Array nonrecursive sets and multiple permitting arguments *

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Abstract

We study a class of permitting arguments in which each positive requirement needs multiple permissions to succeed. Three natural examples of such constructions are given. We introduce a class of r. e. sets, the array nonrecursive sets, which consists of precisely those sets which allow enough permission for these constructions be performed. We classify the degrees of array nonrecursive sets and so classify the degrees in which each of these constructions can be performed.

1 Introduction

Permitting is the name given to a class of techniques for constructing an r. e. set B which is recursive is some fixed r. e. set A. In a permitting argument, enumeration into B is allowed or "permitted" only if some event related to the enumeration of A occurs. For example, in Yates permitting (often called permitting or simple permitting), we allow x to be enumerated in B at stage $s+1$ only if some integer $y \leq x$ is enumerated in A at stage $s+1$. It is obvious that this ensures that $B \leq_T A$. Various notions of permitting can be found in the literature corresponding to various classes of sets A and various types of requirements which appear in the specification of B. Obviously, permitting functions as a negative requirement on B and a notion of permitting may or may not cohere with a positive requirement desired for the enumeration of B. For example, Yates permitting described above and the standard positive requirements for constructing a simple set are compatible, producing the theorem that every r. e. degree bounds an r. e. degree containing a simple set. The most common notions of permitting are Yates permitting, Martin (or high) permitting, and prompt permitting. Each corresponds to a natural class of r. e. degrees and for each there is a large class of constructions that can be done precisely in those degrees.

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In this paper we analyze a class of permitting arguments which are characterized by the fact that each positive requirement requires multiple permissions to succeed. To see how our notion of multiple permitting differs from other standard notions of permitting, and to place all these methods in a common framework, we now review the basic (Yates) permitting method and make some remarks about it. We do this by means of the following theorem.

Theorem 1.1 *If A is r. e. and nonrecursive, then there is a simple set B such that $B \leq_T A$.*

Proof. We construct B to be coinfinite and to meet for every $e \in N$ the requirement

$$\mathbf{P}_e: \quad W_e \text{ infinite } \Rightarrow W_e \cap B \neq \emptyset.$$

Given an enumeration $\{A_s\}_{s \in N}$ of A, we enumerate B in stages. The requirement that $B \leq_T A$ is met as follows. We say that a number x is permitted by A at stage $s+1$ if $(\exists y \leq x)[y \in A_{s+1} - A_s]$. We enumerate x into B at stage $s+1$ only if x is permitted by A at stage $s+1$. This guarantees that $B \leq_T A$ for if s is a stage such that $(\forall y \leq x)[y \in A_s \leftrightarrow y \in A]$, then $x \in B_s \leftrightarrow x \in B$.

CONSTRUCTION.

Stage $s+1$
For every $e < s$, if $W_{e,s} \cap B_s = \emptyset$, and $(\exists x)[x \in W_{e,s}, x > 2e$, and x is permitted by A at $s+1]$, enumerate the least such x in B.

By permitting, $B \leq_T A$. The clause $x > 2e$ guarantees that B is coinfinite. To see that $\mathbf{P}_e$ is satisfied, suppose that W_e is infinite but that $W_e \cap B = \emptyset$. We argue that A is recursive, contrary to hypothesis. To determine if $y \in A$, enumerate W_e until a stage s and integer x are discovered such that $x > y$, $x > 2e$, and $x \in W_{e,s}$. Since $\mathbf{P}_e$ is not satisfied, x is never enumerated in B. This implies that x is never permitted by A after stage s. In particular, $y \in A \leftrightarrow y \in A_s$. ■

We notice the following key features of $\mathbf{P}_e$ which allows the above permitting argument to succeed.

(1) If W_e is infinite, there are infinitely many potential witnesses for $\mathbf{P}_e$. (Any $x \in W_e$ such that $x > 2e$ will do.)

(2) The construction requires that only one witness for $\mathbf{P}_e$ needs to be permitted only once for $\mathbf{P}_e$ to succeed.

(3) Witnesses, once discovered, do not disappear and are available at any later stage. (In this case, if $x \in W_{e,s}$, then $x \in W_{e,t}$ for all $t \geq s$.)

Any set of positive requirements satisfying these three properties can be combined with simple permitting in the manner of Theorem 1.1. Different notions of permitting arise from positive requirements which do not have one or the other of the features above. The two most important examples are high permitting and prompt permitting.

The high permitting method of Martin [M] results from replacing (2) above by

$(2)_{\text{high}}$ The construction requires that cofinitely many of the witnesses for $\mathbf{P}_e$ be permitted each once.

As the name suggests, the method of permitting which results from (1), $(2)_{\text{high}}$, and (3) can only be used with sets of high r. e. degree. Martin used it to show that maximal r. e. sets exist in all high r. e. degrees.

The prompt permitting method of Maass, [MSS,AJSS] results from keeping (1) and (2) but replacing (3) by

$(3)_{\text{prompt}}$ Witnesses, once discovered, need to be permitted immediately (promptly) if they are to be used in satisfying $\mathbf{P}_e$.

The method of permitting which results from (1), (2), and $(3)_{\text{prompt}}$ can only be used with r. e. sets of promptly simple degree. The class of promptly simple degrees is a filter in the upper semilattice of all r. e. degrees which contains low degrees but not all high degrees. Thus prompt permitting is up to degree a different notion than high permitting or standard Yates permitting.

The notion of permitting that we study here arises from positive requirements that satisfy (1) and (3) but in which we modify (2) to

$(2)_{\text{mp}}$ At least one witness x needs to be permitted $f(x)$ times; f is some fixed recursive function.

Note that $(2)_{\text{mp}}$ is a stronger requirement than

$(2)_n$ At least one witness x needs to be permitted n times; n a fixed positive integer.

It is easily seen that $(2)_n$ is no harder to guarantee than (2).

We study arguments which have positive requirements with the characteristics (1), $(2)_{\text{mp}}$, and (3) in a somewhat indirect manner. We first introduce a class of r. e. sets, the array nonrecursive sets.

The array nonrecursive sets are defined as follows. Recall that a sequence of finite sets $\{F_n\}_{n\in N}$ is called a *strong array* if there is a recursive function f such that $F_n = D_{f(n)}$ for every $n \in N$ where D_y denotes the finite set with canonical index y.

Definition 1.2 A strong array $\{F_n\}_{n\in N}$ is a *very strong array* (v. s. a.) if

(4) $$\bigcup_{n\in N} F_n = N,$$

(5) $$F_n \cap F_m = \emptyset \text{ if } n \neq m, \text{ and}$$

(6) $$0 < |F_n| < |F_{n+1}| \text{ for all } n \in N.$$

Definition 1.3 An r. e. set A is *array nonrecursive with respect to* $\{F_n\}_{n\in N}$ (F-a. n. r.) if

(7) $$(\forall e)(\exists n)[W_e \cap F_n = A \cap F_n].$$

Definition 1.4 An r. e. set A is *array nonrecursive* (a. n. r.) if there is a v. s. a. $\{F_n\}_{n\in N}$ such that A is F-a. n. r.

Definition 1.5 An r. e. degree $\mathbf{a}$ is *array nonrecursive* if there is an r. e. set $A \in \mathbf{a}$ such that A is array nonrecursive.

We note the following facts about these definitions. First, if A is a. n. r., then A is nonrecursive. Second, F-a. n. r. sets exist for any v. s. a. $\{F_n\}_{n \in N}$, since $A = \bigcup_{e \in N} W_e \cap F_e$ is F-a. n. r. Finally, (7) is equivalent to

$$(8) \qquad (\forall e)(\overset{\infty}{\exists} n)[W_e \cap F_n = A \cap F_n].$$

The condition (7) translates to a notion of multiple permitting in roughly the following way. Suppose that A is F-a. n. r. and that we are constructing an r. e. set $B \leq_{\mathrm{T}} A$ using Yates permitting. If we enumerate an r. e. set V we are entitled to assume that $(\exists n)[V \cap F_n = A \cap F_n]$. Since we enumerate V, for this n equality implies that we can force up to $|F_n|$ many integers all less than $\max(F_n)$ to enter A. This gives several Yates permissions for a large enough number. (If all that is assumed is that A is nonrecursive, Yates permitting guarantees a single permission on a large enough number.) The simplest example of such a multiple permitting argument is Theorem 2.5 below.

We show in Section 4 that such multiple permitting arguments arise naturally in recursion theory by showing that three constructions from elsewhere in recursion theory can be carried out precisely below those r. e. degrees which are array nonrecursive. These theorems are as follows.

Theorem 1.6 *Let f be a strictly increasing recursive function. Then an r. e. degree $\mathbf{a}$ is a. n. r. iff there is a degree $\mathbf{b} \leq \mathbf{a}$ (not necessarily r. e.) such that some set B of degree $\mathbf{b}$ is not f-r. e. (A Δ^0_2 set is f-r. e. if it has a recursive approximation $\{B_s\}_{s \in N}$ as a Δ^0_2 set such that $|\{s | B_s(x) \neq B_{s+1}(x)\}| \leq f(x)$ for all x.)*

The next theorem arises from a construction performed by Jockusch and Soare [JS, Theorem 1] to show that every degree which contains a consistent extension of Peano arithmetic bounds an incomparable pair of degrees. In that proof, sets B_0, C_0, B_1, C_1 were constructed satisfying the conditions in part (c) of Theorem 1.7.

Theorem 1.7 *For r.e. sets A, the following are equivalent:*

- *(a) A has a. n. r. degree,*
- *(b) there are disjoint r.e. sets B and C each recursive in A such that $B \cup C$ is coinfinite and no set of degree $\mathbf{0}'$ separates B and C,*
- *(c) there exist two disjoint pairs of r.e. sets B_0, C_0 and B_1, C_1 such that $B_i \cup C_i$ is coinfinite for $i = 0, 1$, each set B_i, C_i is recursive in A, and each set which separates (B_0, C_0) is Turing incomparable with each which separates (B_1, C_1).*

The third major theorem concerns a class of r. e. theories called the Martin Pour-El theories. To define this class let Q be the free countable, atomless Boolean algebra and let $\{p_n | n \in N\}$ be a set of generators for it. Then a theory T can be identified with a filter of Q. We call such a theory *well-generated* if there are sets B and C such that T is generated by a set of the form $\{p_n | n \in B\} \cup \{\neg p_n | n \in C\}$. An r. e. theory T is *Martin–Pour-El* if it is well-generated, essentially undecidable, and every r. e. theory $W \supseteq T$ is principal over T. The existence of such theories is due to Martin and Pour-El [MP, Theorem I]. They have been extensively studied by Downey [D1].

Theorem 1.8 *An r. e. degree* **a** *is a. n. r. iff there is a theory* T *of degree* **a** *which is Martin Pour-El.*

In sections 2 and 3 we initiate an investigation into the properties of a. n. r. sets and degrees. Of particular interest because of Theorems 1.6, 1.7, and 1.8 is the classification of a. n. r. degrees. Our principal results are as follows.

- The array nonrecursive degrees are closed upwards in **R**, the class of all r. e. degrees (Corollary 2.8).
- There are low a. n. r. degrees (Theorem 2.1).
- All r. e. degrees **a** such that $\mathbf{a}'' > \mathbf{0}''$ are a. n. r. (Corollary 4.3).
- There exist promptly simple degrees which are not a. n. r. Thus, since promptly simple degrees are noncappable and the non-a. n. r. degrees are closed upwards, every nonzero r. e. degree bounds a nonzero r. e. degree which is not a. n. r. (Corollary 2.11).
- Every a. n. r. degree bounds a low a. n. r. degree (Corollary 3.8, due to Cameron Smith).
- The r. e. weak-truth-table degrees containing no a. n. r. set form an ideal in the upper-semilattice of r. e. wtt-degrees (Corollary 3.14).

Our notation is standard; a reference is Soare [S]. All sets and degrees are r. e. unless otherwise noted. The principal exceptions to this convention are in Theorems 1.6 and 1.7.

2 Basic Existence Theorems

Given a very strong array $\{F_n\}_{n\in N}$, the F-a. n. r. set $A = \bigcup_{n\in N} W_n \cap F_n$ is clearly Turing-complete and, in fact, is creative. The next theorem shows that low F-a. n. r. sets exist. It also clearly exhibits the construction of an a. n. r. set as a finite injury priority argument.

Theorem 2.1 *Let* $\{F_n\}_{n\in N}$ *be a very strong array. Then there is an r. e. set* A *of low degree such that* A *is* F*-a. n. r.*

Proof. To make A F-a. n. r., it suffices to meet for every $e \in N$ the requirement

$$\mathbf{R}_e : \qquad (\exists n)[W_e \cap F_n = A \cap F_n].$$

The requirements to make A of low degree are

$$\mathbf{N}_e : \qquad (\overset{\infty}{\exists}\, s)[\{e\}_s^{A_s}(e)\downarrow] \Rightarrow \{e\}^A(e)\downarrow .$$

Recall that the requirement $\mathbf{N}_e$ is met by preserving the restraint function $r(e,s) = u(A_s, e, e, s)$ at all but finitely many stages s. Let $q(e,s) = \max\{r(i,s)|i \leq e\}$. To meet $\mathbf{R}_e$, we reserve the sets $F_{\langle e,0\rangle}, F_{\langle e,1\rangle}, \ldots$. The construction assigns priority to the requirements in the order $\mathbf{N}_0$, $\mathbf{R}_0$, $\mathbf{N}_1$, $\mathbf{R}_1$, ...

CONSTRUCTION.

Stage $s+1$
Requirement $\mathbf{R}_e$ *requires attention* at stage $s+1$ if there is $i \in N$ such that

$$\min(F_{\langle e,i\rangle}) > q(e,s), \text{ and} \tag{9}$$
$$(\forall j \leq i)[W_{e,s+1} \cap F_{\langle e,j\rangle} \neq A_s \cap F_{\langle e,j\rangle}]. \tag{10}$$

Let e be least such that $\mathbf{R}_e$ requires attention and let i be least such that (9) and (10) hold. Enumerate all of $W_{e,s+1} \cap F_{\langle e,i\rangle}$ into A. This ends the construction.

Note that the construction ensures that $A_s \cap F_{\langle e,i\rangle} \subseteq W_{e,s} \cap F_{\langle e,i\rangle}$ for every e, i, and s, so that if $\mathbf{R}_e$ receives attention at stage $s+1$ and i is the least integer satisfying (9) and (10), then $W_{e,s+1} \cap F_{\langle e,i\rangle} = A_{s+1} \cap F_{\langle e,i\rangle}$. It is now easy to show by simultaneous induction on e that

(a) $\mathbf{N}_e$ is satisfied,

(b) $\lim_s q(e,s) < \infty$,

(c) $\mathbf{R}_e$ is satisfied, and

(d) $\mathbf{R}_e$ receives attention only finitely often. ■

A. Kučera has pointed out that the following extension of Theorem 2.1 holds: For any very strong array $\{F_n\}_{n\in N}$, there is a complete extension T of Peano arithmetic of low degree such that there is an F-a. n. r. set A recursive in T.

The next two results, Theorems 2.2 and 2.5, clarify the role of the very strong array $\{F_n\}_{n\in N}$ in the definition of array nonrecursive sets. In particular, Theorem 2.5 shows that up to degree, the notion of array nonrecursveness is independent of the choice of very strong array. It will also be used in the proof of many subsequent results.

