Some Open Questions and Recent Results on Computable Banach Spaces*

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Abstract. We discuss some open questions and results in the geometry of computable Banach spaces.

1 Introduction

1.1 Banach Spaces

Recall that a normed vector space has associated with it a distance $d(x, y) = \|x - y\|$. If this is a complete metric space, it is called a Banach space. Banach spaces are fundamental to the field of functional analysis, and have extensive applications. The modern theory of computable Banach spaces likely began with the work of Pour-El and Richards [PER83] who showed how the effective theory gave insight into issues from classical physics. Brattka [Bra16] looked at the effective content of basic results from the area including the Open Mapping Theorem, the Closed Graph Theorem, and the Banach-Steinhaus Theorem, and Brattka [Bra16] and earlier Metakides-Nerode-Shore [MNS85] and others studied the important Hahn-Banach Theorem's computable content.

In this paper we will highlight some recent work concerning the algorithmic content of work around the geometry of Banach spaces, specifically those associated with bases, and decompositions.

We remark that the questions provide a fascinating "logician's eye view" of classical constructions, in that it seems that *all* of the classical constructions are insufficient to answer some of the basic questions such as the complexity of finding a Schauder basis.

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1.2 Computable Banach Spaces

Going back to Turing, the fundamental concept of computable analysis is that of a computable real number r, which is one for which there is a computable sequence of rationals (q_i) such that $|r - q_i| < 2^{-i}$. That is, there is a computable *fast Cauchy sequence* with limit r. Such a sequence is also known as a *Cauchy name* of r. The reader unfamiliar with modern computable analysis, might guess that a computable function on the reals is one effectively taking computable reals to computable reals, and this was Turing's [Tur36] intuition, but the modern "type 2" definition of a computable function on the reals, is one that acts effectively on *all* reals: it is a function $f\colon \mathbb{R} \to \mathbb{R}$, induced by a computable functional acting on fast Cauchy sequences, taking any Cauchy name of a real r and producing a Cauchy name of $f(r)$. Note that this definition means that all computable functions on the reals must be continuous, and, indeed, g is continuous iff it is computable relative to some oracle.

The notion of a *computable metric space* is a natural generalization of this approach. This is a complete metric space (X, d), equipped with a sequence of points (q_i), dense in X, restricted to which the metric is computable: that is, the reals $d(q_i, q_j)$ are computable, uniformly given i and j. Using the points (q_i) as analogs of the rational numbers, we can similarly define Cauchy names of points of X, and computable functions between computable metric spaces. Using this notion of computability, we can now define:

Definition 1. *A* computable Banach space *is a computable metric space equipped with a compatible, computable normed vector space structure. That is, addition, scalar multiplication, and the norm, are all computable functions.*

1.3 Generalized computable Banach spaces

One of the guiding principles in the study of effective (computable) structure theory is that most *natural* structures studied in classical mathematics have natural computable presentations. Here "natural" is vaguely defined, but we mean structures arising in, for example, applied mathematics or physics.

This is true for many Banach spaces. For example, Hilbert spaces have computable representations. Other examples include the spaces ℓ^p_n and ℓ^p for computable $p \geq 1$ ($\mathbb{R}^n$ equipped with the p-norm, and the space of p-summable infinite sequences of reals); more generally, $L^p(\Omega)$ spaces for a variety of measure spaces Ω; the space c_0 of infinite sequences of reals converging to 0; the space $C[0,1]$ of continuous functions from the unit interval to $\mathbb{R}$ (equipped with the supremum norm; here as analogs of the rational numbers we can take a suitable sequence of polynomials). There are many other examples.

However, some very natural Banach spaces are missing from this list, starting with the space ℓ^∞ of bounded sequences of reals, equipped with the supremum norm. The problem is that this space is not *separable*, so even the underlying metric space cannot be given a computable structure, using the definition above. In some sense this does point at a deficiency in the definition, in that ℓ^∞ is surely a "natural" space. Researchers in computable analysis have defined more general representations of computable spaces (see Weihrauch [Wei00]). However, all continuous representations are necessarily restricted to separable spaces. Indeed, Brattka [Bra16, Prop. 15.3] observed that there is no representation of ℓ^∞ providing the expected notion of computable points, and for which vector addition is computable. Brattka proposed to omit the norm, and rather, concentrate on convergence. He defined the notion of a *general computable normed space*, which is a represented space in which the operation taking (names of) fast converging Cauchy sequences to (a name of) the limit of the sequence, is a computable function on names. The natural representation of ℓ^∞ is a general computable normed space.

The reason this notion is particularly interesting is that the theory of Banach spaces is replete with results involving the *dual* space. If B is a computable and hence separable Banach space, then its dual is not necessarily separable, but as Brattka [Bra16] showed, it is always a general computable Banach space. The non-computability of the dual space is a great impediment to the development of theory of computable Banach spaces. It means that alternative methods must be found to replace classical arguments using the dual, as we will see, for instance, in the proof of Theorem 8.

We remark that sometimes, the dual space is computable, such as in the finite dimensional case, and more generally, when the space has a well-behaved basis. We will discuss bases in §3 below. We mention here that Brattka and Dillhage [BD07] have a number of results when a space has a nice computable basis ("shrinking" for instance; see [BD07, Cor. 5.9]). We believe that this area is rife with interesting questions.

