ARRAY NONRECURSIVE DEGREES AND GENERICITY

Rod Downey, Carl G. Jockusch, and Michael Stob

Abstract. A class of r.e. degrees, called the array nonrecursive degrees, previously studied by the authors in connection with multiple permitting arguments relative to r.e. sets, is extended to the degrees in general. This class contains all degrees which satisfy $\mathbf{a}'' > (\mathbf{a} \cup \mathbf{0}')'$ (i.e. $\mathbf{a} \in \overline{GL_2}$) but in addition there exist low r.e. degrees which are array nonrecursive (a.n.r.). Many results for $\overline{GL_2}$ degrees extend to the a.n.r. degrees and thus to certain low degrees. A new notion of genericity (called pb-genericity) is introduced which is intermediate between 1-genericity and 2-genericity. It is shown that the upward closure of the pb-generic degrees is the set of a.n.r. degrees.

1. Introduction

This paper is a sequel to [5], which was a study of certain recursively enumerable sets called *array nonrecursive sets.* A number of characterizations of the degrees of these sets were obtained, showing that they are precisely those r.e. degrees relative to which various "multiple permitting constructions" may be performed. (In such a construction an attempt to meet a requirement may require a finite, but recursively bounded, number of permissions from the oracle.) For example, it was shown that, for any increasing recursive function f the array nonrecursive degrees are precisely those r.e. degrees $\mathbf{a}$ such that some function of degree $\mathbf{a}$ is not f-r.e. (A function is f-r.e. if it has a recursive approximation which changes at most $f(n)$ times on argument n for each n.) Subsequently Kummer [10, Theorem 3.2] characterized the array nonrecursive degrees as those r.e. degrees which contain an r.e. set A which is "complex" in the sense of Kolmogorov complexity. (This means that there is a constant c such that, for infinitely many n, the Kolmogorov complexity of the first n bits of the characteristic function of A is at least $2 \log n - c$.)

In the current paper we extend the notion of array nonrecursiveness from r.e. degrees to degrees in general. This is a conservative extension in the sense that the r.e. degrees which are array nonrecursive in the sense of

---

1991 *Mathematics Subject Classification.* Primary 03D30; Secondary 03D25.

Jockusch was supported by NSF grant DMS 92-02833.

[5] coincide with the r.e. degrees which are array nonrecursive in the sense of the current paper, as we will show. It will be clear from our definition that every degree $\mathbf{a} \in \overline{GL_2}$ $(= \{\mathbf{a} : \mathbf{a}'' > (\mathbf{a} \cup \mathbf{0}')'\})$ is array nonrecursive. However, the array nonrecursive (a.n.r.) degrees properly include the $\overline{GL_2}$ degrees, and in fact there are low r.e. degrees which are a.n.r. by [5, Theorem 2.1]. On the other hand, we will show that a.n.r. degrees have many of the properties of $\overline{GL_2}$ degrees. For example, we will show that every a.n.r. degree $\mathbf{a}$ bounds a 1-generic degree (extending Jockusch-Posner [8]) and that every recursive lattice with distinct least and greatest elements can be embedded in $\mathcal{D}_{\leq}(\mathbf{a})$ preserving the lattice operations and the least and greatest elements (extending Fejer [6]). Downey [4] previously showed that every r.e. degree which is a.n.r. in the sense of [5] is the supremum of a minimal pair of (not necessarily r.e.) degrees, and our result extends this with a much easier proof.

We will define a new sort of genericity, called pb-genericity. This concept will be strictly intermediate in strength between 1-genericity and 2-genericity. The upward closure of the pb-generic degrees will be shown to be the a.n.r. degrees. To compare this notion of genericity with 1-genericity, recall that, roughly speaking, a 1-generic set is one which has all properties which can be guaranteed by a Kleene-Post construction with an effective list of requirements $\{R_n\}$ such that, for each requirement R_n and each string σ, either every set extending σ satisfies R_n, or a certain effective search yields a string $\tau \supseteq \sigma$ such that every set extending τ satisfies R_n. The intuitive idea of pb-genericity is that instead of a single effective search as above, there may be a finite sequence of effective searches. The number of such searches must be bounded in advance by a recursive function of n and σ. The first search simply produces σ, and no subsequent search begins until the previous one has terminated. The final terminating search yields a string $\tau \supseteq \sigma$ such that any set extending τ satisfies the requirement. Thus, this paper concerns multiple search arguments, rather than multiple permitting arguments as in [5]. We use the pb-generic sets as a convenient tool to study the cupping properties of a.n.r. degrees. (For background on genericity see, for example, [7] or [9].)

Our notation is standard and follows that of [14].

We are indebted to Peter Fejer and Martin Kummer for helpful corrections and suggestions.

Our new definition of array nonrecursiveness is based on domination properties of functions. We first recall that $f \leq_{wtt} A$ (for f a function and A a set) means that there is a number e and a recursive function b such that, for all n, $f(n) = \{e\}^A(n)$ and, furthermore, for each n, the use of the computation of $\{e\}^A(n)$ does not exceed $b(n)$. Let K be the usual complete r.e. set. It is easily seen that $f \leq_{wtt} K$ iff there are recursive functions

$h(.,.)$ and $p(.)$ such that, for all n, $f(n) = \lim_s h(n,s)$ and $|\{s : h(n,s) \neq h(n,s+1)\}| \leq p(n)$.

