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AUTOMORPHISMS OF SUPERMAXIMAL SUBSPACES

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§1. Introduction. An infinite-dimensional vector space V_∞ over a recursive field F is called *fully effective* if V_∞ is a recursive set identified with ω upon which the operations of vector addition and scalar multiplication are recursive functions, identity is a recursive relation, and V_∞ has a *dependence algorithm*, that is a uniformly effective procedure which when applied to $x, a_1, \ldots, a_n \in V_\infty$ determines whether or not x is an element of $\{a_1, \ldots, a_n\}^*$ (the subspace generated by $\{a_1, \ldots, a_n\}$). The study of V_∞, and of its lattice of r.e. subspaces $L(V_\infty)$, was introduced in Metakides and Nerode [15]. Since then both V_∞ and $L(V_\infty)$ (and many other effective algebraic systems) have been studied quite intensively. The reader is directed to [5] and [17] for a good bibliography in this area, and to [15] for any unexplained notation and terminology.

In [15] Metakides and Nerode observed that a study of $L(V_\infty)$ may in some ways be modelled upon a study of $L(\omega)$, the lattice of r.e. sets. For example, they showed how an e-state construction could be modified to produce an r.e. maximal subspace, where $M \in L(V_\infty)$ is *maximal* if $\dim(V_\infty/M) = \infty$ and, for all $W \in L(V_\infty)$, if $W \supset M$ then either $\dim(W/M) < \infty$ or $\dim(V_\infty/W) < \infty$.

However, some of the most interesting features of $L(V_\infty)$ are those which do *not* have analogues in $L(\omega)$. Our concern here, which is probably one of the most striking characteristics of $L(V_\infty)$, falls into this category. We say $M \in L(V_\infty)$ is *supermaximal* if $\dim(V_\infty/M) = \infty$ and for all $W \in L(V_\infty)$, if $W \supset M$ then $\dim(W/M) < \infty$ or $W = V_\infty$. These subspaces were discovered by Kalantari and Retzlaff [13].

These subspaces appeared to be the true analogue of maximal sets in the sense that they have the thinnest lattice of r.e. superspaces. However, it has since become clear that their behavior differs markedly from that of maximal sets. For example, Remmel showed that there exist r.e. supermaximal subspaces of arbitrary nonzero Turing degrees and dependence degrees, whereas Martin [14] observed that maximal sets may only have high degrees. One of the major results for $L(\omega)$ is that of Soare [19], where he shows that for each pair M_1, M_2 of maximal sets there exists an automorphism Φ of $L(\omega)$ with $\Phi(M_1) = M_2$.

However, in his Ph.D. thesis [11] Guichard showed that each automorphism of

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$L(V_\infty)$ is induced by a recursive semilinear transformation[1] of V_∞. Thus the result of Remmel [18] implies that there exist supermaximal $M_1, M_2 \in L(V_\infty)$ such that no automorphism Φ of $L(V_\infty)$ takes M_1 to M_2. By diagonalizing over all the recursive semilinear transformations of V_∞, Guichard [10] then modified Remmel [18] to show that there exist supermaximal M_1 and M_2 such that no automorphism takes M_1 to M_2 and $d(M_2) = d(M_1) = d(D(M_1)) = d(D(M_2))$. Nerode and Remmel [17], [17a] have proved similar results with M_1 and M_2 having a specified dependence degree "structure"; also Downey [7] has shown that M_1 and M_2 may have the same co-r.e., complementary subspace.

In this paper we examine strengthenings of the concept of supermaximality. We examine classes of supermaximal subspaces with clear lattice-theoretic distinctions, and thus may deduce Guichard's results above without appealing to the diagonalization methods of Guichard. We also blend certain "effectivity" conditions with the notion of supermaximality with a view to perhaps producing a notion of supermaximality which defines a single orbit of the group of automorphisms of $L(V_\infty)$. In particular, this allows us to show that the analogue of Cohen and Jockusch's main result in [4] (namely that no strongly effectively simple set is contained in a maximal set) fails to hold in $L(V_\infty)$.

Finally, we observe that a result of this work is a new and an extremely easy construction of an r.e. supermaximal subspace, namely that if we are somewhat careful with an analogue of the usual simple set construction we immediately construct an r.e. supermaximal subspace.

