THE THEORY OF THE ZIEGLER DEGREES

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Abstract. The Ziegler degrees were introduced to characterize definability in group theory. In [JLSta], the authors show that the first order-theory of the Ziegler degrees (as a partial order) is undecidable. We improve this result, showing that the theory of the Ziegler degrees is bi-interpretable with true second-order arithmetic.

1. Introduction

Ziegler reducibility arose in Martin Ziegler's investigation into the structure of existentially closed groups [Zi80]. In this landmark study, he showed that Ziegler reducibility between word problems of finitely generated groups characterizes quantifier-free parameter definability, which in turn characterizes relative omitting types in existentially closed groups. In addition, recent work shows that Ziegler reducibility also characterizes a notion of definability in subshifts [NCS26].

The restriction of Ziegler reducibility to c.e. sets had appeared in its own right earlier under the name of *quasireducibility* or *Q-reducibility*. Rogers attributes the definition of Q-reducibility to Tennenbaum (see [Ro67], p. 159), who studied it in the context of Post's problem. Belegradek formulated the analogue of Ziegler's theorem for finitely presented groups, which is in terms of Q-reducibility, in [Be74]. The degree theory of Q-reducibility has attracted renewed attention in recent years, for example in [BDS26, KNta].

We postpone a formal definition of Ziegler reducibility until Definition 2.1, but we describe the motivation here. Ziegler called the reducibility "star-reducibility" or "$\leq^*$" ([Zi80, Def. III.1.1]). Here we denote it as $\leq_Z$ in analogy with standard notation for reducibilities and in recognition of growing interest around it. The definition is unusual in that it is a "total reducibility" – in the sense that $A \leq_Z B$ implies that both the positive ($a \in A$) and the negative ($a \notin A$) information about A are "computed" from positive and negative information about the oracle B – but the reducibility is not symmetric in the positive and negative information. The positive information $a \in A$ is determined by finitely many positive bits of B (via *enumeration reducibility*, which will also be defined in Definition 2.1). On the other hand, the negative information $a \notin A$ is determined by finitely many positive bits and a *single negative bit*. This reflects the syntactic provability relation for atomic sentences in group theory over some parameter set: Let U and V be sets of words, and $T = \{u(\overline{g}) = 1, v(\overline{g}) \neq 1 \mid u \in U, v \in V\}$ be a set of atomic sentences

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in the letters $\overline{g}$. Let $T^+ = \{u(\overline{g}) = 1 \mid u \in U\}$ be the *positive part* of T and $T^- = \{v(\overline{g}) \neq 1 \mid v \in V\}$ be the *negative part* of T. Then for any word $w(\overline{g})$, we have

$$\begin{aligned} T \vdash w(\overline{g}) = 1 &\iff T^+ \vdash w(\overline{g}) = 1 \\ &\iff \text{there is a finite subset } T_0^+ \subseteq T^+ \text{ with } T_0^+ \vdash w(\overline{g}) = 1. \end{aligned}$$

On the other hand, for any word $w(\overline{g})$, we have

$$\begin{aligned} T \vdash w(\overline{g}) \neq 1 &\iff \text{for some } v(\overline{g}) \neq 1 \in T^-,\ T^+ \cup \{v(\overline{g}) \neq 1\} \vdash w(\overline{g}) \neq 1 \\ &\iff \text{there is a finite } T_0^+ \subseteq T^+ \text{ and } v(\overline{g}) \neq 1 \in T^- \text{ with} \\ &\qquad\quad T_0^+ \cup \{v(\overline{g}) \neq 1\} \vdash w(\overline{g}) \neq 1. \end{aligned}$$

The structure of Ziegler reducibility parallels this relationship.

In this paper, we investigate the complexity of the first-order theory of the Ziegler degrees. This was undertaken for the Turing degrees in classical work of Simpson, in which he showed that the theory is computably isomorphic to true second-order arithmetic [Si77]. Since the poset of the Turing degrees is interpretable in second-order arithmetic, this is a priori as complicated as possible. The first-order theory of the enumeration degrees is also recursively isomorphic to true second-order arithmetic, which again is as complicated as possible [SW97]. In [JLSta], the authors show that the first-order theory of the Ziegler degrees, is undecidable – indeed, they show this undecidability already holds for the $\forall\exists\forall$-theory. Write $\mathcal{D}_Z$ for the poset of Ziegler degrees. In this paper, we strengthen this result by showing that the first-order theory of $\mathcal{D}_Z$ is recursively isomorphic to true second-order arithmetic.

Following the framework of Nerode and Shore [NS80], we prove two theorems about the algebraic structure of the poset of the Ziegler degrees; namely, that every countable ideal has an exact pair and that every countable distributive lattice embeds as an initial segment.

Theorem 1.1. *Every countable ideal $\mathcal{I}$ in $\mathcal{D}_Z$ has an exact pair; i.e., a pair of degrees* $\deg(A)$ *and* $\deg(B)$ *such that* $\deg(C) \in \mathcal{I}$ *iff* $C \leq_Z A$ *and* $C \leq_Z B$.

Theorem 1.2. *Every countable distributive lattice embeds as an initial segment in $\mathcal{D}_Z$.*

The main theorem follows from these results via the work of [NS80].

Theorem 1.3 (Main Theorem)**.** *$Th(\mathcal{D}_Z)$ is bi-interpretable with true second-order arithmetic.*

The proof of Theorem 1.1 is essentially identical to the argument of the same result in the Turing degrees, in [Sp56], and is presented in Section 4. The proof of Theorem 1.2, given in Section 5, occupies the majority of the paper. Our proof follows the outline of [La68], which first proved the corresponding result in the Turing degrees. However, we make several modifications to guarantee that the reductions are Ziegler reductions. We start with the relevant definitions in Section 2.

2. Notation

Let D_u be the uth canonical finite set, and $(W_e)_e$ a computable enumeration of the c.e. sets.

Definition 2.1. Let $A, B \subseteq \omega$. Then:

(1) A is *enumeration reducible* to B, written $A \leq_e B$, if there is a c.e. set W_e such that
$$a \in A \iff \exists \langle a, u\rangle \in W_e\ (D_u \subseteq B).$$
In this case, we write $A = \Phi_e[B]$.

(2) $A \leq_e^1 B$ if there is a c.e. set W_i such that
$$a \in A \iff \exists \langle a, u, v\rangle \in W_i\ (D_u \subseteq B \wedge D_v \cap B = \emptyset \wedge |D_v| \leq 1).$$
In this case, we write $A = \Psi_i[B]$.

(3) A is *Ziegler reducible* to B, written $A \leq_Z B$, if
   (a) $A \leq_e B$, and
   (b) $\overline{A} \leq_e^1 B$.
   If $A = \Phi_e[B]$ and $\overline{A} = \Psi_i[B]$, then we write $A = (\Phi_e, \Psi_i)[B]$.

Compare the preceding definition to the motivation for Ziegler reducibility in the introduction: (1) captures that positive information about A is determined by positive information about B; and (2) (applied to $\overline{A}$) captures that negative information about A is determined by positive information and "one bit of negative information at a time" about B.