Theorem 2.2 *For every r. e. set A there is a very strong array $\{F_n\}_{n\in N}$ such that A is not F-a. n. r.*

Proof. If A is recursive, then A is not F-a. n. r. for any F. If A is not recursive, let R be an infinite recursive subset of A. Choose a v. s. a. $\{F_n\}_{n\in N}$ such that $F_n \cap R \neq \emptyset$ for every $n \in N$. Let $W = \overline{R}$. Then for every $n \in N$, $W \cap F_n \neq A \cap F_n$ witnessing that A is not F-a. n. r. ■

The following definition and lemma will be used in the proof of Theorem 2.5 and elsewhere.

Definition 2.3 Suppose that A is r. e. with a given enumeration $\{A_s\}_{s\in N}$ and $\{F_n\}_{n\in N}$ is a strong array. *A F-permits y at stage $s+1$* if

$$(\exists z \leq y)(\exists x \leq \max(F_z))[x \in A_{s+1} - A_s].$$

Lemma 2.4 *Suppose that A is r. e. with a given enumeration $\{A_s\}_{s\in N}$ and $\{F_n\}_{n\in N}$ is a strong array. Suppose that f is a recursive function. Suppose that B is an r. e. set with enumeration $\{B_s\}_{s\in N}$ such that for every x, $x \in B_{s+1} - B_s$ only if A F-permits $f(x)$ at stage $s+1$. Then $B \leq_{\mathrm{T}} A$. In fact $B \leq_{\mathrm{wtt}} A$.*

Theorem 2.5 *Suppose that $\{F_n\}_{n\in N}$ and $\{E_n\}_{n\in N}$ are very strong arrays, that A is F-a. n. r., and that $\mathbf{b}$ is an r. e. degree such that $deg(A) \leq \mathbf{b}$. Then there is $B \in \mathbf{b}$ such that B is E-a. n. r.*

Proof. Fix a set $\hat{B} \in \mathbf{b}$. To ensure that $\hat{B} \leq_{\mathrm{T}} B$, we will reserve the sets $E_{\langle 0,i\rangle}$, $i \in N$, for coding $\hat{B}$. Namely, we will enumerate all of $E_{\langle 0,i\rangle}$ in B if and only if $i \in \hat{B}$. The requirements $\mathbf{R}_e$ to make B E-a. n. r. are similar to those of Theorem 2.1:

$$\mathbf{R}_e: \quad (\exists n)[W_e \cap E_n = B \cap E_n].$$

We will reserve the sets $E_{\langle e+1,0\rangle}, E_{\langle e+1,1\rangle}, \ldots$ for meeting $\mathbf{R}_e$. To aid in meeting $\mathbf{R}_e$ we shall also enumerate an r. e. set V_e and since A is F-a. n. r., (8) guarantees that

$$(11) \qquad (\overset{\infty}{\exists}\, n)[V_e \cap F_n = A \cap F_n].$$

For each e, let $n(e)$ be the least integer n such that $|F_n| > |E_{\langle e+1,0\rangle}|$. For each $n \geq n(e)$ let $g(e,n)$ be the greatest pair of the form $\langle e+1, i\rangle$ such that

$$(12) \qquad |F_n| > |E_{g(e,n)}|.$$

Note that if $e \neq f$, then $g(e,n) \neq g(f,m)$ for all n,m. However it is possible that $g(e,n) = g(e,m)$ for some $n \neq m$. However, for every x, the set $\{n \mid g(e,n) = x\}$ is finite (uniformly in x).

We will replace the requirement $\mathbf{R}_e$ with the following requirements $\mathbf{R}_{e,n}$ for $n \geq n(e)$:

$$\mathbf{R}_{e,n}: \quad V_e \cap F_n = A \cap F_n \Rightarrow W_e \cap E_{g(e,n)} = B \cap E_{g(e,n)}.$$

To ensure that action taken for $\mathbf{R}_{e,n}$ does not interfere with the requirement to make $B \leq_{\mathrm{T}} \hat{B}$ we will allow $x \in E_{g(e,n)}$ to enter B at stage $s+1$ only if A F-permits n at stage $s+1$. By Lemma 2.4, we have that $\bigcup_{e,i\in N} E_{\langle e+1,i\rangle} \cap B \leq_{\mathrm{T}} A \leq_{\mathrm{T}} \hat{B}$.

Before giving the construction, which is quite simple, we describe the strategy for one requirement. This strategy is the same one that is used throughout the paper when it is necessary to construct a set recursive in some given a. n. r. set A. It essentially captures the notion of multiple permitting allowed by an a. n. r. set.

Fix e and $n \geq n(e)$. $\mathbf{R}_{e,n}$ is met if either $V_e \cap F_n \neq A \cap F_n$ or $W_e \cap E_{g(e,n)} = B \cap E_{g(e,n)}$. We view our attempts to establish this disjunction as a two-state finite automaton. At any stage s of the construction, we say that requirement $\mathbf{R}_{e,n}$ is in state S_1 if $W_{e,s} \cap E_{g(e,n)} = B_s \cap E_{g(e,n)}$ and in state S_2 otherwise. The construction is intended to ensure that if $\mathbf{R}_{e,n}$ is in state S_2 at stage s then $V_{e,s} \cap F_n \neq A_s \cap F_n$ as indicated in Figure 1.

Suppose that stage $s+1$ is such that $\mathbf{R}_{e,n}$ is in state S_1 at stage s but not at stage $s+1$. Since we enumerate B, this is because an element of $E_{g(e,n)}$ is enumerated in W_e at stage $s+1$ (and thus there are at most $|E_{g(e,n)}|$ such stages). To guarantee that the

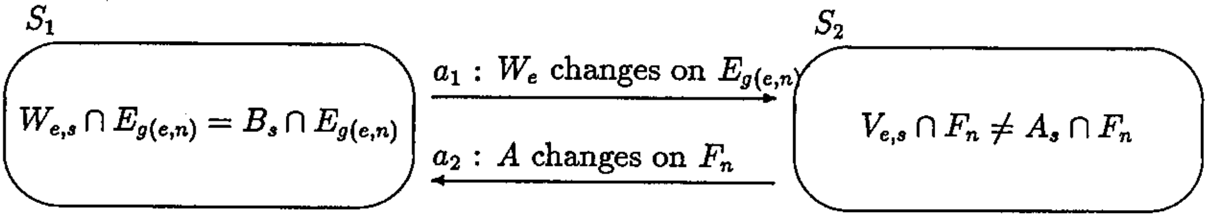

Figure 1: State diagram of the construction

condition of state S_2 holds at stage $s+1$, we enumerate, if necessary, one element of F_n into V_e to cause $V_{e,s+1} \cap F_n \neq A_{s+1} \cap F_n$. This constitutes the action of arrow a_1. Since this is the only action which causes us to enumerate elements of F_n into V_e and since $|F_n| > |E_{g(e,n)}|$, it is always possible to perform this action at such a stage $s+1$.

We also need to guarantee that while $\mathbf{R}_{e,n}$ remains in state S_2, the condition $V_{e,s} \cap F_n \neq A_s \cap F_n$ continues to hold. Thus, let s be a stage such that $V_{e,s} \cap F_n \neq A_s \cap F_n$ but $V_{e,s+1} \cap F_n = A_{s+1} \cap F_n$. It must be the case that an element of F_n is enumerated in A at stage $s+1$. This is just the condition that A F-permits n at stage $s+1$. Thus at stage $s+1$ we may enumerate all of $W_{e,s+1} \cap E_{g(e,n)}$ into B, thereby guaranteeing that $\mathbf{R}_{e,n}$ is in state S_1 at stage $s+1$. This constitutes the action of arrow a_2. Note that to perform such action, we must require that $B_s \cap E_{g(e,n)} \subseteq W_{e,s} \cap E_{g(e,n)}$ for all e, n, s. This is guaranteed by the construction described above and by the fact that $E_{g(e,n)}$ is disjoint from $E_{g(f,m)}$ if $e \neq f$. Because of this stipulation, there are no conflicts between the various requirements. We now give the formal details of the construction.

CONSTRUCTION.

Stage $s+1$

Step 1. (Coding.) If $i \in \hat{B}_{s+1} - \hat{B}_s$, enumerate all of $E_{\langle 0,i\rangle}$ into B.

Step 2. (Arrow a_2.) For every e and $n \geq n(e)$, if A F-permits n, enumerate all of $W_{e,s+1} \cap E_{g(e,n)}$ into B.

Step 3. (Arrow a_1.) For every e and $n \geq n(e)$ if

$$B_s \cap E_{g(e,n)} = W_{e,s} \cap E_{g(e,n)} \quad \text{and} \quad B_{s+1} \cap E_{g(e,n)} \neq W_{e,s+1} \cap E_{g(e,n)} \tag{13}$$

then enumerate one element of $F_n - V_{e,s}$ into V_e, if necessary, to cause $A_{s+1} \cap F_n \neq V_{e,s+1} \cap F_n$. There is such an element since $|F_n| > |E_{g(e,n)}|$ and an element of F_n is enumerated in V_e only if (13) holds; i. e., if arrow a_1 is traversed.

Lemma 2.6 $B \equiv_{\mathrm{T}} \hat{B}$.

Proof. $B \leq_{\mathrm{T}} \hat{B}$ since B restricted to $\cup_{i \in N} E_{\langle 0,i\rangle}$ is recursive in $\hat{B}$ by step 1 of the construction and B restricted to $\cup_{e,i \in N} E_{\langle e+1,i\rangle}$ is recursive in A by step 2 and Lemma 2.4 (applied to the array $\{F_n\}_{n \in N}$ and the function f where $f(x)$ is the greatest n such that $x \in E_{g(e,n)}$). □

Lemma 2.7 *For each* $e \in N$, $\mathbf{R}_e$ *is satisfied.*

Proof. It is enough to show that for every e and $n \geq n(e)$, that $\mathbf{R}_{e,n}$ is satisfied. At every stage of the construction, $\mathbf{R}_{e,n}$ is either in state S_1 or state S_2. Thus, since $\mathbf{R}_{e,n}$ changes state finitely often, $\mathbf{R}_{e,n}$ is in state S_1 at cofinitely many stages of the construction or $\mathbf{R}_{e,n}$ is in state S_2 at cofinitely many stages of the construction. If the former holds, $W_e \cap E_{g(e,n)} = B \cap E_{g(e,n)}$. If the latter holds $V_e \cap F_n \neq A \cap F_n$ since the construction guarantees that if $\mathbf{R}_{e,n}$ is in state S_2 at stage s, then $V_{e,s} \cap F_n \neq A_s \cap F_n$. ∎

The following are easy corollaries of Theorem 2.5.

Corollary 2.8 *Suppose that* $\mathbf{a}$ *is a. n. r. and that* $\mathbf{b} \geq \mathbf{a}$. *Then* $\mathbf{b}$ *is a. n. r. That is, the a. n. r. degrees form a filter in the upper semilattice of the r. e. degrees. Since the Turing reductions employed in Theorem 2.5 are weak-truth-table reductions, this result holds also for the weak-truth-table degrees.*

Corollary 2.9 *Suppose that* $\{F_n\}_{n\in N}$ *is a very strong array and that* A *is a. n. r. Then there is a set* B *of the same weak-truth-table degree as* A *such that* B *is* F*-a. n. r. That is, up to (weak-truth-table) degree, the notion of array nonrecursiveness is independent of array.*

We turn now to existence theorems for array recursive sets and degrees; a set (degree) is *array recursive* just in case it is not array nonrecursive. The following result shows that our notion of multiple permitting is strictly stronger than ordinary (Yates) permitting.

Theorem 2.10 *There is an r. e. degree* $\mathbf{a} > \mathbf{0}$ *such that* $\mathbf{a}$ *is array recursive.*

Proof. Fix a very strong array $\{F_n\}_{n\in N}$. By Corollary 2.9, it suffices to prove that no set of degree $\mathbf{a}$ is F-a. n. r. (We will actually prove that no set of degree less than or equal to that of $\mathbf{a}$ is F-a. n. r., which is equivalent by Corollary 2.9.)

Let $(\Phi_e, B_e)_{e\in N}$ be an effective listing of all pairs (Φ, B) of recursive functionals Φ and r. e. sets B. We will enumerate r. e. sets V_e, $e \in N$, satisfying the following requirement for every $e \in N$ and for every $n > e$:

$$\mathbf{R}_{e,n}: \qquad \Phi_e(A) = B_e \Rightarrow V_e \cap F_n \neq B_e \cap F_n.$$

Requirements $\mathbf{R}_{e,n}$ for $n > e$ suffice to make B not F-a. n. r. by (8). To make A nonrecursive we have for every $e \in N$ the requirement

$$\mathbf{P}_e: \qquad A \neq \overline{W}_e.$$

We use the following priority ordering of the requirements: $\mathbf{R}_{01}$, $\mathbf{P}_0$, $\mathbf{R}_{02}$, $\mathbf{R}_{12}$, $\mathbf{P}_1$, $\mathbf{R}_{03}$, $\mathbf{R}_{13}$, $\mathbf{R}_{23}$, $\mathbf{P}_2$, ... The key fact about this priority ordering is that $\mathbf{R}_{e,n}$ can only be injured by $\mathbf{P}_i$ for $i \leq n-2$ or at most $n-1$ times.

The strategy for meeting $\mathbf{R}_{e,n}$ is as follows. Wait until $l(e,s) > \max(F_n)$ (where $l(e,s)$ measures the length of agreement between the computation $\Phi_{e,s}(A_s)$ and the set $B_{e,s}$). Cause $V_{e,s+1} \cap F_n$ to be unequal to $B_{e,s} \cap F_n$ (by enumerating at most one element of F_n into V_e). Restrain A on the use of the computations involved in establishing that length of agreement. Thus, if $\mathbf{R}_{e,n}$ is not injured by a higher priority requirement, either $V_e \cap F_n \neq B_e \cap F_n$ or $\Phi_e(A) \neq B_e$ and, in either case, $\mathbf{R}_{e,n}$ imposes only a fixed finite restraint on A for the rest of the construction. Since $\mathbf{R}_{e,n}$ can be injured at most $n-1$ times, at most $n-1$ attempts of the above form need be made and this can be done since $|F_n| \geq n$. Note that the requirements $\mathbf{R}_{e,n}$ are purely negative requirements on A (although positive on V_e) and so do not conflict with each other. We omit further details of the construction and its verification. ■

The next two corollaries follow by making the obvious modifications to the construction suggested in the proof above. Alternatively, the second follows from the first, Corollary 2.8, and the fact that no promptly simple degree is half of a minimal pair [MSS, Theorem 1.11].

Corollary 2.11 *There is a promptly simple degree* **a** *which is array recursive.*

Corollary 2.12 *For every r. e. degree* $\mathbf{b} > \mathbf{0}$*, there is an r. e. degree* **a** *such that* $\mathbf{0} < \mathbf{a} < \mathbf{b}$ *and* **a** *is array recursive.*

To state the final theorem of this section, we need the following definition.