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§2. Preliminaries. In [12] Hird proved a general recursive model-theoretic result, one of whose consequences was the existence of a new type of supermaximal subspace:

DEFINITION 2.1 (HIRD). We say $V \in L(V_\infty)$ is *strongly supermaximal* if $\dim(V_\infty/V) = \infty$ and for all r.e. independent sets I if $I \cap V = \varnothing$ then $\dim(I^*/V) < \infty$.

Subsequently, the authors observed that there is a very easy and natural construction of a strongly supermaximal subspace:

THEOREM 2.2 (HIRD, 1981). *There exists an r.e. strongly supermaximal subspace V.*

PROOF. Let $\{I_e \mid e \in \omega\}$ enumerate the r.e. independent subsets of V_∞. We build $V = \bigcup_s V_s$, and an r.e. set $J = \bigcup_s J_s$ with $V_s = J_s^*$ in stages. Let $B = \{a_0 < a_1 < \cdots\}$ list in order a recursive basis of V_∞. At each stage s we enumerate in order $\{b_i^s \mid i \in \omega\}$, a subset of B and a basis of V_∞ over V_s. Define, for convenience,

[1] We say $\Phi: V_\infty \to V_\infty$ is a semilinear transformation of V_∞ if $\Phi(u + v) = \Phi(u) + \Phi(v)$ and $\Phi(\lambda u) = F(\lambda)\Omega(u)$, where F is a field automorphism and Ω is a group automorphism of V_∞. In his Ph.D. thesis [13a], Kalantari showed that each automorphism of $L(V_\infty)$ is induced by a semilinear transformation, and Guichard modified a technique from Kalantari's thesis (namely that if an automorphism is induced by a permutation P of an r.e. basis, then P is recursive) to show that each automorphism is induced by a recursive invertible semilinear transformation.

$g(s,x) = \max_i\{b_i^s \in \mathrm{supp}_s(x)\}$ if $x \notin V_s^*$ and $g(s,x) = -1$ otherwise, where $\mathrm{supp}_s(x)$ denotes the support of x relative to $J_s \cup \{b_i^s \mid i \in \omega\}$. Our requirements are

$P_e: I_e \cap V = \varnothing \rightarrow I_e \subset (V \cup \{b_0, \ldots, b_e\})^*$;

$N_e: \lim_s b_e^s = b_e$ exists.

We say P_e *requires attention* at stage $s+1$ if $\exists x \in I_e^s$ such that $x \notin (V_s \cup \{b_0^s, \ldots, b_e^s\})^*$ and $I_e^s \cap V_s = \varnothing$ and e is the least such.

Construction. Stage 0. Set $b_i^0 = a_i$ for all $i \in \omega$, and set $J_0 = \varnothing$.

Stage $s+1$. If no P_e requires attention, do nothing. If P_e requires attention, find the least x for this e, set $J_{s+1} = J_s \cup \{x\}$ and set

$$b_i^{s+1} = \begin{cases} b_i^s & \text{for } i < g(s,x), \\ b_{i+1}^s & \text{for } i \geq g(s,x). \end{cases}$$

Set $J = \bigcup_s J_s$ and $V = J^*$. This completes the construction.

To complete the proof we observe that (as in the simple set construction) each b_e^s changes at most e times, so $\lim_s b_e^s = b_e$ exists. That all the P_e are met is immediate. □

REMARK 2.3. We observe that the space V we constructed in Theorem 2.8 has the following property. If $W_e \cap V = \{\vec{0}\}$ ($W_e = I_e^*$, the eth r.e. subspace), then $\dim(W_e) \leq e$.

REMARK 2.4. In fact, if $I_e \cap V = \varnothing$ then $I_e \subset (V \cup \{a_0, \ldots, a_{2e}\})^*$, and if $W_e \cap V = \{\vec{0}\}$ then $W_e \subset \{a_0, \ldots, a_{2e}\}^*$. We build on these observations later.