Definition 1.1. *A degree* $\mathbf{a}$ *is array nonrecursive (a.n.r.) if for each* $f \leq_{wtt} K$ *there is a function* g *recursive in* $\mathbf{a}$ *such that* $g(n) \geq f(n)$ *for infinitely many* n.

It is stated in [8, Lemma 1] that an arbitrary degree $\mathbf{a}$ is in $\overline{GL_2}$ iff for each function f recursive in $\mathbf{a} \cup \mathbf{0}'$ there is a function g recursive in $\mathbf{a}$ such that $g(n) \geq f(n)$ for infinitely many n. From this we immediately obtain the following:

Proposition 1.2. *For any degree* $\mathbf{a}$*, if* $\mathbf{a} \in \overline{GL_2}$*, then* $\mathbf{a}$ *is a.n.r.*

As we will show, many results for $\overline{GL_2}$ degrees extend to a.n.r. degrees. The following result gives some characterizations of the a.n.r. degrees. As in [5], we define a *very strong array* to be a sequence $\{F_n\}$ of nonempty, pairwise disjoint finite sets of strictly increasing cardinality with union ω such that the canonical index of F_n is a recursive function of n. Fix a recursive enumeration $\{K_s\}$ of K, and define $m_K(n)$ to be the least s such that $K \restriction n = K_s \restriction n$, where $A \restriction n = \{i < n : i \in A\}$.

Theorem 1.3. *Let* $\mathbf{a}$ *be a degree, and let* $\{F_n\}$ *be a very strong array. Then the following three conditions are equivalent:*

(i) $\mathbf{a}$ *is a.n.r.*

(ii) There is a function h *recursive in* $\mathbf{a}$ *such that* $h(n) \geq m_K(n)$ *for infinitely many* n.

(iii) There is a function r *recursive in* $\mathbf{a}$ *such that for all* e *there exists* n *with* $W_e \cap F_n = W_{e,r(n)} \cap F_n$.

Proof. To prove $(ii) \to (i)$, let f be given with $f \leq_{wtt} K$, and let h satisfy (ii). We must find g recursive in $\mathbf{a}$ with $g(n) \geq f(n)$ for infinitely many n. Fix e and a recursive function b such that $f(n) = \{e\}^K(n)$ with use at most $b(n)$ for all n. We may assume without loss of generality that h and b are increasing. To compute $g(n)$, let s be minimal such that $s > h(b(n+1))$ and $\{e\}_s^{K_s}(n)\downarrow$ with use at most $b(n)$, and let $g(n) = \{e\}_s^{K_s}(n)$. Clearly g is recursive in $\mathbf{a}$. Let n and k be such that $b(n) \leq k \leq b(n+1)$ and $h(k) \geq m_K(k)$, and let s be as in the definition of $g(n)$. We have $s \geq h(b(n+1)) \geq h(k) \geq m_K(k) \geq m_K(b(n))$, so K_s and K are the same below the use of $\{e\}_s^{K_s}(n)$. Hence $g(n) = f(n)$. Since there are infinitely many j with $h(j) \geq m_K(j)$, there are infinitely many n for which k exists as described above, and hence $g(n) = f(n)$ for infinitely many n.

The implication $(i) \to (iii)$ is obtained by applying (i) to the function $f(n) = (\mu s)(\forall e \leq n)[W_e \cap F_n = W_{e,s} \cap F_n]$.

It remains only to show that $(iii) \to (ii)$. Let r witness (iii). Then for each e there are infinitely many n with $W_e \cap F_n = W_{e,r(n)} \cap F_n$. (If this fails, one obtains a contradiction by defining an index e' so that there is no n with $W_{e'} \cap F_n = W_{e',r(n)} \cap F_n$. If $x \in F_n$ for one of the finitely many n such that $W_e \cap F_n = W_{e,r(n)} \cap F_n$, let $\phi_{e'}(x)$ converge in strictly more than $r(n)$ steps, and otherwise let $\phi_{e'}(x)$ converge (if ever) in at least the same number of steps as $\phi_e(x)$.) Hence it suffices to show that there is an e such that, for all n, $\mu s[W_{e,s} \cap F_n = W_e \cap F_n] \geq m_K(n)$, since then it follows that r also witnesses (ii). The set W_e will be $V = \cup_s V_s$, where V_s is defined as follows. The idea is to make V change on F_n whenever an element $< n$ enters K. Let $V_0 = \emptyset$. Given V_s, let $c_{n,s}$ be the least element (if any) of $F_n - V_s$, and let

$$V_{s+1} = V_s \cup \{c_{n,s} : (\exists z < n)[z \in K_{s+1} - K_s]\}.$$

Note that $|F_n \cap V| \leq |\{s : (\exists i < n)[i \in K_{s+1} - K_s]\}| \leq n < |F_n|$ so that $c_{n,s}$ is defined for all n and s. It follows that $(\mu s)[V_s \cap F_n = V \cap F_n] \geq m_K(n)$. By the proof of the slowdown lemma [14, page 284] (which requires only that the sets $V_{e,s}$ be recursive uniformly in e and s and not necessarily finite), there exists e such that $W_e = V$ and, for all s, $W_{e,s} \subseteq V_s$. Hence for all n, $\mu s[W_{e,s} \cap F_n = W_e \cap F_n] \geq m_K(n)$, and the proof is complete. $\square$

Since the truth of (i) of Theorem 1.3 does not depend on the choice of the very strong array $\{F_n\}$, it follows that the truth of (iii) is also independent of the choice of $\{F_n\}$. A similar result on independence of array occurs as Theorem 2.5 of [5]. The following result which shows in particular that the notion of array nonrecursiveness in Definition 1.1 is equivalent for r.e. degrees to the definition of array nonrecursiveness in [5].