Question 1 Suppose that B is a computable Banach space. Under what circumstances is the dual of B computable? Suppose that X is a general computable Banach space. Under what circumstances is it isomorphic to the dual of a computable Banach space? More generally, develop the theory of general computable Banach spaces.

We also remark that while the norm of a dual space may not be computable, the dual of a computable Banach space has a natural representation in which the norm of an element is (uniformly) left-c.e. in the name. This is because the unit sphere of a computable Banach space is a computable closed (located) set. It may be interesting to investigate this as an alternative or an added requirement to general computable spaces.

2 Some classical effectivity results

Some of the best known results in computable Banach space theory are due to Pour-El and Richards e.g. [PER83,PER89]. One of the classic results was to effectivize the classical theorem that an operator on a Banach space is continuous iff it is bounded.

Theorem 1 (Pour-El and Richards [PER83]). *Let X, Y be computable Banach spaces, and (e_i) be a computable sequence in X whose linear span is dense. Let $T : X \to Y$ be a linear operator with closed graph whose domain contains $\{e_i\}$ and such that the sequence $(T(e_i))$ is computable in Y. Then T maps every computable element of its domain onto a computable element of Y if and only if T is bounded.*

Theorem 1 has many applications. For example, it shows that the indefinite integral of a computable function $f \in C[a, b]$ is computable. It also can be used to give a proof of a Theorem of Myhill that there exists computable functions in $C[a, b]$ which have continuous derivatives, but whose derivatives are not computable.

From the point of computable structure theory, being an analytic structure defined via a computable dense sequence means that we can code up the structures via countable (computable) information, and hence the usual methods and questions from computable structure theory (such as e.g. Ash-Knight [AK00]) apply. Indeed, this is the thesis of a recent book [DMNar], which gives a unified view of computable structure theory, both countable and analytic.

For example, now we can think of computable Banach spaces as c.e. sets and hence associate indices to the structures. We can then look at, for example, the complexity of isomorphism and classification. Whilst this is not the main business of the present paper, we mention some recent results of this ilk.

As well as Banach spaces, computable metric spaces, computable locally compact topological groups, and the like, have been investigated. For example, Melnikov and Nies [MN13] showed that *compact* computable metric spaces could be classified by a Π^0_3 effective formula and all were Δ^0_3 categorical, and hence were relatively simple to classify logically, whereas Nies and Solecki [NS15] proved that the characterisation problem for computable *locally* compact metric spaces is Π^1_1-complete, meaning that it is as hard as *any* isomorphism problem for countable structures. Associated results are reported in the survey [DM20].

Various families of Banach spaces have been studied in this way. For example, computable Lebesgue spaces have a Π^0_3 characterization ([BMM])), and $C[0, 1]$ also has an arithmetical characterization ([FHD$^+$20]). The general classification problem is hard.

Theorem 2 (Downey and Melnikov [DM23]). *The isomorphism problem for computable Banach spaces is Σ^1_1-complete.*

Proof. (sketch) The upper bound is Σ^1_1 since it is sufficient to state that there is an isometry that works for special points, maps zero to zero, and is, furthermore, surjective (these properties are closed). The well-known Mazur-Ulam theorem asserts that every isometry with these properties has to be linear. Completeness follows from the Σ^1_1-completeness for Boolean algebras, as follows. First, uniformly produce the computably compact Stone space $\widehat{B}$ of a given Boolean algebra B, and then consider $C(\widehat{B}; \mathbb{R})$ whose computable Banach space structure can be produced uniformly effectively ([BHTM21]) from the compact presentation of the space. It is well-known that the homeomorphism type of the compact domain determines the linear isomorphism type of the resulting space, and vice versa (this is Banach-Stone duality). This gives the Σ^1_1-completeness. $\square$

We remark that Ferenczi, Louveau and Rosendal [FLR09] showed a similar result in the context of Borel equivalence relations. Their construction is direct.

Question 2 (Melnikov) For each n, is there Δ^0_{n+1}-categorical but not Δ^0_n-categorical Banach space? Same for Polish groups.

A somewhat related question concerns the (Anderson-)Kadets (Kadec) Theorem [Kad66] which states that *any two infinite dimensional separable Banach spaces are homeomorphic as* topological *spaces, and hence homeomorphic to* $\mathbb{R}^{\mathbb{N}}$. The result is also true for a more general class called Fréchet spaces.

Question 3 Is Kadets' Theorem true effectively?

The published proofs all involve complex methods involving duality and the effectivity is by no means clear.

3 The geometry of computable Banach spaces

We turn to the main concern of this paper. The theory of finite dimensional vector spaces revolves around the notion of a basis, specifically a *Hamel basis* where every element is a *finite* linear combination of basis elements. In Banach spaces, the picture is again murky. If B is an infinite dimensional Banach space then every Hamel basis must be uncountable. But spaces like ℓ^p are in some sense coded by countable information. One of the main basis notions for Banach spaces is the following.