THEOREM 2.5. *If $V \in L(V_\infty)$ is strongly supermaximal then V is supermaximal.*

PROOF. Suppose V is not supermaximal. Choose $Q \in L(V_\infty)$ with $Q \supset V$, $\dim(Q/V) = \infty$ and $Q \neq V_\infty$. Now find some $a \in V_\infty - Q$, and a recursive (or r.e.) basis $\{q_i \mid i \in \omega\}$ of Q. Then $I = \{a + q_i \mid i \in \omega\}$ is an r.e. independent set with $I \cap V = \varnothing$ and $\dim(I/V) = \infty$. □

It is not too difficult to show that if $V \in L(V_\infty)$, $\dim(V_\infty/V) = \infty$, and V is a recursive set, then there exists a recursive basis B of V_∞ with $B \cap V \neq \varnothing$. Consequently, no strongly supermaximal r.e. subspace is a recursive set. Now Metakides and Nerode [16] have shown that if F (the field of scalars) is infinite then V_∞ contains a recursive supermaximal subspace; thus there are r.e. supermaximal subspaces which are not strongly supermaximal. In §3 we observe that this result implies that not all are supermaximal automorphic. If F was finite it may be the case that the types of supermaximality may coincide; however we have:

THEOREM 2.6. *For any field of scalars F, V contains an r.e. supermaximal subspace V which is not strongly supermaximal. Moreover, given any r.e. degree $\delta \neq 0$ we may construct V to have dependence degree $\leq \delta$ and ensure that $B \cap V \neq \varnothing$ for some recursive basis B of V_∞.*

We leave the proof of this result to the reader (cf. [9]), remarking only that it may be obtained by direct modification of the Kalantari-Retzlaff [13] construction, and that the only difficulties are when the field F is finite.

§3. Automorphisms. In this section we examine automorphisms of $L(V_\infty)$. We shall use the following result:

THEOREM 3.1 (GUICHARD [10]). *Every automorphism of $L(V_\infty)$ is induced by an invertible recursive semilinear transformation of V_∞.*

By producing a pair of r.e. supermaximal subspaces of the same Turing degree and dependence degree, one of which is strongly supermaximal and one not, we infer:

THEOREM 3.2 (GUICHARD [10]). *There exist r.e. supermaximal subspaces* M_1, M_2 *with* $d(M_1) = d(D(M_1)) = d(D(M_2)) = d(M_2)$ *such that no automorphism of* $L(V_\infty)$ *takes* M_1 *to* M_2.

Theorems 2.2 and 2.6 depended only on the fact that V_∞ is a regular (federated) Steinitz system (cf. Metakides and Nerode [16] or Baldwin [3]). Thus, for example, if we can show that automorphisms of $L(F_\infty)$ are recursive in nature, Theorem 2.2 and 2.6 will combine to produce supermaximal algebraically closed subfields with the same degree structure while not in the same orbit.

We may extend Theorem 3.2 to r.e. supermaximal subspaces which have recursive bases of V_∞ in their (set theoretic) complement by using the fact that every automorphism of $L(V_\infty)$ is induced by a recursive semilinear transformation. For example if the field is finite, in which case the Turing degree and dependence degree of an r.e. subspace are equal, we may use Theorem 2.6 to produce $\aleph_0$ r.e. supermaximal subspaces of different Turing and dependence degrees. If the field is infinite, we can also produce $\aleph_0$ recursive supermaximal subspaces (with bases in their complements) each with a differing dependence degree. In each of these degrees, we may diagonalize (as in Guichard [10]) to produce nonautomorphic spaces with the same degree structure. We note that at this stage, however, it is unclear how to impose finer degree restrictions. In the case of strongly supermaximal we have the following.

THEOREM 3.3. *Suppose* $\delta \neq 0$ *is any r.e. degree. There exist* $M_1, M_2 \in L(V_\infty)$ *and a co-r.e. subset R of a recursive basis B such that*

(i) $d(M_1) = d(M_2) = d(D(M_1)) = d(D(M_2)) = \delta$,

(ii) $M_1 \oplus R = M_2 \oplus R = V_\infty$,

(iii) *both* M_1 *and* M_2 *are strongly supermaximal, and*

(iv) *no automorphism* Φ *of* $L(V_\infty)$ *has* $\Phi(M_1) = M_2$.