It is not hard to check that $\leq_Z$ is reflexive and transitive and hence induces an equivalence relation on 2^ω, which we denote by $\equiv_Z$. We write $\mathcal{D}_Z$ for the induced partial order $2^\omega / \equiv_Z$.

Notice that Ziegler reducibility is a common refinement of both enumeration- and Turing-reducibility, and is in turn refined by m-reducibility. It is not hard to show, although we do not do it here, that all of these refinements are strict.

Proposition 2.2. *$\mathcal{D}_Z$ is an upper semilattice with joins given by $A \oplus B = \{2a \mid a \in A\} \cup \{2b+1 \mid b \in B\}$ and least element $\mathbf{0}$. Furthermore, $\mathbf{0}$ consists exactly of the computable sets.*

3. The degree of $\mathcal{D}_Z$

We recall the main theorem of this paper:

Theorem 1.3. *$Th(\mathcal{D}_Z)$ is bi-interpretable with true second-order arithmetic.*

Our proof of Theorem 1.3 is based on the following classical result of Nerode and Shore.

Theorem 3.1 (Nerode/Shore [NS80, Theorem 3.2 and Remark 3.3])**.** *If $\mathcal{P}$ is a partial order satisfying the following three conditions:*

(3.1) *$\mathcal{P}$ is an upper semilattice with least element 0,*

(3.2) *Every countable ideal in $\mathcal{P}$ has an exact pair, and*

(3.3) *Every countable distributive lattice $\mathcal{L}$ with a least element is isomorphic to some initial segment of $\mathcal{P}$,*

then true full second-order arithmetic is 1-reducible to the first-order theory of $\mathcal{P}$.

Proof of Theorem 1.3. Clearly the first-order theory of $\mathcal{D}_Z$ is 1-reducible to second-order arithmetic. The other direction follows immediately by Theorem 3.1: Condition (3.1) that $\mathcal{D}_Z$ is an upper semilattice is Proposition 2.2, and Conditions (3.2) and (3.3) are Theorems 1.1 and 1.2 above, respectively. □

4. An exact pair theorem for the Ziegler degrees

In this section, we prove an exact pair theorem for the Ziegler degrees. Our proof follows Spector [Sp56], who proved the corresponding result for the Turing degrees. Case [Ca70, Theorem 4.3] showed that a similar proof works for the enumeration degrees.

Theorem 1.1. *Every countable ideal $\mathcal{I}$ in $\mathcal{D}_Z$ has an exact pair; i.e., a pair of degrees* $\deg(A)$ *and* $\deg(B)$ *such that* $\deg(C) \in \mathcal{I}$ *iff* $C \leq_Z A$ *and* $C \leq_Z B$*.*

Proof. Let $\mathcal{I}$ be a countable ideal in $\mathcal{D}_Z$. Let $A_0 \leq_Z A_1 \leq_Z \cdots$ represent a cofinal sequence of degrees in $\mathcal{I}$. Let $A = \bigoplus_n A_n$ (viewing A as a function from ω to $\{0,1\}$). We will build B to satisfy the following requirements:

$$R_{\langle e,i,j,k\rangle} : (\Phi_e, \Psi_i)[A] = (\Phi_j, \Psi_k)[B] = X \implies \exists n\, (X \leq_Z A_n)$$

and ensure at the same time that $A_n \leq_Z B$ for all n.

(We note that Posner's trick does not seem to work in the Ziegler degrees – one would need to check negative information about the oracle in order to determine which computation to follow, which breaks Ziegler reducibility's very parsimonious use of negative information.)

The strategy is to code each A_n into the nth column of B, up to a finite set. The construction will fix B (viewed as a function from ω to $\{0,1\}$) on $\omega^{[<s]}$ by stage s. At stage s, we will extend B on $\omega^{[\geq s]}$ by a finite amount to ensure that $(\Phi_e, \Psi_i)[A] \neq (\Phi_j, \Psi_k)[B]$ if possible. If not, then, using that B is fixed on $\omega^{[<s]}$ and $*$-computable from A_s, we will have that if $(\Phi_e, \Psi_i)[A]$ is a valid Ziegler reduction then $(\Phi_e, \Psi_i)[A] \leq_Z A_s$.

Construction.
Stage $s = 0$: We set $B_0 = \emptyset$.
Stage $s + 1$: Suppose that B_s has been defined such that $\mathrm{dom}(B_s) \supseteq \omega^{[<s]}$ and $B_s =^* A^{[<s]}$. Consider $R_s = R_{\langle e,i,j,k\rangle}$. If there are a finite partial function τ and $x \in \omega$ with $\mathrm{dom}(\tau) \cap \mathrm{dom}(B_s) = \emptyset$ and $(\Phi_e, \Psi_i)[A](x) \neq (\Phi_j, \Psi_k)[B_s \cup \tau](x)$, then set $C_{s+1} = B_s \cup \tau$. Else set $C_{s+1} = B_s$. We now need to fill the rest of the sth column with A_s without disturbing the diagonalization. Let A_s^0 be the subset of the function A_s with domain $\omega \setminus \mathrm{dom}(C_{s+1})^{[s]}$. Then set $B_{s+1} = C_{s+1} \cup (\{s\} \times A_s^0)$.

Verification. Firstly, since $B^{[s]} =^* A_s$, we have that $A_s \leq_Z B$ for every s (non-uniformly in s).

Moreover, if τ is found at stage $s + 1$, then R_s is immediately satisfied since $B_s \cup \tau$ is preserved in B and thus $(\Phi_e, \Psi_i)[A] \neq (\Phi_j, \Psi_k)[B]$. If no τ is found at stage $s + 1$ and $(\Phi_e, \Psi_i)[A] = (\Phi_j, \Psi_k)[B] = X$, then we argue that $X \leq_Z A_s$. Indeed, if $x \in X = (\Phi_e, \Psi_i)[A]$ and we did not find τ at stage $s + 1$, then any convergent computation of (Φ_j, Ψ_k) from an extension of B_s must put x into X. Furthermore, since $x \in (\Phi_j, \Psi_k)[B]$, there is some such convergent computation. Thus, $X \leq_e B_s$ via the set $\{\langle x, D'\rangle \mid \exists \langle x, D\rangle \in \Phi_j\ (D' = D \cap \mathrm{dom}(B_s))\}$. On the other hand, $x \notin X$ iff $\Psi_i[A](x) = 1$. Again, since no τ was found at stage $s + 1$, any convergent computation of (Φ_j, Ψ_k) from an oracle extending B_s must agree with $x \notin X$; moreover, such a convergent computation exists since $\Psi_k[B](x) \downarrow = 1$. As before, this implies that $\overline{X} \leq_e^1 B_s$. Hence, using that if $Y =^* Z$, then $Y \leq_e Z$ and $Y \leq_e^1 Z$, we have that

$$X \leq_e B_s \equiv_e B^{[<s]} \equiv_e A_0 \oplus \cdots \oplus A_{s-1}$$

and

$$X \leq_e^1 A_0 \oplus \cdots \oplus A_{s-1}.$$

Thus,

$$X \leq_Z A_0 \oplus \cdots \oplus A_{s-1} \leq_Z A_s,$$

as required. □

5. Embedding all countable distributive lattices with least element

In this section, we adapt Lachlan's argument that every countable distributive lattice with least element embeds as an initial segment of the degree structure $\mathcal{D}_T$ of the Turing degrees to our setting:

Theorem 1.2. *Every countable distributive lattice L with least element embeds as an initial segment of $\mathcal{D}_Z$.*

Proof. We follow Lachlan's proof in [La68], using somewhat updated notation so that the reader can follow more easily.