Definition 2 (Schauder [Sch28]). *Let X be a Banach space. A sequence $(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$ is a* Schauder basis *of X if for all $x \in X$, there is a unique sequence of coefficients $(a_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that*

$$\sum_{i=1}^{\infty} a_i x_i = x$$

A sequence that is the Schauder basis of the closure of its linear span is called a basic sequence.

We emphasise that a Schauder basis is a *sequence* so that order counts. The standard unit vectors for ℓ^p give a Schauder basis as is every orthonormal basis of a Hilbert space. Haar [Haa10] gave a Schauder basis for $L^p(0,1)$ for $1 \le p < \infty$.

Note that every Banach space with a basis must be separable, and in his famous book [Ban32], Banach asked if every separable Banach space has a Schauder basis. There was a huge effort towards solving the basis problem. As a part of the effort, many important properties regarding the geometry of Banach spaces were identified; especially those that were implied by the existence of a Schauder basis. In this paper we will look at some of these concepts and questions of effectivity concerning these geometric considerations.

It was only after 40 years that Banach's question was solved by Per Enflo [Enf73], and he did this by showing that there was a Banach space without something called *the approximation property* (Definition 6), which is a consequence of having Schauder basis. In his PhD Thesis, Bosserhof proved that Enflo's example can be made computable.

Theorem 3 (Bosserhof [Bos08]). *There is a computable copy of Enflo's example, and hence there is a computable Banach space without the approximation property and hence without a Schauder basis.*

Our fundamental question is the following:

Question 4 What is the complexity of having a basis? Specifically, what is the complexity of the index set of computable Banach spaces that have a basis?

To establish an upper bound on this complexity, we need the following fundamental fact about Schauder bases.

Lemma 1 (Banach e.g. in [Ban32]). *Let X be a Banach space and $(x_i)_{i\in\mathbb{N}} \subseteq X$ a sequence of nonzero elements. Then (x_i) is a basis of X if and only if:*

1. *There is a constant $K \in \mathbb{R}$ such that for all $n, m \in \mathbb{N}$ with $m < n$, for all sequences of scalars $(a_i)_{i\in\mathbb{N}}$, we have*
$$\left\|\sum_{i=1}^{m} a_i x_i\right\| \leqslant K \left\|\sum_{i=1}^{n} a_i x_i\right\|$$
2. *The finite linear span of $(x_i)_{i\in\mathbb{N}}$ is dense in X.*

The proof of the harder direction of Lemma 1 consists of considering the projections $\{S_i\}_{i\in\mathbb{N}}$ associated with the basis (x_i), defined by $S_k\left(\sum_{i=0}^{\infty}\alpha_i x_i\right) = \sum_{i=0}^{k}\alpha_i x_i$. Then (1) is equivalent to requiring the value $\sup_i \|S_i\|$ to be finite. To show the lemma, define the alternate norm $\|\cdot\|_b$ on X by $\|\sum_{i=0}^{\infty}\alpha_i x_i\|_b = \sup_n \|\sum_{i=0}^{n}\alpha_i x_i\|$. Note that this is well-defined as $(\sum_{i=0}^{n}\alpha_i x_i)_n \to \sum_{i=0}^{\infty}\alpha_i x_i$, so $\|\cdot\|_b$ is finite on any $v \in X$. Furthermore, $\|\cdot\|_b$ is indeed a norm on X, and $\|v\| \leq \|v\|_b$ for all $v \in X$. In fact, it is not hard to show that $(X, \|\cdot\|_b)$ is complete as well. An application of the open mapping theorem then proves that the norms $\|\cdot\|, \|\cdot\|_b$ are equivalent. Lemma 1 leads to the following fundamental concept.

Definition 3. *Let X be a Banach space and $(x_i)_{i\in\mathbb{N}}$ be a basis of X, and $\{S_i\}_{i\in\mathbb{N}}$ its associated sequence of projections. The* basis constant *of (x_i), denoted as $bc((x_i))$, is the value $\sup_i \|S_i\|$. Note that $bc((x_i))$ is equivalent to the infimum of all K that satisfies the requirements of Lemma 1. The basis constant of the space X, denoted $bc(X)$, is the infimum of basis constants across all of its bases. We set $bc(X) = \infty$ if X has no basis.*

The reader unfamiliar with Banach spaces might think that, like Hilbert spaces, there is always a Schauder basis with constant 1 if there is a basis. Such a basis is called *monotone*. Unfortunately, Szarek [Sza83] showed that there is a *finite* dimensional space which does *not* has a basis with basis constant 1, and in fact, there are finite-dimensional spaces with arbitrarily large basis constant. Recently, Ruofei Xie proved that Szarek's [Sza87] construction can be made effective.

Theorem 4 (Xie [Xie24]). *For each k, for sufficiently large n, there is a computable norm on $\mathbb{R}^n$ whose associated basis constant is greater than k.*

For finite dimensional spaces things are somewhat nice:

Lemma 2 (Bosserhof [Bos08]). *Let X be a computable Banach space, and $\{x_0, \ldots, x_n\}$ be a computable sequence of independent points. Then $bc(x_0, \ldots, x_n)$ is computable, uniformly in $\{x_0, \ldots, x_n\}$.*

Proof. Let $[x_0, \ldots, x_n]$ denote the space spanned by the points. Since the basis constant is the maximum of the norms of the associated projections, it suffices to observe that given an operator on a finite-dimensional computable Banach space, we can compute its norm. To do this, we use the fact that the unit ball of a finite-dimensional Banach space is compact, and if the space is computable, then the unit ball is computably compact. The maximum of a real-valued function on a computably compact set is computable, uniformly. □

The following improves an earlier result of Bosserhof [Bos08] who observed that basis constants of finite dimensional spaces are right c.e.

