Banach spaces, and so alternative arguments must be found. The concept of a generalised computable space is an attempt to overcome this problem. In such a space, the computable metric structure and norm are replaced by an effective convergence condition. This concept hasn't been thoroughly explored.

1.2. **Bases, complexity, and index sets.** As mentioned above, a main motivating question is: "how hard is it to tell whether a given Banach space has a basis?" Here we use the following definition, which is a generalisation of the notion of an orthonormal basis in a Hilbert space.

Definition 1.3 (Schauder [Sch28])**.** Let X be a Banach space. A sequence $\bar{x} = (x_i) \in X^{\mathbb{N}}$ is a *Schauder basis* of X if for all $x \in X$ there is a unique sequence of coefficients $(a_i) \in \mathbb{R}^{\mathbb{N}}$ such that

$$\sum_{i=0}^{\infty} a_i x_i = x.$$

We emphasise that a Schauder basis is a *sequence*: order counts, as convergence may be conditional. This notion of basis is preferred to that of a *Hamel basis*, for which we only take finite linear combinations. A Hamel basis ignores the metric structure of the space. Further, outside the finite-dimensional case, Hamel bases are uncountable, and not Borel. Below we refer to a Schauder basis simply as a basis. Examples are the standard unit vectors in the sequence spaces ℓ_p, and Haar systems [Haa10] in the spaces $L_p(0,1)$.

A Banach space with a basis must be separable, as the collection of finite linear combinations of basis elements is dense in the space. As discussed, Banach asked in [Ban32] whether every separable Banach space has a basis. It was only after 40 years that Banach's question was solved, in the negative, by Per Enflo [Enf73]. Thus, the motivating question above is non-trivial.

How is the question formalised? In this paper we will use the point of view of computability theory, and so investigate the complexity of the *index set* of computable Banach spaces with bases; this complexity will be measured with respect to the arithmetic and analytic hierarchies. To motivate these notions, for readers who are not familiar with these concepts, we will first discuss how the "complexity of having a basis" question is formalised in descriptive set theory. The computable tools that we shall use are, in a fundamental way, a refinement of the following.

Polish spaces, and the Borel and projective hierarchies. Descriptive set theory provides tools for measuring the complexity of subsets of Polish spaces, which are separable and completely metrizable topological spaces. To apply these tools to the motivating question, we start with a space of separable Banach spaces. That is, a Polish space B, in which every point codes a space in some natural way, so that every separable Banach space is coded (up to isomorphism). One way is to take a universal separable Banach space (say $C[0,1]$), and use the collection of all of its closed subspaces, equipped with the Effros topology (see [Kec95, Thm. 12.6]). Using this space, we can examine the subset consisting of the points that code spaces that have bases.

Complexity of subsets of Polish spaces is measured using the *Borel* and the *projective* hierarchies. Recall that the Borel sets of a space are those in the σ-algebra generated by the open sets. The Borel hierarchy measures complexity of Borel sets by how many operations of countable unions or intersections are needed to obtain the set if we start with the open and closed sets. Thus, $\mathbf{\Sigma}^0_1$ denotes the collection of open sets, $\mathbf{\Pi}^0_1$ the closed sets, $\mathbf{\Sigma}^0_2$ the unions of countably many closed sets (also known as F_σ sets), $\mathbf{\Pi}^0_2$ the intersections of countably many open sets (the G_δ), $\mathbf{\Sigma}^0_3$ the unions of countably many $\mathbf{\Pi}^0_2$ sets, and so on. Beyond the Borel, the class $\mathbf{\Sigma}^1_1$ denotes the continuous images of Borel sets (also called *analytic* sets), $\mathbf{\Pi}^1_1$ their complements, $\mathbf{\Sigma}^1_2$ the continuous images of $\mathbf{\Pi}^1_1$ sets, and so on. These classes are semi well-ordered by inclusion: $\mathbf{\Sigma}^0_n \cup \mathbf{\Pi}^0_n \subseteq \mathbf{\Sigma}^0_{n+1} \cap \mathbf{\Pi}^0_{n+1}$, and all Borel sets are $\mathbf{\Sigma}^1_1$.

To measure the complexity of a set A, we try to find the simplest (=smallest) class $\mathbf{\Gamma}$ that it belongs to. This involves two calculations. An upper bound calculation is an argument showing that A belongs to a class $\mathbf{\Gamma}$. A lower bound calculation is an argument showing that A cannot belong to a class strictly contained in $\mathbf{\Gamma}$. The latter is done by showing that A is *hard* for $\mathbf{\Gamma}$, which means that every $B \in \mathbf{\Gamma}$ is continuously *reducible* to A, meaning that B is a continuous pre-image of A: $B = f^{-1}[A]$ for some continuous function f, called the *reduction*. If A is $\mathbf{\Gamma}$-hard then it is not an element of the dual class of complements of sets in $\mathbf{\Gamma}$, and so is not in any smaller class. This is because the classes we are using are *non-self-dual* (are not equal to their dual), and have universal sets. In the happy case that A is both in $\mathbf{\Gamma}$ and is $\mathbf{\Gamma}$-hard, then we say that A is $\mathbf{\Gamma}$-*complete*; this completely determines the complexity of A.

Index sets, and the arithmetic and analytic hirerarchies. As mentioned above, Bosserhof [Bos08, Bos09] showed that Enflo's example has a computable copy. This enables us to apply techniques of computability theory to formalise the question of complexity of having a basis. The key notion here is of an *index set*. This notion goes back to the dawn of computability theory. Fixing some acceptable numbering (φ_e) of the partial computable functions, an index set is simply a set of indices of partial computable functions that depends only on the function and not on the choice of algorithm for computing the function (coded by the index). In other words, a set $I \subseteq \mathbb{N}$ such that for all i and j, if $\varphi_i = \varphi_j$ then $i \in I \iff j \in I$. For example, the totality set $\{e : \operatorname{dom} \varphi_e = \mathbb{N}\}$ is an index set. For the purposes of our question, we use indices of partial computable functions to code computable structures, and so obtain an effective list (M_e) of all partial metric structures in the signature of Banach spaces. We can then ask about the complexity of the index set

$$\texttt{Basis} = \{e : M_e \text{ is a total Banach space which has a basis}\}.$$

The complexity of subsets of $\mathbb{N}$, such as index sets, is measured using the *(hyper)arithmetic* and *analytic* hierarchies. These are effective analogues of the Borel and projective hierarchies. We work in the setting of computable metric spaces, and use "lightface" notation for the computable parts of the classes introduced above. So Σ^0_1 denotes the collection of *effectively* open sets, defined above; and Π^0_1 denotes the collection of their complements, the effectively closed sets. Then, Σ^0_2 denotes the collection of *effective* unions of Π^0_1 sets; these are unions $\bigcup_n P_n$ where (P_n) are *uniformly* Π^0_1, meaning $\bigoplus_n P_n = \{(n, a) : a \in P_n\}$ is a Π^0_1 subset of the product space. Using these ideas we can then define Σ^0_n and Π^0_n sets for all n. We can then define the class Σ^1_1 of images of arithmetic (equivalently, Π^0_1) sets under *computable* functions between spaces, and Π^1_1 their complements. Another way of characterising these hierarchies is by definability in the two-sorted structure $(\mathbb{N}, \mathbb{N}^{\mathbb{N}})$ for second-order arithmetic. An existential number quantifier corresponds to effective countable unions, and so the Σ^0_n sets are those that are defined by formulas in the language of arithmetic of the form $\exists x_1 \forall x_2 \exists x_3 \cdots Q x_n\, \varphi$, where φ contains only bounded quantifiers, and x_i are number variables; a Σ^1_1 set is one defined by a formula $\exists f \forall x\, \varphi$ where f is a function quantifier. Analogously with the pointclasses of descriptive set theory, the "lighface" classes Σ^0_n and Σ^1_n are closed under taking *computable* pre-images, i.e., under computable reductions. The notions of Γ-hardness and Γ-completeness are defined using computable reductions; as in the "boldface" case above, Γ-hardness of a set A ensures that A does not belong to any class simpler than Γ.

The set of natural numbers $\mathbb{N}$, equipped with the discrete topology, is a computable metric space. Descriptive set theory cannot distinguish between subsets of $\mathbb{N}$, since they are all both closed and open. The classes in the arithmetic hierarchy, on the other hand, have interesting characterisations, when we consider subsets of $\mathbb{N}$, via Post's theorem: a subset of $\mathbb{N}$ is Σ^0_1 if and only if it is computably enumerable, and in general, Σ^0_{n+1} if and only if it is computably enumerable relative to $\varnothing^{(n)}$, the n^{th} iteration of the Turing jump.

We recall that Rice's Theorem [Ric53] states that the computable index sets are the trivial ones $\varnothing$ and $\mathbb{N}$, and in general, we will usually only consider complexities starting at the level Π^0_2, since the index set of indices e for which M_e is a total structure and is a Banach space is Π^0_2.

We remark that complexity as measured by the Borel hierarchy on the one hand, and the effective complexity of index sets on the other hand, are closely related but not always perfectly aligned, because they depend on the particular choice of coding Banach spaces by points in a space of Banach spaces, or by algorithms (indices) computing Banach spaces. For example, in the set-theoretic setting, we cannot always obtain a dense sequence from a point coding a space. In some instances, completeness results for index sets do rely on the fact that the space of indices $\mathbb{N}$ is naturally well-ordered. Thus, such results do not always translate between the settings, and the two ways of calculating complexities should be seen as complementary.

We should emphasise though that the gulf between the arithmetic classes Σ^0_n and the analytic class Σ^1_1 should be considered as vast. In particular, a Σ^1_1-hardness result for a class of structures indicates that it is not possible to give computationally useful invariants. For example, Downey and Montalbán [DM08] showed that the index set of pairs of isomorphic torsion free computable abelian groups is Σ^1_1-complete. Hence no classification of isomorphism is possible which is simpler that saying that an isomorphism exists. Similarly, Downey and Melnikov proved the following:

Theorem 1.4 ([DM23])**.** *The isomorphism problem for computable Banach spaces is Σ^1_1-complete.*

So again, there are no useful invariants that capture isomorphism of computable Banach spaces. Ferenczi, Louveau and Rosendal [FLR09] showed a similar result in the context of Borel equivalence relations.

Regarding our original question, we will show that the index set `Basis` of computable Banach space with bases is Σ^1_1, and is not simpler than Π^0_3. This leaves a large gap. As indicated above, we believe that known constructions are insufficient for closing this gap. This is because these constructions are in some sense "too strong". For example, Enflo's construction gives a space that lacks what is called the *bounded approximation property*. Every space with a basis has this property, however, we know that the reverse implication does not hold. As we will see, the index-set `BAP` of spaces with the bounded approximation property is arithmetic, indeed, it is Σ^0_4. Thus, Enflo's construction cannot be used to prove that sets more complicated than Σ^0_4 are reducible to `Basis`. Similarly, a different construction due to Szarek [Sza87] gives a space that lacks a *local basis structure*, another property weaker than having a basis, and whose index set is Σ^0_3-complete; so Szarek's construction cannot show hardness of `Basis` beyond Σ^0_3. For Σ^1_1-completeness of `Basis`, we will need a fine construction — a construction of a space that has no basis, but has the bounded approximation property, and a local basis structure, and similar properties. In turn, the complexity of having each of these properties is interesting in its own right, and in most cases is still open.

Remark 1.5. Often, a reduction that shows that a set A is hard for a class Γ, yields a function that gives a stronger result, if the image of f outside A is contained in a set B smaller than the complement of A. This shows that B is hard for the dual class $\check{\Gamma}$ (of complements of sets in Γ), but also that *any* set C with $A \subseteq C \subseteq B^{\complement}$ is hard for Γ. If D is a set complete for Γ, A and B are disjoint sets, and there is a computable function f with $f[D] \subseteq A$ and $f[D^{\complement}] \subseteq B$, we write $(\Gamma, \check{\Gamma}) \leqslant (A, B)$. For an example, see Remark 3.8 below.

1.3. **Further areas of study.** In this paper we focus on the complexity of index sets such as `Basis`, and related properties of computable Banach spaces. This is only one part of recent investigations into the structure of computable Banach spaces. Another example is the (Anderson-)Kadets' (Kadec) theorem [Kad66] which states that any two infinite dimensional separable Banach spaces are homeomorphic (as topological spaces), and hence homeomorphic to $\mathbb{R}^{\mathbb{N}}$. The result is also true for a more general class called Fréchet spaces. A recent result by Downey, Franklin and Melnikov [DFM] shows that this theorem is close to effective, in that the halting problem $\varnothing'$ is sufficiently strong to compute a homeomorphism between two infinite-dimensional computable Banach spaces. It is unknown if the oracle $\varnothing'$ can be removed, to obtain computable homeomorphisms. Other questions go beyond the class of Banach spaces, for example, to ask about complexity of isomorphisms between computable topological groups.