Let L be a countable distributive lattice with least element 0_L. Without loss of generality, we may assume that it also has a greatest element 1_L. We let $(L_s)_{s<\omega}$ be an increasing approximation to L such that L_0 is the two-element lattice and L_{s+1} is a *simple extension* of L_s, i.e., L_{s+1} is generated, under join and meet, by L_s and a single new element a_s.

5.1. **Definitions.** When considering a finite (distributive) sublattice L' of L, we will fix the following lattice-theoretic notation:

Definition 5.1. Let L' be a finite distributive lattice and a a (nonzero) join-irreducible element in L'. Then we let $S_a = \{b \in L' \mid b \not\geq_{L'} a\}$. (By distributivity, S_a has a greatest element b_a.)

As usual, we will produce a degree $\mathbf{f}_1 = \mathbf{f}_{1_L}$, represented by some $f_1 = f_{1_L} \in 2^\omega$, such that $\mathcal{D}_Z(\leq \mathbf{f}_1) \cong L$ by forcing with f-trees. When embedding a finite distributive lattice L', we may fix in advance a representation of L' as a subset of a finite Boolean algebra, thought of as a subset of $\mathcal{P}(S)$ for some initial segment S of ω. Here, however, we will have to modify the representation as L_s grows. This will be kept track of in the forcing conditions.

We first recall some definitions from Jacobsen-Grocott/Lempp/Scott [JLSta].

Definition 5.2.
(1) An *f-tree* is a map $\mathcal{T} : 2^{<\omega} \to 2^{<\omega}$ such that for all $\pi, \rho \in 2^{<\omega}$, $\pi \subset \rho$ iff $\mathcal{T}(\pi) \subset \mathcal{T}(\rho)$, and $\pi <_{\text{lex}} \rho$ iff $\mathcal{T}(\pi) <_{\text{lex}} \mathcal{T}(\rho)$. (We also allow the *degenerate f-tree* $\mathcal{T} : 2^{<\omega} \to 2^{<\omega}$ where $\mathcal{T}(\rho) = \langle 0^{|\rho|} \rangle$ for all $\rho \in 2^{<\omega}$.)
(2) The *full binary f-tree* is the f-tree $\mathcal{U}$ defined by setting $\mathcal{U}(\rho) = \rho$ for all $\rho \in 2^{<\omega}$.
(3) We denote the set of *infinite paths* through an f-tree $\mathcal{T}$ by $[\mathcal{T}] = \{f \in 2^\omega \mid \exists^\infty \rho\, (\mathcal{T}(\rho) \subset f)\}$.
(4) A *sub-f-tree* of an f-tree $\mathcal{T}$ is an f-tree $\mathcal{S}$ with $\text{ran}(\mathcal{S}) \subseteq \text{ran}(\mathcal{T})$. We then write $\mathcal{S} \subseteq \mathcal{T}$. (In particular, the degenerate f-tree is a sub-f-tree of every f-tree $\mathcal{T}$ with $0^\omega \in [\mathcal{T}]$.)
(5) An f-tree $\mathcal{T}$ *has no explicit $\langle e, i\rangle$-contradictions* if, for every $\tau \in \text{ran}(\mathcal{T})$ and every $x \in \omega$, it is not the case that $\Phi_e^\tau(x) = 1$ and $\Psi_i^\tau(x) = 1$ (i.e., x is not enumerated into both the set being computed and its complement).

(6) A *very uniform* f-tree is an f-tree $\mathcal{T}$ satisfying:
- For all $\pi, \rho \in 2^{<\omega}$, $|\pi| = |\rho|$ implies $|\mathcal{T}(\pi)| = |\mathcal{T}(\rho)|$;
- For all $l \in \omega$ and $k < 2$, there is τ_k^l such that for all $\rho \in 2^l$, $\mathcal{T}(\rho\widehat{\ }\langle k\rangle) = \mathcal{T}(\rho)\widehat{\ }\langle \tau_k^l\rangle$; and
- for all $l \in \omega$, $k < 2$, and $x < |\tau_k^l|$, if $\tau_0^l(x) \neq \tau_1^l(x)$ then $\tau_0^l(x) < \tau_1^l(x)$. (Thus, in particular, any disagreements on any fixed τ_k^l "agree" with each other.)

(7) τ_0 and τ_1 *$\langle e, i\rangle$-split σ on $\mathcal{T}$* if $\tau_0, \tau_1, \sigma \in \operatorname{ran}(\mathcal{T})$; $\tau_0, \tau_1 \supset \sigma$; and

$$x \notin (\Phi_e, \Psi_i)[\tau_0] \text{ and } x \in (\Phi_e, \Psi_i)[\tau_1].$$

If in addition $\tau_0 = \mathcal{T}(\pi_0)$, $\tau_1 = \mathcal{T}(\pi_1)$, $|\pi_0| = |\pi_1|$, and π_0 and π_1 differ in exactly one bit j, say, then we say that τ_0 and τ_1 *minimally $\langle e, i\rangle$-split σ*. In a context with a fixed $n < \omega$, we may in addition specify that the minimal splitting is *via j'*, where $j' \in [0, n)$ and $j' \equiv j \pmod n$. (In the proof, we will be working with a lattice embedding $\eta : L' \to \mathcal{P}([0, n))$ for some n. This will be where n arises from.)

(8) A very uniform f-tree $\mathcal{T}$ is *$\langle e, i\rangle$-splitting* if, for every $\rho \in \operatorname{dom}(\mathcal{T})$, $\mathcal{T}(\rho\widehat{\ }\langle 0\rangle)$ and $\mathcal{T}(\rho\widehat{\ }\langle 1\rangle)$ $\langle e, i\rangle$-split $\mathcal{T}(\rho)$ on $\mathcal{T}$.

We first note the following key observation.