Definition 2.13 An r. e. set is *semirecursive* if there is a recursive function $f : N^2 \to N$ such that

$$f(x,y) \in \{x,y\} \tag{14}$$

$$f(x,y) \in A \Rightarrow \{x,y\} \subseteq A \tag{15}$$

Thus, the function f of Definition 2.13 chooses of x and y the one "least likely" to be an element of A.

Theorem 2.14 *If r. e. set A is semirecursive, then A is not a. n. r.*

Proof. Let $\{F_n\}_{n\in N}$ be a very strong array and let A be semirecursive with f the recursive function satisfying (14) and (15). We enumerate V so that if $|F_n| \geq 2$, $V \cap F_n \neq A \cap F_n$. To do this, for each n such that $|F_n| \geq 2$, we wait for a stage such that for some pair $\{x_n, y_n\} \subseteq F_n$, we have that $x_n \neq y_n$ and $f(x_n, y_n)$ converges. We then enumerate $f(x_n, y_n)$ and no other element of F_n into V. Thus $V \cap F_n = \{f(x_n, y_n)\}$ but if $f(x_n, y_n) \in A$, $A \cap F_n \supseteq \{x_n, y_n\} \neq V \cap F_n$. ■

Corollary 2.15 *Every r. e. truth-table degree contains an array recursive set.*

Proof. This is immediate from Theorem 2.14 since every r. e. truth-table degree contains a semirecursive r. e. set [J1, Corollary 3.7(ii)]. ■

3 Properties of a. n. r. sets and degrees

The first two theorems in this section locate the array nonrecursive sets in the hierarchy of simplicity properties.

Theorem 3.1 *If A is a. n. r., then*

(a) A is not dense simple, and

(b) A is not strongly hypersimple.

Proof. (a). An r. e. set A is dense simple if $p_{\bar{A}}$, the principal function of the complement of A, dominates every recursive function. We use an alternate characterization of dense simplicity due to Robinson [R, Theorem 3]. Namely, A is dense simple if and only if for every strong array $\{F_n\}_{n\in N}$ of disjoint sets,

(16) $$(\exists m)(\forall n \geq m)[|F_n \cap \bar{A}| < n].$$

Now suppose that A is F-a. n. r. Using $W_e = \emptyset$ and the characterization of F-a. n. r. in (8), we have

(17) $$(\overset{\infty}{\exists}\, n)[A \cap F_n = \emptyset].$$

But for any such n, $|F_n \cap \bar{A}| \geq n$, and thus by (16) A is not dense simple. □

(b). A is strongly hypersimple if for every weak array, $\{W_{f(n)}\}_{n\in N}$, of disjoint sets such that $\bigcup_{n\in N} W_{f(n)} = N$ there is an n such that $W_{f(n)} \subseteq A$. Now suppose again that A is F-a. n. r. Define $W_{f(n)}$ for all $n \in N$ as follows. Given F_m, enumerate the least element of F_m in $W_{f(0)}$, the next least in $W_{f(1)}$, and so forth. Obviously because $\{F_n\}_{n\in N}$ is a very strong array, $\bigcup_{n\in N} W_{f(n)} = N$ and the sets $W_{f(n)}$, $n \in N$, are disjoint. By (17), $W_{f(n)} \cap \bar{A} \neq \emptyset$ for every n. (In fact, $W_{f(n)} \cap \bar{A}$ is infinite). Thus A is not strongly hypersimple. ■

Corollary 3.2 *No array nonrecursive set is maximal, hyperhypersimple, or r-maximal.*

The following theorem shows that Theorem 3.1 is the best possible as far as the standard list of simplicity properties is concerned.

Theorem 3.3 *There is an r. e. set A such that A is array nonrecursive and finitely strongly hypersimple.*

Proof. Fix a v. s. a. $\{F_n\}_{n\in N}$. As usual, the requirements to make A F-a. n. r. are

$$\mathbf{R}_e \qquad (\exists n)[W_e \cap F_n = A \cap F_n].$$

The requirements to make A finitely strongly hypersimple are

$\mathbf{Q}_e$: the sets $W_{\{e\}(n)}, n \in N$ are not disjoint or
$\bigcup_{n\in N} W_{\{e\}(n)} \neq N$ or
$(\exists n)[W_{\{e\}(n)}$ is infinite$]$ or
$(\exists n)[W_{\{e\}(n)} \subseteq A]$.

For ease of notation, we will write V_n^e for $W_{\{e\}(n)}$ and $V_{n,s}^e$ for $W_{\{e\}_s(n),s}$ (where we understand that $V_{n,s}^e = \emptyset$ if $\{e\}_s(n)$ does not converge). The strategy for meeting $\mathbf{Q}_e$ while respecting $\mathbf{R}_0$, $\mathbf{R}_1$, $\ldots \mathbf{R}_{e-1}$ is as follows. Wait for a stage s so that $\bigcup_{n\in N} V_{n,s}^e$ contains all the elements of each set F_m assigned to any requirement $\mathbf{R}_i$, $i < e$. If such a stage does not exist then $\bigcup_{n\in N} W_{\{e\}(n)} \neq N$ and $\mathbf{Q}_e$ is satisfied. At stage s, choose n such that $V_{n,s}^e$ does not contain any element of any such F_m. Assign V_n^e to $\mathbf{Q}_e$. We then attempt to meet $\mathbf{Q}_e$ by enumerating all of V_n^e into A. However this threatens to interfere with requirements $\mathbf{R}_i$, $i \geq e$, as this V_n^e may contain elements from almost every F_m. To avoid this conflict, we enumerate an element of V_n^e into A only if we discover a new F_m which we can certify is disjoint from V_n^e (by virtue of being entirely contained in the union of other sets V_i^e for $i \neq n$). Thus if $V_n^e \cap A$ is infinite the requirement is met since V_n^e is infinite but also we are assured of having infinitely many sets F_m not interfered with by $\mathbf{Q}_e$ and so available for use by requirements $\mathbf{R}_i$, $i \geq e$.

The details of combining the strategies for various $\mathbf{Q}_e$ are straightforward and are omitted. ∎

The next four theorems and their corollaries concern degree-theoretic and set-theoretic splitting properties of array nonrecursive sets.

Theorem 3.4 *For every array nonrecursive set A there are disjoint array nonrecursive sets A_0 and A_1 such that $A = A_0 \cup A_1$.*

Proof. Suppose that A is array nonrecursive with respect to the very strong array $\{F_n\}_{n\in N}$. For each $e \in N$ and $i \in \{0,1\}$ we have the requirement

$$\mathbf{R}_{e,i}: \qquad (\exists n)[W_e \cap F_n = A_i \cap F_n].$$

To meet $\mathbf{R}_{e,i}$ we will enumerate a certain set $V_{e,i}$ and use the fact that

$$(18) \qquad (\overset{\infty}{\exists}\, n)[V_{e,i} \cap F_n = A \cap F_n].$$

During the course of the construction, we will reserve certain n for $\mathbf{R}_{e,i}$. Each n may be reserved for at most one requirement $\mathbf{R}_{e,i}$ at any one stage, but the reservation may be cancelled at a later stage for the purpose of reserving n for a requirement of higher priority. (The intention of these reservations is that there will be some n which is reserved for $\mathbf{R}_{e,i}$ and for which $W_e \cap F_n = A \cap F_n$.) The priority order of the requirements $\mathbf{R}_{e,i}$ is in order of increasing $\langle e, i\rangle$.

CONSTRUCTION.

Stage $s+1$

Step 1. For each $x \in A_{s+1} - A_s$ let n be the integer such that $x \in F_n$. If n is reserved for the requirement $\mathbf{R}_{e,i}$, then enumerate x in A_i. If n is not reserved for any requirement, enumerate x in A_0.

Step 2. For each x and e, if $x \in W_{e,s+1} - W_{e,s}$, $x \in F_n$, and n is reserved for a requirement $\mathbf{R}_{e,i}$, then enumerate x in $V_{e,i}$.

Step 3. $\mathbf{R}_{e,i}$ *requires attention* at stage $s+1$ if

(19) $(\forall n)[n$ is reserved for $\mathbf{R}_{e,i} \Rightarrow W_{e,s} \cap F_n \neq A_{i,s} \cap F_n]$, and
(20) $(\exists n)[A_s \cap F_n = \emptyset$ and n is not reserved for any $\mathbf{R}_{f,j}$ such that $\langle f,j\rangle \leq \langle e,i\rangle]$

If such a pair e, i exists, choose the pair such that $\langle e,i\rangle$ is least and let n be the least integer satisfying (20) for e, i. Perform the following actions for these fixed e, i, n. Reserve n for $\mathbf{R}_{e,i}$. Cancel any other reservation of n. Enumerate all of $W_{e,s+1} \cap F_n$ into $V_{e,i}$. This ends the construction.

Lemma 3.5 *If n is reserved for $\mathbf{R}_{e,i}$, and that reservation is never cancelled, then $W_e \cap F_n = V_{e,i} \cap F_n$ and $A_i \cap F_n = A \cap F_n$.*

Proof. The first clause of the conclusion is by steps (2) and (3) of the construction. To see that $A_i \cap F_n = A \cap F_n$, notice that at the stage that n is first reserved for $\mathbf{R}_{e,i}$, $A_{i,s} \cap F_n = A_s \cap F_n$ $(= \emptyset)$ by (20). Step (1) guarantees that this equality is maintained for all later stages. □

Lemma 3.6 *If $V_{e,i} \cap F_n \neq \emptyset$, then n is reserved for $\mathbf{R}_{e,i}$ or some requirement of higher priority at cofinitely many stages.*

Lemma 3.7 *Each requirement $\mathbf{R}_{e,i}$ receives attention only finitely often and is satisfied.*

Proof. Given e, i, let s_0 be such that if $\langle f,j\rangle < \langle e,i\rangle$, $\mathbf{R}_{f,j}$ does not receive attention after s_0. By (18), there are infinitely many n such that $V_{e,i} \cap F_n = A \cap F_n$. Let n be any such n which is not reserved for $\mathbf{R}_{f,j}$ for any $\langle f,j\rangle < \langle e,i\rangle$. There are two cases.

Case (i): n is reserved for $\mathbf{R}_{e,i}$ at some stage of the construction. Then by Lemma 3.5, $W_e \cap F_n = V_{e,i} \cap F_n = A \cap F_n = A_i \cap F_n$. Thus $\mathbf{R}_{e,i}$ is satisfied. Let s_1 be a stage such that $W_{e,s_1} \cap F_n = W_e \cap F_n$ and $A_{i,s} \cap F_n = A_i \cap F_n$. Then by (19), $\mathbf{R}_{e,i}$ never receives attention after stage s_1.

Case (ii): n is never reserved for $\mathbf{R}_{e,i}$. Then by Lemma 3.6, $V_{e,i} \cap F_n = \emptyset$. Thus $A \cap F_n = \emptyset$. Thus (20) applies to n at cofinitely many stages of the construction. Since n is never reserved for $\mathbf{R}_{e,i}$, it must be that $\mathbf{R}_{e,i}$ receives attention only finitely often and that at cofinitely many stages of the construction (19) fails. This implies the existence of m such that $W_e \cap F_m = A_i \cap F_m$ and hence that the requirement is satisfied. ■

It is clear that the requirements to make each set A_0 and A_1 of low r. e. degree can be combined with the construction of Theorem 3.4. Thus we have the following corollary which was first proved (directly) by Cameron Smith.

Corollary 3.8 *For every array nonrecursive degree* $\mathbf{a}$ *there is an array nonrecursive degree* $\mathbf{b} < \mathbf{a}$ *such that* $\mathbf{b}$ *is low.*

It is not true that if A is a. n. r. and A is the disjoint union of sets A_0 and A_1, then at least one of A_0 or A_1 is anr. However this result is true up to degree. In fact we have the stronger result of the next theorem.

Theorem 3.9 *Suppose that $A \leq_{\text{wtt}} A_0 \oplus A_1$ and that A is array nonrecursive. Then there are r. e. sets B_0 and B_1 such that $B_i \leq_{\text{wtt}} A_i$ and one of B_0 or B_1 is array nonrecursive.*

Proof. Let $\{F_n\}_{n\in N}$ and $\{E_n\}_{n\in N}$ be very strong arrays such that $|E_n| > 2|F_{\langle i,n\rangle}|$ for every i and $n > i$. We first show that we may assume that A is E-a. n. r. and $A = A_0 \cup A_1$. To see this we first notice that since A is array non-recursive, the wtt-degree of A contains an array-nonrecursive set $\hat{A}$. This follows from Corollary 2.9. We next rely on the following lemma of Lachlan [L].

Lemma 3.10 *Suppose that B, B_0, and B_1 are r. e. sets such that $B \leq_{\text{wtt}} B_0 \oplus B_1$. Then there are r. e. sets C_0 and C_1 such that $C_0 \leq_{\text{wtt}} B_0$, $C_1 \leq_{\text{wtt}} B_1$, and $B = C_0 \cup C_1$.*

Applying the lemma with $B = \hat{A}$ gives sets $\hat{A}_0$ and $\hat{A}_1$ such that $\hat{A} = \hat{A}_0 \cup \hat{A}_1$ and $\hat{A}_i \leq_{\text{wtt}} A_i$. The sets B_i which result from the proof of the theorem satisfy $B_i \leq_{\text{wtt}} \hat{A}_i$ and thus $B_i \leq_{\text{wtt}} A_i$. We shall also assume that A, A_0, and A_1 are enumerated so that

$$A_s = A_{0,s} \cup A_{1,s}. \tag{21}$$

We will meet the following requirements for every $e, j \in N$:

$$\mathbf{R}_{e,j}: \quad (\exists n)[W_e \cap F_n = B_0 \cap F_n \text{ or } W_j \cap F_n = B_1 \cap F_n].$$

(These requirements suffice to make one of B_0 or B_1 F-a. n. r. since if e is such that there is no n with $W_e \cap F_n = B_0 \cap F_n$ then the satisfaction of $\mathbf{R}_{e,j}$ for all $j \in N$ implies that B_1 is F-a. n. r.) As in Theorem 2.5, we will reserve the sets $F_{\langle i,0\rangle}, F_{\langle i,1\rangle}, \ldots$ for requirement $\mathbf{R}_{e,j}$ where $i = \langle e,j\rangle$. We will use the fact that A is a. n. r. by enumerating r. e. sets V_i and assuming that

$$(\overset{\infty}{\exists}\, n)[V_i \cap E_n = A \cap E_n].$$

To insure that $B_i \leq_{\text{wtt}} A_i$ we will use permitting as follows. We allow $y \in F_{\langle i,n\rangle}$ to enter B_0 (B_1) at stage $s+1$ only A_0 (A_1) E-permits n at stage $s+1$.