Lemma 3. *Let X be a computable Banach space, and $\{x_0, \ldots, x_n\}$ be a computable sequence of linearly independent points. Then $bc([x_0, \ldots, x_n])$ is computable. Furthermore, this is uniform in $\{x_0, \ldots, x_n\}$.*

Note that here we are computing the basis constant of the space, not of the particular basis.

Proof. Denote $D = [x_0, \ldots, x_n]$, and let $(v_i)_{i \leq n}$ be an arbitrary sequence of elements in D. By definition, we may write $v_i = \sum_{j=0}^{n} \alpha_{i,j} x_j$, so the sequence $(v_i)_{i \leq n}$ is uniquely characterised by the sequences of coefficients

$$\alpha_{0,0}, \alpha_{0,1}, \ldots, \alpha_{0,n}, \alpha_{1,0}, \ldots, \alpha_{n,0}, \ldots, \alpha_{n,n}$$

Furthermore, as scalar scaling preserves the basis constant of $(v_i)_{i \leq n}$, we can assume without loss of generality that $\sum_{i=0}^{n} \sum_{j=0}^{n} |\alpha_{i,j}| = 1$. Consider the natural mapping $f : (\mathbb{R}^{n \times n}, \|\cdot\|_1) \to D^n$ given by $f\left((\alpha_{i,j})_{i,j \leq n}\right) = \left(\sum_{j=0}^{n} \alpha_{i,j} x_j\right)_i$. Under this mapping, we can naturally regard each basis of D as an element in the image. Therefore, the basis constant of D is equivalent to the minimum of basis constants on f's image. Now note that $\sum_{i=0}^{n} \sum_{j=0}^{n} |\alpha_{i,j}| = 1$ is an effectively compact subset of $(\mathbb{R}^{n \times n}, \|\cdot\|_1)$ and that f is a computable mapping. As with maxima, the minimum of a real-valued computable function on a computably compact set is computable. □

The proof of Lemma 1 combined with 3 shows that if I give you a computable Schauder basis of a Banach space, then we can approximate the basis constant via the sequence of finite dimensional projections, and hence have the following.

Lemma 4. *Let X be a computable Banach space and $(x_i)_{i\in\mathbb{N}}$ a computable basis of X; then $bc((x_i))$ is a left-c.e. real.*

This result has an easy converse.

Theorem 5. *For any $\alpha \in \mathbb{R}$ that is left-c.e and $\alpha \geq 1$, there is Banach space X with basis $(e_i)_{i\in\mathbb{N}}$ such that $bc((e_i)_{i\in\mathbb{N}}) = \alpha$.*

Proof. In fact, we will show that it is sufficient to have $X = c_0$. Let $(e_i)_{i\in\mathbb{N}}$ denote the standard basis, the idea is to replace blocks of $\{e_i, e_{i+1}\}$ by $\{e_i + e_{i+1}, e_i + \beta_i e_{i+1}\}$, where β_i is some parameter in $\mathbb{Q}$. And since $\mathrm{bc}(e_i + e_{i+1}, e_i + \beta_i e_{i+1})$ is simply a computable function continuous in β_i, we can choose β_i so that $\mathrm{bc}(e_i + e_{i+1}, e_i + \beta_i e_{i+1}) = \alpha_i$, where (α_i) is a computable sequence of rationals increasing to α. Since the blocks are disjoint, the basis constants of the prefixes of the modified basis will form the sequence $\{\alpha_0, \alpha_1, \ldots\}$. $\square$

It is also possible to use a coding argument to show that if a computable Banach space X has a basis of Turing degree $\mathbf{a}$ then it has one of every degree $\geq \mathbf{a}$. Roughly speaking we can prove this by showing that a Schauder basis can be replaced by one using the ideal points defining the underlying computable metric space structure.

Question 5 Let X be a computable Banach space with basis. What is the complexity of $\mathrm{bc}(X)$? What if X has a computable basis?

Theorem 6 (Bosserhof [Bos08]). *There is a computable Banach space with a Schauder basis, but no computable Schauder basis.*

Question 6 Suppose that computable X has no computable Schauder basis but does have a basis. What complexity basis does it have?

Question 7 (Bosserhof [Bos08]) Suppose that a computable Banach space X has a monotone Schauder basis. Must X have a computable Schauder basis?

Bosserhof's construction gives a computable presentation of a Banach space with a basis and no computable one. It leaves open the question:

Question 8 Is there a computable Banach space X with a basis such that no computable presentation of X has a computable basis? Is having a computable basis presentation dependent amongst computable presentations?