Lemma 5.3. *Let $\mathcal{T}$ be a very uniform f-tree which has no explicit $\langle e, i\rangle$-contradictions, and suppose there are $\tau_0 = \mathcal{T}(\pi_0)$ and $\tau_1 = \mathcal{T}(\pi_1)$ which $\langle e, i\rangle$-split σ on $\mathcal{T}$. Then there are $\tau_0' = \mathcal{T}(\pi_0')$ and $\tau_1' = \mathcal{T}(\pi_1')$ which minimally $\langle e, i\rangle$-split σ on $\mathcal{T}$. Furthermore, if $|\tau_0| = |\tau_1|$, then π_0' and π_1' can be chosen to differ at some j at which π_0 and π_1 also differ.*

Proof. Suppose that τ_0 and τ_1 $\langle e, i\rangle$-split σ on $\mathcal{T}$ at some argument x, say, by symmetry, $x \in \Psi_i[\tau_0]$ via a finite set $D \subseteq \tau_0$ and some y with $\tau_0(y) = 0$, and $x \in \Phi_e[\tau_1]$ via a finite set $E \subseteq \tau_1$. Without loss of generality, we may assume that $|\pi_0| = |\pi_1|$ and so $|\tau_0| = |\tau_1|$. But then $\tau_1(y)$ is defined, and furthermore, $\tau_0(y) = 0$ is caused by $\pi_0(j) = 0$ for some j. Now if $\tau_1(y) = 0$, then we can set $\pi'(k) = \max(\pi_0(k), \pi_1(k))$ for all $k < |\pi_0|$, and so $\tau' = \mathcal{T}(\pi')$ will witness an $\langle e, i\rangle$-contradiction via x. Thus we must have $\tau_1(y) = 1$, and we can set $\pi_0'(k) = \max(\pi_0(k), \pi_1(k))$ for all $k \neq j$, and $\pi_1'(k) = \max(\pi_0(k), \pi_1(k))$ for all $k < |\pi_0|$. But then $\tau_0' = \mathcal{T}(\pi_0')$ and $\tau_1' = \mathcal{T}(\pi_1')$ minimally $\langle e, i\rangle$-split σ on $\mathcal{T}$ as desired. $\square$

We also need the following notation.

Notation 5.4. Given a string $\rho \in 2^{<\omega}$, integers $k \geq l > 0$ and a strictly increasing sequence $(h_0, h_1, \ldots, h_{l-1})$ of integers $< k$, we define $\rho[k; h_0, h_1, \ldots, h_{l-1}] \in 2^{<\omega}$ by setting, for $p \in \omega$ and $q < l$:

$$\rho[k; h_0, h_1, \ldots, h_{l-1}](pl + q) = \rho(pk + h_q),$$

and similarly for infinite paths f.

So $\rho[k; h_0, h_1, \ldots, h_{l-1}]$ is obtained from ρ by deleting, from every k consecutive bits, the bits not in positions $h_0, h_1, \ldots, h_{l-1}$. Note that the map $\rho \mapsto \rho[k; h_0, h_1, \ldots, h_{l-1}]$ is surjective onto $2^{<\omega}$.

To illustrate, consider the following example: Suppose $\rho = (0, 1, 1, 1, 0, 1, 0, 0, 1, 0)$ and let $(0, 1, 3)$ be a the strictly increasing sequence of integers. Then

$$\rho[5; 0, 1, 3] = (\mathbf{0}, \mathbf{1}, 1, \mathbf{1}, 0, \mathbf{1}, \mathbf{0}, 0, \mathbf{1}, 0) = (0, 1, 1, 1, 0, 1).$$

5.2. **The forcing partial order.**

Definition 5.5. Let $\mathbb{P}$ be the set of pairs or *conditions* $(\eta, \mathcal{T})$ where

(1) η is a lattice embedding from a finite sublattice $L' \subseteq L$ into $\mathcal{P}(\omega)$ (ordered by inclusion) where $\eta(0_L) = \emptyset$ and $\eta(1_L) = [0, n)$ for some $n \in \omega$.
(2) $\mathcal{T}$ is a map from L' to computable very uniform f-trees such that $\mathcal{T}(a)$ is degenerate iff $a = 0_L$.

Before we define the ordering on $\mathbb{P}$, we introduce some further notation: Given $(\eta, \mathcal{T}) \in \mathbb{P}$ and $a \geq_{L'} b$, we need a map $\Theta_{a,b}$ which maps nodes and paths of $\mathcal{T}(a)$ to nodes and paths of $\mathcal{T}(b)$. This will ultimately guarantee the Ziegler reduction between $\mathbf{f}_a$ and $\mathbf{f}_b$, the degrees corresponding to a and b respectively.

Notation 5.6. Let $(\eta, \mathcal{T}) \in \mathbb{P}$ and $L' \subseteq L$ be the domain of η. Suppose that $\eta(1_L) = [0, n)$. For $a, b \in L'$ with $a \geq_{L'} b > 0_L$, we have $\eta(a) \supseteq \eta(b)$ are nonempty subsets of $[0, n)$; say, $\eta(a) = \{i_0, i_1, \dots, i_{k-1}\}$ and $\eta(b) = \{j_0, j_1, \dots, j_{l-1}\} = \{i_{h_0}, i_{h_1}, \dots, i_{h_{l-1}}\}$, all listed in increasing order. Then we define

$$\Theta_{a,b}(\mathcal{T}(a)(\rho)) = \mathcal{T}(b)\left(\rho[k; h_0, h_1, \dots, h_{l-1}]\right).$$

This is a $\subset$-preserving map from $\mathrm{ran}(\mathcal{T}(a))$ onto $\mathrm{ran}(\mathcal{T}(b))$ which naturally induces a surjection $[\Theta_{a,b}] : [\mathcal{T}(a)] \to [\mathcal{T}(b)]$. For each $a \in L'$, we define

$$\Theta_{a,0_L}(\mathcal{T}(a)(\rho)) = \mathcal{T}(0_L)(\rho),$$

which induces the trivial surjection $[\Theta_{a,0_L}] : [\mathcal{T}(a)] \to [\mathcal{T}(0_L)]$.

For $a \in L'$ and $f_1 = f_{1_L} \in [\mathcal{T}(1_L)]$, we set $f_a = [\Theta_{1_L,a}](f_1)$. Similarly, for $\sigma = \sigma_{1_L} = \mathcal{T}(1_L)(\rho)$, we define $\sigma_a = \Theta_{1_L,a}(\sigma)$.

Write Θ for the collection of such maps corresponding to $(\eta, \mathcal{T})$. (Analogously, we write Θ^* for the collection of such maps corresponding to $(\eta^*, \mathcal{T}^*)$, and similarly for other sub/superscripts.)

Definition 5.7. The ordering $\preceq$ on $\mathbb{P}$ is defined by letting $(\eta^*, \mathcal{T}^*) \preceq (\eta, \mathcal{T})$ iff

(1) $\mathrm{dom}(\eta^*) \supseteq \mathrm{dom}(\eta)$ and for every $a \in \mathrm{dom}(\eta)$, we have $\eta(a) = \eta^*(a) \cap \eta(1_L)$,
(2) $\mathcal{T}^*(a) \subseteq \mathcal{T}(a)$ for every $a \in \mathrm{dom}(\eta)$, and
(3) for all $a \geq_{L'} b$ in $L' = \mathrm{dom}(\eta)$, we have $[\Theta^*_{a,b}] = [\Theta_{a,b}] \restriction [\mathcal{T}^*(a)]$.