PROOF. We modify the construction of Theorem 2.2. We build $K = \bigcup_s J_s$ and $K = \bigcup_s K_s$ in stages. At each stage s we ensure that $\{b_i^s \mid i \in \omega\}$ is a co-basis for both J_s^* and K_s^*. We have a recursive one-to-one total function f with $f(\omega) \equiv_T \delta$. At each stage s we enumerate one of $b^s_{4f(s)}, b^s_{4f(s)+1}, b^s_{4f(s)+2}, b^s_{4f(s)+3}$ into *both* J_s and K_s to ensure that $J, K_T \geq f(\omega)$.

We enumerate x into J_s (or K_s) only if $\mathrm{supp}_{J_s}(x) \geq 4f(s)$ (resp. $\mathrm{supp}_{K_s}(x) \geq 4f(s)$), where $\mathrm{supp}_{J_s}(x)$ denotes the support of x relative to $J_s \cup \{b_i^s \mid i \in \omega\}$. This controls the dependence degree of J_s and K_s. Our requirements are:

$N^1\colon J_T \geq f(\omega), K_T \geq f(\omega)$,

$N^2\colon d(D(J^*)) \leq_T f(\omega), d(D(K^*)) \leq_T f(\omega)$,

$N_e\colon \lim_s b_e^s = b_e$ exists,

$Q\colon J^* \oplus \{b_e \mid e \in \omega\}^* = K^* \oplus \{b_e \mid e \in \omega\}^* = V_\infty$,

$p_e^J\colon I_e \cap J^* = \varnothing \to \exists n(e) < \omega(I_e \subset (J \cup \{b_j \mid j < n(e)\})^*)$,

$p_e^K\colon I_e \cap K^* = \varnothing \to \exists m(e) < \omega(I_e \subset (K \cup \{b_j \mid j < m(e)\})^*)$.

We have the final requirement concerning the automorphisms of $L(V_\infty)$. Suppose $\{\phi_e\}_{e \in \omega}$ is an enumeration of the partial recursive functions. We meet the following requirements (as in Guichard [10]):

R_e: If ϕ_e is a recursive semilinear transformation of V_∞ then either $\exists j < \omega(\phi_e(b_j) \in K^*)$ or $\exists c \in J(\phi_e(c) \notin K^*)$.

During the construction we place markers upon elements of $\{b_i^s \mid i \in \omega\}$, associated with the ϕ_e. We denote these by $H(e,s)$. At each stage s there are only a finite number of markers. Define the following restraint function:

$$r(e,s) = \max(\{j \mid b_j^s = H(i,s) \text{ for } i < e\} \cup \{e, 4r(s)\}).$$

Our priority ranking is $N^1, N^2, Q, N_0, P_0^J, P_0^K, R_0, N_1, \ldots$.

We attack the requirement of highest priority which requires attention at stage $s+1$ according to this priority ranking and the following rules. (In this case we say a requirement *demands attention* if it is one of R_e, P_e^J or P_e^K.) We say P_e^J *requires attention at stage* $s+1$ if $\exists x \in I_e^s(g(s,x) \geq r(e,s))$, and e is the least number less than s with this property. Similarly, P_e^K *requires attention.* Finally we say R_e *requires attention at stage* $s+1$ if (i) $H(e,s)$ is undefined, (ii) $\exists k > r(e,s)$ $(\phi_e^s(b_k^s)\downarrow$ and either $\phi_e^s(b_k^s) \in K_s^*$ or $\phi_e^s(b_j^s) \in J_s^*)$, and (iii) e is least with respect to (i) and (ii).

Construction. Stage 0. Set $J_0 = K_0 = \emptyset$, $b_i^0 = a_i$ for all $i \in \omega$, and declare $H(e,0)$ as being undefined for all $e \in \omega$.