Fix e and j and let $i = \langle e,j\rangle$. Requirement $\mathbf{R}_{e,j}$ is split into the following subrequirements for all $n > \langle e,j\rangle$.

$$\mathbf{R}_{e,j,n}: \quad V_i \cap E_n = A \cap E_n \Rightarrow [W_e \cap F_{\langle i,n\rangle} = B_0 \cap F_{\langle i,n\rangle} \text{ or } W_j \cap F_{\langle i,n\rangle} = B_1 \cap F_{\langle i,n\rangle}].$$

We describe the construction for $\mathbf{R}_{e,j,n}$ as a two-state automaton as in Theorem 2.5. As in Theorem 2.5, we say that $\mathbf{R}_{e,j,n}$ is in state S_1 at stage s if the condition for state S_1 in Figure 2 holds. Otherwise $\mathbf{R}_{e,j,n}$ is in state S_2 at stage s and the construction guarantees that if this happens, the condition in the diagram for state S_2 holds. In order to accomplish this, the action corresponding to arrow a_1 is the same as that of Theorem 2.5. That is, if $\mathbf{R}_{e,j,n}$ is in state S_1 at stage s but not at stage $s+1$, we enumerate an element of E_n into V_i if necessary to cause the condition of state S_2 to hold. Since this happens only if an element of $F_{\langle i,n\rangle}$ is enumerated in W_e or W_j at stage $s+1$, this action need only be performed at most $2|F_{\langle i,n\rangle}|$ many times. Since $|E_n| > 2|F_{\langle i,n\rangle}|$ if $n > i$, we will be able to perform this action. Similarly, if s is such that the condition of state S_2 holds at s but fails at $s+1$, we must be able to ensure that the condition

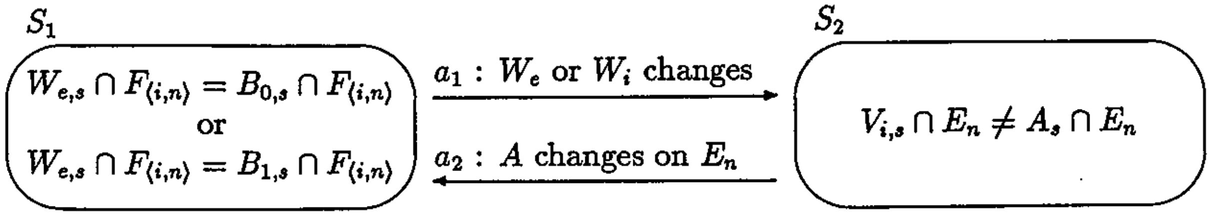

Figure 2: State diagram of the construction.

of state S_1 holds at stage $s+1$. For such an s, it must be the case that an element of E_n is enumerated into A at stage $s+1$, and hence by (21), that element is enumerated in either A_0 or A_1 at stage $s+1$. By our condition on permitting, this allows us to enumerate elements of $F_{\langle i,n\rangle}$ into either B_0 or B_1 at stage $s+1$, thereby guaranteeing that $\mathbf{R}_{e,j,n}$ is in state S_1 at stage $s+1$.

Construction.

Stage $s+1$

Step 1. (Arrow a_2.) For every triple e, j, n such that $\langle e,j\rangle < n$, if $W_{e,s} \cap F_{\langle\langle e,j\rangle,n\rangle} \neq B_{0,s} \cap F_{\langle\langle e,j\rangle,n\rangle}$ and A_0 E-permits n at stage $s+1$, enumerate all of $W_{e,s+1} \cap F_{\langle\langle e,j\rangle,n\rangle}$ into B_0 and similarly for W_j, A_1 and B_1 in place of W_e, A_0 and B_0.

Step 2. (Arrow a_1.) For each triple e, j, n, if

(a) $W_{e,s+1} \cap F_{\langle\langle e,j\rangle,n\rangle} \neq B_{0,s+1} \cap F_{\langle\langle e,j\rangle,n\rangle}$, and

(b) $W_{j,s+1} \cap F_{\langle\langle e,j\rangle,n\rangle} \neq B_{1,s+1} \cap F_{\langle\langle e,j\rangle,n\rangle}$, but

(c) $W_{e,s} \cap F_{\langle\langle e,j\rangle,n\rangle} = B_{0,s} \cap F_{\langle\langle e,j\rangle,n\rangle}$ or $W_{j,s} \cap F_{\langle\langle e,j\rangle,n\rangle} = B_{1,s} \cap F_{\langle\langle e,j\rangle,n\rangle}$,

then enumerate one element of $E_n - V_{\langle e,j\rangle,s}$, if necessary, into $V_{\langle e,j\rangle}$ so that $V_{\langle e,j\rangle,s+1} \cap E_n \neq A_{s+1} \cap E_n$. (Such an element will exist by the construction.) This ends the construction.

The relevant lemmas, parallel in statement and proof (which is omitted), to those of Theorem 2.5 are

Lemma 3.11 $B_0 \leq_{\mathrm{wtt}} A_0$; $B_1 \leq_{\mathrm{wtt}} A_1$.

Lemma 3.12 *For every* e, j, $\mathbf{R}_{e,j}$ *is satisfied.* ∎

The following corollary follows directly from the Theorem and Corollary 2.8.

Corollary 3.13 *Suppose that* $A \leq_{\mathrm{wtt}} A_0 \oplus A_1$ *and that* A *is array nonrecursive. Then the weak-truth-table degree of either* A_0 *or* A_1 *contains an array nonrecursive set.*

An immediate consequence of the preceding corollary is the following.

Corollary 3.14 *The array recursive wtt-degrees form an ideal in the uppersemilattice of r. e. wtt-degrees.*

Proof. By the corollary, the array recursive wtt-degrees are closed under join. By Corollary 2.8, the array recursive wtt-degrees are closed downward. ■

The analogue of Corollary 3.13 and hence of Corollary 3.14 is not available for the Turing degrees as we now show in Theorem 3.15.

Theorem 3.15 *There are r. e. degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}'$ *and* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *are array recursive.*

Proof. Fix a v. s. a. $\{F_n\}_{n\in N}$ such that $|F_n| > 2^{n^2}$ for all $n \in N$. We construct sets A_0 and A_1 of array recursive degree by showing that every set recursive in either is not F-a. n. r. To do this, as in the proof of Theorem 2.10, we enumerate sets V_e and U_e so that for every e and $n > e$ the following requirements are satisfied.

$$\mathbf{R}_{e,n}: \quad \Phi_e(A_0) = B_e \Rightarrow V_e \cap F_n \neq B_e \cap F_n$$

$$\mathbf{Q}_{e,n}: \quad \Phi_e(A_1) = B_e \Rightarrow U_e \cap F_n \neq B_e \cap F_n$$

Here $(\Phi_e, B_e)_{e\in N}$ enumerates all pairs (Φ, B) of reductions Φ and r. e. sets B. To guarantee that $K \leq_{\mathrm{T}} A_0 \oplus A_1$, we will define a recursive function $\gamma : N^2 \to N$ such that

(22) $$\lim_s \gamma(x, s) \text{ exists;}$$

(23) $\gamma(x, s+1) \neq \gamma(x, s)$ only if $(\exists y \leq \gamma(x,s))[y \in A_{0,s+1} - A_{0,s}$ or $y \in A_{1,s+1} - A_{1,s}]$

(24) if $x \in K_{s+1} - K_s$ then $(\exists y \leq \gamma(x,s))[y \in A_{0,s+1} - A_{0,s}$ or $y \in A_{1,s+1} - A_{1,s}]$.

The existence of such a function γ implies that $K \leq_{\mathrm{T}} A_0 \oplus A_1$; the fact that γ depends on s makes this a Turing reduction rather than a weak-truth-table reduction which is prohibited by Theorem 3.9. We define $\gamma(x, 0) = x$ for all $x \in N$.

The two-state automaton corresponding to requirement $\mathbf{R}_{e,n}$ is in Figure 3.

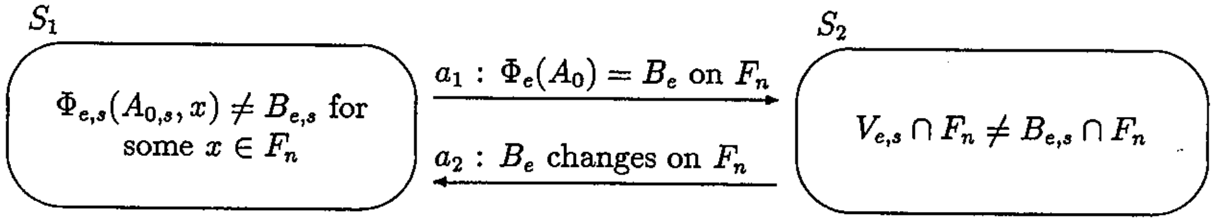

Figure 3: State diagram of the construction.

Arrow a_1 is traversed at any stage $s+1$ such that $\Phi_{e,s+1}(A_{0,s+1}, x) = B_{e,s+1}(x)$ for all $x \in F_n$. At this stage, we enumerate as usual into V_e to cause $V_{e,s+1} \cap F_n \neq B_{e,s+1} \cap F_n$. We also take further action to attempt to preserve all the computations $\Phi_{e,s+1}(A_{0,s+1}, x)$ for $x \in F_n$. Suppose that it is possible to preserve these computations forever and suppose there is a stage $t+1 > s+1$ at which the condition of state S_2 fails. This implies that an integer $x \in F_n$ is enumerated in B_e at stage $t+1$. But then we have that $\Phi_{e,t+1}(A_{0,t+1}) = \Phi_{e,s+1}(A_{0,s+1}) = B_{e,s+1} \neq B_{e,t+1}$ and this disagreement is preserved forever. Thus requirement $\mathbf{R}_{e,n}$ remains in state S_1 forever and is satisfied. The bound on $|F_n|$ above reflects the fact that in taking action a_1 we will not always be able to preserve all computations because of the requirements for coding K. We will ensure that the action a_1 is injured fewer that 2^{n^2} times and thus that arrow a_1 requires traversal at most 2^{n^2} times.

CONSTRUCTION.

Stage $s+1$

Step 1. Let n be the least element of $K_{s+1} - K_s$. Enumerate $\gamma(n,s)$ into A_0. Define $\gamma(y,s+1) = \gamma(y+s,s)$ for all $y \geq n$.

Step 2. (Arrow a_1.) Requirement $\mathbf{R}_{e,n}$ ($\mathbf{Q}_{e,n}$) *requires attention* at stage $s+1$ if

(25)$\Phi_{e,s+1}(A_{0,s+1},x) = B_{e,s+1}(x) \quad (\Phi_{e,s+1}(A_{1,s+1},x) = B_{e,s+1}(x))$ for all $x \in F_n$, and

$$V_{e,s} \cap F_n = B_{e,s+1} \cap F_n \quad (U_{e,s} \cap F_n = B_{e,s+1} \cap F_n). \tag{26}$$

Let n be least and e least for n such that either $\mathbf{R}_{e,n}$ or $\mathbf{Q}_{e,n}$ requires attention. If $\mathbf{R}_{e,n}$ requires attention do the following. Let u be the maximum element of A_0 used in the computations mentioned in (25). If $\gamma(n,s) \leq u$, enumerate $\gamma(n,s)$ into A_1 and define $\gamma(y,s+1) = \gamma(y+s,s)$ for all $y \geq n$. (By the usual conventions on the use function of a computation, $\gamma(y,s+1) > u$ for all $y \geq n$. Thus this step has the effect of clearing the computations of (25) of lower priority markers.) Also, choose $z \in F_n - V_{e,s}$ (such will exist) and enumerate $z \in V_e$. If instead $\mathbf{Q}_{e,n}$ requires attention but $\mathbf{R}_{e,n}$ does not, attend to $\mathbf{Q}_{e,n}$ just as $\mathbf{R}_{e,n}$ but with U_e, A_0, and A_1 in place of V_e, A_1, and A_0 respectively. This ends the construction.

Lemma 3.16 *For every $e,n \in N$ such that $n > e$, requirements $\mathbf{R}_{e,n}$ and $\mathbf{Q}_{e,n}$ receive attention at most 2^{n^2} times and are satisfied.*

Proof. We assume the lemma is true for all pairs e', n' such that $n' < n$ or $n' = n, e' < e$ and give the proof for $\mathbf{R}_{e,n}$. The proof for $\mathbf{Q}_{e,n}$ is identical. Suppose that $\mathbf{R}_{e,n}$ receives attention at stage $s+1$ and there is $z \in F_n - V_{e,s}$. Then $V_{e,s+1} \cap F_n \neq B_{e,s+1} \cap F_n$. Furthermore, by (25) $\Phi_{e,s+1}(A_{0,s+1},x) = B_{e,s+1}(x)$ for all $x \in F_n$ so that if these computations are never injured, either $V_e \cap F_n \neq B_e \cap F_n$ or $\Phi_e(A_0) \neq B_e$ and $\mathbf{R}_{e,n}$ never requires attention after stage $s+1$. Now by the definition of $\gamma(y,s+1)$ for $y \geq n$, the computation in (25) can be injured at a later stage $t+1$ only if $\gamma(y,t+1) = \gamma(y,s)$ enters A_0 for some $y < n$. This happens only if such a number y enters K at stage $t+1$ or because a requirement $\mathbf{R}_{e',y}$ or $\mathbf{Q}_{e',y}$ for some e' such that $e' < y < n$ receives attention at stage $t+1$. Therefore there can be at most $n + \sum_{0<y<n} 2^{y^2}$ many stages $s+1$ at which $\mathbf{R}_{e,n}$ receives attention and is later injured. Thus $\mathbf{R}_{e,n}$ receives attention at most $1 + n + \sum_{0<y<n} 2^{y^2} \leq 2^{n^2}$ times. Since $|F_n| > 2^{n^2}$, $F_n - V_e \neq \emptyset$. Thus, if $\Phi_e(A_0) = B_e$, $\mathbf{R}_{e,n}$ will receive attention enough times to enumerate V_e to make $V_e \cap F_n \neq B_e \cap F_n$. □

Lemma 3.17 $K \leq_T A_0 \oplus A_1$.

Proof. The definition of γ satisfies (24) by step (1) of the construction. (23) is satisfied since $\gamma(y,s) \neq \gamma(y,s+1)$ only if some $\gamma(n,s)$ for $n \leq y$ is enumerated in either A_0 or A_1 at stage $s+1$, and γ is increasing in its first argument. To see that (22) is satisfied, note that $\gamma(y,s+1) \neq \gamma(y,s)$ only if some $n \leq y$ enters K at stage $s+1$ or some requirement $\mathbf{R}_{e,n}$ or $\mathbf{Q}_{e,n}$ receives attention for some $n < y$. Because of Lemma 3.16, there are only finitely many such stages and thus (22) is satisfied. ∎

4 Natural multiple permitting arguments

In this section we prove the three main theorems, Theorems 1.6, 1.7, and 1.8 promised in Section 1. In each, we show that a certain construction from elsewhere in recursion theory can be done below precisely the a. n. r. degrees. Thus, besides characterizing the degrees which admit these constructions by a simple recursion theoretic property, these constructions show that the notion of multiple permitting considered here is quite natural.