Theorem 7 ([Qia21]). *The index set of computable Banach spaces with computable Schauder bases is Σ^0_3-complete.*

A theorem attributed to Mazur shows that every infinite dimensional Banach space (separable or otherwise) has a an infinite dimensional subspace with a Schauder basis. If we restrict ourselves to the linear structure, the computable analogue of Mazur's theorem fails: there is a computable, infinite-dimensional vector space, all of whose computable independent subsets are finite (Metakides and Nerode [MN77]). However, in the normed context, Mazur's theorem has a computable version.

Theorem 8. *Let X be an infinite dimensional computable Banach space, then there is a computable basic sequence in X.*

The proof of this theorem (given in the appendix) relies on methods quite distinct from the classical case, which heavily uses duality. Note that it leaves the following question open.

Question 9 Suppose that X is a general computable infinite dimensional Banach space. Does X have an infinite basic sequence? More generally, how complicated are the basic sequences in X?

Returning to the general basis question, the characterisation in terms of basis constants shows that the index-set of computable Banach spaces with bases is Σ^1_1. Is this set Σ^1_1-complete? We can prove Π^0_3-hardness, but this leaves an enormous gap.

One of the reasons this question is difficult, is that the known constructions of spaces without bases do so by producing spaces without other properties, that follow from having a basis, but are each weaker than having a basis. In most cases, these properties are known to be arithmetical, and so these constructions cannot be used to show Σ^1_1-completeness of having a basis. In turn, the complexity of having each of these properties is interesting in its own right, and in most cases is still open. We mention three such properties here; for more details, see [Qia21,JL01].

Definition 4 (Schauder decomposition). *Let X be a Banach space. A* Schauder decomposition *(SD) of X is an infinite sequence $(Z_i)_{i\in\mathbb{N}}$ of closed subspaces of X such that for all $x \in X$, there exists an unique sequence $(z_i)_{i\in\mathbb{N}}$, $z_i \in Z_i$ such that*

$$x = \sum_{i=1}^{\infty} z_i.$$

A Schauder decomposition where the spaces Z_i are all finite dimensional is called a finite dimensional Schauder decomposition *(FDD).*

If a Banach space X has a Schauder basis $(e_i)_{i\in\mathbb{N}}$, we can think of X being decomposed into one-dimensional spaces of the form $X = \mathrm{span}(e_0) \oplus \mathrm{span}(e_1) \oplus \ldots$. Schauder decompositions are then equivalent to requiring X to be decomposed into *closed subspaces* in the form $X = M_1 \oplus M_2 \oplus M_3 \oplus \ldots$, where the spaces M_i are no longer required to be one-dimensional. Finite dimensional Schauder decompositions simply enforces the spaces $\{M_i\}$ to be finite dimensional. Szarek [Sza87] proves these properties are strictly weaker than having a Schauder basis.

Definition 5 (Local basis structure). *Let X be a Banach space. X is said to have the* local basis structure *(LBS) if there is some constant $K \in \mathbb{R}$ such that for any finite dimensional subspace $B \subset X$, there exists a finite dimensional space $L \subset X$ such that $B \subseteq L$ and $bc(L) \leq K$.*

X having the local basis structure means it can be *approximated* by a sequence of finite dimensional subspaces, where each one of them have a "nice" basis of low basis constant. It accords with the intuition that we can build a Schauder basis by finite extension, in the same way we build a Hamel basis in the finite dimensional case. It is not unreasonable to wonder if LBS in fact *equivalent* to having a basis. Since it might seem that we can always build a basis using LBS by inductively extending the current "basis elements" $\{b_0, \ldots, b_n\}$ to a bigger space $E \supseteq \text{span}\{b_0, \ldots, b_n\}$ which still has a bounded basis constant. However, the problem with this line of reasoning is that while we are guaranteed $\text{bc}(E) \leq K$ for some universal constant K, this only means that *some* basis of E has a low basis constant. It might be the case that no basis of E which extends the current "candidate basis" $\{b_0, \ldots, b_n\}$ has its basis constant bounded by K. As it turns out, this is indeed the case as shown by the original construction by Enflo in [Enf73], which has LBS yet lacks any basis. The locality of LBS is the reason that the associated index set is Σ^0_3, and indeed, this simplicity, together with the techniques required for Theorem 4, gives us the only known completeness result in this area:

Theorem 9 (Xie [Xie24]). *The index-set of computable Banach spaces with the local basis structure is Σ^0_3-complete.*

Definition 6 (Approximation property).

1. *Let X be a Banach space. X is said to have the* approximation property *(AP) if for all compact sets K, for all $\epsilon > 0$, there is a finite rank operator T on X such that $(\forall x \in K)\, (\|Tx - x\| < \epsilon)$.*
2. *Let X be a Banach space. X is said to have the* bounded approximation property *(BAP) if there is a $\lambda \geq 1$ such that for all compact sets K, for all $\epsilon > 0$, there is a finite rank operator T on X such that $(\forall x \in K)\, (\|Tx - x\| < \epsilon)$ and $\|T\| \leq \lambda$.*

Szarek's construction produces a space with the bounded approximation property (and a finite-dimensional decomposition) but which does not have the local basis structure.

For index-sets, we have the following bounds. For a property X, let X_I denote the index-set of computable Banach spaces with property X.