Proposition 1.4. *Let* **a** *be an r.e. degree and let* $\{F_n\}$ *be a very strong array. Then the following are equivalent:*

(i) **a** *is a.n.r.*

(ii) There is an r.e. set A *of degree* **a** *such that* $(\forall e)(\exists n)[W_e \cap F_n = A \cap F_n]$.

Proof. To prove $(ii) \rightarrow (i)$, we assume that (ii) holds and show that (iii) of Theorem 1.3 holds with $r(n) = (\mu s)[A_s \cap F_n = A \cap F_n]$, where $\{A_s\}$ is a recursive enumeration of A. To do this, for each e let $V_e = \{x : (\exists s)[x \in W_{e,s} - A_s]\}$. Then V_e is an r.e. set so by (ii) there exists n such that $A \cap F_n = V_e \cap F_n$. It is then easily seen that $W_e \cap F_n = W_{e,r(n)} \cap F_n$ as needed to prove (iii) of Theorem 1.3.

For the converse, assume that **a** is a.n.r. as defined in Definition 1.1. Let $\{F_n\}$ be any very strong array. We shall construct an r.e. set A recursive in **a** such that $(\forall e)(\exists n)[W_e \cap F_n = A \cap F_n]$. (This suffices to prove (ii) by [5, Corollary 2.8].) Let $f(n) = (\mu s)(\forall e \leq n)[W_{e,s} \cap F_{\langle e,n\rangle} = W_e \cap F_{\langle e,n\rangle}]$. Clearly $f \leq_{wtt} K$, so there exists g of degree at most **a** with $g(n) \geq f(n)$ for infinitely many n. By the Modulus Lemma, there is a recursive function

$h(n,s)$ and a function p recursive in $\mathbf{a}$ such that $g(n) = h(n,s)$ for all $s \geq p(n)$. We now define the r.e. set A. It suffices to ensure that if $n \geq e$ and $g(n) \geq f(n)$ then $A \cap F_{\langle e,n\rangle} = W_e \cap F_{\langle e,n\rangle}$. Whenever $h(n,s) \neq h(n,s+1)$ and $e \leq n \leq s$, put all elements of $W_{e,h(n,s+1)} \cap F_{\langle e,n\rangle}$ into A at stage s, and let A be the set of all numbers obtained in this fashion. If $x \in A \cap F_{\langle e,n\rangle}$, then $x \in A_{p(n)}$, so A is recursive in p and hence A has degree at most $\mathbf{a}$. Suppose now that $n \geq e$ and $g(n) \geq f(n)$. It follows from the definition of f that $W_{e,g(n)} \cap F_{\langle e,n\rangle} = W_e \cap F_{\langle e,n\rangle}$. Choose s as large as possible so that $h(n,s) \neq h(n,s+1)$. (There is no loss of generality in assuming there is at least one such $s \geq n$.) Then $h(n,s+1) = g(n)$ and so $A_{s+1} \cap F_{\langle e,n\rangle} = W_{e,h(n,s+1)} \cap F_{\langle e,n\rangle} = W_{e,g(n)} \cap F_{\langle e,n\rangle} = W_e \cap F_{\langle e,n\rangle}$. Furthermore, by the maximality of s, no elements of $F_{\langle e,n\rangle}$ enter A after $s+1$, so $A \cap F_{\langle e,n\rangle} = W_e \cap F_{\langle e,n\rangle}$, as needed to complete the proof. $\square$

If references to recursive enumerability are deleted from Proposition 1.4, then $(i) \to (ii)$ still holds but $(ii) \to (i)$ fails. In fact, for any very strong array $\{F_n\}_{n\in\omega}$ there is a set A such that $(\forall e)(\exists n)[W_e \cap F_n = A \cap F_n]$ and every function recursive in A is majorized by a recursive function, so that $\deg(A)$ fails badly to be a.n.r. This may be proved by modifying the standard construction of a hyperimmune-free degree so that the recursive perfect trees used are compatible with the condition that $(\forall e)(\exists n)[W_e \cap F_n = A \cap F_n]$. Specifically, one constructs a descending sequence $\{T_e\}_{e\in\omega}$ of recursive perfect trees and obtains A as their unique common branch. (Here a recursive perfect tree is a recursive function T from $2^{<\omega}$ to $2^{<\omega}$ such that, for any string σ, $T(\sigma 0)$ and $T(\sigma 1)$ are incompatible extensions of $T(\sigma)$.) The additional condition imposed on each T_e is that there exist infinitely many n such that if σ is any string of length $\min F_n$ extendible to a string in the range of T_e, then any string τ which extends σ and has length $(\max F_n) + 1$ is also extendible to a string in the range of T_e.