Stage $s+1$. If no P_e^J, P_e^K or R_e for $e \leq s$ demands attention, define $J_s^1 = J_s$ and $K_s^1 = K_s$. If P_e^J demands attention via x with x least for e, set $J_s^1 = J_s \cup \{x\}$ and $K_s^1 = K_s \cup \{b_g^s(s,x)\}$. If P_e^K demands attention via x, set $J_s^1 = J_s \cup \{b_g^s(s,x)\}$ and $K_s^1 = K_s \cup \{x\}$. Finally if R_e demands attention via k, if $\phi_e^s(b_k^s) \in K^*$ set $J_s^1 = J_s$ and $K_s^1 = K_s$. If $\phi_e^s(b_k^s) \notin K_s^*$ there are two cases.

Case (i). $\phi_e^s(b_k^s) \notin (K_s \cup \{b_k^s\})^*$. In this case set $J_s^1 = J_s \cup \{b_k^s\}$ and $K_s^1 = K_s \cup \{b_k^s\}$.

Case (ii). $\phi_e^s(b_k^s) \in (K_s \cup \{b_k^s\})^*$. In this case find $m > r(e,s) + k + 3$ such that $\phi_e^s(b_k^s) \notin (K_s \cup \{b_k^s + b_m^s\})^*$ and set $J_s^1 = J_s \cup \{b_k^s\}$ and $K_s^1 = K_s \cup \{b_k^s + b_m^s\}$.

In all of the cases where R_e demands attention define $H(e,s) = b_k^s$ and declare as undefined all the $H(j,s)$ for $j > e$, and set $H(i,s+1) = H(i,s)$ for $i < e$. Now we add one of $b_{4f(s)+3}^s$, $b_{4f(s)}^s$, $b_{4f(s)+1}^s$ or $b_{4f(s)+2}^s$ to J_s^1 and K_s^1 for J_{s+1} and K_{s+1}. If any requirement P_e^J, P_e^K or R_e demands attention at stage $s+1$ via $y = x$ or $y = b_k^s$ (as above) we say $g(s,y)$ is *poisoned* at stage $s+1$. Find the least j satisfying the following:

(i) $j = 4f(s), 4f(s)+1, 4f(s)+2$ or $4f(s)+3$,

(ii) j is not poisoned, and

(iii) if defining $J_{s+1} = J_s^1 \cup \{b_j^s\}$ and $K_{s+1} = K_s^1 \cup \{b_j^s\}$ injures any requirement R_t it injures the requirement R_t with the largest t (define this t as $t_{(j)}$) (i.e. the one of lowest priority).

Now set $K_{s+1} = K_s^1 \cup \{b_j^s\}$ and $J_{s+1} = J_s^1 \cup \{b_j^s\}$ and declare as undefined $H(i,s)$ for all $i > t_{(j)}$ (if $t_{(j)}$ in (iii) exists). As in Guichard [10] we observe that we do not injure the requirement of highest priority which is threatened. This completes the construction.

Now, we can show that $\lim_s b_e^s = b_e$ exists, as a b_i^s may only change if $i \geq 4f(s)$ and $i \geq e$, which can happen at most finitely often. It is easy to show by induction that each R_e is injured at most finitely often; as each R_e may only be injured by a P_t^J or P_t^K or an R_k for $t \leq e$ and $k < e$, each P_j^J or P_j^K requires attention at most once and the coding strategy specifically protects the requirement of highest priority which is threatened. That is, either each $H(i,s)$ is undefined for all $s > t$ for some t, or $\lim_s H(i,s) = H(i)$ exists. We show that each R_e is met as follows: Suppose e is least such that R_e is not met. Go to a stage s' where $s' > t$ and $\forall i < e \, \forall s > s' \, [(b_i^s = b_i \text{ for } i \le e) \,\&\, (f(s) \ge e) \,\&\, (\forall t > s' \; (H(i,t) \text{ is undefined}) \vee (H(i,s) = H(i,s') = H(i)) \,\&\, \forall j \le e \, (P_j^J$ and P_j^J and R_i do not demand attention at stage $s))]$. Find a stage $s'' \ge s'$ such that

$$\forall s > s''(((b_i^s = H(j) \text{ for some } j < i) \to b_i^s = b_i^{s''}) \,\&\, (f(s) > \max_{j<e}\{i \mid b_i^s = H(j)\})).$$

