Degree Spectra of Relations on Boolean Algebras

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October 26, 2000

Abstract

We show that every computable relation on a computable Boolean algebra is either definable by a quantifier-free formula with constants from $\mathfrak{B}$ (in which case it is obviously intrinsically computable) or has infinite degree spectrum.

Computable mathematics has been the focus of a large amount of research in the past few decades. Computable model theory in particular has seen vigorous and varied activity, leading to the discovery and intensive investigation of a number of central recurring themes. Among these is the study of the computability-theoretic properties of the images of a relation on a structure in different computable copies of the structure.

In this paper, we investigate computable relations on Boolean algebras from this point of view. Boolean algebras are very interesting to computable model theorists because, like linear orderings, they are a natural, nontrivial, and well-studied class of structures that exhibits much more structure than is present in the general case. Thus, studying computable Boolean algebras can give us insight into the nature of computation under constraints.

We will define the relevant concepts from computable model theory below. A valuable recent reference covering a wide range of topics in computable mathematics is the handbook [4]. For basic notions of computability theory, model theory, and (computable) Boolean algebras, the reader is referred to [14], [10], and [5], respectively.

The first and second authors' research was partially supported by the Marsden Fund of New Zealand.

When doing computable mathematics, we must abandon the idea that isomorphic structures are essentially identical. For example, under the standard ordering of the natural numbers, the successor relation is computable, but it is not hard to construct a computable linear ordering of type ω in which the successor relation is not computable. From a computability-theoretic point of view, these two computable copies of the same structure are very different. This leads us to study structures up to *computable* isomorphism, and gives rise to the notion of a computable presentation. (We always assume we are working with computable languages.)

1. Definition. A structure $\mathcal{A}$ is *computable* if both its domain $|\mathcal{A}|$ and the atomic diagram of $\langle \mathcal{A}, a \rangle_{a \in |\mathcal{A}|}$ are computable. (In other words, the domain, constants, functions, and relations of $\mathcal{A}$ are uniformly computable.)

An isomorphism from a structure $\mathcal{M}$ to a computable structure is called a *computable presentation* of $\mathcal{M}$. (We often abuse terminology and refer to the image of a computable presentation as a computable presentation.)

One way in which we may hope to understand the differences between computable presentations of the same structure is to look at the images in these presentations of a relation on the domain of the structure. The study of relations on computable structures began with the work of Ash and Nerode [1], and has proved to be not only a useful way to study the differences between the various computable presentations of a structure, but also a fruitful independent area of research with ties to several other parts of computable model theory, as well as with other branches of logic and computer science.

In [1], Ash and Nerode were concerned with relations that maintain some degree of effectiveness in different computable presentations of a structure.

2. Definition. Let U be a relation on the domain of a computable structure $\mathcal{A}$ and let $\mathfrak{C}$ be a class of relations. U is *intrinsically* $\mathfrak{C}$ on $\mathcal{A}$ if the image of U in any computable presentation of $\mathcal{A}$ is in $\mathfrak{C}$.

Ash and Nerode gave syntactic characterizations of the intrinsically c.e. and intrinsically computable relations under an extra decidability condition (which cannot be dropped in general), and there has been a large amount of research following this approach. For instance, in [12], Moses gave a syntactic characterization of the intrinsically computable relations on computable linear orderings.

3. Theorem (Moses). *Let R be a computable relation on the domain of a computable linear ordering $\mathfrak{L}$. Then R is either definable by a quantifier-free formula with constants from $\mathfrak{L}$ (in which case it is intrinsically computable) or not intrinsically computable.*

Another approach to the study of relations on computable structures is to examine their degree spectra, a concept which was first defined in Harizanov's dissertation [6], although its study dates back at least to the work of Remmel [13].

4. Definition. Let U be a relation on the domain of a computable structure $\mathcal{A}$. The *degree spectrum* of U on $\mathcal{A}$, $\mathrm{DgSp}_{\mathcal{A}}(U)$, is the set of (Turing) degrees of the images of U in all computable presentations of $\mathcal{A}$.

In the general case, there are no known restrictions on the sets of degrees that can be realized as degree spectra of relations on computable structures, other than the ones that follow from the fact that the set of images of a relation on the domain of a computable structure in different computable presentations of the structure is (by definition) Σ^1_1. Furthermore, there are many natural special cases that turn out to be no less restrictive than the general case. Hirschfeldt, Khoussainov, Shore, and Slinko [9] showed that the sets of degrees that can be realized as degree spectra of relations on computable structures do not change if we restrict ourselves to structures in any of the following classes: symmetric, irreflexive graphs; partial orderings; lattices; rings (with zero-divisors); integral domains of arbitrary characteristic; commutative semigroups; and 2-step nilpotent groups.