2. Working below a.n.r. degrees

A number of results about initial segments determined by $\overline{GL_2}$ degrees extend easily to a.n.r. degrees. This is illustrated by the following two results.

Theorem 2.1. *If* $\mathbf{a}$ *is any a.n.r. degree, then there is a 1-generic degree* $\mathbf{b} \leq \mathbf{a}$. *Hence, no a.n.r. degree is minimal.*

Proof. It is shown in [8 , Lemma 3] that there is a function f recursive in $\mathbf{0}'$ such that for any g with $g(n) \geq f(n)$ for infinitely many n there is a 1-generic set B recursive in g. It suffices to show that the function f specified in the proof of that lemma satisfies $f \leq_{wtt} K$. This is clear since f has the form $f(n) = \max\{\psi(e,\sigma) : e \leq n, |\sigma| = n, \psi(e,\sigma)\downarrow\}$, for a certain partial recursive function ψ. (Here σ ranges over binary strings.) Thus the natural recursive approximation to $f(n)$ changes at most once for each pair (e, σ), with $e \leq n$ and $|\sigma| = n$. $\square$

Downey [4, Theorem 1.1] proved that every r.e. degree $\mathbf{a}$ which is a.n.r. in the sense of [5] is the sup of a minimal pair of (possibly non-r.e.) degrees. Below, this result is extended to a.n.r. degrees in general and also (applying the construction of Fejer [6]) to a more general class of lattices. (Another extension to a.n.r. degrees in general will be given in Theorem 2.5.)

Theorem 2.2. *Let* $\mathbf{a}$ *be any a.n.r. degree. Then any recursively presented lattice with distinct least and greatest elements can be embedded in* $\mathcal{D}_{\leq}(\mathbf{a})$ *with least and greatest elements preserved.*

Proof. This is established by the proof of Theorem 6 of [6], once it is shown that the function F defined in that proof satisfies $F \leq_{wtt} K$. The function F has the form $F(n) = \max\{f(\sigma, e) : (\sigma, e) \in D_{g(n)}\}$, where g is recursive, D_k is the finite set with canonical index k, and $f(\sigma, e)$ is a certain function. (Here σ ranges over $\omega^{<\omega}$.) The definition of f is quite involved, but makes it clear that f has a natural recursive approximation which changes at most 5 times on each argument. Thus $f \leq_{wtt} K$, so $F \leq_{wtt} K$. $\square$

T. Slaman [12] proved that there is a nonzero r.e. degree $\mathbf{b}$ which is not the sup of a minimal pair of (not necessarily r.e.) degrees. Downey [4, §1] then answered a question raised by Slaman by showing that there is a low r.e. degree $\mathbf{a}$ such that every r.e. degree $\mathbf{b} \geq \mathbf{a}$ is the sup of a minimal pair. (This used the existence of a low r.e. degree which is a.n.r. and the result [4, Theorem 1.1] cited above.) The following corollary removes the restriction that $\mathbf{b}$ be r.e. from Downey's result and also gives a simpler proof.

Corollary 2.3. *There is a low r.e. degree* $\mathbf{a}$ *such that every degree* $\mathbf{b} \geq \mathbf{a}$ *is the sup of a minimal pair.*

Proof. Let $\mathbf{a}$ be any low r.e. a.n.r. degree and apply Theorem 2.2 (or Theorem 2.5, to follow) and the obvious fact that the a.n.r. degrees are upward closed. It is shown in [5, Theorem 2.1] that there is a low r.e. degree which is a.n.r. in the sense of [5], but such degrees are also a.n.r. in the sense of the current paper by Proposition 1.4. $\square$

It is shown in [8, Lemma 6] that if $\mathbf{a} \in \overline{GL_2}$ and $\{S_n\}$ is a sequence of dense sets of strings uniformly recursive in $\mathbf{a} \cup \mathbf{0}'$, then there is a function F recursive in $\mathbf{a}$ which meets each S_n. It is easily seen that the converse holds. (If the conclusion holds for $\mathbf{a}$, then for each f recursive in $\mathbf{a} \cup \mathbf{0}'$, we can let $S_n = \{\sigma : (\exists k \geq n)[p_\sigma(k) \downarrow > f(k)]\}$ to show that there is a function g recursive in $\mathbf{a}$ with $g(k) \geq f(k)$ for infinitely many k. Here p_σ enumerates $\{n : \sigma(n) = 1\}$ in increasing order.) It follows from [8, Lemma 1] that $\mathbf{a} \in \overline{GL_2}$. Thus we have an example of a result for $\overline{GL_2}$ degrees which does not extend to a.n.r. degrees. In fact, there is an a.n.r. degree

$\mathbf{a}$ and a sequence of dense sets $\{S_n\}$ uniformly recursive in $\mathbf{0}'$ such that no function recursive in $\mathbf{a}$ meets each S_n. (Let $\mathbf{a}$ be a low a.n.r. degree, which exists by [5, Theorem 2.1] and Proposition 1.4.) Nonetheless, in the following result, we prove an appropriate analogue of [8, Lemma 6]. Call a sequence $\{S_n\}$ of sets of strings *uniformly wtt-dense* if there is a function $d \leq_{wtt} K$ such that for all strings $\sigma \in 2^{<\omega}$, $d(n,\sigma) \supseteq \sigma$ and $d(n,\sigma) \in S_n$.