Then it follows by construction that as R_e is not met

$$\forall s > s'' \; (\phi_e^s(b_x^s)\downarrow \; \to x < r(e,s))$$

and so $\forall s > s'' \; (\phi_e^s(b_x^s)\downarrow \to x < 4f(s))$ and thus by the usual permitting argument

$$\forall s > s'' \; (\phi_e^s(b_x^s)\downarrow \; \to \forall t > s(x < 4f(t))),$$

and so $f(\omega)$ is recursive contrary to hypothesis. Therefore all the R_e are met. Similar arguments will show all the P_e^J and P_e^K are met, and as in Remmel [18] or Guichard [10] we ensure that $\delta \le_T J$ and $\delta \le_T K$ by the coding and that $d(D(J^*)) \le_T \delta$ and $d(D(K^*)) \le_T \delta$ by the fact that b_e^s only changes at stage s if $e > f(s)$ and so $\{b_i^s \mid i \in \omega\} \le_T \delta$ (see, in particular, Remmel [18]), and the result follows. □

§4. Effective strong supermaximality. In this section we examine other classes of supermaximal subspaces. In particular we examine subspaces with properties stronger than that of strong supermaximality. Our starting points are Remarks 2.3 and 2.4 of §2. These suggest the following definitions analagous to those of $L(\omega)$:

DEFINITON 4.1. (i) Suppose $W \in L(V_\infty)$, $\dim(V_\infty/W) = \infty$ and there exists a recursive function f such that if $W_e \cap W = \{\vec{0}\}$ then $\dim(W_e) \le f(e)$. Then we say that W is *effectively simple.*

(ii) We say $W \in L(V_\infty)$ is *effectively supermaximal* if $\dim(V_\infty/W) = \infty$ and there exists a recursive function f such that if $I_e \cap W = \varnothing$ then $\dim((I_e \cup W)^*/W) \le f(e)$.

DEFINITION 4.2 (ANALOGUE OF COHEN AND JOCKUSCH [4]). We say $W \in L(V_\infty)$ is *strongly effectively simple* if $\dim(V_\infty/W) = \infty$ and there exists a recursive function f such that if $W_e \cap W = \{\vec{0}\}$ then $W_e \subset \{v \mid v \in V_\infty \,\&\, v \le f(e)\}^*$. Finally $W \in L(V_\infty)$ is *strongly effectively supermaximal* if $\dim(V_\infty/W) = \infty$ and there exists a total recursive function f such that if $I_e \cap W = \varnothing$ then $I_e \subset (W \cup \{v \in V_\infty \mid v < f(e)\})^*$.

Henceforth, we will write $V_\infty[v]$ for $\{x \in V_\infty \mid x \le v\}^*$. In Remarks 2.3 and 2.4 we observed that the object constructed in the proof of Theorem 2.2 was not only strongly supermaximal but, in fact, strongly effectively supermaximal. The existence of such a space shows that the natural analogue of the following theorem concerning $L(\omega)$ fails for $L(V_\infty)$.

THEOREM 4.3 (COHEN AND JOCKUSCH [4]). *An r.e. strongly effectively simple set is contained in no r.e. maximal set.*

We show that, in general, strong supermaximality does not imply strong effective supermaximality via the following:

THEOREM 4.4. *Suppose W is effectively simple; then $d(D(W)) = 0'$, that is, W has complete dependence degree.*

PROOF. Let $B = \{b_0 < b_1 < \cdots\}$ be a recursive basis of V_∞. Let K be a complete subset of B, and define $m(s) = \mu s(x \in K_s)$ if $x \in K$ and let $m(s)$ be undefined otherwise. Suppose W is effectively simple via f, with $W = \bigcup_s W_s$. Inductively define a sequence $R_s = \{a_i^s \mid i \in \omega\}$ by the following:

Stage 0. $a_i^0 = b_i$ for all $i \in \omega$.

Stage $s + 1$. $a_0^{s+1} = \mu a_i^s(a_i^s \notin W_{s+1}^*)$, and in general

$$a_j^{s+1} = \mu a_i^s(a_i^s \notin (W_{s+1} \cup \{a_k^{s+1} \mid k < j\})^*).$$

Define $R = \bigcap_s R_s$.