1.4. **Preliminaries: located and computably compact sets.** We list here some basic definitions and facts about computable metric spaces. Above, we defined the notion of an effectively open subset of a computable metric space. The complement of such a set is called *effectively closed*, or sometimes co-c.e. closed. This is equivalent to the distant function from the set being lower semi-computable. We will need a dual notion. A closed subset P of a computable metric spaces is called *located*, or *c.e. closed*, if the collection of basic open balls that intersect P is c.e.; equivalently, if the distance function $d(-, P)$ is upper semi-computable. We will use the following characterisation by Brattka and Presser, see [BP03, Cor. 3.14]:

Lemma 1.6. *Let X be a computable metric space. A closed set $P \subseteq X$ is located if and only if there is a computable sequence (x_n) of points such that $\{x_n\} \subseteq P$ and $\{x_n\}$ is dense in P.*

A consequence is that if $P \subseteq X$ is a located set, then P is a computable metric space in its own right. Further, this extends to a normed vector space structure, yielding:

Lemma 1.7. *Let X be a computable Banach space. If $\bar{x}$ is a computable sequence of points in X, then the closure of the linear span of $\bar{x}$ is a computable Banach space, uniformly given $\bar{x}$.*

We will use is the:

Lemma 1.8. *For any computable Banach space X, the closed unit ball of X is both effectively closed and located.*

The next, key notion that we will use is that of *computable compactness*. This notion has many equivalent definitions. We mention two; the following equivalent conditions are the definition of a space being computably compact.

Proposition 1.9. *The following are equivalent for a computable metric space X:*

(1) *Uniformly, given $\varepsilon > 0$, we can produce a finite collection of basic open balls, each of radius $\leqslant \varepsilon$, which together cover X.*
(2) *Uniformly, given a c.e. collection of basic open balls that cover X, we can produce a finite sub-cover.*

Computable compactness guarantees the effective analogues of classical properties of compact sets. For example:

- If X is computably compact, then any computable $f\colon X \to Y$ has a computable modulus of uniform continuity: there is a computable function $g\colon \mathbb{Q}^{>0} \to \mathbb{Q}^{>0}$ such that for all rational $\varepsilon > 0$, for all $x, z \in X$, if $d_X(x,z) < g(\varepsilon)$ then $d_Y(f(x), f(z)) < \varepsilon$.
- If X is computably compact and $f\colon X \to Y$ is computable and surjective, then Y is computably compact.
- If X is computably compact and $P \subseteq X$ is effectively closed and located, then P is computably compact. (see Brattka and Presser [BP03, Cor. 4.14]).

The most useful for us will be the following, due to Pour-El and Richards [PER83]:

Proposition 1.10. *If X is computably compact and $f\colon X \to \mathbb{R}$ is computable then* $\max \operatorname{range} f$ *is computable. This is uniform: for example, If X is computably compact, Y is a computable metric space, and $f\colon Y \times X \to \mathbb{R}$ is computable, then the function*

$$y \mapsto \max \{f(y,x) \,:\, x \in X\}$$

from Y to $\mathbb{R}$ is computable.

Of course, the proposition applies to minima as well.

Proof. This is standard; we give some details, as a reader unfamiliar with this area may find the ideas instructive. If you are only familiar with algorithms over finite objects you anticipate an algorithm which halts in finite time and gives the output. As we are in the continuous setting, our goal is to give a Cauchy name for $\max \operatorname{range} f$. That is, find better and better *approximations* to the value; equivalently, build a collection of quickly shrinking balls whose limit is the desired value.

As mentioned, since X is computably compact, f is uniformly continuous with a computable modulus of uniform continuity. Now to compute $\max \operatorname{range} f$ to within 2^{-n}, we first compute some δ sufficiently small so that $\|x - y\| < \delta$ implies $\|f(x) - f(y)\| < 2^{-n-1}$ for all $x, y \in X$. Then we find a finite cover of X by balls of radius δ (Proposition 1.9(1)). For each ball $B(x,r)$ in this cover, we compute $f(x)$ with precision 2^{-n-1}. The maximum of these values will be within 2^{-n} of $\max \operatorname{range} f$. □

Proposition 1.10 will often be utilised in conjunction with the following.

Proposition 1.11. *If X is a computable, finite-dimensional Banach space, then any bounded, effectively closed and located subset of X is computably compact. In particular, the unit ball B_X of X is computably compact.*

We give an application.

Lemma 1.12. *Let X be a computable Banach space. For each $n \geqslant 1$, the collection of n-tuples $(y_1, \dots, y_n) \in X^n$ which are linearly independent in X is an effectively open subset of X^n (uniformly in n).*

Proof. A tuple $(y_1, \dots, y_n)$ is linearly independent if and only if

$$\inf \left\{ \| \sum_{i \leqslant n} \| \lambda_i y_i \,:\, \lambda_i \in \mathbb{R}, \sum_{i \leqslant n} |\lambda_i| = 1 \right\} > 0.$$

The function $(\lambda_i), (y_i) \to \| \sum \|_{i \leqslant n} \lambda_i y_i$ is a computable function from $\mathbb{R}^n \times X^n$ to $\mathbb{R}$, and the set $\{(\lambda_i) \,:\, \sum_i |\lambda_i| = 1\}$ is an effectively compact subset of $\mathbb{R}^n$, and so $(y_i) \mapsto \min \left\{ \| \sum_{i \leqslant n} \| \lambda_i y_i \,:\, \lambda_i \in \mathbb{R}, \sum_{i \leqslant n} |\lambda_i| = 1 \right\}$ is computable. The inverse image of the effectively open set $(0, \infty) \subseteq \mathbb{R}$ under this computable function is effectively open. □

Proposition 1.13. *Let X be a computable Banach space. There is a computable sequence $\bar{e} = (e_i) \in X^{\mathbb{N}}$ which is linearly independent, and such that the linear span of $\bar{e}$ is dense in X. The sequence $\bar{e}$ can be chosen as a computable subsequence of the sequence of special points that determines the computable structure on X.*

Proof. Let $\bar{x}$ be the sequence of special points. We will choose $\bar{e} = (e_0, e_1, \dots)$ as a subsequence of $\bar{x}$ by recursion. At step s, suppose that we have already determined $(e_0, \dots, e_{s-1})$. For each i, in turn, we run the enumeration of the collection of independent tuples in X^{s+1} given by Lemma 1.12, as well as an enumeration of the rational linear combinations of $e_0, \dots, e_{s-1}$, until we find that either $(e_0, \dots, e_{s-1}, x_i)$ is linearly independent, or we find a rational linear combination w of $(e_0, \dots, e_{s-1})$ such that $\|w - x_i\| < 2^{-s}$. In the latter case, we repeat the process with x_{i+1}. In the former case, we set $e_s = x_i$, and move to step $s + 1$. □

We emphasise that a sequence $\bar{e}$ as in Proposition 1.13 need not be a basis. The point is that for any $y \in X$, for all $k \in \mathbb{N}$ there is some sequence of scalars $\bar{\lambda}^k = (\lambda_i^k)_{i \in \mathbb{N}}$, almost all 0, such that $\|y - \sum_i \lambda_i^k e_i\| < 2^{-k}$. But we cannot put these together to obtain some fixed sequence (λ_i) such that $y = \sum_i \lambda_i e_i$; for a fixed i, as k increases, the "coefficient" λ_i^k does not converge to a limit, and can very well be unbounded with k, so we cannot even choose a convergent subsequence. Furthermore, the linear independence of $\bar{e}$ is finitely-based, and so does not ensure the uniqueness of presentations of points as infinite sums even when such presentations exist: we can have $\sum \lambda_i e_i = 0$ without $\bar{\lambda} = \bar{0}$, since every finite partial sum may be nonzero.

This cautionary tale should give the reader the beginning of an appreciation of the subtlety of some of the notions in the theory of Banach spaces, and see why we need to abandon using finitely dimensional spaces, or Hilbert spaces, for intuition. This "deviation from intuition" is encapsulated in the concept of a *basis constant*, which we introduce shortly.

2. The basis constant

To establish a non-trivial upper bound on the complexity of the index set `Basis`, we use the following characterisation of bases.

Lemma 2.1 (Banach)**.** *Let X be a Banach space. A sequence (x_i) of nonzero elements of X is a basis of X if and only if:*

(i) *There is a constant K such that for all $n < m$ in $\mathbb{N}$, for all sequences of scalars $(\lambda_i)_{i \leqslant m}$, we have*
$$\Big\| \sum_{i=0}^{n} \lambda_i x_i \Big\| \leqslant K \Big\| \sum_{i=0}^{m} \lambda_i x_i \Big\|.$$

(ii) *The finite linear span of (x_i) is dense in X.*

The proof (originally in [Ban32]) is relatively straightforward, see for example [LT77, Prop. 1.a.3]. Let us sketch briefly a more constructive argument, in light of the discussion above regarding Proposition 1.13. Suppose that the conditions of the lemma hold for $\bar{x} = (x_i)$. Let $z \in X$. For each k find some $\bar{\lambda}^k$ (almost all zero) such that $\|z - w_k\| < 2^{-k}$, where $w_k = \sum_i \lambda_i^k x_i$. Since $\|w_k\| \to \|z\|$, there is a bound M on all $\|w_k\|$. This implies that $|\lambda_0^k|$ are all bounded by KM (where K is the constant from the lemma). In turn, compactness implies that $\{\lambda_0^k\}$ has a limit point, λ_0. Restricting to a subsequence of k such that $\lambda_0^k \to \lambda_0$, we now repeat the argument with λ_1, then λ_2, etc., to build a representation $z = \sum \lambda_k x_k$. Uniqueness of representation is also immediately implied by (i).

Corollary 2.2. `Basis` *is* Σ_1^1.

Proof. Both conditions of Lemma 2.1 are arithmetic properties of the sequence (x_i). To see this, observe that by continuity, in (i) we may restrict to rational scalars (a_i). Similarly, for (ii), with the countable dense sequence (y_i) given by the presentation of the space X, it suffices to show that each ball $B(y_i, r)$ for rational $r > 0$ contains a finite linear combination of the x_i's. □

The characterisation of bases given by Lemma 2.1 indicates why `Basis` is not "obviously" simpler than Σ_1^1. If we try to build a basis step-by-step, in the same way that bases of vector spaces are constructed, then we may run into "dead ends": finite sequences satisfying (i) (for some fixed constant K) that cannot be further extended to sequences satisfying the same, and the collection of finite sequences extendible to infinite ones may be Σ_1^1, and not simpler. This, however, is speculation. For all we know, there may be some other property that will allow us to build bases recursively. This would be ruled out by a proof that `Basis` is Σ_1^1-complete. However, we will see that currently known lower bounds are much lower than Σ_1^1.

2.1. **The associated projections.** If (x_i) is a basis of X, then for $n \in \mathbb{N}$ we let $S_n \colon X \to X$ be the projection mapping $\sum_{i=0}^{\infty} a_i x_i$ to $\sum_{i=0}^{n} a_i x_i$. Then Lemma 2.1(i) implies that each S_k is a bounded operator, in fact uniformly.

A linear operator on a Banach space is continuous if and only if it is bounded. Thus, if $\bar{x} = (x_i)$ is a basis of a Banach space X, then Lemma 2.1 implies that the associated projections S_k are all continuous. This is effective; Brattka and Dillhage [BD07, Prop. 3.3] showed that if $\bar{x}$ is a computable basis of a computable Banach space X, then the associated projections are all computable (uniformly). Indeed, they can be computed by a simple search: given a (name of a) point $z \in X$ and $\varepsilon > 0$, we search for a finite sequence of rationals (α_i) such that $\|z - \sum_i \alpha_i x_i\| < \varepsilon$. Then letting $y = \sum_i \alpha_i x_i$, the point $S_k(y) = \sum_{i=0}^{k} \alpha_i x_i$ is computable given the data, and $\|S_k(z) - S_k(y)\| \leqslant K\|z - y\| < K\varepsilon$, where K is as in Lemma 2.1(i), so $S_k(y)$ is a good approximation of $S_k(z)$.

Note that this implies that if $\bar{x}$ is a computable basis, then the coordinate functions $\sum \alpha_i x_i \mapsto \alpha_k$ are computable as well (uniformly), as $\alpha_k x_k = S_k(z) - S_{k-1}(z)$ where $z = \sum_i \alpha_i x_i$.