Theorem 2.4. *Let* $\mathbf{a}$ *be an a.n.r. degree, and let* $\{S_n\}$ *be uniformly wtt-dense. Then there is a set* A *recursive in* $\mathbf{a}$ *which meets each* S_n.

Proof. Let $\mathbf{a}$ and S_n be as in the hypothesis of the theorem, and let $d(n,\sigma)$ witness the wtt-density of $\{S_n\}$. Let $\hat{d}(n,\sigma,s)$ and $b(n)$ be recursive functions such that $\lim_s \hat{d}(n,\sigma,s) = d(n,\sigma)$ and $|\{s : \hat{d}(n,\sigma,s) \neq \hat{d}(n,\sigma,s+1)\}| \leq b(n)$. Let $h(n,\sigma) = (\mu s)(\forall t \geq s)[\hat{d}(n,\sigma,t) = \hat{d}(n,\sigma,s)]$. Let $f(n) = \max\{h(e,\sigma) : e \leq n, |\sigma| \leq n\}$. We have $f \leq_{wtt} K$ since $d \leq_{wtt} K$. Fix a function g recursive in $\mathbf{a}$ such that $g(n) \geq f(n)$ for infinitely many n.

We obtain A as $\cup_s \sigma_s$, where $|\sigma_s| = s$. The method of proof is familiar from [8], and we give only an informal sketch. The idea for making A meet S_e is to wait for a stage $n \geq e$ where $\hat{d}(e,\sigma_n,g(n)) \supseteq \sigma_n$ and then to set $\sigma_t = \hat{d}(e,\sigma_n,g(n)) \restriction t$ for $n < t \leq |\hat{d}(e,\sigma_n,g(n))|$. However, this process may be interrupted if there is action for some S_m with $m < e$ before it is completed, and in this case we start over on S_e. Also, we say that the attack is *discredited* at stage u ($u > n$), if for some t, $g(n) < t \leq g(u)$ we have $\hat{d}(e,\sigma_n,t) \neq \hat{d}(e,\sigma_n,g(n))$. If this happens, we also start over on S_e.

To see that this strategy works and only acts finitely often, assume this for all $e' < e$, and consider $n > e$ with $g(n) \geq f(n)$ such that we do not act for any $e' < e$ at stage n or any subsequent stage. If we act as above for e at stage n, the process is clearly never interrupted or discredited, and it is clear that A meets S_e and that we never act again for S_e after the process is completed. If we did not act as above for e at stage n, there must have existed a stage $m < n$ such that we attacked e at stage m and the attack was not interrupted or discredited at any stage k, $m < k \leq n$. Since $g(n) \geq f(n) \geq f(m)$, it follows that the attack starting at stage m will never be discredited or interrupted. Thus also in this case, A meets S_e and we act only finitely often to achieve this. $\square$

The following application of the method of proof of Theorem 2.4 implies Theorem 2.1 and a special case of Theorem 2.2, as well as the fact that no degree is minimal among the a.n.r. degrees. The method of proof is basically that used by Cooper [1, Theorem 1] to show that there is a minimal pair of degrees with sup $\mathbf{0}'$.

Theorem 2.5. *Every a.n.r. degree is the supremum of two 1-generic a.n.r. degrees which form a minimal pair.*

Proof. Let $\mathbf{a}$ be a given a.n.r. degree and let A be a set of degree $\mathbf{a}$ such that A is recursive in every infinite subset of A. (The last condition entails no loss of generality by [3, P2 and T2].) We will construct a set $B = B_0 \oplus B_1$ so that the degrees of B_0 and B_1 have the desired properties. To make $A \leq_T B_0 \oplus B_1$ we require that $B_0 \cap B_1 \subseteq A$, and that $B_0 \cap B_1$ be infinite. Thus we must satisfy the following requirements for all $e \in \omega$ and all $i \leq 1$:

$G_e^i : (\exists \sigma \subseteq B_i)(\sigma \in W_e \vee (\forall \tau \supseteq \sigma)[\tau \notin W_e])$
$N_e : \{e\}^{B_0} = \{e\}^{B_1} = f, f$ total $\to f$ recursive
$A_e^i : (\exists k > e)[p_{B_i}(k) > m_K(k)]$
$P_e : |B_0 \cap B_1| \geq e$

Effectively list the requirements other than the P_e's as $R_0, R_1, \ldots$. We will define a function d such that $d(n, \sigma) \supseteq \sigma$ for all n and σ, $d \leq_{wtt} K$, and R_n is satisfied whenever $B \supseteq d(n, \sigma)$ for any string σ. It would then be a direct application of Theorem 2.4 to conclude that $\mathbf{a}$ bounds a minimal pair of a.n.r., 1-generic degrees. The requirements P_e are easily combined with these since there is an $\mathbf{a}$-recursive function p such that, for all σ, $p(e, \sigma) \supseteq \sigma$, and any set B which extends $p(e, \sigma)$ satisfies P_e. (The P_e's should be effectively meshed into the list of requirements, and values of p should be used in place of values of $\hat{d}$ when attacking P_e. Such attacks are never discredited but may be interrupted for requirements of higher priority.) However, we must also ensure that $B_0 \cap B_1 \subseteq A$. Call a string σ *admissible* if $\{n : \sigma(2n) = \sigma(2n+1) = 1\} \subseteq A$. To make $B_0 \cap B_1 \subseteq A$, we also require that every string extended by B be admissible. The proof of Theorem 2.4 has the property that every string σ_t chosen during an attack for S_e is extended by $\hat{d}(e, \sigma_s, u)$ for some $s < t$, and some u. We will choose d and $\hat{d}$ so that if σ is admissible, then $\hat{d}(e, \sigma, s)$ is admissible for all e and s. Furthermore, the function p mentioned above can be chosen so that if σ is admissible, then $p(e, \sigma)$ is admissible for all e. Also, in the construction we let σ_0 be the empty string and, if no requirement requires attention when we define σ_{s+1}, we let $\sigma_{s+1} = \sigma_s^* < 0 >$. It is then clear by induction on s that σ_s is admissible.