Theorem 10.

1. *$\Pi^0_3 \leq BASIS_I \leq \Sigma^1_1$.*
2. *$\Pi^0_3 \leq BAP_I \leq \Sigma^0_4$.*
3. *$\Pi^0_3 \leq AP_I \leq \Pi^1_1$.*
4. *$\Pi^0_3 \leq FDD_I \leq \Sigma^1_1$.*

5. $\Pi^0_3 \le SD_I \le \Sigma^1_1$.

The reader can see that there are *many* gaps in the classifications. The open question is to close them.

We remark that there are many varieties of Schauder bases ([Meg98,Sin70,Sin81]), such as monotone, shrinking, absolute, etc, and their complexity is mostly open (see [BD07]). There are also other notions of basis, such as *Markushevich basis*, which seem completely unexplored from a computability-theoretical perspective. For example, every separable Banach space has a Markushevich basis ([Mar43] even a "strong" one Terenzi [Ter94])), but we have no idea of this result's effective content. Hajek et. al. [HSVZ08] is a good reference.

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4 Appendix-Some Proofs

4.1 Proof of Theorem 8

To prove Theorem 8, we will first need the following classical lemma. This proof is taken from [Qia21].

Lemma 5 (Mazur). *Let X be an infinite dimensional Banach space, $B \subset X$ be a finite-dimensional subspace, and $\epsilon > 0$. Then there is an $x \in X$ with $\|x\| = 1$ so that*

$$\|y\| \leq (1+\epsilon)\|y + \lambda x\|$$

for all $y \in B, \lambda \in \mathbb{R}$. In fact, x can be chosen so that this inequality is strict whenever $\|y\|, \lambda \neq 0$.

When working with separable Banach spaces, this lemma can be slightly strengthened so that we only have to deal with the dense elements.

Lemma 6. *In Lemma 5, further suppose that X is a separable Banach space and that $(q_i)_{i\in\mathbb{N}}$ is dense in the unit sphere of X. We can require the desired $x \in X$ to be some element from (q_i).*

Proof. Let X be some separable Banach space, and let $(q_i)_{i\in\mathbb{N}}$ be dense in the unit sphere of X. Let $B \subset X$ be some finite-dimensional subspace and $\epsilon > 0$ be some pre-determined constant. Further denote $x \in X$ to be some element that satisfies the requirements as given by Lemma 5 with $\|x\| = 1$. Note that by homogeneity ($y \in B \iff \frac{y}{\lambda} \in B$) it is sufficient to find some $z \in (q_i)$ which satisfies

$$\|y\| \leq (1+\epsilon)\|y + z\|$$

for all $y \in B$. As $x \notin B$, we have that $\delta_x = \min_{y\in B}\|x + y\|$ is both well-defined and positive. Let $z \in X$ be any element where $\|z\| = 1$, since $\|y + x\| \leq \|y + z\| + \|x - z\|$, we have

$$\delta_x = \min_{y\in B}\|y + x\| \leq \delta_z + \|x - z\|$$

From the inequality above, we can choose some z sufficiently close to x with $\|z\| = 1$ so that $\|x - z\| \leq \epsilon(1+\epsilon)^{-1}\delta_z$, we show that this choice works

$$\|y\| \leq (1+\epsilon)\|y + x\| = (1+\epsilon)\|y + x - z + z\|$$

$$\leq (1+\epsilon)(\|y + z\| + \|x - z\|) \leq (1+\epsilon)\left(\|y + z\| + \epsilon(1+\epsilon)^{-1}\delta_z\right)$$

And by definition of δ_z, we get that

$$(1+\epsilon)\left(\|y + z\| + \epsilon(1+\epsilon)^{-1}\delta_z\right) \leq (1+\epsilon)\|y + z\| + \epsilon\|y + z\|$$

$$= (1+2\epsilon)\|y + z\|$$

Since Lemma 5 works for all values of ϵ, the conclusion follows. In fact, the exact same argument shows that we can always choose the desired $x \in X$ to be some computable point when X is a computable Banach space. $\square$

We are now ready to prove Theorem 8.

Proof (Proof of Theorem 8). In light of Lemmas 6 and 2, we can simply carry out the classical construction. Fix some sequence of computable reals $(\epsilon_i)_{i\in\mathbb{N}}$ such that $\prod_{i=0}^{\infty}(1+\epsilon_i) < \infty$. We will construct a basic sequence $(u_i)_{i\in\mathbb{N}}$ inductively. Having constructed $u_0, \ldots, u_n$, find some x in the effective dense sequence for X such that $\mathrm{bc}(u_0, \ldots, u_n, x) \leq \prod_{i=0}^{n+1}(1+\epsilon_i)$. The existence of such an element is guaranteed by Lemma 6. Furthermore, this process is computable as the basis constants are computable. □

4.2 Complexity of computable basis

Whilst we don't have space to prove all of the claims in the paper, we will give a brief skectch of how to prove Σ_3^0 completeness of the index sets of computable Banach spaces with computable bases. In doing so, we also sketch the ideas used by Bosserhof [Bos08] as per [Qia21]. Below, let BASIS_C denote the index-set of computable Banach spaces that have a computable Schauder basis.