2.2. **The basis constant.** The following concept, of a *basis constant*, turns out to be important.

Definition 2.3. Let X be a Banach space.

(a) If $\bar{x}$ is a basis of X, then $\mathrm{bc}(\bar{x})$ is $\sup_k \|S_k\|$, where S_k are the associated projections.
(b) We let $\mathrm{bc}(X)$ be $\inf \mathrm{bc}(\bar{x})$ as $\bar{x}$ ranges over all bases of X. If X has no basis then we let $\mathrm{bc}(X) = \infty$.

Note that for a basis $\bar{x}$, $\mathrm{bc}(\bar{x})$ is the infimum of the constants K satisfying (i) of Lemma 2.1. If $\bar{x}$ is an orthonormal basis of a Hilbert space, then its basis constant is 1. A basis with constant 1 is called *monotone*. It is difficult to visualise spaces with larger basis constants, however they do exist, even in the finite dimensional case. In general, if X is a Banach space of dimension n, then $\mathrm{bc}(X) \leqslant \sqrt{n}$ (this follows from John's theorem [Joh48]). Spaces realising this upper bound (asymptotically) were first constructed by Gluskin [Glu81]. We will later observe that such examples can be made computable.

Before we return to the complexity of having a basis, and of related properties, we recall some work on the possible complexity of the basis constant itself. Bosserhof [Bos09, Lem. 8] showed that the basis constant of a computable basis in a finite-dimensional space is computable (see [DGQ24, Lem. 2]), and in [DGQ24, Lem. 3] it is shown that the basis constant of a finite-dimensional computable Banach space is computable. It follows that for an infinite-dimensional computable space X, the basis constant of a computable basis is left-c.e. However, the general question about the complexity of a basis constant of a computable Banach space (that has a basis) is open.

This is made complicated by the fact that there are computable Banach spaces that have a basis, but do not have any computable basis. Such a space was constructed by Bosserhof [Bos09]. We will examine his construction in the next section. The result leaves a general question: what is an upper bound on the complexity of some basis, in a computable Banach space with a basis? This can affect the complexity of the index set `Basis`. For suppose, for example, that we could show that if a computable Banach space has a basis, then it has a basis computable from, say, $\varnothing''$. Then `Basis` is arithmetic, since the function quantifier can be replaced by a number quantifier ranging over the reductions to $\varnothing''$.

2.3. **Basic sequences.** Bosserhof's result stands in contrast with the situation concerning basic sequences.

Definition 2.4. Let X be a Banach space. A sequence $\bar{x} = (x_0, x_1, \dots)$ of elements of X is *basic* if it is a basis of the closure of its linear span.

A theorem attributed to Mazur states that that every infinite-dimensional Banach space (separable or otherwise) has a an infinite-dimensional subspace with a Schauder basis. In other words, that every infinite-dimensional Banach space contains an infinite basic sequence. In the setting without norms, that is, of (countable) computable vector spaces, the computable analogue of Mazur's theorem fails: there is a computable, infinite-dimensional vector space, all of whose computable independent subsets are finite (Metakides and Nerode [MN77b]); indeed Simpson [Sim09] showed that there is a computable infinite dimensional vector space in which every infinite independent set computes $\mathbf{0}'$. However, in the normed context, Mazur's theorem has a computable version: every infinite-dimensional Banach space contains an infinite, computable basic sequence. See [DGQ24, Thm. 7]. This can be extended to all Turing degrees:

Proposition 2.5. *Let X be an infinite-dimensional computable Banach space. For any Turing degree $\mathbf{a}$, there is a basic sequence $\bar{y}$ in X of Turing degree $\mathbf{a}$.*

Proof. Let $A \in 2^\omega$ have degree $\mathbf{a}$. By [DGQ24, Thm. 7], let $\bar{x} = (x_k)$ be a computable basic sequence in X. Let c be a rational upper bound on $\mathrm{bc}(\bar{x})$. Let $\bar{u} = (u_j)$ be the sequence of special points giving the presentation of X. For each k, let $y_{k,0}$ and $y_{k,1}$ be the first two *distinct* elements of $\bar{u}$ with $\|x_k - y_{k,i}\| < 2^{-k-1}/2c$ for both $i = -0, 1$; let $z_k = y_{k,A(k)}$. Since $\bar{x}$ is computable, $\bar{z} \leqslant_{\mathrm{T}} A$; an in the other direction, given any name for z_k we can tell which $y_{k,i}$ is closer to z_k, and so compute $A(k)$; so $\bar{z}$ has Turing degree $\mathbf{a}$.

Now a classic lemma ([KMR40], see also [LT77, Prop. 1.a.9]) gives that $\bar{z}$ is basic, using the fact that $\sum_k \|z_k - x_k\| < 1/2c$. The argument is brief so we give it. Let Y be the closure of the span of $\bar{x}$. Let $u = \sum_k \alpha_k x_k$ in Y. For each n, let $u_n = \sum_{k \leqslant n} \alpha_k x_k$ and $v_n = \sum_{k \leqslant n} \alpha_k z_k$. Since $c \geqslant \mathrm{bc}(\bar{x})$, for all k, $|\alpha_k| \leqslant 2c\|u\|$. Then for all $n < m$,

$$\|v_m - v_n\| \leqslant \|u_m - u_n\| + \sum_{k=n}^{m} |\alpha_k| \cdot \|z_k - x_k\| \leqslant \|u_m - u_n\| + 2c\|u\| \cdot 2^{-n}/2c,$$

showing that (v_n) is a Cauchy sequence, and so converges to some $v \in X$. A similar calculation shows that $\|v - u\| < 1$. The map T sending u to v is linear, and $\|T - I\| < 1$ (where I is the identity map on Y), so T is injective, and so maps the basic sequence $\bar{x}$ to a basic sequence. □

We remark that not all infinite sequences of elements of X will have a Turing degree. That is, the Turing degrees of the names of the sequence will not contain a least degree. This is even true for points in infinite-dimensional spaces such as $\mathbb{R}^{\mathbb{N}}$. To measure the complexity of such points, J. Miller introduced continuous reducibility ([Mil04]). It would be interesting to extend Proposition 2.5 to all continuous degrees.

3. Schauder decompositions and the approximation property

Following Enflo [Enf73], Davie [Dav73] constructed a separable Banach space that does not have a basis. For a detailed exposition see, for example, [Lou05]. Bosserhof [Bos09, § 2] showed that Davie's space has a computable copy. We fix such a copy Z. We use the following property:

Lemma 3.1. *There is a computable sequence $\bar{z} = (z_k)$ in Z such that:*

(i) *The linear span of $\bar{z}$ is dense in Z; and*
(ii) *There is an infinite computable set $L \subseteq \mathbb{N}$ and a constant C such that for all $n \in L$, the linear span $[z_0, \dots, z_n]$ of $\{z_0, \dots, z_n\}$ has basis constant $< C$.*

The second fact we will need regards the approximation property. Recall that a bounded operator $T \colon X \to Y$ between Banach spaces is *finite rank* if the image of T has finite dimension; it is *compact* if the image $T[B_X]$ under T of the unit ball of X is a compact subset of Y. Every finite-rank operator is compact.

Definition 3.2 (Banach)**.** A Banach space X has the *approximation property* if for any space Y, every compact operator $T\colon Y \to X$ is the limit of finite-rank operators (in the operator norm).

Grothendieck ([Gro55], see also [LT77, Thm. 1.e.4]) gave a characterisation that is easier to work with.

Proposition 3.3. *A Banach space X has the approximation property if and only if the identity operator can be approximated by finite rank operators on compact sets: for every compact set $K \subset X$, for every $\varepsilon > 0$, there is a finite rank operator T on X such that $\|Tx - x\| < \varepsilon$ for all $x \in K$.*

Every Banach space that has a basis has the approximation property. Davie used the following to show that Z does not have a basis:

Lemma 3.4. *Z does not have the approximation property.*

In this section we use these tools to give a lower bound on the complexity of the index set `Basis`, and study a generalisation of having a basis. The results are from [Qia21].

3.1. **Sums of spaces.** Let $X_0, X_1, \dots$ be a sequence of Banach spaces. We let $(\bigoplus_k X_k)_{c_0}$ denote the collection of sequences $\bar{x} \in \prod_k X_k$ such that $\|x_k\|_{X_k} \to 0$. This set is endowed with the expected vector space structure, and the norm $\|\bar{x}\| = \sup_k \|x_k\|_{X_k}$, making it a Banach space. Note that we can also use this construction to define a direct sum of finitely many spaces. For two spaces we write $X \oplus_{c_0} Y$ to specify the norm, however other choices of norms on $\mathbb{R}^2$ will result in equivalent structures, and so we usually just write $X \oplus Y$. Observe that for all n,

$$\Big(\bigoplus_k X_k\Big)_{c_0} \cong X_n \oplus \Big(\bigoplus_{k \neq n} X_k\Big)_{c_0},$$

so X_n is *complemented* in $(\bigoplus_k X_k)_{c_0}$.[1]

Lemma 3.5. *Let $X_0, X_1, \dots$ be Banach spaces, and suppose that $\sup_k \operatorname{bc}(X_k) < \infty$. Then $(\bigoplus_k X_k)_{c_0}$ has a basis.*

Proof. Let $M = \sup_k \operatorname{bc}(X_k)$; for each k, let $\bar{b}_k = (b_{k,i})$ be a basis of M with $\operatorname{bc}(\bar{b}_k) \leqslant M$; note that X_k may be finite or infinite-dimensional. Using the usual pairing function, we assume that $\bigcup_k \bar{b}_k$ is ordered in order-type ω; to avoid confusion, we write $\bar{\beta} = (\beta_i)$, and the important property is that for all k, the elements of $\bar{b}_k$ are listed in $\bar{\beta}$ in order. We show that $\bar{\beta}$ is a basis for $(\bigoplus_k X_k)_{c_0}$. To do so, we apply Lemma 2.1.

It is clear that the linear span of $\bar{\beta}$ is dense in $(\bigoplus_k X_k)_{c_0}$. We show that $\operatorname{bc}(\bar{\beta}) \leqslant M$. Let $n \leqslant m$, and let $(\lambda_i)_{i \leqslant m}$ be scalars. Let $v = \sum_{i \leqslant m} \lambda_i \beta_i$ and $u = \sum_{i \leqslant n} \lambda_i \beta_i$. We need to show that $\|u\| \leqslant M\|v\|$. Write $u = \sum_k u_k$ with $u_k \in X_k$ (all but finitely many u_k will be finite); similarly write $v = \sum v_k$. By the ordering requirement

[1] Recall that a subspace Y of a Banach space X is complemeneted in X if and only if there is a bounded projection $P\colon X \to Y$, meaning a bounded operator satisfying $P\restriction Y = \mathrm{id}_Y$. The Hahn-Banach theorem implies that every finite-dimensional subspace of a Banach space X is complemented in X.

on $\bar{\beta}$, each u_k is the result of applying to v_k a projection associated with the basis $\bar{b}_k$, so $\|u_k\| \leqslant M\|v_k\|$. The result follows. $\square$

There is a Banach space X with a basis, which has a complemented subspace that does not have a basis. This is why we need the approximation property.

Lemma 3.6. *If a Banach space X has the approximation property, then every complemented subspace of X has the approximation property.*

Proof. Let $Q\colon X \to Y$ be a projection showing that Y is complemented in X; let $J\colon Y \to X$ be the inclusion map. Let K be a compact subset of Y. Then $J[K]$ is a compact subset of X. Let $\varepsilon > 0$. Let T be a finite rank operator on X with $\|T(x) - x\| < \varepsilon$ for all $x \in J[K]$. Then PTJ is as required for K and $\varepsilon\|P\|$. $\square$

3.2. **A lower bound for `Basis`.**

Proposition 3.7. *The index set* `Basis` *is Π^0_3-hard.*

Proof. Let C be the constant, $\bar{z}$ be the sequence, and L be the computable set, all from Lemma 3.1. Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be the increasing enumeration of L. For each n let $Z_n = [z_0, z_1, \dots, z_{\sigma(n)}]$ (the subspace spanned by these elements). So $\mathrm{bc}(Z_n) \leqslant C$ for all n.

A complete Π^0_3 set is the collection of indices e such that for every n, the section (column) $W_e^{[n]}$ of the e^{th} c.e. set W_e is finite. Thus, to show the proposition, it suffices to show how to uniformly, given a c.e. set W, produce a computable Banach space X such that X has a basis if and only if for all n, $W_e^{[n]}$ is finite. Fix a c.e. set W.