Theorem 4.1 *Let f be a strictly increasing recursive function. Then the r. e. degree* $\mathbf{a}$ *is a. n. r. iff there is a degree* $\mathbf{b} \leq \mathbf{a}$ *(not necessarily r. e.) such that some set B of degree* $\mathbf{b}$ *is not f-r. e.*

Proof. (only if). Let $\{F_n\}_{n\in N}$ be a very strong array such that $|F_n| > f(\langle e,n\rangle)$ for every n and $e \leq n$. Let $A \in \mathbf{a}$ such that A is F-a. n. r. We shall define B by giving a recursive approximation $\{B_s\}_{s\in N}$ of B so that for every pair $\langle e,n\rangle$, $B_{s+1}(\langle e,n\rangle) \neq B_s(\langle e,n\rangle)$ only if A F-permits n. Then, by a suitable analogue to Lemma 2.4, $B \leq_{\mathrm{T}} A$. The requirements to make B not f-r. e. are as follows. Let $\{\phi_e\}_{e\in N}$ be a recursive enumeration of the partial recursive binary functions.

$$\mathbf{R}_e: \quad \text{if } \lim_y \phi_e(x,y) = B(x) \text{ then } (\exists x)[|\{y : \phi_e(x,y) \neq \phi_e(x,y+1)\}| > f(x)].$$

To meet $\mathbf{R}_e$, as usual we enumerate sets V_e and use (8):

$$(27) \qquad (\overset{\infty}{\exists}\, n)[V_e \cap F_n = A \cap F_n].$$

For the witness x mentioned in requirement $\mathbf{R}_e$, we use the numbers $\langle e,0\rangle, \langle e,1\rangle, \ldots$. We recast $\mathbf{R}_e$ as the following sequence of requirements $\mathbf{R}_{e,n}$ for $n \geq e$.

$$\mathbf{R}_{e,n}: \quad V_e \cap F_n = A \cap F_n \Rightarrow \langle e,n\rangle \text{ witnesses } \mathbf{R}_e.$$

In light of (27), the requirements $\mathbf{R}_{e,n}$ for $n \geq e$ are enough.

The strategy for meeting $\mathbf{R}_{e,n}$ is represented by a two-state automaton. For the purpose of describing the machine, we make the following definition. We say that ϕ_e is *correct on* $\langle e,n\rangle$ *at stage* s if $\phi_{e,s}(\langle e,n\rangle, y_s) = B_s(\langle e,n\rangle)$ where y_s is the greatest integer, if such exists, such that $\phi_{e,s}(\langle e,n\rangle, y_s)$ converges. The machine is given by Figure 4. Arrow a_1 is implemented in the usual way. That is, when ϕ_e is not correct on $\langle e,n\rangle$ at s but is correct at $s+1$, we enumerate an element of F_n into V_e if necessary to enter state S_2. For arrow a_2, we define B_{s+1} so that ϕ_e is not correct on $\langle e,n\rangle$ at $s+1$. Since

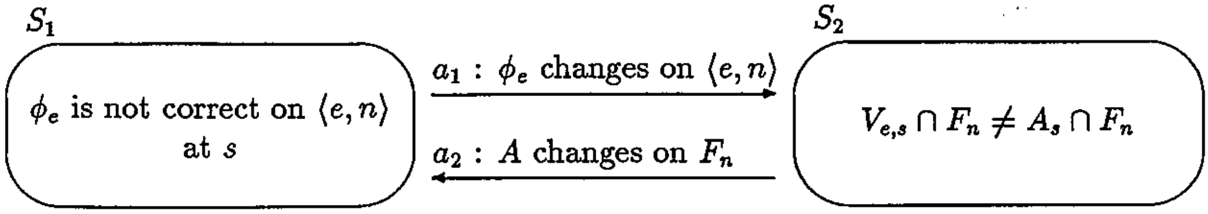

Figure 4: State diagram for construction.

arrow a_2 is traversed only if A F-permits n, we are allowed by our permitting condition to do this. The important thing to notice is that for each complete traversal of the machine from state S_1 to S_2 and back to S_1 again, there must exist a new y such that $\phi_e(\langle e,n\rangle,y) \neq \phi_e(\langle e,n\rangle,y+1)$. Since $|F_n| > f(\langle e,n\rangle)$, we will be able to force that there are more than $f(\langle e,n\rangle)$ such y if $V_e \cap F_n = A \cap F_n$ and $\lim_y \phi_e(\langle e,n\rangle,y) = B(\langle e,n\rangle)$.

CONSTRUCTION.

Stage $s+1$

Step 1. (Arrow a_2) For every e and $n \geq e$ such that A F-permits n at $s+1$ do the following. Let y_{s+1} be maximal, if such exists, such that $\phi_{e,s+1}(\langle e,n\rangle,y_{s+1})$ converges. Define $B_{s+1}(\langle e,n\rangle)$ so that $B_{s+1}(\langle e,n\rangle) \neq \phi_{e,s+1}(\langle e,n\rangle,y_{s+1})$. For all x such that $B_{s+1}(x)$ has not otherwise been defined in this step, define $B_{s+1}(x) = B_s(x)$.

Step 2. (Arrow a_1) For each e and $n \geq e$, if ϕ_e is correct on $\langle e,n\rangle$ at $s+1$ and $V_{e,s}\cap F_n = A_{s+1}\cap F_n$, enumerate the least element of $F_n - V_{e,s}$ into V_e.

To see that $\mathbf{R}_{e,n}$ is satisfied, suppose that $V_e \cap F_n = A \cap F_n$ and that $\lim_y \phi_e(\langle e,n\rangle,y) = B(\langle e,n\rangle)$. Step (2) then implies that arrow a_1 is traversed $|F_n|$ many times. Let $s_1 < s_3$ be stages such that consecutive traversals of arrow a_1 are made at s_1+1 and s_3+1. Let s_2+1 be the intervening stage at which arrow a_2 is traversed. Thus $\phi_{e,s_1+1}(\langle e,n\rangle,y_{s_1+1}) = B_{s_1+1}(\langle e,n\rangle)$, $\phi_{e,s_3+1}(\langle e,n\rangle,y_{s_3+1}) = B_{s_3+1}(\langle e,n\rangle)$, and $\phi_{e,s_2+1}(\langle e,n\rangle,y_{s_2+1}) \neq B_{s_2+1}(\langle e,n\rangle)$. This implies the existence of y such that $y_{s_1+1} \leq y < y_{s_3+1}$ such that $\phi_e(\langle e,n\rangle,y) \neq \phi_e(\langle e,n\rangle,y+1)$. The existence of $|F_n| > f(\langle e,n\rangle)$ such y implies that $\mathbf{R}_{e,n}$ is satisfied. $\square$

(if) Suppose that $B \in \mathbf{b}$ is not f-r. e., and $B \leq_T A$. Let Γ and γ be such that $B = \Gamma(A)$ with use function γ; i.e., $\gamma(x,s)$ is the use of the computation $\Gamma_s(A_s,x)$ at stage s if the computation converges. We may assume that $\gamma(x,s)$ is increasing in x. By speeding up the enumeration of Γ and A, we may also assume that $\Gamma_s(A_s,x)$ is defined for all $s > x$. We construct $C \leq_T A$ such that C is array nonrecursive. Fix a v. s. a. $\{F_n\}_{n\in N}$. The requirements are

$$\mathbf{R}_e: \quad (\exists n)[W_e \cap F_n = C \cap F_n].$$

We devote the sets $F_{\langle e,0\rangle}, F_{\langle e,1\rangle}, \ldots$ to $\mathbf{R}_e$.

To meet $\mathbf{R}_e$, we will construct a recursive approximation $\{B^e_s\}_{s\in N}$ which threatens to witness that B is f-r. e. Define $I_{\langle e,n\rangle} = \{x \mid |F_{\langle e,n\rangle}| < x \leq |F_{\langle e,n+1\rangle}|\}$. We split $\mathbf{R}_e$ into the following requirements

$$\mathbf{R}_{e,n}: \quad W_e \cap F_{\langle e,n\rangle} = C \cap F_{\langle e,n\rangle} \text{ or } B^e \text{ works on } I_{\langle e,n\rangle}$$

where B^e works on $I_{\langle e,n\rangle}$ means that for all $x \in I_{\langle e,n\rangle}$, we have that $|\{s \mid B^e_s(x) \neq B^e_{s+1}(x)\}| \leq f(x)$ and $\lim_s B^e_s(x) = B(x)$. Note that for a fixed e, the sets $I_{\langle e,n\rangle}$ are finite, disjoint, and have cofinite union. It is clear that the requirements $\mathbf{R}_{e,n}$ are sufficient to meet $\mathbf{R}_e$.

The strategy for meeting $\mathbf{R}_{e,n}$ is given by the two-state automaton of Figure 5.

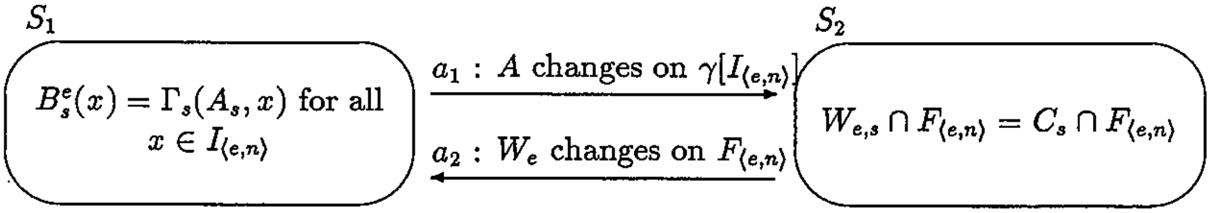

Figure 5: State diagram for the construction.

In Figure 5, $\gamma[I_{\langle e,n\rangle}]$ denotes $\max\{y \mid y \leq \gamma(x,s)$ for some $x \in I_{\langle e,n\rangle}\}$. We begin the strategy for $\mathbf{R}_{e,n}$ at stage s_0 such that $s_0 = \max I_{\langle e,n\rangle} + 1$. At this stage, $\Gamma_{s_0}(A_{s_0}, x)$ converges for all $x \in I_{\langle e,n\rangle}$ and we set $B^e_t(x) = \Gamma_{s_0}(A_{s_0}, x)$ for all $t \leq s_0$. Thus, at stage s_0, we are in state S_1 of figure 5. While $\Gamma_s(A_s, x) = \Gamma_{s+1}(A_{s+1}, x)$ for all $x \in I_{\langle e,n\rangle}$, we set $B_{s+1}(x) = B_s(x)$ and remain in state S_1 (without any changes in our approximation to B). Suppose that $s \geq s_0$ is such that $\Gamma_{s+1}(A_{s+1}, x) \neq \Gamma_s(A_s, x)$ for some $x \in I_{\langle e,n\rangle}$. Then it must be the case that $(\exists y \leq \gamma(x,s))[y \in A_{s+1} - A_s]$. We take this to be our permitting condition and thus are allowed to follow arrow a_1 at stage $s+1$ and set $C_{s+1} \cap F_{\langle e,n\rangle} = W_{e,s+1} \cap F_{\langle e,n\rangle}$. While in state S_2, we also cause $B_{s+1}(x) = B_s(x)$ for all $x \in I_{\langle e,n\rangle}$. We remain in state S_2 unless there is s such that W_e changes on $F_{\langle e,n\rangle}$ at stage $s+1$. For such an s, we pass to state S_1 by defining $B_{s+1}(x) = \Gamma_{s+1}(A_{s+1}, x)$ for all $x \in I_{\langle e,n\rangle}$. Notice that the above construction requires us to change the approximation to B at a stage $s+1$ on $x \in I_{\langle e,n\rangle}$ only if arrow a_2 is traversed at stage $s+1$. This can occur only $F_{\langle e,n\rangle}$ times. Since $f(x) > |F_{\langle e,n\rangle}|$ for all $x \in I_{\langle e,n\rangle}$, our approximation B_e to B does not change too often. Furthermore, if the construction for $\mathbf{R}_{e,n}$ ends in state S_1, we have that $\lim_s B^e_s(x) = B(x)$ for all $x \in I_{\langle e,n\rangle}$ and $\mathbf{R}_{e,n}$ is satisfied. On the other hand, if the construction for $\mathbf{R}_{e,n}$ ends in state S_2, the requirement $\mathbf{R}_e$ is satisfied (on $F_{\langle e,n\rangle}$). Note also that our permitting condition guarantees that $C \leq_{\mathrm{T}} A$. There are no conflicts among the various requirements $\mathbf{R}_{e,n}$. We omit the description of the whole construction as it is now quite routine. ∎

We now use Theorem 4.1 to improve the classification begun in Section 3 of the a. n. r. degrees in terms of the jump operator. We use the following result of Jockusch [J2, Theorem 1].

Lemma 4.2 *If* $\mathbf{a}$ *is any r. e. degree then* $\mathbf{a}' \geq \mathbf{0}''$ *iff the recursive sets are uniformly of degree* $\leq \mathbf{a}$.

We now have the following Corollary of Theorem 4.1.

Corollary 4.3 *If* $\mathbf{a}$ *is an r. e. degree such that* $\mathbf{a}'' > \mathbf{0}''$*, then* $\mathbf{a}$ *is array nonrecursive.*

Proof. Suppose that $\mathbf{a}$ is array recursive. By the theorem, this implies that if f is a strictly increasing function, then every set B such that $deg(B) \leq \mathbf{a}$ is f-r. e. It is easy to see that this implies that the sets recursive in $\mathbf{a}$ are uniformly $\leq_{tt} K$ and hence of degree $\leq \mathbf{0}'$. By the relativization of Lemma 4.2 to $\mathbf{a}$, we have that $\mathbf{0}'$ is high over $\mathbf{a}$; i. e., $\mathbf{0}'' = \mathbf{a}''$. ∎

The result of Corollary 4.3 is best possible since Downey [D2, Theorem 1.3] has shown that there are array recursive degrees that are low_2 but not low. The easiest way that we know to construct such a degree is indirect. First, it is possible to construct an r. e. 1-topped degree which is array recursive. This is done by modifying a construction of Downey and Jockusch [DJ, Theorem 2.1]. Then we use the fact, also proved in [DJ, Theorems 3.1, 3.2], that all nonzero 1-topped r. e. degrees are complete or low_2 but not low.

A. Kučera and the authors have observed that Theorem 4.1 can be used in conjunction with other results to give a new proof of Theorem 2.10. First, observe that a straightforward modification of the proof of the low basis theorem [JS, Theorem 2.1] shows that every nonempty, recursively bounded Π_1^0 class has an element A such that, for some recursive function f, every set B r. e. in A is f-r. e. Applying this to such a Π_1^0 class which contains only sets of fixed-point-free degree (see [K, Remark 1]), there is a set A of fixed-point-free degree and a function f such that every set B r. e. in A is f-r. e. Then by Kučera's result that every fixed-point-free degee below $\mathbf{0}'$ bounds a nonzero r. e. degree [K, Theorem 1], there is a nonrecursive r. e. set C recursive in A. The degree of C is array recursive by Theorem 4.1. Corollaries 2.11 and 2.12 of 2.10 follow by the same argument, since every fixed-point-free degree below $\mathbf{0}'$ bounds a promptly simple degree by [K, Remark 2].

In [JS, Theorem 1], Jockusch and Soare show that every degree which contains a consistent extension of Peano arithmetic bounds an incomparable pair of degrees. Used in the proof of that theorem is a construction of two pairs B_0, C_0 and B_1, C_1 of r. e. sets such that $B_0 \cap C_0 = B_1 \cap C_1 = \emptyset$ and whenever S separates B_0 and C_0 and T separates S_1 and C_1, then S and T are Turing incomparable. We now show that this construction can be done below precisely the array nonrecursive degrees.