We note the following obvious properties, all immediate by Notation 5.4 and our definition of f-trees:

Remark 5.8. In the notation of Definition 5.5, we have:

(1) For $a \geq_{L'} b \geq_{L'} c >_{L'} 0_L$, we have $\Theta_{a,c} = \Theta_{b,c} \circ \Theta_{a,b}$ (when considered as maps on strings).
(2) For $a \geq_{L'} b \geq_{L'} c \geq_{L'} 0_L$, we have $[\Theta_{a,c}] = [\Theta_{b,c}] \circ [\Theta_{a,b}]$ (when considered as maps on paths).
(3) Recall that $\eta(1_L) = [0, n)$. For a (nonzero) join-irreducible element $a \in L'$, fix b_a as in Definition 5.1. Then $\eta(a) \not\subseteq \eta(b_a)$, and for each $i \in \eta(a) - \eta(b_a)$, each $j \in \omega$ with $j \equiv i \pmod{n}$ and each $\pi \in 2^j$, we have $\Theta_{1,b_a}(\mathcal{T}(1_L)(\pi^\frown\langle 0\rangle)) = \Theta_{1,b_a}(\mathcal{T}(1_L)(\pi^\frown\langle 1\rangle))$, but $\Theta_{1,a}(\mathcal{T}(1_L)(\pi^\frown\langle 0\rangle)) \neq \Theta_{1,a}(\mathcal{T}(1_L)(\pi^\frown\langle 1\rangle))$.

We note that Remark 5.8(3) takes the place of the notion of "alternating f-trees" in Jacobsen-Grocott/Lempp/Scott [JLSta].

The key property of $\Theta_{a,b}$ is the following:

Lemma 5.9. *Let $(\eta, \mathcal{T})$ be a condition with $L' = \operatorname{dom}(\eta)$ and $a \geq_{L'} b$ in L'. Suppose $f_a \in [\mathcal{T}(a)]$ and $f_b \in [\mathcal{T}(b)]$ with $[\Theta_{a,b}](f_a) = f_b$. Then $f_b \leq_Z f_a$.*

In fact, the proof will show that indeed $f_b \leq_m f_a$.

Proof. To check whether $x \in f_b$, we first check whether there is $i < 2$ such that $\sigma(x) = i$ for every σ on $\mathcal{T}(b) \upharpoonright x + 1$, in which case $f_b(x) = i$. (Notice that since $\mathcal{T}(b)$ is computable, this can be "built into" the reduction and thus does require use of the oracle.) Else, there is some $\rho \in 2^{<\omega}$ such that $f_b(x) = i$ iff f_b extends $\mathcal{T}(b)(\rho^\frown\langle i\rangle)$. In this case, there is $\pi \in 2^{<\omega}$ with $\Theta_{a,b}(\mathcal{T}(a)(\pi^\frown\langle i\rangle)) = \mathcal{T}(b)(\rho^\frown\langle i\rangle)$. Since $\mathcal{T}(a)$ is very uniform, there is some y such that $f_a(y) = i$ iff f_a extends $\mathcal{T}(a)(\pi^\frown\langle i\rangle)$. Thus, summarizing the argument above, we have that $x \in f_b$ iff $y \in f_a$. So $f_b \leq_Z f_a$. □

5.3. **Four propositions.** Here we lay out the four propositions that form the basis of Lachlan's proof, adapted to our setting. The first allows us to "stretch" out the f-tree so as to allow extending the finite lattice.

Proposition 5.10 (following Lachlan [La68, Proposition 1])**.** *Let $(\eta, \mathcal{T})$ be a condition. Then there is a condition $(\eta^*, \mathcal{T}^*) \preceq (\eta, \mathcal{T})$ such that*

(1) $\operatorname{dom}(\eta^*) = \operatorname{dom}(\eta)$ and $\forall S \in \operatorname{ran}(\eta^*)\,(|S|$ is even) and
(2) $\mathcal{T}^* = \mathcal{T}$.

Proof. For (1), define $\eta^*(a) = \eta(a) \cup \{j + n \mid j \in \eta(a)\}$ where $n = |\eta(1_L)|$. Then (2) follows easily. To finish, we check that $[\Theta^*_{a,b}] = [\Theta_{a,b}] \upharpoonright [\mathcal{T}^*(a)]$. Suppose that $|\eta(a)| = k$ and $|\eta(b)| = l$, and $\eta(b)$ consists of the h_0th, …, h_{l-1}th elements of $\eta(a)$. Then $|\eta^*(a)| = 2k$ and $|\eta^*(b)| = 2l$, and $\eta^*(b)$ consists of the h_0th, …, h_{l-1}th, h_lth, …, h_{2l-1}th elements of $\eta^*(a)$. By the definition of η^*, for each $q \geq l$ we have that $h_q = h_{q-l} + k$. Hence, for every string ρ, we have that $\rho[k; h_0, \dots h_{l-1}] = \rho[2k; h_0, \dots h_{l-1}, h_l, h_{2l-1}]$ and thus, since $\mathcal{T}^* = \mathcal{T}$, we have that $[\Theta^*_{a,b}] = [\Theta_{a,b}] \upharpoonright [\mathcal{T}^*(a)]$. □

We can now specify how we modify the f-tree when we expand the lattice.

Proposition 5.11 (following Lachlan [La68, Proposition 2])**.** *Let $(\eta, \mathcal{T})$ be a condition such that $\forall S \in \operatorname{ran}(\eta)\,(|S|$ is even). Let L' be a simple extension of $\operatorname{dom}(\eta)$. Then there is a condition $(\eta^*, \mathcal{T}^*) \preceq (\eta, \mathcal{T})$ such that*

(1) $\operatorname{dom}(\eta^*) = L'$ and $\eta^* \supseteq \eta$ and
(2) $\mathcal{T}^* \upharpoonright \operatorname{dom}(\eta) = \mathcal{T}$.

Proof. Say L' generated by $\operatorname{dom}(\eta)$ and $a' \in L'$. Let

$$a^- = \bigvee \{a \in \operatorname{dom}(\eta) \mid a <_{L'} a'\} \text{ and}$$
$$a^+ = \bigwedge \{a \in \operatorname{dom}(\eta) \mid a >_{L'} a'\}.$$

Then $|\eta(a^-)| + 2 \leq |\eta(a^+)|$, so fix $j \in \eta(a^+) - \eta(a^-)$. Setting $\eta^*(a') = \eta(a^-) \cup \{j\}$ determines η^* uniquely to satisfy Definition 5.5 and (1). Furthermore, since $\eta^*(a) = \eta(a)$ and $\mathcal{T}^*(a) = \mathcal{T}(a)$ for every $a \in \operatorname{dom}(\eta)$, we have that $[\Theta^*_{a,b}] = [\Theta_{a,b}] \upharpoonright [\mathcal{T}^*(a)]$ whenever $a \geq_{L'} b$ in $\operatorname{dom}(\eta)$. □

Next, we show that we can diagonalize or force partiality.