For each n, let $X_n = Z_k$ if $|W_e^{[n]}| = k$; if $W_e^{[n]}$ is infinite, let $X_n = Z$ (the original Davie space, which recall is $[z_0, z_1, \dots]$). Then the spaces (X_n) are uniformly computable, as we can produce, given n, a computable sequence of points in Z whose closure is X_n (then use Lemma 1.7).

We then let $X = (\bigoplus_n X_n)_{c_0}$, which is computable, uniformly given and index for W. If every $W_e^{[n]}$ is finite, then for all n, $\mathrm{bc}(X_n) \leqslant C$; by Lemma 3.5, X has a basis. Suppose that some $W_e^{[n]}$ is infinite; so $X_n = Z$. By Lemma 3.4, Z does not have the approximation property. Since $Z = X_n$ is complemented in X, by Lemma 3.6, X does not have the approximation property, and so does not have a basis. $\square$

Remark 3.8. In the proof of Proposition 3.7, in the Σ^0_3 case (some $W_e^{[n]}$ is infinite), we produced a space X that not only does not have a basis, it lacks the approximation property. Using the notation mentioned in Remark 1.5, we showed that $(\Pi^0_3, \Sigma^0_3) \leqslant (\texttt{Basis}, \texttt{AP}^\complement)$, where `AP` is the index set of computable Banach spaces that have the approximation property. Thus, `AP` is Π^0_3-hard, and every intermediate property between having a basis and having the approximation property is Π^0_3-hard as well.

For a constant C, let $\texttt{Basis}_C$ be the index set of computable Banach spaces that have basis constant $\leqslant C$. In the proof above we implicitly used the fact that $(\Sigma^0_2, \Pi^0_2) \leqslant (\texttt{Basis}_C, \texttt{AP}^\complement)$ to show, in fact, that $(\Pi^0_3, \Sigma^0_3) \leqslant (\texttt{Basis}_C, \texttt{AP}^\complement)$. The argument shows that for any n, if for some C, $(\Sigma^0_n, \Pi^0_n) \leqslant (\texttt{Basis}_C, \texttt{AP}^\complement)$, then $(\Pi^0_{n+1}, \Sigma^0_{n+1}) \leqslant (\texttt{Basis}_C, \texttt{AP}^\complement)$, and in particular, that `Basis` is Π^0_{n+1}-hard.

3.3. **Schauder decompositions.**

Definition 3.9. Let X be a Banach space. A *Schauder decomposition* of X is an infinite sequence $\bar{Z} = (Z_i)$ of closed subspaces of X such that for all $x \in X$, there exists an unique sequence $\bar{z} = (z_i)$ with $z_i \in Z_i$ such that $x = \sum_{i=1}^{\infty} z_i$.

A Schauder decomposition where the spaces Z_i are all finite dimensional is called a *finite dimensional Schauder decomposition* (FDD).

If a Banach space X has a basis (x_i), then it has a finite dimensional Schauder decomposition, as witnessed by the 1-dimensional spaces $Z_i = [x_i]$. A Schauder decomposition presents X as the sum $\bigoplus_i Z_i$, i.e., as decomposed into closed subspaces that are not necessarily 1-dimensional. Szarek [Sza87] showed that there is a space with a finite-dimensional Schauder decomposition that does not have a basis. On the other hand, every space with FDD as the approximation property. Hence, as discussed in Remark 3.8:

Proposition 3.10. *The index set* `FDD` *of computable Banach spaces that have finite-dimensional Schauder decompositions is* Π^0_3*-hard.*

An analogue of Banach's Lemma 2.1 holds for FDDs. See, for example, [Mar69, P. 93].

Lemma 3.11. *A sequence* $\bar{Z} = (Z_i)$ *of closed subspaces of a Banach space* X *is a Schauder decomposition of* X *if and only if:*

(i) *There is a constant* K *such that for all* $n < m$ *in* $\mathbb{N}$*, for any sequence* $(z_i)_{i \leqslant m}$ *with* $z_i \in Z_i$*,* $\|\sum_{i \leqslant n} z_i\| \leqslant K \|\sum_{i \leqslant m} z_i\|$.

(ii) *The linear span of* $\bigcup_i Z_i$ *is dense in* X*.*

As for Corollary 2.2, we obtain:

Corollary 3.12. `FDD` *is* Σ^1_1.

3.4. **Computable bases and decompositions.**

Bosserhof [Bos09] constructed a computable Banach space that has a basis, but has no *computable* basis. Elaborating on his construction, Qian [Qia21] showed that the index set `CompBasis` of Banach spaces that have a computable basis is Σ^0_3-complete; the proof was also given in [DGQ24, Thm. 8].

Qian showed that a similar argument gives the Σ^0_3-completeness of having a computable FDD. We give some details here, however first, we need to define the notion of a computable FDD. The "natural" definition is not immediately obvious. Certainly we require the spaces (Z_i) in a computable decomposition to be uniformly computable. However, this condition is not quite as strong as would appear, recalling (Lemma 1.7) that closed subspaces that are not necessarily effectively closed can form computable spaces, and this was used in the arguments above: in some sense, we can describe computable spaces in a c.e. way. The following definition requires that the spaces (Z_i) are given in a computable way, in that their bases are given as finite sets (sometimes known as "strong indices").

Definition 3.13. Let X be a computable Banach space. A finite-dimensional Schauder decomposition (Z_i) of X is *computable* if there is a computable sequence (B_i) of finite sets such that each B_i is a basis of Z_i.

Note that the definition implies that the sequence of dimensions $(\dim Z_i)$ is computable. If $\bar{x}$ is a computable basis of X, then the 1-dimensional spaces $[x_i]$

form a computable FDD of X. Just like for computable bases, Lemma 3.11 implies that having a computable FDD is a Σ^0_3 property.

Toward showing the Σ^0_3-completeness of having a computable FDD, we recall the tools giving the Σ^0_3-completeness of having a computable basis. We again use the computable copy Z of Davie's space, and the associated constant C and sequence $\bar{z}$ given by Lemma 3.1; and again let σ be the increasing enumeration of the set L given by that lemma. We again let $Z_n = [z_0, \dots, z_{\sigma(n)}]$, so $\mathrm{bc}(Z_n) \leqslant C$. Following Bosserhof, we let

$$Y_\infty = \Big(\oplus_k Z \Big)_{c_0}$$

be the sum of infinitely many copies of Z. For any function $f\colon \mathbb{N} \to \mathbb{N}$ we let

$$Y_f = \Big(\oplus_k Z_{f(k)} \Big)_{c_0},$$

which we regard as a subspace of Y_∞. Note that Lemma 3.5 implies that each Y_f has a basis; Lemma 3.6 implies that Y_∞ does not have the approximation property, and so does not have a basis. Bosserhof noted [Bos09, Lem. 13] that if f is lower semi-computable (has a non-decreasing approximation) then Y_f is a computable Banach space, and the inclusion map of Y_f in Y_∞ is computable. Bosserhof showed, by direct diagonalisation, that there is some lower semi-computable f such that Y_f has no computable basis.

[Bos09, Cor. 12] holds for FDD. For each k, let θ_k be the embedding of Z as the k^{th} component of Y_∞. If $X \subseteq Y_\infty$ is a subspace, then each $\theta_k[Z] \cap X$ is complemented in X, and so, if X has the approximation property, then $\theta_k[Z] \cap X$ cannot be all of $\theta_k[Z]$. If X has an FDD then it has the approximation property, so the conclusion applies: if $X \subseteq Y_\infty$ has an FDD then for all k, $\theta_k[Z] \nsubseteq X$.

In the following, let (φ_e) effectively list the partial computable functions; we regard each value $\varphi_e(m)$ as a code giving a finite list $\bar{q}_1, \dots, \bar{q}_{n_{e,m}} = \bar{q}^{e,m}_1, \dots, \bar{q}^{e,m}_{n_{e,m}}$ of partial computable sequences of special points of Y_∞. If $\bar{q}^{e,m}_j$ is a total Cauchy name for Y_∞ then we let $y^{e,m}_j$ denote the named point; if each $\bar{q}^{e,m}_j$ is a total Cauchy name, then we let $X^{e,m}$ be the subspace $[y^{e,m}_1, \dots, y^{e,m}_{n_{e,m}}]$. If each $X^{e,m}$ is defined, then we let X^e be the closure of the linear span of $\bigcup_m X^{e,m}$.

The following are all Σ^0_1 properties of a triple (e, m, c):

- Some $\bar{q}_j = \bar{q}^{e,m}_j$ fails to be a partial Cauchy sequence, i.e., for some $i < i'$, $\|q_{j,i'} - q_{j,i}\| > 2^{-i}$;
- There is enough convergence to ensure that if each X^e_m is defined, then the decomposition constant of the sequence $\bar{X}^e = (X^e_m)_m$ is greater than c.

If neither condition above materialises, and each $\bar{q}^{e,m}_j$ is total, then X^e is defined and is a space with computable FDD, with constant $\leqslant c$. If X is a subspace of Y_∞ with computable FDD then $X = X^e$ for some e such that $(X^{e,m})_m$ has finite decomposition constant (is an FDD of X^e).

Therefore, given e, c and k, if the two conditions above do not hold, then we can iteratively, for $l = 1, 2, \dots$, search for enough convergence of various $\bar{q}^{e,m}_j$ to see that each of $\theta_k(z_0)$, $\theta_k(z_1)$, dots, $\theta_k(z_{\sigma(l)})$ are within 2^{-l} of a point in X^e. This process does not allow l to go to ∞, since that would ensure that X^e is defined, has decomposition constant $\leqslant c$, and that $\theta_k[Z] \subseteq X^e$, which as we noted is impossible. We thus get an extension of [DGQ24, Lem. 10]:

Lemma 3.14. *There is a lower semi-computable function g such that for all e, c and k, either*

- *X^e is undefined, or $(X^{e,m})$ has decomposition constant $> c$; or*
- *$\theta_k[Z_{g(e,c,k)}] \nsubseteq X^e$.*

The rest of the construction is as in [DGQ24]. Using a movable marker argument, given a c.e. set W, we uniformly obtain a lower semi-computable function h such that:

- If some $W^{[n]}$ is infinite, then for all e and c there is some k such that $h(k) > g(e,c,k)$, and so $Y_h \nsubseteq X^e$;
- If every $W^{[n]}$ is finite, then h is computable.

By [DGQ24, Lem. 11], if h is computable then Y_h has a computable basis, and so computable FDD. We thus obtain:

Proposition 3.15. *The index set* CompFDD *of computable Banach spaces that have a computable FDD is Σ^0_3-complete. In fact, $(\Sigma^0_3, \Pi^0_3) \leqslant (\mathtt{CompBasis}, \mathtt{CompFDD}^{\complement})$.*

3.5. **The complexity of the approximation property.** We now turn to an upper bound on the complexity of the index set AP of computable Banach spaces that have the approximation property. We will show that AP is Π^1_1. The main quantifier is the quantification over all compact sets (in the formulation from Proposition 3.3). To do that in a way that also allows us to uniformly enumerate dense subsets of said compact sets, we use the following characterisation of compact subsets of Banach spaces.

For a sequence $\bar{y} = (y_i)$ of points in a Banach space, we let $\mathrm{Conv}(\bar{y})$ denote the closure of the convex hull of $\bar{y}$. If $y_i \to 0$, then

$$\mathrm{Conv}(\bar{y}) = \Big\{ \sum_{i=0}^{\infty} \lambda_i y_i \,:\, \lambda_i \geqslant 0, \sum_{i=0}^{\infty} \lambda_i \leqslant 1 \Big\}$$

(the condition $y_i \to 0$ ensures that each such sum $\sum_i \lambda_i y_i$ converges in X). The following is [LT77, Prop. 1.e.2].

Lemma 3.16. *Let X be a Banach space. A closed subset $K \subseteq X$ is compact if and only if there is a sequence $\bar{y}$ of points in X such that $y_i \to 0$ and $K \subseteq \mathrm{Conv}(\bar{y})$.*

Before we calculate the upper bound, we need to recall that if X and Z are computable Banach spaces, and $\dim Z < \infty$, then the space $L(Z,X)$ of bounded linear operators $T\colon Z \to X$, equipped with the operator norm, is a computable Banach space. Fixing a basis $z_1, \ldots, z_n$ of Z, the special points of $L(Z,X)$ are the operators mapping each basis element z_i to a special point of X, and so can be coded by n-tuples of special points of X, and therefore by n-tuples of natural numbers. What we need to note is that the operator norm is computable on these special points. This is because the closed unit ball B_Z of Z is computably compact (this is where we use that Z is finite-dimensional), and each computable operator T defines a computable function from Z to X, uniformly in T; so we apply Propositions 1.10 and 1.11.