Thus it suffices to define $p(e, \sigma), d(n, \sigma)$, and $\hat{d}(n, \sigma, s)$ as described above. The definition of p is left to the reader. We define d below and in all cases let $\hat{d}$ be the "natural" recursive approximation to d.

We now define $d(n, \sigma)$. Let $\sigma = \sigma_0 \oplus \sigma_1$, so $\sigma_j(n) = \sigma(2n + j)$ for $j \leq 1$. Suppose first that R_n is a genericity requirement G_e^0. If there is a string $\tau_0 \in W_e$ such that $\tau_0 \supseteq \sigma_0$ choose the first such τ_0 in an effective enumeration of W_e and let $d(n, \sigma) = \tau_0 \oplus \tau_1$, where τ_1 is obtained by concatenating σ_1 with an appropriate number of 0's. If no such τ_0 exists, let $d(n, \sigma) = \sigma$. Now suppose that R_n is a minimal pair requirement N_e. If there is no e-split pair (τ_0, μ_0) of extensions of σ_0, let $d(n, \sigma) = \sigma$. Otherwise, consider the first one

(τ_0, μ_0) in an effective enumeration of all such e-splittings, and let k be the least argument on which τ_0, μ_0 e-split. We may assume $|\tau_0| = |\mu_0| = q$, and let τ_1 be an extension of σ_1 of length q obtained by concatenating σ_1 with an appropriate number of $0's$. If there is no δ_1 extending τ_1 with $\{e\}^{\delta_1}(k)\downarrow$, let $d(n,\sigma) = \tau_0 \oplus \tau_1$. Otherwise, choose the first such δ_1. If τ_0 and δ_1 are e-split, let γ_0 be the extension of τ_0 of the same length as δ_1 obtained by concatenating 0's to τ_0, and set $d(n,\sigma) = \gamma_0 \oplus \delta_1$. Otherwise, μ_0 and δ_1 are e-split, and $d(n,\sigma)$ is defined similarly with μ_0 in place of τ_0. We leave the case where R_n is A^i_e to the reader.

Note that $d(n,\sigma)$ is always defined so that it does no new coding, i.e. if $d(n,\sigma)(2k) = d(n,\sigma)(2k+1) = 1$, then $2k+1 < |\sigma|$. We can easily arrange that this is also true of the approximations $\hat{d}(n,\sigma,s)$, so we are able to make all σ_s admissible as described above.

$\square$

3 Genericity and cupping

We saw in the previous section that every a.n.r. degree bounds a 1-generic degree. On the other hand, 1-genericity is not a sufficiently strong form of genericity for all of our purposes. In this section we define a notion of genericity intermediate between 1-genericity and 2-genericity which corresponds more precisely to the a.n.r. property. Using this notion for convenience we prove that every a.n.r. degree can be nontrivially cupped to all larger degrees.

First, let $f \leq_{pb} C$ mean that f can be computed from oracle C by a reduction procedure with a primitive recursive bound on the use function. Note that $f \leq_{pb} K$ iff there is a recursive function $\hat{f}(n,s)$ and a primitive recursive function p such that $\lim_s \hat{f}(n,s) = f(n)$ and $|\{s : \hat{f}(n,s) \neq \hat{f}(n,s+1)\}| \leq p(n)$. It is then easy to see that the various functions $f \leq_{wtt} K$ mentioned in §2 actually satisfy $f \leq_{pb} K$. If S is a set of strings, call S *pb-dense* if there is a function $f \leq_{pb} K$ such that, for all strings σ, $f(\sigma) \supseteq \sigma$ and $f(\sigma) \in S$.

Definition 3.1. *A set A is pb-generic if A meets every pb-dense set of strings.*

For each e, $\{\sigma : \sigma \in W_e \vee (\forall \tau \supseteq \sigma)[\tau \notin W_e]\}$ is pb-dense, so each pb-generic set is 1-generic. If $f \leq_{pb} K$, then the range of f is Σ_2, so every 2-generic set is pb-generic.

Theorem 3.2. *(i) If* $\mathbf{a}$ *is a.n.r., there is a pb-generic set A recursive in* $\mathbf{a}$.
(ii) If A is pb-generic, then $deg(A)$ is a.n.r.