Proposition 5.12 (following Lachlan [La68, Proposition 3])**.** *Let $(\eta, \mathcal{T})$ be a condition; let $a, b \in \operatorname{dom}(\eta) = L'$ with $a \not\leq_{L'} b$; and let $e, i \in \omega$. Then there is a condition $(\eta^*, \mathcal{T}^*) \preceq (\eta, \mathcal{T})$ such that*

(1) $\eta^* = \eta$,
(2) for every $c \in \operatorname{dom}(\eta)$, $\mathcal{T}^*(c) \subseteq \mathcal{T}(c)$, and
(3) for every $f_1 \in [\mathcal{T}(1_L)]$, we have $(\Phi_e, \Psi_i)[f_b] \neq f_a$.

Proof. Recall our notation that $\sigma_a = \Theta_{1_L,a}(\sigma)$. Let $k = |\eta(a)|$ and $l = |\eta(b)|$. First check whether there are $x < k$ and $\sigma \in \operatorname{ran}(\mathcal{T}(a))$ such that

- $x \in \Phi_e[\sigma]$ and $x \in \Psi_i[\sigma]$, or
- for all $\tau \in \operatorname{ran}(\mathcal{T}(a))$ extending σ, neither $x \in \Phi_e[\tau]$ nor $x \in \Psi_i[\tau]$.

If so, then fix π with $\sigma = \big(\mathcal{T}(1_L)(\pi)\big)_a$, where without loss of generality we may assume that $|\pi|$ is a multiple of n. Then we set $\mathcal{T}^*(1_L)(\rho) = \mathcal{T}(1_L)(\pi\frown\rho)$ for all $\rho \in 2^{<\omega}$; and now the condition that $[\Theta^*_{1_L,c}] = [\Theta_{1_L,c}] \restriction [\mathcal{T}^*(1_L)]$ fully determines $\Theta^*(c)$ for all $c \in L'$. Note that since $|\pi|$ is a multiple of n, we have that $[\Theta^*_{c,d}] = [\Theta_{c,d}] \restriction [\mathcal{T}^*(c)]$ whenever $c \geq_{L'} d$ in L'. Moreover, now $(\Phi_e, \Psi_i)[f_b](x)$ is not a valid computation for every f_1 in $[\mathcal{T}(1_L)]$, so in particular is not equal to $f_a(x)$.

Otherwise, fix $p > 0$ and $\rho \in 2^{pl}$ such that $(\Phi_e, \Psi_i)[\mathcal{T}(b)(\rho)] \restriction |\mathcal{T}(a)(\langle 0^k \rangle)|$ is defined, i.e., that for each $x < |\mathcal{T}(a)(\langle 0^k \rangle)|$, $x \in \Phi_e(\mathcal{T}(b)(\rho))$ or $x \in \Psi_i(\mathcal{T}(b)(\rho))$ (and not both). Since $a \not\leq_{L'} b$, there is some $j \in \eta(a) - \eta(b)$. Hence we can find $\pi', \pi'' \in 2^{<\omega}$ such that $\pi'_b = \pi''_b$ and so

$$\Theta_{1_L,b}(\mathcal{T}(1_L)(\pi')) = \Theta_{1_L,b}(\mathcal{T}(1_L)(\pi'')) = \mathcal{T}(b)(\rho).$$

But $\pi'(j) \neq \pi''(j)$, so $\pi'_a \neq \pi''_a$ and thus

$$\Theta_{1_L,a}(\mathcal{T}(1_L)(\pi')) \restriction |\mathcal{T}(a)(\langle 0^k \rangle)| \neq \Theta_{1_L,a}(\mathcal{T}(1_L)(\pi'')) \restriction |\mathcal{T}(a)(\langle 0^k \rangle)|.$$

Again we may assume without loss of generality that $|\pi'| = |\pi''|$ is a multiple of n. And now we can set $\pi = \pi'$ or $\pi = \pi''$ such that $\sigma_1 = \mathcal{T}(1_L)(\pi)$ while ensuring that $\sigma_b = \mathcal{T}(b)(\rho)$ as well as

$$(\Phi_e, \Psi_i)[\sigma_b] \restriction |\mathcal{T}(a)(\langle 0^k \rangle)| \neq \sigma_a \restriction |\mathcal{T}(a)(\langle 0^k \rangle)|,$$

which will establish (3).

As before, we set $\mathcal{T}^*(1_L)(\nu) = \mathcal{T}(1_L)(\pi\frown\nu)$ for all $\nu \in 2^{<\omega}$; and again the condition that $[\Theta^*_{1_L,c}] = [\Theta_{1_L,c}] \restriction [\mathcal{T}^*(1_L)]$ fully determines $\mathcal{T}^*(c)$ for all $c \in L'$. This implies (2) and $[\Theta^*_{a,b}] = [\Theta_{a,b}] \restriction [\mathcal{T}^*(a)]$, using that $|\pi|$ is a multiple of n. ☐

The last proposition is the core of the argument, allowing us to conclude that we can force the degrees below **f** (the path we are building in $\mathcal{T}(1_L)$) to form exactly the lattice L – and not a superstructure.

Proposition 5.13 (following Lachlan [La68, Proposition 4])**.** *Let $(\eta, \mathcal{T})$ be a condition with $\operatorname{dom}(\eta) = L'$, and let $e, i \in \omega$. Then there are a condition $(\eta^*, \mathcal{T}^*) \preceq (\eta, \mathcal{T})$ and $a_0 \in L'$ such that*

(1) $\eta^* = \eta$,
(2) for every $a \in \operatorname{dom}(\eta)$, $\mathcal{T}^*(a) \subseteq \mathcal{T}(a)$, and
(3) there is $a_0 \in L$ such that for every $f_1 \in [\mathcal{T}^*(1_L)]$, if $(\Phi_e, \Psi_i)[f_1]$ is a valid computation, then $(\Phi_e, \Psi_i)[f_1] \equiv_Z f_{a_0}$.

Proof. Fix n so that $\eta(1_L) = [0, n)$.

We distinguish two cases: If there are $x \in \omega$ and $\pi \in 2^{<\omega}$ such that, for $\sigma = \mathcal{T}(1_L)(\pi)$,

- $x \in \Phi_e[\sigma]$ and $x \in \Psi_i[\sigma]$, or
- for all $\tau \in \operatorname{ran}(\mathcal{T}(a))$ extending σ, neither $x \in \Phi_e[\tau]$ nor $x \in \Psi_i[\tau]$,

then we may again assume without loss of generality that $|\pi|$ is a multiple of n. As before, we set $\mathcal{T}^*(1_L)(\rho) = \mathcal{T}(1_L)(\pi\hat{\ }\rho)$ for all $\rho \in 2^{<\omega}$; and again the condition that $[\Theta^*_{1_L,a}] = [\Theta_{1_L,a}] \restriction [\mathcal{T}^*(1_L)]$ fully determines $\mathcal{T}^*(a)$ for all $a \in L'$.

Otherwise, we will first need to refine some notions from Definition 5.2 as follows:

Definition 5.14. Suppose that $\tau = \mathcal{T}(1_L)(\pi)$ and $\tau' = \mathcal{T}(1_L)(\pi')$ such that $|\tau| = |\tau'|$, and that τ and τ' $\langle e, i\rangle$-split $\sigma = \mathcal{T}(1_L)(\rho)$ on $\mathcal{T}(1_L)$.