On the other hand, the class of possible degree spectra of "natural" relations appears much more restricted. Thus it becomes quite interesting to look for natural families of structures that give rise to more restricted classes of possible degree spectra of relations than in the general case. Research in this direction not only clarifies the computability-theoretic structure of these classes of structures, but also provides insight into the nature of computation under constraints.

One particularly interesting "pathological" phenomenon that can happen in general but not in certain particular cases is that of computable relations whose degree spectra are finite but not singletons. Such relations have been known to exist since the work of Harizanov [7], but seem to require complicated computability-theoretic constructions.

In [8], Hirschfeldt proved the following theorem.

5. Theorem (Hirschfeldt). *Let R be a computable relation on the domain of a computable structure $\mathcal{A}$. If there exists a Δ^0_2 function f such that $f(\mathcal{A})$ is a computable structure and $f(R)$ is not computable then $\mathrm{DgSp}_{\mathcal{A}}(R)$ is infinite.*

Combining this result with the proof of Theorem 3 yields the following theorem.

6. Theorem (Hirschfeldt). *Let R be a computable relation on the domain of a computable linear ordering. Then R either is intrinsically computable or has infinite degree spectrum.*

In this paper, we prove the analog of Theorems 3 and 6 in the case of Boolean algebras. Our main tools will be Theorem 5 and a modification of a result of Moses, which we now explain.

Before proceeding, we need some notation to talk about finite portions of a computable structure $\mathcal{A}$ of (possibly infinite) signature L.

7. Definition. Let $\mathcal{A}$ be a computable structure of signature L and let $m \in \omega$. Let L_m be the language obtained by restricting L to its first m symbols, substituting all j-ary function symbols by $(j+1)$-ary relation symbols in the obvious way, and dropping any constant whose interpretation in $\mathcal{A}$ is not in $[0, m]$. Define $\mathcal{A} \upharpoonright m$ to be the finite structure obtained from $\mathcal{A}$ by restricting the domain to $|\mathcal{A}| \cap [0, m]$ and restricting the language to L_m.

In the proof of Theorem 3, Moses used the following earlier result from [11].

8. Theorem (Moses). *Let $\mathcal{A}$ be a computable structure and let R be a k-ary computable relation on $\mathcal{A}$. Suppose there is a computable binary function f such that for every $m \in \omega$ there is a tuple $\vec{a} \in R$ for which there are infinitely many $s \in \omega$ with embeddings $\varphi : \mathcal{A} \upharpoonright s \to \mathcal{A} \upharpoonright f(m, s)$ with φ the identity on $\mathcal{A} \upharpoonright m$ and $\varphi(\vec{a}) \notin R$. Then R is not intrinsically c.e..*

The original statement of Theorem 8 assumed that $\mathcal{A}$ is a structure in a finite relational language, but the original proof works in general, using the notion of "substructure" given in Definition 7. Furthermore, the proof of Theorem 8 builds a Δ^0_2 function g such that $g(\mathcal{A})$ is a computable structure and $g(R)$ is not computable, and hence Theorem 5 implies that a relation satisfying the hypotheses of Theorem 8 has infinite degree spectrum. It is easy to modify the proof of Theorem 8 to establish the following result. We include a proof sketch for completeness.

9. Theorem. *Let R be a k-ary computable relation on a computable structure $\mathcal{A}$. Suppose there is a computable binary function f such that for every $m \in \omega$ there is a tuple $\vec{a} \in |\mathcal{A}|^k$ for which there are infinitely many $s \in \omega$ with embeddings $\varphi : \mathcal{A} \upharpoonright s \to \mathcal{A} \upharpoonright f(m, s)$ with φ the identity on $\mathcal{A} \upharpoonright m$ and $R(\varphi(\vec{a})) \neq R(\vec{a})$. Then $\mathrm{DgSp}_{\mathcal{A}}(R)$ is infinite.*

Proof sketch. By Theorem 5, it is enough to build a Δ^0_2 function g such that $g^{-1}(\mathcal{A})$ is a computable structure and $g^{-1}(R)$ is not computable. For ease of notation, we assume that R is unary and $|\mathcal{A}| = \omega$.