Theorem 4.4 *For r.e. sets A, the following are equivalent:*

(a) A has a. n. r. degree,

(b) there are disjoint r.e. sets B and C each recursive in A such that $B \cup C$ is coinfinite and no set of degree $\mathbf{0}'$ *separates B and C,*

(c) there exist two disjoint pairs of r.e. sets B_0, C_0 and B_1, C_1 such that $B_i \cup C_i$ is coinfinite for $i = 0, 1$, each set B_i, C_i is recursive in A, and each set which separates (B_0, C_0) is incomparable with each which separates (B_1, C_1).

Proof. (c)⇒(b) is easy; let $B = B_0$, $C = C_0$.

(a)⇒(c). We first review the construction of B_0, C_0, B_1, and C_1 when there is no requirement that any of these sets be recursive in A. Let $\mathbf{R}_{2e+j}$ ($j = 0$ or $j = 1$) be the requirement that if S is any separating set for B_j, C_j, and T is any separating set for B_{1-j}, C_{1-j}, then $S \neq \{e\}^T$. The basic strategy for $\mathbf{R}_{2e+j}$ is to choose a witness w and wait for a stage s such that for some set T with $\max(T) < s$, T separates $B_{1-j,s}$, $C_{1-j,s}$, and $\{e\}^T_s(w) = 0$ or 1. Let u be the use in the computation $\{e\}^T_s(w)$. At stage $s+1$, we enumerate all elements of T into B_{1-j} and all elements of $\overline{T}$ which are less than u into C_{1-j}. This insures that $T[u] = T'[u]$ for any set T' which separates $B_{1-j,s+1}$ and $C_{1-j,s+1}$ and hence for any set T' which separates B_{1-j} and C_{1-j}. Now if $\{e\}^T_s(w) = 0$ we enumerate w into B_j and if $\{e\}^T_s(w) = 1$ we enumerate w into C_j. This meets the requirement forever.

To combine the requirements, we use many witnesses for each requirement. Specifically, if k witnesses are assigned to requirements $\mathbf{R}_m$ for $m < n$, we use 2^k witnesses for $\mathbf{R}_n$; i.e., one for every subset D of the set W of witnesses for the requirements R_m, for $m < n$. If witness w_D corresponds to $D \subseteq W$, it is handled as above except that one considers only separating sets T with $T \cap W = D$, and $\mathbf{R}_{2e+j}$ does not cause any elements of W to be enumerated into B_{1-j} or C_{1-j}. Whenever a witness w for $\mathbf{R}_{2e+j}$ is enumerated into B_j or C_j by a requirement $\mathbf{R}_{2i+(1-j)}$ of higher priority, then w is replaced by a new witness w' for $\mathbf{R}_{2e+j}$ which is not yet in B_j or C_j. Thus requirements $\mathbf{R}_{2e+j}$ of lower priority than $\mathbf{R}_{2i+j}$ must consider new possibilites for sets D contained in the witnesses assigned to requirements (such as $\mathbf{R}_{2i+j}$) of higher priority than $\mathbf{R}_{2e+j}$. Nonetheless, it is easy to compute a recursive upper bound $w(n)$ for the number of witnesses ever assigned to $\mathbf{R}_n$. This is not important for the basic existence result we have been discussing but it is crucial to carrying it out below a given array nonrecursive degree.

To make the sets we construct recursive in a given array nonrecursive set A, we first choose a very strong array $\{F_n\}_{n\in N}$ such that each F_n has sufficiently large cardinality (to be specified later). We shall assume that A is F-a. n. r. since by Theorem 2.5 there is a set $\hat{A}$ of the same degree as A which is F-a. n. r. If X is any one of the sets B_0, C_0, B_1, C_1, we shall guarantee that $X \leq_{\mathbf{T}} A$ by the condition: $x \in X_{s+1} - X_s$ only if A F-permits x at stage $s+1$. The requirements $\mathbf{R}_k$ above, are replaced by infinitely many requirements

$$\mathbf{R}_{k,i}: \quad A \cap F_i = V_k \cap F_i \Rightarrow \mathbf{R}_k \text{ holds}$$

where V_k is an auxiliary r. e. set enumerated during the construction. We meet $\mathbf{R}_{k,i}$ only for $i \geq k$ which suffices by (8). We assign priorities to the requirements $\mathbf{R}_{k,i}$ in increasing order of $\langle k, i\rangle$.

We now describe the strategy for meeting $\mathbf{R}_{k,i}$ which is similar but not identical to that for $\mathbf{R}_k$. The basic idea is to make $V_{k,s} \cap F_i \neq A_s \cap F_i$ by enumerating an element of F_i whenever permission is needed to enumerate an element $\geq i$ into one of B_0, C_0, B_1, or C_1. If $A \cap F_i = V_k \cap F_i$, then the desired permission must occur. The set F_i will be of sufficiently large cardinality so that an element of F_i will always be available. Let W be the set of numbers which are less than i or are witnesses for requirements of higher priority than $\mathbf{R}_{k,i}$ (and thus either cannot be forced to be permitted by the above method or cannot be enumerated by $\mathbf{R}_{k,i}$). Actually, W depends on the stage. For each set $D \subseteq W$, assign a witness $w_D \geq i$ to $\mathbf{R}_{k,i}$. Suppose that $k = 2e+j$. Wait for a stage s such that there is a set T with $\max(T) < s$, T separates $B_{1-j,s}$ and $C_{1-j,s}$, and $\{e\}_s^T(w_D) = 0$ or 1. Now restrain B_{1-j} and C_{1-j} through the use of this computation. (Notice that this restraint was not involved in the original strategy for $\mathbf{R}_k$. It is needed now to ensure that T is still a separating set when, if ever, we get permission for the desired enumerations.) If $V_{k,s} \cap F_i = A_s \cap F_i$, enumerate the least element of $F_i - V_{k,s}$ into V_k. As usual, this element will exist by the construction and the choice of the F_i. Assign new witnesses to all requirements of lower priority than $\mathbf{R}_{k,i}$. If there is a stage $t > s$ such that A F-permits i at stage t, and no requirement of higher priority than $\mathbf{R}_{k,i}$ has acted between s and t, then the obvious enumerations should be made to meet $\mathbf{R}_{k,i}$ for all separating sets T' with $T' \cap W = D$. Specifically, let T be, as before, a set separating $B_{1-j,s}$ and $C_{1-j,s}$ with $\max T < s$ and $\{e\}_s^T(w_D) = 0$ or 1 with use u and $T \cap W = D$. Enumerate all elements of $T - W$ into B_{1-j} and all elements of $\overline{T} \cap \overline{W}$ into C_{1-j}. If $\{e\}_s^T(w_D) = 0$, put w_D into B_j, and otherwise put w_D into C_j. Assign new witnesses to all requirements of lower priority than $\mathbf{R}_{k,i}$.

It is a standard finite injury argument to see that the above procedure works. The main point is that we can in advance choose the sets F_i to be of sufficiently large cardinality. We first define, by recursion on $\langle k, i\rangle$, a recursive bound $w(k,i)$ on the number of witnesses ever assigned to $\mathbf{R}_{k,i}$. Let $c = \sum\{w(k',i') | \langle k',i'\rangle < \langle k,i\rangle\}$, so that c bounds the total number of witnesses ever assigned to requirements of higher priority than $\mathbf{R}_{k,i}$. Since any requirement acts at most twice using any given witness, there are at most $2c$ stages at which requirements of higher priority than $\mathbf{R}_{k,i}$ act. At any such stage, at most 2^{i+c} witnesses are assigned to $\mathbf{R}_{k,i}$, so we may set $w(k,i) = 2c2^{i+c}$. Finally, it suffices for the cardinality of F_i to be at least the number of witnesses for $R_{k,i}$ for each $k \le i$, since each witness can causes at most one element of F_i to enter V_k. We thus require that $|F_i| = \max\{w(k,i) | k \le i\}$. □

(b)⇒(a). Let $\{F_n\}_{n\in N}$ be a very strong array. We construct $A \le_{\mathrm{T}} B \oplus C$ to meet

$$\mathbf{R}_e: \quad (\exists n)[W_e \cap F_n = A \cap F_n].$$

As usual, we reserve the sets $F_{\langle e,i\rangle}$, $i \in N$ for meeting $\mathbf{R}_e$. We first give the construction of A. We show that if the construction fails to meet $\mathbf{R}_e$ for some e, then there is a Δ_2^0 set X such that $K \le_{\mathrm{T}} X$ and X separates the pair B, C.

Define a recursive function g by the two conditions

$$\begin{aligned} g(e,0) &= 1 + |F_{\langle e,0\rangle}| \quad \text{for every } e \in N \\ g(e,i+1) &= g(e,i) + 1 + |F_{\langle e,i+1\rangle}| \quad \text{for every } e, i \in N. \end{aligned}$$

At any stage in the construction, let $d_{0,s} < d_{1,s} < d_{2,s} \dots$ be the elements of $\overline{B_s \cup C_s}$. To ensure that $A \le_{\mathrm{T}} B \oplus C$, we require that if $x \in A_{s+1} - A_s$ and $x \in F_{\langle e,i\rangle}$ then $d_{g(e,i+1),s} \ne d_{g(e,i+1),s+1}$. Let d_n denote $\lim_s d_{n,s}$.

CONSTRUCTION.

Stage $s+1$

For every e and i, if $W_{e,s} \cap F_{\langle e,i\rangle} \ne A_s \cap F_{\langle e,i\rangle}$ and $d_{g(e,i+1),s} \ne d_{g(e,i+1),s+1}$, then enumerate all of $W_{e,s+1} \cap F_{\langle e,i\rangle}$ into A at stage $s+1$. This ends the construction of A.

To see that each requirement $\mathbf{R}_e$ is satisfied, suppose otherwise and fix e such that $\mathbf{R}_e$ is not satisfied; i. e., that $W_e \cap F_{\langle e,i\rangle} \neq A \cap F_{\langle e,i\rangle}$ for all $i \in N$. We shall define a Δ_2^0 set X such that $K \leq_T X$, $X \supseteq B$ and $X \cap C = \emptyset$. We specify X by giving a recursive approximation $\{X_s\}_{s \in N}$.

We first describe the idea behind the construction of X. Let $D_0 = \{d_0, d_1, \ldots, d_{g(e,0)}\}$ and let $D_i = \{d_{g(e,i-1)+1}, \ldots, d_{g(e,i)}\}$ for all $i > 0$. Since $d_y \in \overline{B \cup C}$ for every y, we are free to enumerate elements of D_i in or out of X as we wish. Observe that our choice of g guarantees that $|D_i| = 1 + |F_{\langle e,i\rangle}|$ for all i. We will use the set D_i to code into X whether $i \in K$ and also to code into X the set D_{i+1}. (Thus X will be able to compute inductively for each integer n whether $n \in K$.) The coding into D_i will be tied to the attempt to meet R_e via $F_{\langle e,i\rangle}$ so that the fact that $W_e \cap F_{\langle e,i\rangle} \neq A \cap F_{\langle e,i\rangle}$ will allow the coding to be successful. Figure 6 gives the two-state diagram for the coding for i.

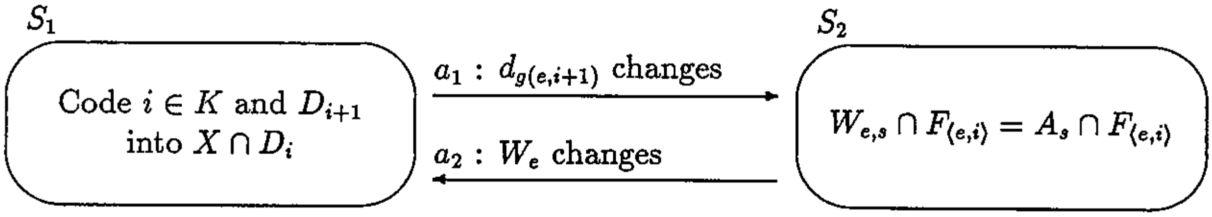

Figure 6: State diagram for the construction of X.

We call Figure 6 the i-module. The action that we take for the i-module at any stage $s+1$ is predicated on the assumption that the current approximation to D_i is correct. Arrow a_1 of the i-module corresponds exactly to the construction of A given above. Also, if a_1 is traversed at stage $s+1$, this indicates that the hypothesis of the j-module for all $j > i$ is false for all stages $\leq s$ so we restart each of these modules at stage $s+1$. This is the intent of Step 1 of the construction of X below. If arrow a_2 is traversed or i is enumerated in K at stage $s+1$, step 2 of the construction below codes this event in X by using an element of (the current approximation to) D_i. This coding is enough for X to recover D_{i+1} from D_i. We now give the formal construction of X and the verification.

CONSTRUCTION.

Define $X_0 = \emptyset$. Having defined X_s, we define X_{s+1} in steps. (For convenience, if we do not specify whether $n \in X_{s+1}$, then $n \in X_{s+1}$ if and only if $n \in X_s$.)

Stage $s+1$

Step 1. Let i be least, if any, such that $d_{g(e,i),s+1} \neq d_{g(e,i),s}$. Remove from X all integers $y > d_{g(e,i-1),s}$ ($y \geq 0$ if $i = 0$).

Step 2. Let i be least, if any, such that $i \in K_{s+1} - K_s$ or such that $W_{e,s+1} \cap F_{\langle e,i\rangle} \neq A_{s+1} \cap F_{\langle e,i\rangle}$ but $W_{e,s} \cap F_{\langle e,i\rangle} = A_s \cap F_{\langle e,i\rangle}$. Let j be least such that $g(e,i-1) < j \leq g(e,i)$ and $d_{j,s+1} \notin X_s$. (We will argue in Lemma 4.7 that such a j exists.) Enumerate $d_{j,s+1}$ into X.

Step 3. Enumerate each y in B_{s+1} into X.

Lemma 4.5 $\lim_s X_s$ *exists.*

Proof. Only step 1 causes removal of any element of X. It is clear from step 1 that any integer y may be removed finitely often. □

Lemma 4.6 $X \supseteq B$ *and* $X \cap C = \emptyset$.

Proof. $X \supseteq B$ by step 3 of the construction. Suppose that $y \in C$ is enumerated into X at some stage $s+1$. Then by the construction, $y = d_{j,s+1}$ for some j. Since $y \in C$, there is $t \geq s+1$ such that $d_{j,t} \neq d_{j,t+1}$. Thus by step 1, y is removed from X at stage $t+1$ (if not before). □

Lemma 4.7 $K \leq_T X$.

Proof. Let s_i be the least stage such that $d_{g(e,i),s_i} = d_{g(e,i)}$. By the construction, all integers in D_i are removed from X at stage s_i. Now for any such integer, d_j, d_j is enumerated into X at a stage $s+1 \geq s_i$ only if step 2 applies to i at stage $s+1$; i. e. if $i \in K_{s+1} - K_s$ or W_e changes on $F_{\langle e,i\rangle}$ at stage $s+1$. This can happen at most $1 + |F_{\langle e,i\rangle}|$ times. This proves the claim in the construction that j exists with $g(e,i-1) < j \leq g(e,i)$ and $d_{j,s+1} \notin X_s$.