Proposition 3.17. *The index set* AP *of computable Banach spaces that have the approximation property is Π^1_1.*

Proof. Let X be a computable Banach space. Let $\bar{e}$ be a computable, linearly independent sequence given by Proposition 1.13. For each k, let $X_k = [e_0, e_1, \dots, e_k]$ be the subspace generated by $e_0, \dots, e_k$. So $\bigcup_k X_k$ is dense in X.

Consider the following statement:

(∗): For every compact set $K \subset X$ and every $\varepsilon > 0$ there is some m and some bounded operator $T\colon X \to X_m$ such that $\|T(x) - x\| \leqslant \varepsilon$ for all $x \in K$.

We claim that X has the approximation property if and only if (∗) holds. In the non-trivial direction, suppose that X has the approximation property. Let $K \subset X$ be compact, and let $\varepsilon > 0$. Let C be a bound on $\|y\|$ for all $y \in K$. Let $Z \subseteq X$ be a finite-dimensional subspace, and let $T\colon X \to Z$ be a bounded operator satisfying $\|T - I\|_K < \varepsilon$, that is, $\|T(x) - x\| < \varepsilon$ for all $x \in K$. Since Z is finite-dimensional and $\bigcup_m X_m$ is dense in X, for each $\delta > 0$ there is some m and some linear $Q\colon Z \to X_m$ such that $\|Q(x) - x\| \leqslant \delta\|x\|$ for all $x \in Z$. We choose $\delta = \varepsilon/(C + \varepsilon)$. Let $y \in K$. Then $\|T(y)\| \leqslant C + \varepsilon$, and so $\|QT(y) - y\| \leqslant \varepsilon + \delta(C + \varepsilon) \leqslant 2\varepsilon$, so QT is as required for (∗).

To obtain the complexity bound, we need to explain why the quantifier over T is equivalent to an arithmetic property. For m and k, call a linear $S\colon X_k \to X_m$ *special* if it is one of the special points of the computable structure we gave $L(X_k, X_m)$. For each k, let A_k be the collection of rational linear combinations of $e_0, \dots, e_k$ (so is dense in X_k). If $\bar{y} \in X^{\mathbb{N}}$ and $y_i \to 0$, then we let $B(\bar{y})$ be the collection of all points of the form $\sum \lambda_i y_i \in \mathrm{Conv}(\bar{y})$ such that the λ_i are all rational and all but finitely many are 0. So $B(\bar{y})$ is dense in $\mathrm{Conv}(\bar{y})$, and is countable (and enumerable given $\bar{y}$).

Consider the following property:

(∗∗): For every sequence $\bar{y} \in X^{\mathbb{N}}$ such that $y_i \to 0$, for every $\varepsilon > 0$, there is some m and some $D \in \mathbb{N}$ such that for all k, there is some special $S\colon X_k \to X_m$ such that $\|S\| \leqslant D$, and such that for every $w \in A_k$ and every $u \in B(\bar{y})$,

$$\|S(w) - w\| \leqslant 2^{-k}\|w\| + D\|w - u\| + \varepsilon.$$

The property (∗∗) is certainly Π^1_1, as other than the quantification over $\bar{y}$ (which is of course quantification over sequences of names for points of X), all other quantifiers are number quantifiers, and involve arithmetic properties: for example, $y_i \to 0$ is an arithmetic property of $\bar{y}$, and $\|S\| \leqslant D$ is an arithmetic (indeed, Π^0_1) property of S and D). So it remains to show that (∗) and (∗∗) are equivalent.

First, suppose that (∗) holds. Let $\bar{y} \in X^{\mathbb{N}}$ with $y_i \to 0$, and let $\varepsilon > 0$. Let $K = \mathrm{Conv}(\bar{y})$, which is compact by Lemma 3.16. Let m and T be given by (∗) for K and ε. Let $D = \|T\| + 1$. Fix k. Then $T{\upharpoonright}X_k \in L(X_k, X_m)$; there is a special $S \in L(X_k, X_m)$ such that $\|S - T{\upharpoonright}X_k\| \leqslant 2^{-k}$. Since $\|T{\upharpoonright}X_k\| \leqslant \|T\|$ and $\|S - T{\upharpoonright}X_k\| \leqslant 1$, $\|S\| \leqslant D$. Let $w \in X_k$ and $u \in K$. Then

$$\|S(w) - w\| \leqslant \|S(w) - T(w)\| + \|T(w) - T(u)\| + \|T(u) - u\| + \|u - w\| \leqslant$$
$$2^{-k}\|w\| + \|T\| \cdot \|w - u\| + \varepsilon + \|u - w\| = 2^{-k}\|w\| + D\|w - u\| + \varepsilon,$$

as required for (∗∗).

In the other direction, suppose that (∗∗) holds. Let $K \subset X$ be compact, and let $\varepsilon > 0$. By Lemma 3.16, let $\bar{y} \in X^{\mathbb{N}}$ with $y_i \to 0$ such that $K \subseteq \mathrm{Conv}(\bar{y})$. Let m and D be given by (∗∗) for $\bar{y}$ and ε. For each k, fix some $S_k\colon X_k \to X_m$ with the properties guaranteed by (∗∗).

Now by induction on k, we define infinite sequences $\bar{n}^k = (n_i^k)_{i\in\mathbb{N}}$ of natural numbers, starting with $n_i^0 = i$, so that each $\bar{n}^{k+1}$ is a subsequence of $\bar{n}^k$; these are chosen so that $\lim_i S_{n_i^k}\restriction X_k$ is an operator $T_k\colon X_k \to X_m$ with $\|T_k\| \leqslant D$, and T_{k+1} extends T_k. Suppose that these have been defined for k. Since $\|S_i\| \leqslant D$ for all i, $\left\{S_{n_i^k}(e_{k+1}) \,:\, i \in \mathbb{N}\right\}$ is bounded; since X_m is finite-dimensional, this set has a limit point, and we choose $\bar{n}^{k+1}$ to be a subsequence of $\bar{n}^k$ so that $S_{n_i^{k+1}}(e_{k+1})$ converges to some z_{k+1}; we let T_{k+1} extend T_k by setting $T_{k+1}(e_{k+1}) = z_{k+1}$. Since $\bar{n}^{k+1}$ is a subsequence of $\bar{n}^k$, $T_{k+1} = \lim_i S_{n_i^{k+1}}\restriction X_{k+1}$ pointwise. Since X_k is finite-dimensional, it follows that $\|T_k\| \leqslant D$. We let $T_\infty = \bigcup_k T_k$, which is a bounded linear operator from $\bigcup_k X_k$ to X_m; we have $\|T_\infty\| \leqslant D$. Further, by continuity, each T_k has the property that the S_i have in $(**)$. Since $\bigcup_k X_k$ is dense in X, we let $T\colon X \to X_m$ be the unique extension of T_∞ to a bounded operator on X; $\|T\| \leqslant D$.

Let $y \in K$. We show that $\|T(y) - y\| \leqslant \varepsilon$. Let C be a bound on $\|y\|$ for all $y \in \operatorname{Conv}(\bar{y})$. Let $\delta > 0$. There are $u \in B(\bar{y})$, some $k \in \mathbb{N}$ and some $w \in A_k$ such that $2^{-k} < \delta$, $\|u - y\| < \delta$ and $\|u - w\| < \delta$. Then

$$\|T(y) - y\| \leqslant \|T(y) - T(w)\| + \|T(w) - w\| + \|w - y\| \leqslant \\ 2\delta D + 2^{-k}\|w\| + D\|w - u\| + \varepsilon + 2\delta \leqslant (4D+2)\delta + \delta(C+\delta) + \varepsilon;$$

since C and D are constant, we let $\delta \to 0$ to obtain the required bound. □

3.6. **The bounded approximation property.** Proposition 3.17 shows that Davie's construction cannot be used to show the Σ^1_1-completeness of `Basis`: suppose that A is reducible to (`Basis`, $\mathtt{AP}^\complement$). Then both A and its complement are Σ^1_1, meaning that A is Δ^1_1, and so hyperarithmetic. The *bounded* approximation property is a property stronger than the approximation property and yet weaker than having a basis; Proposition 3.20 will imply that Davie's construction cannot be used to prove anything more than Σ^0_4-hardness of `Basis`.

Definition 3.18. A Banach space X has the *bounded approximation property* if it has the approximation property with a uniform bound on the norms of the witnessing operators. That is, there is some C such that for every compact $K \subset X$, for every $\varepsilon > 0$, there is a finite-rank T on X with $\|T\| \leqslant C$ and $\|T(y) - y\| < \varepsilon$ for all $y \in K$.

The norm bound allows us to simplify the definition quite a bit. For example, we can replace "compact Y" by "finite Y": if Y is compact and $\varepsilon > 0$, we find some finite $Y_0 \subseteq Y$ which is ε/C-dense in Y, that is, every point in Y is within ε/C-distance from a point in Y_0; if T witnesses the BAP for Y_0 and ε, then it witnesses the same for Y and 3ε, since if $y \in Y$, $y_0 \in Y_0$ and $\|y - y_0\| \leqslant \varepsilon/C$, then

$$\|T(y) - y\| \leqslant \|T(y) - T(y_0)\| + \|T(y_0) - y_0\| + \|y_0 - y\| \leqslant C\frac{\varepsilon}{C} + \varepsilon + \frac{\varepsilon}{C}.$$

Because of the quantification over all $\varepsilon > 0$, we can then "normalise" the property, as follows:

Lemma 3.19. *A Banach space X has the bounded approximation property if and only if there is some C such that for every finite-dimensional $W \subseteq X$ and every $\varepsilon > 0$ there is some finite-rank T on X with $\|T\| \leqslant C$ such that $\|T(w) - w\| \leqslant \varepsilon\|x\|$ for all $w \in W$.*

For more equivalent formulations of the bounded approximation property see [JRZ71] or [LT77]; for example, we can require $T(x) = x$ for $x \in Z$.

Let BAP be the index set of computable Banach spaces that have the bounded approximation property. Since $\texttt{Basis} \subseteq \texttt{BAP} \subseteq \texttt{AP}$, Remark 3.8 shows that BAP is Π^0_3-hard. We cannot quite show completeness, but the gap for BAP is much smaller than for the other two properties.

Proposition 3.20. BAP *is* Σ^0_4.

Proof. The argument goes along the line of that of Proposition 3.17, however the fact that we only need to check finite-dimensional subspaces simplifies things significantly. Let X be a computable Banach space. Use a sequence $\bar{e}$ as above (Proposition 1.13); define $X_k = [e_0, \dots, e_k]$ as above. Consider:

$(*)$: There is some C such that for every m and every $\varepsilon > 0$ there is some $k \geqslant m$ and some $T\colon X \to X_k$ such that $\|T\| \leqslant C$ and $\|T(x) - x\| \leqslant \varepsilon\|x\|$ for all $x \in X_m$.

We claim that X has the bounded approximation property if and only if $(*)$ holds. In one direction, suppose that X has the bounded approximation property; let C be a witness. Let $m \in \mathbb{N}$ and $\varepsilon > 0$. Let $Z \subseteq X$ be finite-dimensional and let $T\colon X \to Z$ with $\|T\| \leqslant C$ and $\|T(x) - x\| \leqslant \varepsilon\|x\|$ for all $x \in X_m$. As in the previous proof, there is some $k \geqslant m$ and some $Q\colon Z \to X_k$ with $\|Q(z) - z\| \leqslant \delta\|z\|$ for all $z \in Z$, where δ is as small as we require. Then $\|QT\| \leqslant C(1 + \delta)$, and for $x \in X_m$,

$$\|QT(x) - x\| \leqslant \|T(x) - x\| + \|QT(x) - T(x)\| \leqslant \varepsilon\|x\| + \delta\|T(x)\| \leqslant \varepsilon\|x\| + \delta C\|x\|,$$

so setting $\delta = \varepsilon/C$ will give us QT which is as required by $(*)$ for X_m and 2ε.