Then we say τ and τ' *differ in* $a \in L'$ if $\tau_a = \Theta_{1_L,a}(\tau) \neq \Theta_{1_L,a}(\tau') = \tau'_a$ (and so are incomparable since they have the same length); otherwise, τ and τ' *agree in* $a \in \eta(1_L)$. We say that τ and τ' *differ minimally in* $a \in L'$ if they differ in $a \in L'$ and this a is minimal such in L'.

For each $\sigma \in \operatorname{ran}(\mathcal{T}(1_L))$, we set

$$\mathcal{C}(\sigma) = \{a \in L' \mid \exists \tau, \tau' \, (\tau, \tau' \text{ minimally } \langle e, i\rangle\text{-split } \sigma \text{ on } \mathcal{T}(1_L) \text{ and minimally differ in } a)\}.$$

Then we have the following

Claim 5.15. *(1) $\sigma \subset \tau$ in $\operatorname{ran}(\mathcal{T}(1_L))$ implies $\mathcal{C}(\sigma) \supseteq \mathcal{C}(\tau)$.*

(2) There is $\sigma_0 \in \operatorname{ran}(\mathcal{T}(1_L))$ such that $\mathcal{C}(\sigma_0) = \mathcal{C}(\sigma)$ for all $\sigma \supseteq \sigma_0$ in $\operatorname{ran}(\mathcal{T}(1_L))$.

(3) Let σ_0 be as in (2), and $a_0 = \bigvee \mathcal{C}(\sigma_0)$ (with $a_0 = 0_L$ if $\mathcal{C}(\sigma_0) = \emptyset$). Then there are no $\tau, \tau' \supset \sigma_0$ with $\tau, \tau' \in \operatorname{ran}(\mathcal{T}(1_L))$, $|\tau| = |\tau'|$, $\tau_{a_0} = \tau'_{a_0}$, and which $\langle e, i\rangle$-split σ on $\mathcal{T}(1_L)$.

Proof. Item (1) is obvious, and so is (2) since L' is finite.

For (3), assume for a contradiction that there are τ and τ' with $\tau_{a_0} = \tau'_{a_0}$, and such that τ and τ' $\langle e, i\rangle$-split σ_0 on $\mathcal{T}(1_L)$. By Lemma 5.3, we may in addition assume that they minimally $\langle e, i\rangle$-split σ_0 via some $j < n$. Let $a \in L'$ be such that τ and τ' differ minimally in a. Then $j \in \eta(a)$, but this implies $a \in \mathcal{C}(\sigma_0)$. Since $\tau_{a_0} = \tau'_{a_0}$, we have $j \notin \eta(a_0)$. So $a \not\leq_{L'} a_0$, contradicting our choice of a_0. $\square$

We can now define $\mathcal{T}^*$, using the values of a_0 and σ_0 above, where we again assume that $|\pi|$ is a multiple of n for $\sigma_0 = \mathcal{T}(1_L)(\pi)$. We first set $\mathcal{T}^*(1_L)(\langle\rangle) = \sigma_0$. We then proceed by recursion on $m = l \cdot n + j$: Assume that we have already defined $\mathcal{T}^*(1_L)(\rho)$ for all $\rho \in 2^m$, and that for any such ρ, we have defined $\mathcal{T}^*(1_L)(\rho) = \mathcal{T}(1_L)(\pi(\rho))$, where $|\pi(\rho)|$ is a multiple of n. We now distinguish two cases:

Case 1: $j \notin \eta(a_0)$: Then we set, for all $\rho \in 2^m$ and all $k < 2$,

$$\mathcal{T}^*(1_L)(\rho\hat{\ }\langle k\rangle) = \mathcal{T}(1_L)(\pi(\rho)\hat{\ }\langle 0^j\rangle\hat{\ }\langle k\rangle\hat{\ }\langle 0^{n-j-1}\rangle).$$

Case 2: $j \in \eta(a_0)$: Then $j \in \eta(a)$ for some $a \in \mathcal{C}(\sigma_0)$. Hence there are π_0 and π_1 such that $\mathcal{T}(1_L)(\pi(\langle 0^m\rangle)\hat{\ }\pi_0)$ and $\mathcal{T}(1_L)(\pi(\langle 0^m\rangle)\hat{\ }\pi_1)$ minimally $\langle e, i\rangle$-split $\mathcal{T}^*(\langle 0^m\rangle)$ in $\mathcal{T}(1_L)$ and differ minimally in a. Extending if necessary, we may further assume that $|\pi_0| = |\pi_1|$ is divisible by n.

We then set, for all $\rho \in 2^m$ and all $k < 2$,

$$\mathcal{T}^*(1_L)(\rho\hat{\ }\langle k\rangle) = \mathcal{T}(1_L)(\pi(\rho)\hat{\ }\pi_k\hat{\ }\langle 0^j\rangle\hat{\ }\langle k\rangle\hat{\ }\langle 0^{n-j-1}\rangle).$$

And, again, the condition that $[\Theta^*_{1_L,a}] = [\Theta_{1_L,a}] \restriction [\mathcal{T}^*(1_L)]$ fully determines $\mathcal{T}^*(a)$ for all $a \in L'$.

Since $\mathcal{T}$ was very uniform, $\mathcal{T}^*$ remains very uniform.

We now first establish, for Case 2, the following

Claim 5.16. *For all $\rho \in 2^m$, we have that $\mathcal{T}^*(1_L)(\rho^\frown\langle 0\rangle)$ and $\mathcal{T}^*(1_L)(\rho^\frown\langle 1\rangle)$ $\langle e,i\rangle$-split $\mathcal{T}^*(1_L)(\rho)$ on $\mathcal{T}^*(1_L)$ and differ in a_0.*

Proof. By hypothesis, we have that $\mathcal{T}(1_L)(\pi(\langle 0^m\rangle)^\frown\pi_0)$ and $\mathcal{T}(1_L)(\pi(\langle 0^m\rangle)^\frown\pi_1)$ $\langle e,i\rangle$-split $\mathcal{T}^*(1_L)(\langle 0^m\rangle)$ in $\mathcal{T}(1_L)$. Thus there is x with $x \in \Psi_i[\mathcal{T}(1_L)(\pi(\langle 0^m\rangle)^\frown\pi_0)]$ and $x \in \Phi_e[\mathcal{T}(1_L)(\pi(\langle 0^m\rangle)^\frown\pi_1)]$. But then $x \in \Psi_i[\mathcal{T}(1_L)(\pi(\langle 0^m\rangle)^\frown\pi_0)]$ hinges only on $(\pi(\langle 0^m\rangle)^\frown\pi_0)(j') = 0$ for some $j' \in [|\pi(\langle 0^m\rangle)|, |\pi(\langle 0^m\rangle)^\frown\langle\pi_0\rangle|)$. Now this ensures that $x \in \Psi_i[\mathcal{T}(1_L)(\pi(\rho)^\frown\pi_0)]$ and $x \in \Phi_e[\mathcal{T}(1_L)(\pi(\rho)^\frown\pi_1)]$ for all $\rho \in 2^m$, so $\mathcal{T}^*(1_L)(\rho^\frown\langle 0\rangle)$ and $\mathcal{T}^*(1_L)(\rho^\frown\langle 1\rangle)$ $\langle e,i\rangle$-split $\mathcal{T}^*(1_L)(\rho)$ on $\mathcal{T}(1_L)$ and thus also on $\mathcal{T}^*(1_L)$.