Theorem 11. *$BASIS_C$ is Σ_3^0 complete.*

We first introduce the construction used in [Bos08]. Let Z denote the Banach space constructed in [Dav73] that lacks the approximation property. It was proven in [Bos08] that this space is computable and also exhibits the local basis property.

Theorem 12 ([Bos08]). *There exists a computable Banach space without AP but has LBS.*

In particular, this implies that Z can be approximated by a sequence of "nice" subspaces.

Theorem 13 ([Bos08]). *There is a computable linearly independent sequence $(x_i)_{i\in\mathbb{N}} \subseteq Z$, a computable increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ and an universal constant C such that $[x_0, \ldots] = Z$ and*

$$(\forall n \in \mathbb{N})(\mathrm{bc}([x_0, \ldots, x_{\sigma(n)}]) < C)$$

We first need the following definitions.

Definition 7 ([Bos08]). *For any $n \in \mathbb{N}$, Z_n is defined as:*

$$Z_n = [x_0, \ldots, x_{\sigma(n)}]$$

where $(x_i)_{i\in\mathbb{N}}$ is given by Theorem 13. For any $\tau : \mathbb{N} \to \mathbb{N}$, the Banach space Y_τ is defined as:

$$Y_\tau = \left(\oplus_i Z_{\tau(i)}\right)_{c_0}$$

which is the sequence space where norms of elements within each sequence tends to 0, and the norm on the sequence is the supremum norm on the elements.

An important feature of this space is that it has a basis. Intuitively, as the columns have universally bounded basis constants, we can simply "join up" the bases of the columns in the larger space, and the resulting sequence will be a basis.

Lemma 7 ([Bos08]). *The space Y_τ as defined in Definition 7 has a basis for any $\tau : \mathbb{N} \to \mathbb{N}$.*

The key idea is that Y_τ is a Banach space with basis, however each of its components can be made arbitrarily "large" such that no computable sequence can span it. For the sake of simplicity, also denote $Y = (\oplus_i Z)_{c_0}$. The following lemma is crucial.

Lemma 8 ([Bos08]). *For any basic sequence $(y_i)_{i\in\mathbb{N}} \in Y^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have*

$$emb^n(Z) \nsubseteq [y_0, y_1, \ldots]$$

Where $\mathrm{emb}^n : Z \to Y$ is the map defined by

$$\mathrm{emb}^n(x) = (0, \ldots, 0, x, 0, \ldots) \in Y$$

mapping $x \in Z$ to n-th position of a sequence that is otherwise entirely zero.

There is also a natural computability structure on the space Y_τ for certain classes of τ.

Definition 8. *A function $\tau : \mathbb{N} \to \mathbb{N}$ is* lower semicomputable *if there is a c.e set $A \subseteq \mathbb{N}$ such that*

$$\tau(n) = \sup\{k \in \mathbb{N} : \langle n, k\rangle \in A\}$$

for all $n \in \mathbb{N}$.

Lemma 9 ([Bos08]). *For any $\tau : \mathbb{N} \to \mathbb{N}$ that is lower semicomputable, the constructed space Y_τ equipped with the dense set $\{emb^j(x_i)\}_{i\leq\sigma(\tau(j)),j\in\mathbb{N}}$ is a computable Banach space.*

Finally, to construct a computable Banach space without any computable basis, it is sufficient to construct some lower semicomputable τ such that Y_τ does not contain any computable basis. Furthmore, by Lemma 8 and Theorem 13, we can construct τ by directly diagonalising against all computable basic sequences. The following is due to [Bos08], although presented in a slightly different fashion.

Lemma 10 ([Bos08]). *There is a lower semicomputable function $\psi : \mathbb{N}^3 \to \mathbb{N}$ such that for all $n, k, i \in \mathbb{N}$, if ϕ_n computes a basic sequence $(y_i)_{i\in\mathbb{N}} \in Y^{\mathbb{N}}$ with basis constant smaller than k, we have*

$$emb^i(Z_{\psi(n,k,i)}) \nsubseteq [y_0, \ldots]$$

Corollary 1 ([Bos08]). *There exists a computable Banach space without computable basis.*

Proof. By Lemmas 9 and 10, define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(\langle n, k\rangle) = \psi(n, k, \langle n, k\rangle)$$

The resulting space Y_τ is a computable Banach space where $\tau(\langle n, k\rangle)$ is large enough so that $\mathrm{emb}^{\langle n,k\rangle}(Z_{\tau(\langle n,k\rangle)})$ is not spanned by ϕ_n (if it is a basic sequence with basis constant smaller than k). This implies that the space Y_τ cannot be spanned by any computable basic sequence[3], and therefore lacks basis. $\square$

It is worth noting that although the space constructed in Corollary 1 has no computable basis, it is unclear how uncomputable the bases are.

Question 10 Let Y_τ be the space used in the proof of Corollary 1 that was constructed by [Bos08]. What are the corresponding Turing degrees for the bases in this space?

Using Lemma 1, it is easy to see that having a computable basis is Σ^0_3, and hence we need following lemma to show completeness.