In the other direction the argument is similar but simpler. Suppose that $(*)$ holds; let C be a witness. Let $W \subseteq X$ be finite-dimensional; let $\varepsilon > 0$. Choosing some small $\delta > 0$ later, find some m and some $Q\colon W \to X_m$ such that $\|Q(w) - w\| \leqslant \delta\|w\|$ for all $w \in W$. Let $T\colon X \to X_k$ be as given by $(*)$ for X_m and ε. Then since $\|Q\| \leqslant 1 + \delta$, for all $w \in W$,

$$\|T(w) - w\| \leqslant \|TQ(w) - Q(w)\| + \|Q(w) - w\| \leqslant \varepsilon\|Q(w)\| + \delta\|w\| \leqslant (\varepsilon(1+\delta) + \delta)\|w\|,$$

showing that T itself can be taken for W and 2ε under an appropriate choice of δ.

Now we need to unpack the quantification over T, as above:

$(**)$: There is some C such that for every m and every $\varepsilon > 0$ there is some $k \geqslant m$ such that for all $k' \geqslant k$ there is some $T\colon X_{k'} \to X_k$ such that $\|T\| \leqslant C$ and $\|T(x) - x\| \leqslant \varepsilon\|x\|$ for all $x \in X_m$.

The equivalence of $(*)$ and $(**)$ is as in the previous proof, but simpler, since we do not need to approximate elements of $\mathrm{Conv}(\bar{y})$ by elements of X_m. So it remains to show that $(**)$ is Σ^0_4. For that, we observe that the relation "there is some $T\colon X_{k'} \to X_k$ such that $\|T\| \leqslant C$ and $\|T(x) - x\| \leqslant \varepsilon\|x\|$ for all $x \in X_m$" is Π^0_1. To see this, observe that since $X_m \subseteq X_k$ when $m \leqslant k$, the last condition on T is $\|(T - I)\restriction X_m\| \leqslant \varepsilon$. The map $g(t) = \|(T - I)\restriction X_m\|$ is computable (from $L(X_{k'}, X_k)$ to $\mathbb{R}$), and so the condition holds if and only if

$$\min\{g(T) : \|T\| \leqslant C\} \leqslant \varepsilon.$$

The closed ball $\{T : \|T\| \leqslant C\}$ is computably compact (uniformly in C), so the desired result is given by Proposition 1.10. ∎

4. Local basis structure

Enflo's and Davie's constructions left open the question whether the bounded approximation property is equivalent to having a basis. This was answered in the negative by Szarek [Sza87]. Szarek's method yields a space which does not have a basis because it fails yet another property weaker than having a basis: having a local basis structure. Recall the notion of a basis constant of a space Definition 2.3.

Definition 4.1. A Banach space X has a *local basis structure* if there is some C such that for every finite-dimensional $W \subseteq X$ there is a finite-dimensional Z with $W \subseteq Z \subseteq X$ and $\mathrm{bc}(Z) \leqslant C$.

Certainly every space with a basis has a local basis structure, as is witnessed by the spaces spanned by initial segments of the basis (and mild perturbations thereof). It would appear to the uninitiated that given a local basis structure for a separable space X, we could build a basis for X step-by-step, always extending to a basis of a larger space with a low basis constant. But here recall that the basis constant of a *space* is the infimum over the basis constants of its various bases. So even if $W \subseteq Z$ both have basis constant $< C$, a basis $\bar{w}$ of W with basis constant $< C$ cannot necessarily be extended to a basis of Z with a similar bound. And indeed, Enflo's and Davie's spaces have a local basis structure (essentially by Lemma 3.1). Thus, having a local basis structure is incomparable with the approximation property (or the bounded version).

The hope that Szarek's construction could be used to provide useful lower bounds for the complexity of `Basis` is dealt a fatal blow by the main result of this section, that the index set `LBS` of computable Banach spaces with a local basis structure is Σ^0_3-complete. We present the details of this not only as a negative result. First, it is interesting to actually have an arithmetical completeness result for the properties related to having a basis. Second, Szarek's construction is of interest because it is (by nature) finitely-based; the pathological space is built as the limit (or sum) of finite-dimensional spaces. Szarek showed how the basis constant of finite-dimensional spaces can be manipulated, increased or decreased on demand, and we hope that this technology will be useful. Beyond that, as mentioned above, for a long time, it was not known whether finite-dimensional spaces can have large basis constants; this was answered by Gluskin [Glu81], whose technique (together with Bourgain's, see [Bou88]) is the basis of Szarek's work. We will show that these unusual finite-dimensional spaces can be chosen to be computable:

Theorem 4.2. *For every k there is a computable, finite-dimensional Banach space X with $\mathrm{bc}(X) > k$.*

We will shortly present a proof of this result, as well as of the completeness result for `LBS`. The proof relies on a presentation of Szarek's work in [MTJ03]. We will quote some results from that paper and concentrate on the effective aspects of the argument. Some of these arguments given in [MTJ03] were expanded to be more accessible to people not already well-versed in the area; these can be found in Xie's thesis [Xie24], but they would be too long to include here.

First, however, we dispense of the upper bound:

Proposition 4.3. *The index set `LBS` is Σ^0_3.*

Proof. The argument relies on a result implicit in the work of Pujara [Puj71] and Bosserhof [Bos08], that for a computable Banach space, having a local basis structure implies having a computable version of the same property, and that the property is essentially equivalent to the property we listed in Lemma 3.1. They proved the following:

(a) A Banach space X has a local basis structure if and only if there is some C, a sequence $\bar{x} \in X^{\mathbb{N}}$ that is dense in X, and an infinite set L such that for all $n \in L$, $\mathrm{bc}([x_0, \dots, x_n]) < C$.

(b) If a computable Banach space X has a local basis structure, then such a sequence $\bar{x}$ and set L can be chosen to be computable, in fact, $\bar{x}$ can be chosen to be a computable sequence of special points.

This gives the upper bound, since the quantification only needs to be performed over (indices of) computable objects. The statement $\mathrm{bc}([x_0, \dots, x_n]) < C$ is Σ^0_1, as the quantification over bases can be restricted to special points. ☐

4.1. **Basic concepts.** As the argument below is fairly complex, we present it in a modular fashion. We will start with the completeness result for `LBS`, based on an effectivity result that we will present later. We start by introducing the various notions that are required for stating and proving the results.

The Banach-Mazur distance.

Definition 4.4. The *Banach-Mazur distance* $d_{\mathrm{BM}}(X, Y)$ between Banach spaces X and Y is

$$\inf \left\{ \|T\|_{L(X,Y)} \|T^{-1}\|_{L(Y,X)} \, : \, T \colon X \to Y \text{ is linear and invertible} \right\}.$$

We write $d_{\mathrm{BM}}(X, Y) = \infty$ if X and Y are not linearly isomorphic.

Note that this is a multiplicative distance: $d(X, Y) \geqslant 1$, and $d(X, Z) \leqslant d(X, Y)d(Y, Z)$. If X and Y are finite-dimensional, then the infimum is realised, and so in this case, $d(X, Y) = 1$ if and only if X and Y are isometric.

For the definition below and for later, recall that for $n \in \mathbb{N}$ and $p \geqslant 1$, l^n_p denotes the Banach space $\mathbb{R}^n$, equipped with the p-norm $\|\cdot\|_p$. The adjective "Euclidean" refers to the Hilbert space l^n_2; we sometimes say that X is "D-Euclidean" if $d_{\mathrm{BM}}(X, l^n_2) \leqslant D$.

Definition 4.5. For a Banach space X and $n \in \mathbb{N}$, we let

$$\partial_n(X) = \sup \left\{ d_{\mathrm{BM}}(E, l^n_2) \, : \, E \subseteq X \text{ is a subspace } \& \ \dim E = n \right\}.$$

Computability of singular values. We will use the notion of the singular values of an operator or matrix. For any linear map $T \colon \mathbb{R}^n \to \mathbb{R}^m$ of rank k, there are orthonormal systems $u_1, \dots, u_k \in \mathbb{R}^n$ and $v_1, \dots, v_k \in \mathbb{R}^m$ and positive scalars $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_k > 0$ such that $Tu_i = \sigma_i v_i$, and $\{u_1, \dots, u_k\}$ span the orthogonal complement of the kernel of T; thus $v_1, \dots, v_k$ span the image of T and T is determined by this information. The orthonormal $\bar{u}$ and $\bar{v}$ are not unique, but the *singular values* are; for $i > k$ we set $\sigma_i = \sigma_i(T) = 0$. When $m = n$ we can also choose $u_{k+1}, \dots u_n$ and $v_{k+1}, \dots, v_n$ so that (u_i) and (v_i) are orthonormal bases of $\mathbb{R}^n$ and $Tu_i = \sigma_i v_i$. In other notation,

$$T = \sum_{i \leqslant n} \sigma_i \langle u_i, \cdot \rangle v_i.$$

This is known as a *polar decomposition* of T.

In terms of matrices, we use the *singular value decomposition*: For any $(m \times n)$-matrix A there are orthogonal matrices $U \in O_m(\mathbb{R})$, $V \in O_n(\mathbb{R})$, and a rectangular diagonal $(m \times n)$-matrix Σ with non-decreasing, non-negative values on its diagonal such that $A = U\Sigma V^T$; these are the singular values of A, which are the same as the singular values of the operator $A \mapsto Ax\colon \mathbb{R}^n \to \mathbb{R}^m$.

Lemma 4.6. *Let $n, m \geqslant 1$. For all $k \leqslant \min\{n,m\}$, the map $T \mapsto \sigma_k(T)$ (from $L(\mathbb{R}^n, \mathbb{R}^m)$ to $\mathbb{R}$) is computable. This is uniform in n, m and k.*

Indeed, there is a computable function that given a matrix A, produces a singular value decomposition of A. An easy way to see the computability of the singular values, is to note that the singular values of T are the square roots of the eigenvalues of T^*T. The latter is self-adjoint and positive semi-definite; so the matrix representing it is symmetric and so diagonalizable.

Mixing operators. The argument that Szarek's space does not have a basis relies on norm estimates of projections (and differences of projections). It is easier to reason about a wider family of operators.

Definition 4.7. Let $\beta, \gamma > 0$. An operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is (β,γ)-*mixing* if there is a subspace $E \subset \mathbb{R}^n$ satisfying:

(1) $\dim E \geqslant \beta n$;
(2) for all $x \in E$, the Euclidean distance between Tx and E (namely $\|P_{E^\perp}Tx\|_2$) is at least $\gamma\|x\|_2$

Here we are using the Euclidean structure on $\mathbb{R}^n$; $E^\perp$ denotes the orthogonal complement of E in $\mathbb{R}^n = l_2^n$, and $P_{E^\perp}$ is the orthogonal projection onto $E^\perp$. Thus, the operator T is "mixing" in the sense that it moves elements of E "far away" from E. Note that in this case $P_{E^\perp}T \restriction E$ is invertible, and the distancing condition is equivalent to saying that $\|(P_{E^\perp T} \restriction E)^{-1}\|_{L(l_2^n)} \leqslant 1/\gamma$. In terms of singular values, we observe that we are asking that $\sigma_k(P_{E^\perp}T \restriction E) \geqslant \gamma$.

The following indicates the connection between projections and mixing operators:

Proposition 4.8. *Let $Q\colon \mathbb{R}^n \to \mathbb{R}^n$ be a projection of rank $k \leqslant n/2$. Then, Q is $(k/n, 1/2)$-mixing.*

For a proof see [MTJ03, p. 1213] or [Xie24, Prop. 5.1.4]. The following is Szarek's key to making spaces with large norms.

Definition 4.9. Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a Banach space. For $\beta > 0$ we let

$$m(X,\beta) = \inf\left\{\|T\|_{L(X)} : T\colon \mathbb{R}^n \to \mathbb{R}^n \text{ is } (\beta,1)\text{-mixing}\right\}.$$

We will let

$$m(X) = m(X, 1/32).$$

It is important to note that $m(X)$ is not an invariant of the isometry-type of X; the notion of a mixing operator relies on the Euclidean structure of $\mathbb{R}^n$. More abstractly, $m(X)$ relies on both the norm of X and a choice of an inner product on the underlying vector space of X.

We note that 4.8 implies:

Proposition 4.10. *For any Banach space X, if $\dim X \geqslant 32$, then $\mathrm{bc}(X) \geqslant m(X)/2$.*

The main result connecting all the notions introduced so far is the following, which is [MTJ03, Thm. 48]. A detailed proof is given in [Xie24, § 5.3].

Theorem 4.11. *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a Banach space with $n \geqslant 8$. For any Banach space W,*

$$bc(X \oplus W) \geqslant c\frac{\sqrt{m(X)}}{\partial_n(W)}$$

(where c is a universal constant, and $\oplus$ denotes the l_2 sum).

4.2. **The main results.** The following is an effective version of [MTJ03, Prop. 41]). It implies Theorem 4.2.