We want to satisfy the following requirement for each $e \in \omega$:

$$Q_e : g^{-1}(R) \neq \Phi_e.$$

At stage 0 we define g_0 to be the empty map. At stage $s+1$ we begin with a finite map g_s and extend it to a map g_{s+1}.

Let $n = \max(\mathrm{rng}(g_s))$. For each $e \leqslant s$, define m_e to be the maximum of e, $\{g_s(i) \mid i \leqslant e\}$, and the last stage at which a requirement Q_i, $i < e$, acted (defined below). Say that $b \in \mathrm{dom}(g_s)$ may be used to attack Q_e if $\Phi_e(b)\downarrow = R(g_s(b))$ and there is an embedding $\varphi : \mathcal{A} \upharpoonright n \to \mathcal{A} \upharpoonright f(m_e, n)$ with φ the identity on $\mathcal{A} \upharpoonright m_e$ and $R(\varphi \circ g_s(b)) \neq R(g_s(b))$.

Look for the least $e \leqslant s$ such that Q_e is not currently satisfied (defined below) and there is an $a \in \mathrm{dom}(g_s)$ that may be used to attack Q_e. If no such number exists then define g_{s+1} so that it extends g_s and $g_{s+1}(x) = y$, where x and y are the least numbers not in $\mathrm{dom}(g_s)$ and $\mathrm{rng}(g_s)$, respectively. Otherwise, let φ be as above and define g_{s+1} so that it extends $\varphi \circ g_s$ and $g_{s+1}(x) = y$, where x and y are the least numbers not in $\mathrm{dom}(g_s)$ and $\mathrm{rng}(\varphi \circ g_s)$, respectively. We say that Q_e is satisfied and every Q_i, $i > e$, is unsatisfied.

This completes the construction. It is straightforward to check that $g = \lim_s g_s$ is a well-defined bijection from ω to ω, that $g^{-1}(\mathcal{A})$ is a computable structure, and that the hypotheses of the theorem imply that each Q_e is met. For further details, see the proof of Theorem 1 in [11]. $\square$

The following sufficient condition for a structure to have infinite degree spectrum follows easily from Theorem 9, but will be more convenient for our application to Boolean algebras.

10. Corollary. *Let R be a computable k-ary relation on the domain of a computable structure $\mathcal{A}$. Suppose there exists a Δ^0_2 function h such that for each $m \in \omega$ there is a pair of elements $\vec{x}_0, \vec{x}_1 \in (\mathcal{A} \upharpoonright h(m))^k$ satisfying the following conditions.*

1. *$R(\vec{x}_0) \neq R(\vec{x}_1)$.*

2. *For all $s \geqslant h(m)$ there exist embeddings $g_i : \mathcal{A} \upharpoonright s \to \mathcal{A}$, $i = 0, 1$, such that*

   (a) $g_0(\vec{x}_0) = g_1(\vec{x}_1)$ and

(b) $g_0(j) = g_1(j) = j$ *for all* $j \in \mathcal{A} \upharpoonright m$.

Then $\mathrm{DgSp}_{\mathcal{A}}(R)$ *is infinite.*

Proof. Given $m, s \in \omega$, look for a $t \in \omega$ such that either $h(m)[t] > s$ or there exist $\vec{x}_0, \vec{x}_1 \in (\mathcal{A} \upharpoonright h(m))^k$ and embeddings $g_i : \mathcal{A} \upharpoonright s \to \mathcal{A} \upharpoonright t$ satisfying 1, 2.a, and 2.b above. By the hypotheses, such a t will be found. Define $f(m, s) = t$.

Fix $m \in \omega$. By the hypotheses and the fact that there are only finitely many pairs $\vec{x}_0, \vec{x}_1 \in (\mathcal{A} \upharpoonright h(m))^k$, there exists such a pair for which there are infinitely many $s \in \omega$ and embeddings $g_i^s : \mathcal{A} \upharpoonright s \to \mathcal{A} \upharpoonright f(m, s)$ satisfying 1, 2.a, and 2.b. For each such s we have $R(g_i^s(\vec{x}_i)) \neq R(\vec{x}_i)$ for some $i = 0, 1$. So for some $i = 0, 1$ it must be the case that $R(g_i^s(\vec{x}_i)) \neq R(\vec{x}_i)$ for infinitely many $s \in \omega$. Taking $\vec{a} = \vec{x}_i$ we see that the hypotheses of Theorem 9 are satisfied. □

We are now ready to prove the analog of Theorems 3 and 6 in the case of Boolean algebras.