Now to compute K from X, we assume by induction that we know D_i and show how to compute from X whether $i \in K$ and D_{i+1}. We can assume that we know D_0 since it is a finite set. Given D_i, let s be a stage such that $W_{e,s} \cap F_{\langle e,i\rangle} \neq A_s \cap F_{\langle e,i\rangle}$, $d_{g(e,i),s} = d_{g(e,i)}$, and such that for all $y \leq g(e,i)$, $d_y \in X$ iff $d_y \in X_s$. Then by step 2 of the construction, $i \in K$ iff $i \in K_s$. We claim also that $d_{g(e,i+1),s} = d_{g(e,i+1)}$. For otherwise let $t > s$ be such that $d_{g(e,i+1),t+1} \neq d_{g(e,i+1),t}$. Then $W_{e,t+1} \cap F_{\langle e,i\rangle} = A_{t+1} \cap F_{\langle e,i\rangle}$ by the construction of A. Thus there is $u \geq t+1$ such that $W_{e,u+1} \cap F_{\langle e,i\rangle} \neq A_{u+1} \cap F_{\langle e,i\rangle}$ but $W_{e,u} \cap F_{\langle e,i\rangle} = A_u \cap F_{\langle e,i\rangle}$. Then by step 2 of the enumeration of X, an element of D_i is enumerated in X at stage $u+1$ contrary to the assumption on s. Thus, at stage s we also know D_{i+1}. ■

Let Q denote the free Boolean algebra generated by a fixed recursive set $\{p_i | i \in N\}$ of literals. A (propositional) theory can be viewed as a filter of Q. We consider r. e. theories. An r. e. theory is *well-generated* if it is generated by a pair of sets $\{p_i | i \in B\}$ and $\{\neg p_i | i \in C\}$. It is well-known that if an r. e. theory T is well-generated and the r. e. sets B and C are recursively inseparable, then T is essentially undecidable. We examine such theories which in addition have relatively few r. e. extensions.

Definition 4.8 *An r. e. propositional theory is* Martin–Pour-El *if it is well-generated, essentially undecidable, and every r. e. extension of* T *is a principal extension. (That is, if* $T \subseteq W$ *and* W *is a consistent r. e. theory, then there is* $q \in Q$ *such that* W *is the theory generated by* T *and* q*.)*

Martin and Pour-El [MP, Theorem I] showed the existence of such theories. Downey in his thesis and in [D1], obtained numerous related results, including a number of results on the possible Turing degrees of such theories. The next theorem, together with the results of sections 2 and 3 considerably extend these results.

Theorem 4.9 *An r. e. degree* **a** *is array nonrecursive if and only if there is a Martin–Pour-El theory* T *of degree* **a**.

Proof. (only if) We first review the construction of a Martin–Pour-El theory T without the requirement that T be of a particular a. n. r. degree **a**. Let $\{S_n\}_{n \in N}$ be an enumeration of the r. e. consistent theories. If $F \subseteq Q$, we write F^* for the theory generated by F. We will construct T in stages so that T_s will denote the theory constructed by stage s. For each s, T_{s+1} will be of the form $(T_s \cup F)^*$ where F is a finite set of literals or their negations. At each stage s, we will denote by $d_{0,s} < d_{1,s} < \ldots$ the set $\{p_i | p_i, \neg p_i \notin T_s\}$; the ordering of the literals is that given by $p_0 < p_1 < \ldots$. We will use $\epsilon_i p_i$ to denote either p_i or $\neg p_i$.

For each $e \in N$ we have the requirements:

$$\mathbf{R}_e : \quad 0 \notin (T \cup S_e)^* \Rightarrow (\exists x)[(T \cup \{x\})^* = (T \cup S_e)^*], \text{ and}$$

$$\mathbf{N}_e : \quad \lim_s d_{e,s} \text{ exists.}$$

The requirements $\mathbf{N}_e$ guarantee that T is incomplete and consistent. The requirements $\mathbf{R}_e$ guarantee that every r. e. extension of T is principal over T. Together these requirements and the fact that T is well-generated guarantee that T is essentially undecidable. To meet $\mathbf{R}_e$ we shall construct a finite set Q_e such that $x = \bigwedge Q_e$ is the witness for $\mathbf{R}_e$. Q_e will be constructed in stages; $Q_{e,s}$ is the finite set constructed by stage s and $Q_e = \lim_s Q_{e,s}$.

We say that $\mathbf{R}_e$ *requires attention* at stage $s+1$ if $(\exists y)[y \in S_{e,s},\ y \notin (T_s \cup Q_{e,s})^*$ and $0 \notin (T_s \cup S_{e,s})^*]$. If y is least with this property, we say that $\mathbf{R}_e$ *requires attention at* $s+1$ *via* y. We will assume that S_e consists of elements of the form $\bigvee \epsilon_i p_i$.

CONSTRUCTION.

Stage 0
Let $T_0 = Q_{e,0} = \emptyset$ for every $e \in N$.
Stage $s+1$
Find the least e such that $\mathbf{R}_e$ requires attention and let y be such that $\mathbf{R}_e$ requires attention via y. Define $F = \{\neg \epsilon_i d_{i,s} | \epsilon_i d_{i,s}$ occurs in y and $i \geq e\}$. Let $T_{s+1} = (T_s \cup F)^*$ and $Q_{e,s+1} = Q_{e,s} \cup \{y\}$. For $i \neq e$, let $Q_{i,s+1} = Q_{i,s}$. This ends the construction.

We show that the construction succeeds in two lemmas.

Lemma 4.10 *If* $\mathbf{R}_e$ *receives attention at stage* $s+1$ *via* y, *then there exists a Boolean combination of* $\{d_{0,s}, d_{1,s} \ldots d_{e-1,s}\} = \{d_{0,s+1}, d_{1,s+1} \ldots d_{e-1,s+1}\}$ *such that* $T_{s+1} \vdash y \leftrightarrow x$.

Proof. We write y as a disjunction of the form

$$\bigvee_{i<e} \epsilon_i d_{i,s} \vee \bigvee_{i \geq e} \epsilon_i d_{i,s} \vee \bigvee_{\epsilon_i p_i \in T_s} \epsilon_i p_i \vee \bigvee_{\neg \epsilon_i p_i \in T_s} \epsilon_i p_i$$

Thus y has the form $x \vee z \vee m \vee n$. Since $\vdash x \to y$, it suffices to show that $T_{s+1} \vdash y \to x$. Now if $m \neq 0$, then $y \in T_s$ since $\vdash m \to y$ and $m \in T_s$. But then $\mathbf{R}_e$ does not require attention via y. Thus $m = 0$. Now $\neg n \in T_{s+1}$ by definition of n, and $\neg z \in T_{s+1}$ by construction so it follows that $T_{s+1} \vdash y \to x$ as desired. □

Lemma 4.11 *For every e, Q_e is finite (and hence $\mathbf{R}_e$ is satisfied and $\lim_s d_{e,s}$ exists).*

Proof. Assume by induction that for all $i < e$, $\lim_s d_{i,s}$ exists and let s_0 be such that for all $i < e$, $\lim_s d_{i,s} = d_{i,s_0}$. Suppose that $\mathbf{R}_e$ receives attention via y at a stage $s+1 \geq s_0$. Then by Lemma 4.10, there is x, a disjunction of some of the d_{i,s_0}, $i < e$, and their negations such that $T_{s+1} \vdash y \leftrightarrow x$. Thus $(T_{s+1} \cup Q_{e,s+1})^* \vdash x$ (since $y \in Q_{e,s+1}$). However $(T_s \cup Q_{e,s})^* \not\vdash x$ since otherwise $y \in (T_s \cup Q_{e,s})^*$, contradicting that $\mathbf{R}_e$ requires attention via y. Thus, for each stage $s+1 > s_0$ such that $\mathbf{R}_e$ requires attention, a new disjunction x of the literals d_{i,s_0}, $i < e$, or their negations is used. There are only 2^{2e} such disjunctions. Thus $\mathbf{R}_e$ receives attention only finitely often after s_0, Q_e is finite, and thus $\mathbf{R}_e$ is satisfied. □

Of course the key fact in the above construction is that requirement $\mathbf{R}_e$ requires attention only 2^{2e} times, at most, after requirements $\mathbf{R}_i$, $i < e$, have ceased acting.

We now include the requirements that T be Turing computable from A for a fixed array nonrecursive set A. (We construct $T \leq_{\mathrm{T}} A$. A simple coding strategy similar to that of the previous theorem can be used to make $T \equiv_{\mathrm{T}} A$.) Let f be a recursive function defined by the conditions: $f(0) = 1$ and $f(i+1) = 2^{2i+2}\sum_{j=0}^{i} f(j)$ for all $i \geq 0$. We assume that A is array nonrecursive for a fixed v. s. a. $\{F_n\}_{n\in N}$ such that $|F_i| > f(i)$ for all i. As usual, we conceive of requirement $\mathbf{R}_e$ as consisting of subrequirements

$$\mathbf{R}_{e,i}: \qquad V_e \cap F_i = A \cap F_i \Rightarrow \mathbf{R}_e \text{ holds}$$

where V_e is an auxiliary r. e. set which we enumerate. We need only meet cofinitely many of these requirements for each $e \in N$. (In this construction, unlike previous ones, the confinite set of requirements $\mathbf{R}_{e,i}$ which we meet for a fixed e is specified only by recursive approximation.) Requirement $\mathbf{R}_{e,i}$ follows requirements $\mathbf{N}_0$, $\mathbf{N}_1$,, . . , $\mathbf{N}_{i-1}$ in priority. The permitting condition is

$$d_{i,s} \text{ or } \neg d_{i,s} \in T_{s+1} - T_s \text{ only if } A \text{ } F\text{-permits } i \text{ at stage } s+1.$$

Since requirement $\mathbf{R}_{e,i}$ uses F_i and does not attempt to enumerate $d_{0,s}, \ldots, d_{i-1,s}$ or the negations of these into T, this permitting is appropriate.

The strategy for meeting a single $\mathbf{R}_{e,i}$ in isolation is the natural one. That is, $\mathbf{R}_{e,i}$ waits for a stage $s+1$ such that $\mathbf{R}_e$ requires attention at stage $s+1$ via y in the sense of the above construction. At such a stage, we cause $V_{e,s+1} \cap F_i \neq A_{s+1} \cap F_i$ and enumerate $y \in Q_{e,s}$. At a later stage $t+1$ such that A F-permits i at $t+1$, we enumerate $\neg\epsilon_j d_{j,s}$ into T for every $j \geq i$ such that $\epsilon_j d_{j,s}$ occurs in y. Were there no other requirements, $\mathbf{R}_{e,i}$ would be satisfied since $|F_i| > 2^{2i}$ by exactly the proofs of Lemmas 4.10 and 4.11.

The strategy for $\mathbf{R}_{e,i}$ conflicts with that for $\mathbf{R}_{e,j}$, $j \neq i$, and for other requirements $\mathbf{R}_f$. It may be the case, for instance, that by the stage $t+1$ such that A F-permits i at stage $t+1$, $d_{j,s} \neq d_{j,t}$ and indeed that $\epsilon_j d_{j,s} \in T_t$. Then the action specified for $\mathbf{R}_{e,i}$ results in making T_{t+1} inconsistent.

Two devices serve to relieve these conflicts. First, to minimize the interference of $\mathbf{R}_e$ with $\mathbf{R}_f$ for $e < f$, we do the following. We will insure that $\mathbf{R}_e$ acts only finitely often and at each stage $s+1$ such that $\mathbf{R}_e$ acts at stage $s+1$ we will restart $\mathbf{R}_f$. We will assume at stage $s+1$ the literals $d_{j,s}$ that $\mathbf{R}_e$ is concerned with satisfy $j \leq s$. Thus, we specify that after stage $s+1$, we shall only attempt to meet requirements $\mathbf{R}_{f,j}$ for

$j > s$. Since these requirements do not disturb $d_{j,s}$ for $j \leq s$, this insures that $\mathbf{R}_{f,j}$ will not injure whatever action was wanted for $\mathbf{R}_e$ at stage $s+1$. We may also assume for such stages $s+1$, that $V_{f,s+1} = \emptyset$ (by starting a "new" V_f.) In the construction, $m(f,s)$ will denote the least i such that we are attempting to meet $\mathbf{R}_{f,i}$ at stage s.

The device to insure the cooperation of the requirements $\mathbf{R}_{e,i}$ and $\mathbf{R}_{e,j}$ for $j \neq i$ is the following. Requirement $\mathbf{R}_{e,i}$ works under the assumption that for $j < i$ there will be no further permissions for $\mathbf{R}_{e,j}$. By the above, $\mathbf{R}_{e,i}$ may also work under the assumption that $\mathbf{R}_f$ for $f < e$ has ceased acting. Thus it is the assumption of $\mathbf{R}_{e,i}$ at stage s that $d_{0,s}, \ldots d_{i-1,s}$ have attained their final values. Suppose then that at stage $s+1$, $\mathbf{R}_{e,i}$ requires attention via y. Then at stage $s+1$ we activate not only $\mathbf{R}_{e,i}$, but all requirements $\mathbf{R}_{e,j}$ such that $i \leq j \leq s$ (by causing $V_{e,s+1} \cap F_j \neq A_{s+1} \cap F_j$ for all such j). Let $t+1$ be the least stage beyond $s+1$ such that A F-permits some such j at stage $t+1$. Then $\mathbf{R}_{e,j}$ causes us to enumerate $\neg\epsilon_k d_{k,s}$ into T for all $k \geq j$ such that $\epsilon_k d_{k,s}$ occurs in y. Notice first that $d_{k,t} = d_{k,s}$ for all such k because this is the first stage at which such permission occurs. Thus, this action at stage $t+1$ is permissible. Notice also that this action does not injure $\mathbf{R}_{e,i}$ for this action only enumerates in T terms that requirement $\mathbf{R}_{e,i}$ would enumerate in T, given permission. Of course $\mathbf{R}_{e,j}$ must now be allowed to act again after stage $t+1$ (for a different y) since its assumption is that there will be no further permissions for $\mathbf{R}_{e,k}$, $k < j$. However any later action for $\mathbf{R}_{e,j}$ involves only literals $d_{k,t+1}$ for $k \geq j$. By the construction at t, no such literal occurs in y and so such actions do not interfere with $\mathbf{R}_{e,i}$. What this argument shows is that $\mathbf{R}_{e,j}$ does not interfere with $\mathbf{R}_{e,i}$ if $i < j$ and that $\mathbf{R}_{e,i}$ is satisfied (in 2^{2i} attempts) if its hypothesis is correct.

In the construction below, $n(e,s)$ denotes the greatest integer such that $\mathbf{R}_{e,j}$ is waiting for a permission for all j such that $m(e,s) \leq j \leq n(e,s)$. (For convenience, $n(e,s) = m(e,s) - 1$ denotes that there is no such j.) If defined, $z(e,j,s)$ denotes the term which $\mathbf{R}_{e,j}$ wishes to enumerate in T if permitted (either $d_{j,s}$ or $\neg d_{j,s}$).

CONSTRUCTION.

Stage 0
Define $m(e,0) = e$, $n(e,0) = m(e,0) - 1$ and let $z(e,i,0)$ be undefined for all $e, i \in N$.
Stage $s+1$
Requirement $\mathbf{R}_e$ *requires attention* at stage $s+1$ if

(a) A F-permits $n(e,s)$ and $m(e,s) \leq n(e,s)$ or

(b) there is $y \in S_{e,s}$ such that $y \notin (T_s \cup Q_{e,s})^*$ and $0 \notin (T_s \cup S_{e,s})^*$.