By the recursion theorem define a recursive function h by

$$I_{h(x)} = \begin{cases} \{a_0^{m(x)}, \ldots, a_{fh(x)}^{m(x)}\} & \text{if } x \in K, \\ \varnothing & \text{otherwise.} \end{cases}$$

Now set $r(x) = \mu s(a_{fh(x)}^s = a_{fh(x)})$; then clearly $r \leq_T R$ and by construction $R \equiv_T d(D(W))$. Now if $x \in K$ and $r(x) \leq m(x)$ then $W_{h(x)} \cap W = \{\vec{0}\}$ (as then $I_{h(x)}$ is independent over W and $I_{h(x)}^* = W_{h(x)}$). Therefore $\dim(W_{h(x)}) < fh(x)$ by definition of f and the fact that W is effectively simple. However then, by construction, $\dim(W_{h(x)}) = fh(x) + 1$, a contradiction. Therefore for all $x \in K$, $r(x) > m(x)$, i.e. $x \in K \leftrightarrow x \in K_{r(x)}$, so $K \leq_T d(D(W))$. □

COROLLARY 4.5. *Not every strongly supermaximal subspace is effectively simple, and so, in particular, there exists a strongly supermaximal subspace which is not effectively supermaximal.*

PROOF. In Theorem 2.6 we observed that there exists a strongly supermaximal subspace of arbitrary low dependence degree D. However, for it to be effectively simple, D must be complete. □

Theorem 4.4 might lead one to believe that every r.e. "effectively non-complemented" subspace is complete, as is in the $L(\omega)$ case. However, it is fairly easy to show that if F (the field of scalars) is infinite, then V_∞ contains a recursive effectively simple subspace.

We may hope that strong effective supermaximality for r.e. subspaces may be sufficient to guarantee that they are in the same orbit of the group of automorphisms of $L(V_\infty)$. However, this is not the case (even if F is finite).

THEOREM 4.6. *There exists a pair M_1, M_2 of r.e. strongly effectively supermaximal subspaces such that no automorphism Φ of $L(V_\infty)$ has $\Phi(M_1) = M_2$.*

PROOF. We satisfy similar requirements to Theorem 3.3 via Remarks 2.3 and 2.4. Again set $B = \{a_0 < a_1 < \cdots\}$. We build $M_1 = \bigcup_s M_1^s$ and $M_2 = \bigcup_s M_2^s$. At each stage s, $\{b_{j,e}^s \mid e \in \omega\}$ lists in order a basis of V_∞ over M_j (for $j = 1, 2$). Our requirements are

$N_{1,e}$: $\lim_s b_{1,e}^s = b_{1,e}$ exists,

$N_{2,e}$: $\lim_s b_{2,e}^s = b_{2,e}$ exists,

$P_{1,e}$: $I_e \cap M_1 = \varnothing \to I_e \subset (M_1 \cup \{a_i \mid i \leq 2(e + 1) + 2^{2^e}\})^*$,

$P_{2,e}$: $I_e \cap M_2 = \varnothing \to I_e \subset (M_1 \cup \{a_i \mid i \leq 2(e + 1) + 2^{2^e}\})^*$, and

R_e: if ϕ_e is a recursive semilinear transformation of V_∞ then for some $j \leq 2(e + 1) + 2^{2^e}$ either (i) $a_j = b_{1,k}$ for some k and $\phi_e(a_j) \in M_2$, or (ii) $a_j \in M_1$ and $\phi_e(a_j) \notin M_2$.

We modify the system of markers of 3.3. Those elements a_j with $\phi_e(a_j) \in M_2$ we mark with $H(1,e,s)$. Otherwise we mark with. $H(2,e,s)$. Define for $j = 1,2$ the following:

$$r(j,e,s) = \max(\{i \mid b_i^s = H(j,i,s) \text{ for } i \leq e\} \cup \{e\}).$$