11. Theorem. *Let* R *be a computable relation on the domain of a computable Boolean algebra* $\mathfrak{B}$*. Then* R *either is definable by a quantifier-free formula with constants from* $\mathfrak{B}$ *(in which case it is intrinsically computable) or has infinite degree spectrum.*

Proof. If R is k-ary then we can think of it as a unary relation on the Boolean algebra $\mathfrak{B}^k$ with no change in the degree spectrum. Thus we can assume that R is unary. We assume that R is not definable by a quantifier-free formula with constants from $\mathfrak{B}$ and show that it satisfies the hypotheses of Corollary 10.

Given $m \in \omega$, let $b_0, b_1, \ldots, b_k \in \mathfrak{B}$ be such that $b_i \cap b_j = 0_{\mathfrak{B}}$ for $i \neq j$, $b_0 \cup b_1 \cup \cdots \cup b_k = 1_{\mathfrak{B}}$, and the subalgebra generated by $b_0, b_1, \ldots, b_k$ contains the elements of $\mathfrak{B} \upharpoonright m$. (Here we are using the notation of Definition 7.) Let $\mathfrak{B}_i$ be the subalgebra of $\mathfrak{B}$ consisting of those elements that are less than or equal to b_i (in the Boolean algebra sense) and let $\overline{\mathfrak{B}}_i = \mathfrak{B}/\mathfrak{B}_i$. We identify $\mathfrak{B}$ with $\mathfrak{B}_i \times \overline{\mathfrak{B}}_i$.

Let $P_i = \{(x, y) \mid \exists a \in \overline{\mathfrak{B}}_i \; \exists b, c \in \mathfrak{B}_i - \{0, b_i\} \; [x = a \cup b, \; y = a \cup c]\}$. Let S be the set of all pairs (x, y) (in a fixed ordering of pairs of elements of $\mathcal{B}$) such that, for some $i \leqslant k$, (x, y) is the least pair in P_i with $R(x) \neq R(y)$. Define $h(m) = \max\{x, y \mid (x, y) \in S\}$.

Clearly, h is Δ_2^0, so all we need to show to conclude that R satisfies the hypotheses of Corollary 10 is that, for each $m \in \omega$, there is a pair $x, y \in \mathfrak{B} \upharpoonright h(m)$ such that $R(x) \neq R(y)$ and for all $s \geqslant h(m)$ there exist embeddings $g_j : \mathfrak{B} \upharpoonright s \to \mathfrak{B}$, $j = 0, 1$, such that $g_0(x) = g_1(y)$ and $g_0(l) = g_1(l) = l$ for all $l \in \mathfrak{B} \upharpoonright m$.

Fix $m \in \omega$ and let P_i, S, and $\mathfrak{B}_i$ be as above. Let I be the set of all $i \leqslant k$ such that P_i contains a pair in S. If $\bigcup_{i \in I} \mathfrak{B}_i$ were finite then we could define R by a quantifier-free formula with constants $b_0, \dots, b_k$ and the elements of $\bigcup_{i \in I} \mathfrak{B}_i$. So there is at least one pair $(x, y) \in S$ contained in a P_i such that $\mathfrak{B}_i$ is infinite. Let b and c be the ith coordinates of x and y, respectively, when written as elements of $\mathfrak{B}_0 \times \cdots \times \mathfrak{B}_k$. Let $s \geqslant h(m)$ and let n be such that every element of $\mathfrak{B} \upharpoonright s$ can be represented as the join of an element of $\mathfrak{B}_i \upharpoonright n$ with an element of $\overline{\mathfrak{B}}_i \upharpoonright n$.

Now, $\mathfrak{B}_i$ is infinite, the class of all finite Boolean algebras has the amalgamation property, and any finite Boolean algebra can be embedded into any infinite Boolean algebra. So there exist embeddings $k_j : \mathfrak{B}_i \upharpoonright s \to \mathfrak{B}_i$, $j = 0, 1$, such that $k_0(b) = k_1(c)$. These can clearly be extended to the required embeddings g_0 and g_1 by defining $g_j(a \cup d) = a \cup k_j(d)$ for $a \in \overline{\mathfrak{B}}_i \upharpoonright n$ and $d \in \mathfrak{B}_i \upharpoonright n$. $\square$

It is interesting to consider how far results such as Theorems 6 and 11 may be extended by weakening the hypothesis that the relation is computable.

12. Question. Is there any intrinsically arithmetical relation on a Boolean algebra that is not intrinsically computable but has finite degree spectrum (of cardinality greater than one)? If so, then what if the relation is intrinsically Δ^0_2 or intrinsically c.e.?