Let e be least such that $\mathbf{R}_e$ requires attention at stage $s+1$. For all $f > e$, let $m(f,s+1) = \max\{f, s+1\}$, let $V_{f,s} = \emptyset$, let $n(f,s+1) = m(f,s+1) - 1$, and let $z(f,i,s+1)$ be undefined for all i. For $f < e$ let $m(f,s+1) = m(f,s)$ and let $n(f,s+1) = n(f,s)$.

If $\mathbf{R}_e$ requires attention because of clause (a) above, let i be least such that A F-permits i. For all $j \geq i$ such that $z(e,j,s)$ is defined, enumerate $z(e,j,s)$ into T_{s+1} and let $z(e,j,s+1)$ be undefined. Let $n(e,s+1) = i - 1$. In this case we say that requirements $\mathbf{R}_{e,j}$ for j such that $i \leq j \leq n(e,s)$ *receive permission* at stage $s+1$.

If $\mathbf{R}_e$ requires attention because of (b) above but not (a), let y be least satisfying (b). Enumerate y in $Q_{e,s+1}$. Let y be written as a disjunction

$$y = \bigvee_{i \leq n(e,s)} \epsilon_i d_{i,s} \vee \bigvee_{i > n(e,s)} \epsilon_i d_{i,s} \vee \bigvee_{\epsilon_i p_i \in T_s} \epsilon_i p_i \vee \bigvee_{\neg \epsilon_i p_i \in T_s} \epsilon_i p_i$$

(As we argued in Lemma 4.10, the third term of the disjunction is vacuous.) For each $i > n(e,s)$ such that $\epsilon_i d_{i,s}$ occurs in y, set $z(e,i,s+1) = \neg\epsilon_i d_{i,s}$. Let $n(e,s+1) = s$. For each i such that $n(e,s) < i \leq n(e,s+1)$, enumerate one integer, if necessary, into $V_e \cap F_i$ so as to cause $V_{e,s+1} \cap F_i \neq A_{s+1} \cap F_i$. In this case, we say that requirements $\mathbf{R}_{e,i}$ such that $n(e,s) < i \leq n(e,s+1)$ *receive attention* at stage $s+1$.

If $m(e,s+1)$ is not otherwise specified by this construction, then $m(e,s+1) = m(e,s)$. Similarly, for $n(e,s+1)$ and $z(e,i,s+1)$. This ends the construction.

It suffices to prove the following lemma.

Lemma 4.12 *For every $e \in N$, the requirement $\mathbf{R}_e$ is satisfied and receives attention finitely often.*

Proof. Fix $e \in N$. By induction, let s_0 be the least stage such that for all $f < e$ and $s > s_0$, requirement $\mathbf{R}_f$ does not receive attention at s. Then $m(e,s_0)$ is the final value of $\lambda s m(e,s)$, $n(e,s_0) = m(e,s_0) - 1$, and $V_{e,s_0} = \emptyset$.

By induction on $i \geq m(e,s_0)$, we show that $\mathbf{R}_{e,i}$ receives attention after s_0 fewer than $f(i)$ times and is satisfied. If this is the case, then $\mathbf{R}_e$ is satisfied (via the least $i \geq m(e,s_0)$ such that $V_e \cap F_i = A \cap F_i$) and it is easy to see that this implies that $\mathbf{R}_e$ receives attention only finitely often. Fix $i \geq m(e,s_0)$. We establish the following claim.

Claim Suppose that $t_0 < t_1$ are stages $\geq s_0$ such that no requirement $\mathbf{R}_{e,j}$ for $j < i$ receives attention or permission at any stage s such that $t_0 \leq s \leq t_1$. Then $\mathbf{R}_{e,i}$ receives attention at at most 2^{2i} stages s such that $t_0 \leq s \leq t_1$.

To prove the claim, suppose that $\mathbf{R}_{e,i}$ receives attention at stage $s_1 + 1$ such that $t_0 \leq s_1 + 1 \leq t_1$. Let $s_2 + 1 \leq t_1$ be the least stage, if any, beyond $s_1 + 1$ such that A F-permits i at $s_2 + 1$. Then by induction on s, for all s such that $s_1 + 1 \leq s < s_2 + 1$, we have $n(e,s) \geq i$, $d_{j,s_1} = d_{j,s_2}$ for all $j \leq i$, and $z(e,i,s) = z(e,i,s_1+1)$. Furthermore $z(e,i,s_1+1)$ is enumerated in T at stage s_2+1. By the same argument applied to j such that $i < j \leq n(e,s_1+1)$, we have that each value $z(e,j,s_1+1)$ defined at stage s_1+1 is enumerated in T at or before stage s_2+1. Thus, for the y enumerated in Q_e at stage s_1+1, we have enumerated into T_{s_2+1} all the elements the basic strategy for $\mathbf{R}_e$ would have immediately enumerated into T. Thus, applying the arguments of Lemmas 4.10 and 4.11, we see that requirement $\mathbf{R}_{e,i}$ receives attention at most 2^{2i} times between stages t_0 and t_1.

To see that the claim is enough to prove the lemma, first note that the claim and the choice of f imply that $\mathbf{R}_{e,i}$ receives attention at most $f(i)$ times. Thus, since $|F_i| > f(i)$, an element of $F_i - V_e$ is always available for $\mathbf{R}_{e,i}$ if it requires attention. Therefore, if $V_e \cap F_i = A \cap F_i$ and s is a stage such that no higher priority requirement that $\mathbf{R}_{e,i}$ receives attention after s and such that $V_{e,s} \cap F_i = V_e \cap F_i$ and $A_s \cap F_i = A \cap F_i$, then

$\mathbf{R}_{e,i}$ never receives attention after stage s. This implies both that $\mathbf{R}_e$ is satisfied and that $\mathbf{R}_e$ never requires attention after stage s because of case (b). Thus $\mathbf{R}_e$ receives attention only finitely often. □

(if) Suppose that T is a Martin–Pour-El theory. We construct A array nonrecursive such that $A \leq_{\mathbf{T}} T$. Let a very strong array $\{F_n\}_{n\in N}$ be given. As usual we have the requirements

$$\mathbf{R}_e: \quad (\exists n)[W_e \cap F_n = A \cap F_n].$$

We reserve $F_{\langle e,0\rangle}, F_{\langle e,1\rangle}, \ldots$ for meeting $\mathbf{R}_e$. The proof is quite similar to that of Theorem 4.4, (b) implies (a). As in that theorem, we will first give the construction of $A \leq_{\mathbf{T}} T$ and then show that if the construction fails to meet $\mathbf{R}_e$, then there is an r. e. theory $V \supset T$ which is not principal over T.

As in the proof of the other direction of this theorem, let $d_{0,s} < d_{1,s} < \cdots$ list the p_i in order of increasing i such that neither p_i nor $\neg p_i$ is in T_s. Define a recursive function $g : N^2 \to N$ as follows.

$$\begin{aligned} g(e,0) &= 1 + |F_{\langle e,1\rangle}| \quad \text{for every } e \in N \\ g(e,i) &= g(e,i-1) + 1 + |F_{\langle e,i+1\rangle}| \quad \text{for every } e \in N \text{ and } i > 0. \end{aligned}$$

To insure that $A \leq_{\mathbf{T}} T$, we require that $x \in A_{s+1} - A_s$ and $x \in F_{\langle e,i\rangle}$ implies that $d_{g(e,i),s} \neq d_{g(e,i),s+1}$.

CONSTRUCTION.

Stage $s+1$

For every e and i, if $W_{e,s} \cap F_{\langle e,i\rangle} \neq A_s \cap F_{\langle e,i\rangle}$ and $d_{g(e,i),s} \neq d_{g(e,i),s+1}$, then enumerate all of $W_{e,s+1} \cap F_{\langle e,i\rangle}$ into A at stage $s+1$.

To see that each requirement $\mathbf{R}_e$ is satisfied, suppose that e is a counterexample; i. e., that $W_e \cap F_n \neq A \cap F_n$ for all $n \in N$. We construct an r. e. theory V such that $V \supset T$, V is consistent, but V is not a principal extension of T. To construct V, we shall produce an infinite sequence (not necessarily recursive) z_i of elements such that

$$V = (T \cup \{z_i \mid i \in N\})^*. \tag{28}$$

We will ensure that $z_i \notin (T \cup \{z_j \mid j \neq i\})^*$.

Let $d_n = \lim_s d_{n,s}$ for all n, let $D_0 = \{d_0, \ldots, d_{g(e,0)}\}$, and let $D_i = \{d_{g(e,i-1)+1}, \ldots, d_{g(e,i)}\}$ for all $i \in N$. We will define for each $i \in N$ a nonempty finite set $E_i \subseteq D_i$. Given any finite set $X \subset \{p_i \mid i \in N\}$, let $\hat{X}$ denote the subset of X resulting from removing the literal p_i of greatest index. Then for every $i \in n$, z_i is defined by

$$z_i = \bigvee(\bigcup_{j<i} \hat{E}_j) \vee \bigvee E_i. \tag{29}$$

It is easy to see that if the z_i are defined in this way then V as defined in (28) has the desired properties except possibly for recursive enumerability.

At each stage we will have defined approximations to finitely many of the sets E_i. $E_{i,s}$, if defined, will denote the approximation to E_i at stage s. We will have that $E_{i,s} \subseteq D_{i,s} = \{d_{g(e,i-1)+1,s}, \ldots, d_{g(e,i),s}\}$. The two-state diagram is as follows. Of course

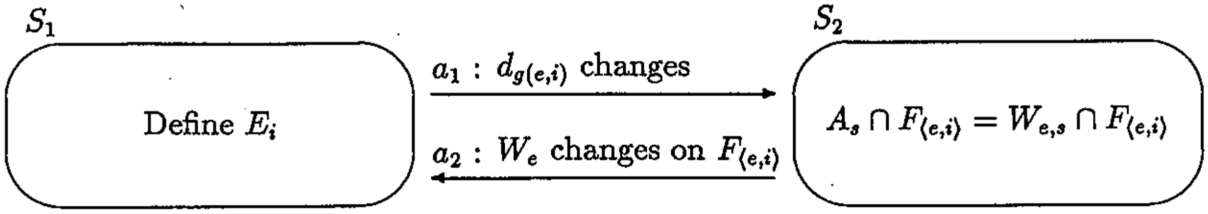

Figure 7: State diagram of the construction.

the i-module pictured in Figure 7 works at a stage s under the assumption that the j-modules for $j < i$ are in state S_1 for every stage after stage s.

The construction of the sets E_i is in stages as follows. First, let s_0 be the least stage such that $D_{0,s_0} = D_0$ and $W_{e,s_0} \cap F_{\langle e,0\rangle} \neq A_{s_0} \cap F_{\langle e,0\rangle}$.

CONSTRUCTION.

Stage s_0
Let $E_{0,s_0} = D_{0,s_0} = D_0$ and E_{i,s_0} be undefined if $i > 0$.
Stage $s + 1 > s_0$.
There are three cases.

Case (a): (Arrow a_1.) There exists i such that $E_{i,s}$ is defined and $d_{g(e,i),s} \neq d_{g(e,i),s+1}$. In this case, let i be least with this property. Let $E_{j,s+1}$ be undefined for $j \geq i$. Let $E_{j,s+1} = E_{j,s}$ for $j < i - 1$. Let $E_{i-1,s+1} = \hat{E}_{i-1,s}$.

Case (b): (Arrow a_2.) If i is least such that $E_{i,s}$ is not defined, then $W_{e,s+1} \cap F_{\langle e,i\rangle} \neq A_{s+1} \cap F_{\langle e,i\rangle}$. (Note that this case cannot happen if case (a) happens.) In this case define $E_{i,s+1} = D_{i,s+1}$ and $E_{j,s+1} = E_{j,s}$ for all $j < i$.

Case (c): Otherwise, let $E_{i,s+1} = E_{i,s}$ for all i such that $E_{i,s}$ is defined.

The following facts can easily be shown by induction on $s \geq s_0$. For each s there is i such that $E_{j,s}$ is defined for $j \leq i$ and undefined for $j > i$. Further, if $E_{j,s}$ is defined, then $W_{e,s} \cap F_{\langle e,j\rangle} \neq A_s \cap W_{\langle e,j\rangle}$ and $E_{j,s} \subseteq D_{j,s}$. Finally, if $E_{j,s+1}$ and $E_{j,s}$ are both defined then $E_{j,s+1} = E_{j,s}$ unless case (a) applies at stage $s + 1$ with $i = j + 1$. In this case $E_{j,s+1} \subset E_{j,s}$ and $|E_{j,s} - E_{j,s+1}| = 1$.

Fix i and let s_i be the least stage such that for all $s \geq s_i$, E_{i,s_i} is defined. (This is consistent with our definition of s_0.) At s_i, E_{i,s_i} is defined by case (b) (unless $i = 0$) and $E_{i,s_i} = D_{i,s_i}$. Thus $|E_{i,s_i}| = |F_{\langle e,i+1\rangle}| + 1$. Furthermore, $d_{g(e,i),s_i} = d_{g(e,i)}$ so that $D_{i,s_i} = D_i$. Now for $s \geq s_i$, $E_{i,s+1} \neq E_{i,s}$ only if $E_{i+1,s}$ is defined but becomes undefined at stage $s + 1$ because case (a) applies to $i + 1$. This can happen at most $|F_{\langle e,i+1\rangle}|$ times since it corresponds to the traversal of arrow a_1 in the $(i + 1)$-module. But if this happens at stage $s + 1$, we have that $E_{i,s+1} \subset E_{i,s}$ and $|E_{i,s} - E_{i,s+1}| = 1$. Thus we have that $\lim_s E_{i,s} = E_i$ exists and is nonempty. Therefore, if z_i is defined as in (29), we have that V as defined in (28) is a nonprincipal extension of T. We need only show that V is r. e.

We now give an enumeration procedure for V. For any s such that $E_{i,s}$ is defined, let

$$z_{i,s} = \bigvee(\bigcup_{j<i} \hat{E}_{j,s}) \vee \bigvee E_{i,s}.$$

$z_{i,s}$ is the natural approximation to z_i. Obviously $\lim_s z_{i,s} = z_i$. We claim that $V = (T \cup \{z_{i,s} \mid i \in N, s \geq s_0, \text{ and } z_{i,s} \text{ is defined}\})^*$. The inclusion from left to right follows from $\lim_s z_{i,s} = z_i$. To see the other inclusion, let i and $s \geq s_0$ be given such that $z_{i,s} \neq z_{i,s+1}$. Then by the definition of $z_{i,s}$, $E_{j,s} \neq E_{j,s+1}$ for some $j \leq i$. Let j be least with this property. Then $z_{j,s+1}$ is defined and $\vdash z_{j,s+1} \rightarrow z_{i,s}$. Thus we have that for all i, s, if $z_{i,s}$ is defined there is $j \leq i$ such that $\vdash z_{j,s+1} \rightarrow z_{i,s}$. This immediately implies that for every i, s such that $z_{i,s}$ is defined, there is $j \leq i$ such that $\vdash z_j \rightarrow z_{i,s}$. This establishes the desired inclusion. ■

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