Lemma 11. *Recall the construction carried out in Lemma 7. If τ is a computable function, then Y_τ contains a computable basis.*

Proof. As the basis constant of $Z_{\tau(i)}$ is uniformly bounded by some constant C, there is some basis $(a_{i,j})_{j\leq\sigma(\tau(i))}$ with basis constant smaller than C for each $Z_{\tau(i)}$. It was proved in [Bos08] that the natural embedding of these bases into Y_τ (i.e. $\{\mathrm{emb}^i(a_{i,j})|i \in \mathbb{N}, j \leq \sigma(\tau(i))\}$) forms a basis for Y_τ. We will show that this is actually computable when τ is computable. If τ is computable, the sequence

$$x_0, x_1, \dots, x_{\sigma(\tau(i))}$$

will be computable as well since $(x_i)_{i\in\mathbb{N}}$ and σ are both computable. Therefore, the rational span of the sequence will be computable as well. By continuity, we can therefore effectively find some basis that lies in the rational span of $(x_i)_{i\leq\sigma(\tau(i))}$ with basis constant smaller than C. As this procedure is uniform, it gives a computable basis in Y_τ. $\square$

We are now ready to prove Theorem 11.

Proof (Proof of Theorem 11). $\mathrm{BASIS}_C \in \Sigma_3$ essentially follows from Lemma 1, it remains to show that BASIS_C is Σ_3 hard. It is a well known fact that for any set $A \in \Sigma_3$, there is a computable function $g : \mathbb{N}^2 \to \mathbb{N}$ such that

$$x \in A \iff (\exists y)(W_{g(x,y)} \text{ is infinite })$$

For all $x \in \mathbb{N}$, we construct a lower semicomputable function $h : \mathbb{N} \to \mathbb{N}$ in stages. Let $\{\psi_s\}$ be some computable enumeration of the function ψ constructed in Lemma 10. We also define the function $C : \mathbb{N} \to \mathbb{N}$, initially $C_0(n) = n$ for all $n \in \mathbb{N}$. $C(n)$ indicates the computable sequence that is diagonalised against at n. Initialise the construction by setting $h_s = 0$. At stage s, the following is carried out for each $n \leq s$.

[3] Note that any computable sequence in Y_τ is also a computable sequence in Y, so it is sufficient to diagonalise against computable sequences in Y.

- If $C(n) = -1$, do nothing. Otherwise:
- Enumerate $W_{g(x,C(n)),s}$. If a new element is enumerated, set $C(k)$ to $C(k-1)$ for all $k > n + |W_{g(x,C(n)),s}|$ and $C(n + |W_{g(x,C(n)),s}|)$ to -1.
- View $C(n)$ as a pair $\langle a, b\rangle$ and set $h_s(n)$ to $\max(h_{s-1}(n), \psi_s(a, b, n))$.

Finally we define h as $h = \lim_{s\to\infty} h_s$. This is the end of the construction, we now verify its validity.

Lemma 12. *The function h constructed is indeed a lower semicomputable function.*

Proof. The constructed sequence $\{h_s\}$ is clearly a computable enumeration of h. So it remains to verify that $\{h_s\}$ converges. For any $n \in \mathbb{N}$, we have $C(n) \leq n$. Therefore $h_s(n) \leq \max_{\langle a,b\rangle \leq n} \psi(a, b, n)$ for all s, and since $(h_s(n))_s$ is monotone, this implies convergence. □

We now show that the constructed h has the desired properties.

Lemma 13. *In addition to h being lower semicomputable, it also exihibit the following properties*

- *If $x \in A$, h is computable (although this might be non-uniform).*
- *If $x \notin A$, Y_h contains no computable basis.*

Proof. Suppose $x \in A$, thus there is some y such that $W_{g(x,C(y))}$ is infinite. By the construction, this means that

$$-1 = C(y + 1) = C(y + 2) = C(y + 3) = \dots$$

Therefore, to compute $h(k)$ for any $k > y$, we just have to run the computable construction for finitely many steps until $C(k) = -1$, in which case the current value of $h(k)$ will be its final value. And since there are only finite many values $h(k)$ for $k \leq y$, this can be computed non-uniformly. Hence, h is a computable function.

Now suppose $x \notin A$, in which case $W_{g(x,y)}$ is finite for all $y \in \mathbb{N}$. We will show that for all $\langle a, b\rangle \in \mathbb{N}$, there is some $n \in \mathbb{N}$ where $C(n) = \langle a, b\rangle$, implying that $h(n) \geq \psi(a, b, n)$ and therefore Y_h cannot contain any computable basis. At each stage s of the construction, there will be some index i_s where $C_s(i_s) = \langle a, b\rangle$. So it suffices to show that $(i_s)_s$ eventually stabilises. But by the construction, i_s can only increase when some new element has been enumerated in $W_{g(x,C(k))}$ for some $C(k) < \langle a, b\rangle$. And since $\{k : C(k) < \langle a, b\rangle\}$ is finite, and each set of the form $W_{g(x,y)}$ is finite as well, i_s can only increase for a finite number of steps until it eventually converges, and the proof is complete. □

Therefore, as the construction of h is uniform in x, we have established a reduction from an arbitrary Σ_3 set to BASIS_C, proving that BASIS_C is indeed Σ_3 hard. □