In the case of linear orderings, we do know a little more. As pointed out in [8], the existence of a maximal d.c.e. degree, proved by Cooper, Harrington, Lachlan, Lempp, and Soare [2], can be used to show that there exists an intrinsically d.c.e. invariant relation on the domain of a Δ^0_2-categorical structure with a two-element degree spectrum.

We can give a similar example in the case of linear orderings, but we need to go one jump higher. Relativizing the proof in [2], we see that there exists a maximal $\mathbf{0}'$-d.c.e. degree, that is, a degree $\mathbf{d}$ such that $\mathbf{d}$ is d.c.e. relative to $\mathbf{0}'$ and there are no $\mathbf{0}'$-d.c.e. degrees in $(\mathbf{d}, \mathbf{0}'')$.

13. Proposition. *There exists a relation U on the domain of a computable linear ordering such that U has a two-element degree spectrum.*

Proof. Let $\mathbf{d}$ be a maximal $\mathbf{0}'$-d.c.e. degree and let $D \in \mathbf{d}$ be $\mathbf{0}'$-d.c.e.. Let $\langle \cdot\,, \cdot \rangle$ be a standard pairing function on ω. Define the binary relation R on the standard presentation of the linear ordering ω as follows. $R(\langle n, i\rangle, \langle n, i+1\rangle)$ holds for all $i < 2n \in \omega$. If $n \in D$ then $R(\langle n, 2n\rangle, \langle n, 0\rangle)$ holds; otherwise, $R(\langle n, 2n\rangle, \langle n, 2n+1\rangle)$ and $R(\langle n, 2n+1\rangle, \langle n, 0\rangle)$ hold. So, thinking of R as defining a directed graph, there is a $(2n+1)$-cycle for each $n \in D$, a $(2n+2)$-cycle for each $n \notin D$, and no other cycles.

Clearly, $\deg(R) = \mathbf{d}$. Now let $\mathcal{L}$ be a computable presentation of ω. It is easy to check that there is a Δ^0_2 isomorphism f from the standard presentation of ω to $\mathcal{L}$, and that this implies that $R^{\mathcal{L}} = f(R)$ is $\mathbf{0}''$-computable and $\mathbf{0}'$-d.c.e.. But D is computable in $R^{\mathcal{L}}$, and thus $\deg(R^{\mathcal{L}}) \in [\mathbf{d}, \mathbf{0}'']$. From this we conclude that the degree spectrum of R is either $\{\mathbf{d}\}$ or $\{\mathbf{d}, \mathbf{0}''\}$.

Now let η be the order type of the rationals and let R' be the binary relation on $(2+\eta)\cdot\omega$ that holds of x and y if and only if x and y are elements of consecutive adjacencies. It is easy to check that the degree spectrum of R' consists of all $\mathbf{0}'$-c.e. degrees. Thus the relation on $(2+\eta)\cdot\omega+\omega$ consisting of R' on $(2+\eta)\cdot\omega$ and R on ω has degree spectrum

$$\{\mathbf{c} \mid \mathbf{c} = \mathbf{a} \vee \mathbf{b},\ \mathbf{a} \in \mathrm{DgSp}_{\omega}(R),\ \mathbf{b} \in \mathrm{DgSp}_{(2+\eta)\cdot\omega}(R')\} = \\ \{\mathbf{c} \mid \mathbf{c} = \mathbf{d} \vee \mathbf{a},\ \mathbf{a} \text{ is } \mathbf{0}'\text{-c.e.}\} = \{\mathbf{d}, \mathbf{0}''\}.$$

□

In light of this example, we ask the following question.

14. Question. Is the degree spectrum of an intrinsically Δ^0_2 relation on a linear ordering always either a singleton or infinite? If not, then what if the relation is intrinsically c.e.?

While on the subject of linear orderings, we mention the question of the possible degree spectrum of the adjacency relation on a linear ordering. Downey and Moses [3] showed that there is a computable linear ordering whose adjacency relation is intrinsically complete, that is, has degree spectrum $\{\mathbf{0}'\}$. Surprisingly little else is known about this question.

15. Question. Is there a computable linear ordering whose adjacency relation is intrinsically incomplete? Can the degree spectrum of an adjacency relation have finite cardinality greater than 2? Can it consist of a single degree other than $\mathbf{0}$ and $\mathbf{0}'$?

We conclude with the following question, which seems to be a natural next step after Theorems 6 and 11.

16. Question. Is the degree spectrum of a computable relation on an Abelian group always either a singleton or infinite?

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