Proof. To show (i), we first observe that the functions $\leq_{pb} K$ are uniformly $\leq_{wtt} K$. Let $p_0, p_1, \ldots$ be a uniformly recursive listing of the primitive recursive functions. Let $g(\langle a,b\rangle,n) = \{b\}^K(n)$ if, for each $i \le n$, $\{b\}^K(i)\downarrow$ with use at most $p_a(i)$. Otherwise, let $g(\langle a,b\rangle,n) = 0$. It is clear that $g \le_{wtt} K$ and that each function $f \le_{pb} K$ has the form $n \mapsto g(e,n)$ for some e. Let $h(e,\sigma) = g(e,\sigma)$ if $g(e,\sigma) \supseteq \sigma$, and otherwise let $h(e,\sigma) = \sigma$. Let $S_n = \{h(n,\sigma) : \sigma \in 2^{<\omega}\}$. Then the sets $S_0, S_1, \ldots$ are uniformly wtt-dense. Also each pb-dense set contains some S_i. By Theorem 2.4, there is a set A recursive in **a** which meets each S_n, and clearly A is pb-generic.

To prove (ii), assume that A is pb-generic. We show that the principal function p_A of A satisfies $p_A(n) \ge m_K(n)$ for infinitely many n. This suffices by Theorem 1.3 to conclude that $deg(A)$ is a.n.r. Let k be given. We define a pb-dense set S of strings so that every set A which meets S is such that $(\exists n \ge k)[p_A(n) \ge m_K(n)]$. Define:

$$S = \{\sigma : (\exists n \ge k)[p_\sigma(n) \ge m_K(n)]\}$$

(Here p_σ enumerates $\{n : \sigma(n) = 1\}$ in increasing order.)

Then S is witnessed pb-dense by f, where $f(\sigma) = \sigma 1^k 0^{m_K(n)} 1$, and $n = |\sigma^{-1}(1)| + k$. Note that if $A \supseteq f(\sigma)$, then $p_A(n) \ge m_K(n)$, where n is as in the definition of $f(\sigma)$ and, in particular, $n \ge k$. $\square$

The following corollary and the remarks after Definition 3.1 show that the concept of pb-genericity is strictly intermediate between 1-genericity and 2-genericity.

Corollary 3.3. *There is a 1-generic degree* **b** *which is not pb-generic. There is also a pb-generic degree* **d** *which is not 2-generic.*

Proof. Let **a** be a nonzero r.e. degree which is not a.n.r. (Such a degree exists by [5, Theorem 2.10] and Proposition 1.4.) Let **b** be a 1-generic degree $\le$ **a** (see [14, Exercise VI.3.9]). Then **b** is 1-generic, but it is not pb-generic by Theorem 3.2 and the upward closure of the a.n.r. degrees. For the other half, it is clear from Theorem 3.2 and the fact that $\mathbf{0}'$ is a.n.r. that there is a pb-generic degree $\mathbf{d} \le \mathbf{0}'$. Clearly **d** is not 2-generic, as any 2-generic degree is 1-generic relative to $\mathbf{0}'$ and hence not below $\mathbf{0}'$. $\square$

Work of S. Kurtz [11] gives further examples of 1-generic degrees which are not pb-generic. Specifically, Kurtz [11, Theorem 4.1] showed that for almost every degree **b** (in the sense of the measure on degrees induced by the usual product measure on 2^ω) there is a 1-generic degree $\mathbf{a} \le \mathbf{b}$. In the other direction, Kurtz [11, Theorem 4.3] showed that there is a fixed function f recursive in $\mathbf{0}'$ such that for almost every degree **b**, every function g recursive in **b** is such that $f(n) \ge g(n)$ for all sufficiently large n. Kurtz's proof actually shows that $f \le_{wtt} K$. It follows that almost no degree is a.n.r., and thus that almost every degree bounds a 1-generic degree but no pb-generic degree.

We close by investigating cupping properties of a.n.r. degrees.

Theorem 3.4. *Let* $\mathbf{a}$ *be an a.n.r. degree, and suppose that* $\mathbf{c} > \mathbf{a}$. *Then there is a degree* $\mathbf{b} < \mathbf{c}$ *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{c}$.

Proof. If we had the additional assumption that $\mathbf{a}$ is n-r.e. for some n, the result would be virtually a direct application of [8, Lemma 7] and Theorem 3.2. (Theorem 3.2 would need to be modified in an entirely trivial way to handle strings in $3^{<\omega}$.) However, to avoid making this assumption, we construct a minimal pair $\mathbf{b}_0, \mathbf{b}_1$ of degrees such that $\mathbf{a} \cup \mathbf{b}_i = \mathbf{c}$ for $i = 0, 1$, so that the theorem holds for $\mathbf{b} = \mathbf{b}_0$ or $\mathbf{b} = \mathbf{b}_1$. To obtain $\mathbf{b}_0$ and $\mathbf{b}_1$, we let $F : \omega \to \{0, 1, 2\}$ be a function which is recursive in $\mathbf{a}$ and pb-generic for sets of strings in $3^{<\omega}$. (The definition of pb-genericity and the proof of Theorem 3.2 are modified in the obvious way for such strings.) Then F has the following properties:

(1) There are infinitely many even n and infinitely many odd n such that $F(n) = 2$.

(2) If $\hat{F}$ is obtained from F by replacing all values of 2 by either 0 or 1 in any way whatever, then the degrees of $\{n : \hat{F}(2n) = 1\}$ and $\{n : \hat{F}(2n+1) = 1\}$ form a minimal pair.