Proposition 4.12. *For any $n \geqslant 128$ and rational $p > 2$ there is a computable Banach space $X_{n,p} = (\mathbb{R}^n, \|\cdot\|_{X_{n,p}})$ satisfying:*

(1) $m(X_{n,p}) \geqslant \alpha n^{1/2 - 1/p}$;
(2) $X_{n,p}$ is a subspace of l_p^{2n}.

Here α is a universal constant, and the computability is uniform: the norm $\|\cdot\|_{X_{n,p}}$ is computable, uniformly in n and p, and we can also uniformly obtain a computable copy $Y_{n,p}$ of l_p^{2n} such that $Y_{n,p} \supset X_{n,p}$.

We delay the proof, and first show how we use it.

Theorem 4.13. *The index set* LBS *of computable Banach spaces with a local basis structure is Σ_3^0-complete.*

By Proposition 4.3, it suffices to show hardness. Following Szarek, we define sequences (n_k) and (p_k) (with $n_k \to \infty$ and (p_k) decreasing to 2) recursively:

- $n_0 = 128$ and $p_0 = 3$;
- For $k > 0$, given n_{k-1} and p_{k-1}, we choose $p_k < p_{k-1}$ sufficiently close to 2 (from above) so that
$$n_{k-1}^{1/2 - 1/p_k} \leqslant 2;$$
- Given n_{k-1} and p_k, we choose $n_k > n_{k-1}$ sufficiently large so that
$$n_k^{1/2 - 1/p_k} > k^2 n_{k-1}.$$

Szarek [Sza86, Sza87] showed that the space

$$\Big(\bigoplus_k X_{n_k,p_k}\Big)_2$$

has no basis. This space is the l_2-sum of the spaces X_{n_k,p_k}, defined similarly to the c_0-sum defined in Section 3 above, but taking those sequences $\bar{x} \in \prod_k X_{n_k,p_k}$ with $\sum_k \|x_k\|^2_{X_{n_k,p_k}} < \infty$ and using the l_2 norm as expected. In general, all infinite sums in this proof are l_2 sums, so we will omit mentioning this.

To establish Σ_3^0 completeness, we need an extension of Szarek's result.

For a set $A \subseteq \mathbb{N}$ we define a Banach space $Z(A)$ as follows. For brevity, we let $X_k = X_{n_k,p_k}$, and we let Y_k be a computable copy $l_{p_k}^{2n_k}$ containing X_k as a subspace. For each k,

- If $k \in A$ let $Z(A)_k = Y_k$;
- If $k \notin A$ let $Z(A)_k = X_k$.

We then let

$$Z(A) = \bigoplus_k Z(A)_k.$$

For any k, we write $Z(A)_{\leqslant k} = \bigoplus_{m \leqslant k} Z(A)_m$, and similarly for $Z(A)_{>k}$ or $Z(A)_{\neq k}$, all identified with subspaces of $Z(A)$.

Lemma 4.14. *For all A and k, $\partial_{n_k}(Z(A)_{\neq k}) \leqslant \sqrt{2n_{k-1}}$.*

Proof. Let $E \subseteq Z(A)_{\neq k}$ have dimension n_k. For each $m \neq k$, we let $E_m = P_{Z(A)_m}[E]$ (the orthogonal projection of E onto the m^{th} component of $Z(A)$), and note that $E \subseteq \bigoplus_{m \neq k} E_m$. So it suffices to show that for every $m \neq k$, $d_{\text{BM}}(E_m, l_2^{\dim E_m}) \leqslant \sqrt{2n_{k-1}}$. Fix $m \neq k$. There are two cases.

If $m < k$, then we use John's theorem ([Joh48], also see [Lew78]) that states that for any N-dimensional Banach space U, $d_{\text{BM}}(U, l_2^N) \leqslant \sqrt{N}$. Since $m < k$, $n_m \leqslant n_{k-1}$, and $\dim E_m \leqslant \dim Z(A)_m \leqslant 2n_m$.

If $m > k$, then we use Lewis's elaboration [Lew78] on John's theorem, which states that if U is an N-dimensional subspace of some l_p^M (for some $p > 2$), then $d_{\text{BM}}(U, l_2^N) \leqslant N^{1/2 - 1/p}$. In this case, $p_m \leqslant p_{k+1}$ and so by the choice of the latter, and the fact that $\dim E_m \leqslant \dim E = n_k$, we have

$$d_{\text{BM}}(W_m, l_2) \leqslant n_k^{1/2 - 1/p_m} \leqslant 2.$$

Since $n_0 \geqslant 2$, we have $2 \leqslant \sqrt{2n_{k-1}}$. □

The completeness is then establishes with the following proposition.

Proposition 4.15.

(a) *If A is c.e. then the space $Z(A)$ is computable (uniformly in a c.e. index of A).*
(b) *If A is cofinite then $Z(A)$ has a basis.*
(c) *If A is not cofinite then $Z(A)$ does not have a local basis structure.*

This clearly suffices to prove Theorem 4.13, as $\{e : W_e \text{ is cofinite}\}$ is Σ_3^0-complete, and every space with a basis has a local basis structure. (In other words, we are proving $(\Sigma_3^0, \Pi_3^0) \leqslant (\texttt{Basis}, \texttt{LBS}^{\complement})$).

Proof. (a) is easy: we start with $\bigoplus_k X_k$; when k enters A we extend X_k to Y_k and update the sum accordingly.

(b): Let $k^* = \max A^{\complement}$. Then $Z(A) = Z(A)_{\leqslant k^*} \oplus \bigoplus_{k > k^*} Y_k$. The first component $Z(A)_{\leqslant k^*}$ is finite-dimensional, and so has a basis. The second component is isometric with $\bigoplus_{k > k^*} l_{p_k}^{2n_k}$. For each k, $\text{bc}(l_{p_k}^{2n_k}) = 1$, and so we can build a basis for $Z(A)_{>k}$ by combining bases of each Y_k, each with basis constant 1 (as is done in the proof of Lemma 3.5).

(c): Let $k \notin A$; let $Z \subseteq Z(A)$ be a subspace with $X_k \subseteq Z$. Then $Z = X_k \oplus W$, where $W \subseteq Z(A)_{\neq k}$. By Lemma 4.14,

$$\partial_{n_k}(W) \leqslant \sqrt{2n_{k-1}}.$$

By Theorem 4.11,

$$\text{bc}(Z) \geqslant c\sqrt{m(X_k)}/\sqrt{2n_{k-1}}.$$

By the choice of X_k and of n_k, we have

$$m(X_k) \geqslant \alpha n_k^{1/2 - 1/p_k} > \alpha k^2 n_{k-1},$$

so $\mathrm{bc}(Z) \geqslant \sqrt{\alpha}ck/\sqrt{2}$. Since $A^{\complement}$ is infinite, the spaces X_k for $k \in A^{\complement}$ witness that $Z(A)$ does not have a local basis structure. ☐

This completes the proof of Theorem 4.13, given Proposition 4.12.

4.3. **Taking duals.** We now start working toward the proof of Proposition 4.12. Szarek's method relies on quotients rather than subspaces, so we need to take duals. We mention that we can handle quotients computably. Let $X = (\mathbb{R}^n, \|\cdot\|_X)$ be a Banach space, and let $q\colon X \to \mathbb{R}^m$ be linear and onto. Then $\mathbb{R}^m$ can be equipped with the quotient norm $\|\cdot\|_{X/K}$ (where K is the kernel of q), defined by

$$\|y\|_{X/K} = \min\{\|x\|_X \,:\, q(x) = y\}.$$

Observe that the $q[B_X] = B_{X/K}$ (the quotient map q maps the unit ball of X onto the unit ball of X/K.)

Lemma 4.16. *If X and q are computable, then $(\mathbb{R}^m, \|\cdot\|_{X/K})$ is a computable Banach space.*

Proof. B_X is computably compact (Proposition 1.11), and so the image $B_{X/K}$ under q is computably compact as well, and so is effectively closed and located. To compute $\|x\|_{X/K}$, we approximate which scalar multiples of x are in $B_{X/K}$. ☐

We will prove:

Proposition 4.17. *For any $n \geqslant 128$ and rational $p \in [1, 2)$ there is a computable Banach space $X_{n,p} = (\mathbb{R}^n, \|\cdot\|_{X_{n,p}})$ satisfying:*

- (i) *$m(X_{n,p}) \geqslant \alpha n^{1/p - 1/2}$;*
- (ii) *$X_{n,p}$ is a quotient of l_p^{2n}.*

Again, all the spaces will be uniformly computable. In particular, the quotient maps from l_p^{2n} to $X_{n,p}$ will be uniformly computable.

For the following, recalling that $m(X)$ depends also on the Euclidean structure of $\mathbb{R}^n$, we use the natural duality between l_2^n and itself.

Lemma 4.18. *For any finite-dimensional Banach space X over $\mathbb{R}^n$, $m(X) = m(X^*)$.*

For a proof see [Xie24, Lem. 5.2.2]. We can now explain why Proposition 4.17 implies Proposition 4.12. We observed above that for any finite-dimensional Banach space X, if X is computable then so is X^*. Also note that if $q\colon \mathbb{R}^m \to \mathbb{R}^n$ is linear and computable, then the inclusion map $q^*\colon (\mathbb{R}^n)^* \to (\mathbb{R}^m)^*$ is computable; this is independent of the norms chosen for $\mathbb{R}^n$ and $\mathbb{R}^m$, i.e., this is pure linear algebra. And of course, we recall that for $p < 2$, $(l_p^N)^*$ is isometric with l_q^N with q satisfying $1/p - 1/2 = 1/2 - 1/q$.

4.4. **Reducing to $p = 1$.** Next, we observe that it suffices to prove Proposition 4.17 for $p = 1$. Fix n, and suppose that $q\colon \mathbb{R}^{2n} \to \mathbb{R}^n$ is a (computable) quotient map, mapping $\|\cdot\|_1$ to $\|\cdot\|_{X_{n,1}}$. For $p \in (1, 2)$, we simply let $\|\cdot\|_{X_{n,p}}$ be the quotient of $l_p^{2n} = (\mathbb{R}^{2n}, \|\cdot\|_p)$ under q.

Lemma 4.19. *For $p \in (1, 2)$, $m(X_{n,p}) \geqslant (2n)^{\frac{1}{p} - 1} m(X_{n,1})$.*

Proof. By the definition of $m(X)$, it suffices to prove that

$$\|T\|_{L(X_{n,p})} \geqslant (2n)^{\frac{1}{p}-1}\|T\|_{L(X_{n,1})}$$

for every operator T. We use the sequence of inclusions

$$(2n)^{1/p-1}\|\cdot\|_1 \leqslant \|\cdot\|_p \leqslant \|\cdot\|_1$$

on $\mathbb{R}^{2n}$, equivalently, the sequence of inclusions of unit balls

$$B_{l_1^{2n}} \subset B_{l_p^{2n}} \subset (2n)^{1-1/p} B_{l_1^{2n}}.$$

Since the quotient map q is linear and maps the unit ball of l_1^{2n} to that of $X_{n,1}$, and the unit ball of l_p^{2n} to that of $X_{n,p}$, we get

$$B_{X_{n,1}} \subseteq B_{X_{n,p}} \subseteq (2n)^{1-1/p} B_{X_{n,1}},$$

in other words,

$$(2n)^{1/p-1}\|\cdot\|_{X_{n,1}} \leqslant \|\cdot\|_{X_{n,p}} \leqslant \|\cdot\|_{X_{n,1}}.$$

This implies that for any operator $T\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$\|T\|_{L(X_{n,p})} = \max_{x\neq 0} \frac{\|Tx\|_{X_{n,p}}}{\|x\|_{X_{n,p}}} \geqslant \max_{x\neq 0} \frac{(2n)^{\frac{1}{p}-1}\|Tx\|_{X_{n,1}}}{\|x\|_{X_{n,1}}} = (2n)^{\frac{1}{p}-1}\|T\|_{L(X_{n,1})}.$$

□

Thus, it remains to prove:

Proposition 4.20. *For each $n \geqslant 128$ there is a computable Banach space $X_n = (\mathbb{R}^n, \|\cdot\|_{X_n})$ satisfying:*
- (i) *$m(X_n) \geqslant \alpha\sqrt{n}$ (where α is a universal constant);*
- (ii) *X_n is a quotient of l_1^{2n}.*

The spaces X_n and the quotient maps $q_n\colon l_1^{2n} \to X_n$ are uniformly computable.