We say $P_{j,e}$ *requires attention at stage* $s+1$ if $P_{j,e}$ has highest priority and (i) $I_e^s \cap M_j^s = \varnothing$, (ii) $\exists x \in I_e^s(x \notin (M_1^s \cup \{b_i^s \mid i \leq r(j,e,s)\})^*)$. As in the earlier constructions, if $P_{j,e}$ requires attention set $M_j^{s+1} = (M_j^s \cup \{x\})^*$ and $M_i^{s+1} = M_i^s$ for $i \neq j$, and set (i) $b_{i,t}^{s+1} = b_{i,t}^s$ for all $t \in \omega$ and (ii) $b_{j,t}^{s+1} = b_{j,t}^s$ for $t < g(i,s,x)$; $b_{j,t}^{s+1} = b_{j,t+1}^s$ for $t \geq g(i,s,x)$ where $g(j,s,x) = \max(t \mid b_{j,t} \in \operatorname{supp}_{j,s}(x)\}$ for $x \notin M_j^s$, and $g(j,s,x) = -1$ if $x \in M_j^s$ where $\operatorname{supp}_{j,s}(x)$ denotes the support of x relative to $\{b_{j,i}^s \mid i \in \omega\}$ over M_j^s.

Finally we deal with the R_e. We say R_e *requires attention* if neither $H(1,e,s)$ nor $H(2,e,s)$ is defined and $\phi_e^s(b_{i,e}^s)\downarrow$. We describe the actions we take to meet R_e. There are three cases.

Case (i). $\phi_e^s(b_{1,e}^s) \in M_2^s$. Define $H(1,e,s) = b_{1,e}^s$ and change nothing else.

Case (ii). $\phi_e^s(b_{1,e}^s) \notin (M_2^s \cup \{b_{2,j}^s \mid j \leq e\})^*$. Define $H(i,e,s) = b_{i,e}^s$ and set $M_2^{s+1} = (M_2^s \cup \{\phi_e^s(b_{1,e}^s)\})^*$. In this case set $M_1^{s+1} = M_1^s$, $b_{1,i}^{s+1} = b_{1,i}^s$ for all $i \in \omega$ and set

$$b_{2,i}^{s+1} = \begin{cases} b_{2,i}^s & \text{for } i < g(2,s,\phi_e^s(b_{1,e}^s)), \\ b_{2,i+1}^s & \text{otherwise.} \end{cases}$$

Case (iii). Otherwise. Set $M_1^{s+1} = (M_1^s \cup \{b_{1,e}^s\})^*$. Define $H(2,e,s) = \phi_e^s(b_{1,e}^s)$, change the appropriate $b_{1,i}^s$'s similarly as in case (ii) and otherwise do nothing. For the construction we attack as described above. For the bounds on the a_i's in the statement of the requirement, one can check on the number of times any positive requirement may be injured. □

By modifying this construction (in a way similar that to that of Theorem 3.3) we can ensure that both M_1 and M_2 have the same fully co-r.e. complement.

COROLLARY 4.7. *There exist a pair* (M_1, M_2) *of r.e. strongly effectively supermaximal subspaces and a co-r.e. subset* R *of a recursive basis of* V_∞ *such that*

(i) $M_1 \oplus R^* = M_2 \oplus R^* = V_\infty$, *and*

(ii) *no automorphism* Φ *of* $L(V_\infty)$ *takes* M_1 *to* M_2.

PROOF. By the remarks above, and Theorem 4.6. □

We close with a couple of questions. The techniques of Guichard, Nerode and Remmel, together with those used here, always seem to produce M_1 *not of the same* 1-*degree* as M_2, or at least $D_k(M_1) \not\equiv_1 D_k(M_2)$ for some k. We ask if it is possible to produce nonautomorphic supermaximal M_1 and M_2 of the same 1-degree dependence structure? Perhaps the techniques introduced here may be useful in answering this question. We remark that evidently this is a necessary condition for M_1 and M_2 to be automorphic. A second question is to ask whether or not all (super) maximals are automorphic under automorphisms of $L^*(V_\infty)$, the lattice of r.e. subspaces modulo finite-dimensional subspaces.

Finally, there is an essential difference between the results of Kalantari and Retzlaff [13] and other nonautomorphism results. In [13] the authors produced a pair M_1, M_2 of maximal subspaces and a first order formula $\gamma(x)$ such that M_1 satisfied γ and M_2 did not. Is there a similar "elementary" property distinguishing different orbits of supermaximal subspaces? We remark that this question has been analysed in [8], and Downey has unpublished material extending [8]. However the question remains open.

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