Suppose that (1) and (2) have been established. To complete the proof of the result, let $n_0, n_1, \ldots$ be the even values of n with $F(n) = 2$ in natural order, and let $m_0, m_1, \ldots$ be the odd values of n with $F(n) = 2$ in natural order. Let C be a set of degree $\mathbf{c}$. Define $\hat{F}(n_k) = \hat{F}(m_k) = C(k)$ and elsewhere let $\hat{F}(n) = F(n)$. Let $B_i = \{n : \hat{F}(2n + i) = 1\}$ for $i \leq 1$. Then the result holds with $\mathbf{b}_i = deg(B_i)$.

The proof of (1) and (2) is similar to the proof of Theorem 2.5. However, in defining $d(n, \sigma)$ when R_n is a minimal pair requirement, one must successively consider all ways of replacing the 2's in σ by 0's and 1's. This can be done without introducing any 2's into $d(n, \sigma)$ not already present in σ. We omit the details, but see [8, Lemma 7] for a similar argument. In fact, it is easily seen that $B_0 \oplus B_1$ is 1-generic. Thus, if $\mathbf{a}$ is any a.n.r. degree and $\mathbf{c} \geq \mathbf{a}$, then there exist $\mathbf{b}_0$ and $\mathbf{b}_1$ such that $\mathbf{b}_0, \mathbf{b}_1$ form a minimal pair, $\mathbf{a} \cup \mathbf{b_0} = \mathbf{a} \cup \mathbf{b_1} = \mathbf{c}$, and $\mathbf{b_0}, \mathbf{b_1}, \mathbf{b_0} \cup \mathbf{b_1}$ are all 1-generic.
□

It follows from the proof of Theorem 3.4 (or from Theorems 3.2 and 3.4) that every pb-generic degree can be nontrivially cupped up to all higher degrees. On the other hand, it is not true that every 1-generic degree can be nontrivially cupped up to all higher degrees. This follows from the theorem of Cooper [2] and Slaman-Steel [13, Theorem 3.1] that there are r.e. degrees $\mathbf{a}, \mathbf{c}$ with $\mathbf{0} < \mathbf{a} < \mathbf{c}$ such that no degree $\mathbf{b} < \mathbf{c}$ satisfies $\mathbf{a} \cup \mathbf{b} = \mathbf{c}$ and the fact that every non-zero r.e. degree bounds a 1-generic degree [14, Exercise VI.3.9].

We do not know whether the set of a.n.r. degrees is definable in the partial ordering of the degrees. One conceivable definition (in view of Theorem 2.5) is that the a.n.r. degrees are exactly those degrees $\mathbf{a}$ such that every degree $\mathbf{b} \geq \mathbf{a}$ is the sup of a minimal pair.

References

[1] S. B. Cooper, *Degrees of unsolvability complementary between recursively enumerable degrees*, Annals of Math. Logic **4** (1972), 31–73.

[2] S. B. Cooper, *The strong anticupping property for recursively enumerable degrees*, J. Symbolic Logic **54** (1989), 527–539.

[3] J. C. E. Dekker and J. Myhill, *Retraceable sets*, Canad. J. Math. **10** (1958), 357–373.

[4] R. Downey, *Array nonrecursive degrees and lattice embeddings of the diamond*, Ill. J. Math. **37** (1993), 349–374.

[5] R. Downey, C. Jockusch, and M. Stob, *Array nonrecursive sets and multiple permitting arguments*, Recursion Theory Week (Proceedings, Oberwolfach 1989) (K. Ambos-Spies, G. H. Müller, and G. E. Sacks, eds.); Lecture Notes in Math., vol. **1432**, Springer–Verlag, 1990, pp. 141–173.

[6] P. Fejer, *Embedding lattices with top preserved below non-GL_2 degrees*, Zeitschr. f. math. Logik und Grundlagen d. Math. **35** (1989), 3–14.

[7] C. Jockusch, *Degrees of generic sets*, Recursion Theory: Its Generalisations and Applications (F. Drake and S. S. Wainer, eds.); Proceedings of Logic Colloquium '79; London Mathematical Society Lecture Notes Series No. 45, Cambridge University Press, 1980, pp. 110–139.

[8] C. Jockusch and D. Posner, *Double jumps of minimal degrees*, J. Symbolic Logic **43** (1978), 714–724.

[9] M. Kumabe, *Degrees of generic sets*, This Volume.

[10] M. Kummer, *Kolmogorov complexity and instance complexity of recursively enumerable sets*, SIAM Journal of Computing (to appear).

[11] S. Kurtz, *Randomness and Genericity in the Degrees of Unsolvability*, Ph. D. thesis, University of Illinois at Urbana-Champaign, 1981.

[12] T. Slaman, *A recursively enumerable degree that is not the top of a diamond in the Turing degrees*, to appear.

[13] T. Slaman and J. Steel, *Complementation in the Turing degrees*, J. Symbolic Logic **54** (1989), 160–176.

[14] R. I. Soare, *Recursively Enumerable Sets and Degrees*, Springer–Verlag, New York, 1987.

Mathematics Department, Victoria University, P.O. Box 600, Wellington, New Zealand

E-mail address: downey@math.vuw.ac.nz

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801

E-mail address: jockusch@math.uiuc.edu

Department of Mathematics, Calvin College, Grand Rapids, MI

E-mail address: stob@calvin.edu