4.5. **The set of mixing operators.** We will need to quantify over mixing operators, and so, we need some effective compactness. For the following proof, we will use a computable structure on the orthogonal group. Recall that $O_n(\mathbb{R})$ is the collection of $(n \times n)$-orthogonal matrices, which we also identified with operators in $L(\mathbb{R}^n)$. For any invertible $A \in \mathrm{GL}_n(\mathbb{R})$, let $\mathrm{GS}(A)$ be the result of applying the Gram-Schmidt process to the columns of A. Then GS is a partial computable function on $L(\mathbb{R}^n)$, as it is produced by composing partial computable functions such as $x \mapsto x/\|x\|_2$ (defined for $x \neq 0$) and $(x, y) \mapsto \langle x, y\rangle$.

Now $\mathrm{GL}_n(\mathbb{Q})$ is a computable subset of the collection $M_n(\mathbb{Q})$ of ideal points of $L(\mathbb{R}^n)$, and so we can view its image under the Gram-Schmidt process as a computable sequence of points in $O_n(\mathbb{R})$. This image is dense because GS is continuous on $\mathrm{GL}_n(\mathbb{R})$ and $O_n(\mathbb{R}) = \operatorname{range} \mathrm{GS}$ (indeed, $\mathrm{GS}^2 = \mathrm{GS}$). Hence, we see that $O_n(\mathbb{R})$ is an effectively closed and located subset of $L(\mathbb{R}^n)$. Since it is bounded, it is computably compact.

Remark 4.21. $O_n(\mathbb{Q})$ is actually dense in $O_n(\mathbb{R})$, but this is a bit more complicated to prove.

For an operator $T \in L(\mathbb{R}^n)$ and $k \leqslant n$, let

$$M(T, k) = \sup\{\gamma : T \text{ is } (k/n, \gamma)\text{-mixing}\}.$$

Lemma 4.22. *The function $T \mapsto M(T,k)$ (from $L(\mathbb{R}^n)$ to $\mathbb{R}$) is computable, uniformly in n and k.*

Proof. Recall that

$$M(T,k) = \sup\left\{\sigma_k(P_{E^\perp}T\restriction E) \,:\, E \subseteq \mathbb{R}^n \text{ is } k\text{-dimensional}\right\}.$$

To quantify over subspaces, we could give a computable metric structure to the collection of all k-dimensional subspaces (known as the Grassmannian). This is possible, but not necessary: we can use the orthogonal group instead. Given $A \in O_n(\mathbb{R})$, let $E_A = E_{A,k}$ be the subspace of $\mathbb{R}^n$ spanned by the first k-columns of A. This is of course k-dimensional, and comes equipped with a basis (those columns of A). The last $(n-k)$-many columns of A are an orthonormal basis of $E_A^\perp$, from which we can find the matrix defining the orthogonal projection $P_{E_A^\perp}$ (using the standard basis of $\mathbb{R}^n$). Given a matrix defining T, we can now compute a matrix defining $P_{E_A^\perp}T\restriction E_A$ (using the given basis for E_A in the domain and the standard basis of $\mathbb{R}^n$ for the range), and so we can compute σ_k of this operator (Lemma 4.6). Of course, it is important to note that every k-dimensional subspace E is E_A for some $A \in O_n(\mathbb{R})$; we managed to quantify over all subspaces, and it does not matter that we "repeated" subspaces ($E_A = E_B$ for distinct A, B). Note that we needed something like $O_n(\mathbb{R})$ (rather than say $M_n(\mathbb{R})$), since given a k-tuple of elements of $\mathbb{R}^n$, we cannot computably tell whether they span a k-dimensional subspace or not.

Since the function $(T,A) \mapsto \sigma_k(P_{E_A^\perp}T\restriction E_A)$ is computable, and $O_n(\mathbb{R})$ is computably compact, we can use Proposition 1.10 to show that $T \mapsto M(T,k)$ is computable as well. ☐

4.6. **Parameterized spaces.** The construction of the spaces X_n guaranteed by Proposition 4.20 is not direct. Rather, it is done by a computable search on a collection of spaces (i.e., a collection of norms on $\mathbb{R}^n$). This is because the existence proof of such spaces is based on the probabilistic method.

Fixing $n \geqslant 128$, let $m = m(n) = \lfloor n/128 \rfloor$. We will build X_n as a quotient of l_1^{n+m}, which is itself a quotient of l_1^{2n}.

For $\mathbf{g} = (g_1, \ldots, g_m) \in (\mathbb{R}^n)^m$, we define

$$q_{\mathbf{g}} : \mathbb{R}^{n+m} \to \mathbb{R}^n$$

be the linear map determined by mapping:

- For $i = 1, \ldots, n$, $e_i \in \mathbb{R}^{n+m} \mapsto e_i \in \mathbb{R}^n$;
- For $j = 1, \ldots, m$, $e_{n+j} \in \mathbb{R}^{n+m} \mapsto g_j \in \mathbb{R}^n$.

We let $\|\cdot\|_{\mathbf{g}}$ be the quotient norm on $\mathbb{R}^n$ using this map. The space $(\mathbb{R}^n, \|\cdot\|_{\mathbf{g}})$ is denoted by $X_{\mathbf{g}}$. The uniformity of Lemma 4.16 shows that the norm $\|\cdot\|_{\mathbf{g}}$ is computable from $\mathbf{g}$, that is, the map

$$(\mathbf{g}, x) \mapsto \|x\|_{\mathbf{g}}$$

(from $(\mathbb{R}^n)^m \times \mathbb{R}^n$ to $\mathbb{R}$) is computable. In particular, if $\mathbf{g}$ is computable, then $\|\cdot\|_{\mathbf{g}}$ and $q_{\mathbf{g}}$ are computable, and this is uniform, in $\mathbf{g}$ and in n.

We let

$$L = L_n = \left\{\mathbf{g} \in (\mathbb{R}^n)^m \,:\, (\forall j \leqslant m)\ 1/2 \leqslant \|g_j\|_2 \leqslant 2\right\}.$$

This set is computably compact.

Lemma 4.23. *for all* $\mathbf{g} \in L$,

$$B_{l_1^n} \subseteq B_{X_{\mathbf{g}}} \subseteq 2B_{l_2^n} \subseteq 2\sqrt{n}B_{l_1^n}.$$

Proof. The unit ball of any Banach space is *absolutely convex*: it is convex and symmetric about the origin. The unit ball $B_{l_1^n}$ is the *absolute convex hull* of the standard unit vectors $\{e_1, \dots, e_n\}$ (the smallest absolutely convex subset of $\mathbb{R}^n$ containing these vectors). This gives the first containment. Similarly, since $B_{l_1^{n+m}}$ is the absolutely convex hull of the unit vectors in $\mathbb{R}^{n+m}$, the definition of $q_{\mathbf{g}}$ shows that $B_{X_{\mathbf{g}}}$ is the absolutely convex hull of $\{e_1, \dots, e_n, g_1, \dots, g_m\}$. Since $\|e_i\|_2, \|g_j\|_2 \leqslant 2$ when $\mathbf{g} \in L$, and $B_{l_2^n}$ is absolutely convex, we obtain the second containment. □

Lemma 4.23 implies that the closed unit ball $B_{X_{\mathbf{g}}}$ is $\mathbf{g}$-computably compact, uniformly in $\mathbf{g}$. Also, as in the proof of Lemma 4.19, Lemma 4.23 implies:

Lemma 4.24. *For all* $\mathbf{g} \in L$*, for every operator* $T \in L(\mathbb{R}^n)$,

$$\|T\|_{L(l_1^n)} \geqslant \|T\|_{L(X_{\mathbf{g}})} \geqslant \frac{1}{2\sqrt{n}}\|T\|_{L(l_1^n)}.$$

What we need is:

Proposition 4.25 (Szarek [Sza86])**.** *There is some* $\alpha > 0$ *such that for all* $n \geqslant 128$ *there is some* $\mathbf{g} \in L = L_n$ *such that*

$$m(X_{\mathbf{g}}) > \alpha\sqrt{n}.$$

We of course may take α be rational. The proof of Proposition 4.25, which as mentioned relies on the probabilistic method, is complicated. For a detailed argument see [Xie24, § 5.4].

We can now prove Proposition 4.20, and so Proposition 4.17, and so Proposition 4.12, and so Theorem 4.13 and Theorem 4.2.

Proof of Proposition 4.20. Fix n; let $k = \lceil n/32 \rceil$. Let B be the closed ball $\overline{B}_{L(l_1^n)}(0, \alpha n)$. Since B is computably compact (uniformly in n), we have a computable modulus of uniform continuity of the computable function $T \mapsto M(T, k)$ on B; so we can compute some $\varepsilon > 0$ such that if $T, S \in B$ and $\|S - T\|_{L(l_1^n)} \leqslant \varepsilon$ then $|M(S, k) - M(T, k)| \leqslant 1/2$. We also ensure that $\varepsilon < \alpha\sqrt{n}/4$. Again since B is computably compact, we can effectively find a finite ε-cover C of B.

We now let

$$W = \left\{ \mathbf{g} \in L \,:\, \forall T \in C \; \left(M(T, k) \geqslant 1/2 \implies \|T\|_{L(X_{\mathbf{g}})} > \alpha\sqrt{n}/2 \right) \right\}.$$

We make three observations:

- W is an effectively open subset of L. This is because $(\mathbf{g}, T) \mapsto \|T\|_{L(X_{\mathbf{g}})}$ is computable.
- $W \neq \varnothing$. This follows from Proposition 4.25: suppose that $\mathbf{g} \in L$ and $m(X_{\mathbf{g}}) > \alpha\sqrt{n}$. Then for all T, if $M(T, k) \geqslant 1/2$ then $M(2T, k) \geqslant 1$ and then $\|2T\|_{L(X_{\mathbf{g}})} > \alpha\sqrt{n}$.
- For all $\mathbf{g} \in W$, $m(X_{\mathbf{g}}) \geqslant \alpha\sqrt{n}/4$. For let $T \in L(\mathbb{R}^n)$ and suppose that $m(T, k) \geqslant 1$. We need to show that $\|T\|_{L(X_{\mathbf{g}})} \geqslant \alpha\sqrt{n}/4$. There are two cases.

  If $T \in B$, then we find some $S \in C$ with $\|S - T\|_{L(l_1^n)} \leqslant \varepsilon$. Then by the choice of ε, we have $M(T, k) \geqslant M(S, k) - 1/2 \geqslant 1/2$. Since $\mathbf{g} \in W$,

$\|S\|_{L(X_{\mathbf{g}})} \geqslant \alpha\sqrt{n}/2$. By Lemma 4.24, $\|S-T\|_{L(X_{\mathbf{g}})} \leqslant \|S-T\|_{L(l_1^n)} \leqslant \varepsilon \leqslant \alpha\sqrt{n}/4$, so $\|T\|_{L(X_{\mathbf{g}})} \geqslant \alpha\sqrt{n}/4$ as required.

If $T \notin B$, i.e., if $\|T\|_{L(l_1^n)} \geqslant \alpha n$, then by Lemma 4.24, $\|T\|_{L(X_{\mathbf{g}})} \geqslant \alpha\sqrt{n}/2$.

To produce $\mathbf{g}$ as required, effectively given n, we simply run the enumeration of W until we see some special point enter that set. □

5. Summary of results

We summarise the index set calculations in this paper.

(a) $\Pi^0_3 \leqslant$ `Basis` $\leqslant \Sigma^1_1$: Definition 1.3, Proposition 3.7,Corollary 2.2.
(b) $\Pi^0_3 \leqslant$ `FDD` $\leqslant \Sigma^1_1$: Definition 3.9, Proposition 3.10, Corollary 3.12.
(c) $\Pi^0_3 \leqslant$ `AP` $\leqslant \Pi^1_1$: Definition 3.2, Remark 3.8, Proposition 3.17.
(d) $\Pi^0_3 \leqslant$ `BAP` $\leqslant \Sigma^0_4$: Definition 3.18, Remark 3.8, Proposition 3.20.
(e) `LBS` is Σ^0_3-complete: Definition 4.1, Proposition 4.3, Theorem 4.13.

We also showed that the index sets `CompBasis` and `CompFDD` (Definition 3.13) are Σ^0_3-complete (Proposition 3.15).

5.1. Further results.

Qian studied further related properties, such as being a π-space, having a local Π-basis structure, and the commuting bounded approximation property. Qian also observed that a construction by Bossard [Bos02] shows that the index set of *reflexive* spaces is Π^1_1-complete. Reflexive spaces, interesting in their own right, are related to strong notions of bases, such as *shrinking* bases, which are in turn useful in constructing bases of dual spaces. For details, see [Qia21].

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