Our definition of $\mathcal{T}^*(1_L)(\rho^\frown\langle k\rangle)$ for $k < 2$ now also guarantees that they differ in a_0. □

Concluding the proof of Proposition 5.13, we now observe that given the definition of $\mathcal{T}^*(1_L)$, item (1) of Proposition 5.13 completely determines η^*. Furthermore, we define, for $a \in L'$ and $\rho \in 2^{<\omega}$, $\mathcal{T}(a)(\rho) = \Theta_{1,a}(\mathcal{T}^*(1_L)(\rho))$. This immediately implies item (2) and that $[\Theta^*_{a,b}] = [\Theta_{a,b}] \upharpoonright [\mathcal{T}^*(a)]$ whenever $a \geq_{L'} b$ in L'.

It remains to verify item (3) of Proposition 5.13. So fix $f_1 \in [\mathcal{T}^*(1_L)]$ such that $(\Phi_e, \Psi_i)[f_1]$ is a valid computation.

We first verify that $(\Phi_e, \Psi_i)[f_1] \leq_Z f_{a_0}$, so fix an argument x. Effectively in f_{a_0}, we can find $\rho \in 2^{<\omega}$ such that, for $\tau = \mathcal{T}(1_L)(\rho)$:

- $\tau_{a_0} = (f_{a_0} \upharpoonright |\tau|)_{a_0}$; and
- $x \in \Phi_e[\tau]$ or $x \in \Psi_i[\tau]$.

Then, by Claim 5.15(3), $(\Phi_e, \Psi_i)[f_1](x) = (\Phi_e, \Psi_i)[\tau](x)$ as desired.

On the other hand, let's now verify that $f_{a_0} \leq_Z (\Phi_e, \Psi_i)[f_1]$. Then f_{a_0} at an argument x', say, is determined by f_1 at an argument x.

If there is $k < 2$ and some ℓ such that every $\sigma \in \operatorname{ran}(\mathcal{T}^*)$ of length ℓ has $\sigma(x) = k$, then $f_1(x) = k$ and so $f_{a_0}(x') = k$. Otherwise, there are $\sigma, \tau \in \operatorname{ran}(\mathcal{T}^*)$ such that $\sigma(x) = 0 < 1 = \tau(x)$. Since $\mathcal{T}^*$ is very uniform, we may in addition assume that there is $\rho \in 2^{<\omega}$ such that $\mathcal{T}^*(\rho^\frown\langle 0\rangle) = \sigma$ and $\mathcal{T}^*(\rho^\frown\langle 1\rangle) = \tau$. Now, by the way we constructed $\mathcal{T}^*$ (in Case 2 of the proof of Proposition 5.13), we know that $(\Phi_e, \Psi_i)[\sigma](y) = 0 < 1(\Phi_e, \Psi_i)[\tau](y)$ for some y, and so $f_{a_0}(x') = f_1(x) = (\Phi_e, \Psi_i)[f_1](y)$ can be Ziegler-computed from $(\Phi_e, \Psi_i)[f_1]$ as claimed. □

5.4. **Proof of Theorem 1.2.** Let the countable distributive lattice L be given by a (usually noncomputable) approximation by finite distributive lattices L_s such that L_0 is the 2-element lattice, and each L_{s+1} either equals L_s or else is a simple extension of L_s. We will define a (usually noncomputable) decreasing sequence of conditions $\{(\eta^s, \mathcal{T}^s)\}_{s\in\omega}$; the unique infinite path f_1 through all f-trees $\mathcal{T}^s$ will represent the desired Ziegler degree.

We also define a (possibly non-effective) enumeration $\{(a_s, b_s, e_s, i_s)\}_{s\in\omega}$ of all $a, b \in L$ and $e, i \in \omega$ such that $a, b \in L_s$, $a \neq 1_L$ is join-irreducible in L_s, and $b = b_a$ (in the sense of L_s, and so $b \in L_s$).

Stage $s = 0$: Let $\eta^0_{1_{L_0}} : L_0 \to \mathcal{P}(\omega)$ be given by $\eta^0_{L_0}(0_{L_0}) = \emptyset$ and $\eta^0_{L_0}(1_{L_0}) = \{0\}$. Let $\mathcal{T}^0(1_L)$ be the full binary f-tree and $\mathcal{T}^0(0_L)$ be the degenerate f-tree.

Stage $4s+1$*:* We obtain condition $(\eta^{4s+1}, \mathcal{T}^{4s+1})$ from $(\eta^{4s}, \mathcal{T}^{4s})$ using Proposition 5.10.

Stage $4s+2$*:* We obtain condition $(\eta^{4s+2}, \mathcal{T}^{4s+2})$ from $(\eta^{4s+1}, \mathcal{T}^{4s+1})$ using Proposition 5.11 applied to the lattices L_s and L_{s+1}.

Stage $4s+3$*:* We obtain condition $(\eta^{4s+3}, \mathcal{T}^{4s+3})$ from $(\eta^{4s+2}, \mathcal{T}^{4s+2})$ using Proposition 5.12 applied to the Ziegler reduction (Φ_e, Ψ_i) and $a, b \in L$ where $(a, b, e, i) = (a_s, b_s, e_s, i_s)$.

Stage $4s+4$*:* We obtain condition $(\eta^{4s+4}, \mathcal{T}^{4s+4})$ from $(\eta^{4s+3}, \mathcal{T}^{4s+3})$ using Proposition 5.13 applied to the Ziegler reduction (Φ_e, Ψ_i) where $\langle e, i\rangle = s$.

Verification. We start by showing that if $a \geq_L b$, then $\mathbf{f}_b \leq_Z \mathbf{f}_a$. Indeed, fix a stage $t = 4s+2$ such that $a, b \in L_s$. Then f_a is on $\mathcal{T}^t(a)$ and f_b is on $\mathcal{T}^t(b)$, which are both computable very uniform f-trees, and, using (3) of Definition 5.7, we have $f_b = [\Theta^t_{a,b}](f_a)$. Hence Lemma 5.9 implies that $f_b \leq_Z f_a$.

At stages $s \equiv 2 \pmod 4$, we guarantee that each $a \in L$ is introduced to the construction at some stage. At stages $s \equiv 3 \pmod 4$, we ensure that $a \not\leq_L b$ implies that $f_a \not\leq^* f_b$. Combined with the previous paragraph, this guarantees that there is an embedding of the lattice L into $\mathcal{D}_Z(\leq \mathbf{f}_1)$. At stages $s \equiv 4 \pmod 4$, we force that if $\mathbf{a} \leq_Z \mathbf{f}_1$, then $\mathbf{a} \equiv_Z \mathbf{f}_a$ for some $a \in L$. This completes the proof of Theorem 1.2. □

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