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Degrees bounding minimal degrees

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1. *Introduction*

A set is called n-generic if it is Cohen generic for n-quantifier arithmetic. A (Turing) degree is n-generic if it contains an n-generic set. Our interest in this paper is the relationship between n-generic (indeed 1-generic) degrees and minimal degrees, i.e. degrees which are non-recursive and which bound no degrees intermediate between them and the recursive degree. It is known that n-generic degrees and minimal degrees have a complex relationship since Cohen forcing and Sacks forcing are mutually incompatible. The goal of this paper is to show.

THEOREM A. *There is a minimal degree* $\mathbf{a} < 0'$ *recursive in no* 1*-generic degree.*

Theorem A concludes a sequence of results. Jockusch [7] showed that for $n > 2$ no n-generic degree bounds a minimal degree. In Chong and Jockusch [3] it is established that if $0 < \mathbf{a} < \mathbf{b} < 0'$ and $\mathbf{b}$ is 1-generic then $\mathbf{a}$ is not minimal (indeed bounds a 1-generic degree), and Haught [6] has improved this to show that if $\mathbf{b} < 0'$ is 1-generic then $\mathbf{a}$ is 1-generic also. Finally it is shown in [1] that there is, however, a minimal degree $\mathbf{a} < 0'$ recursive in a 1-generic degree below $0''$. This left open the question whether every minimal degree below $0'$ is recursive in a 1-generic degree. This is answered by Theorem A.

The main ingredient of the proof of Theorem A is a slight generalization of the notion of a Σ_1 *dense set of strings* introduced in [1] which is defined there as follows. If Q is recursive in $\emptyset'$, then by the limit lemma there is a recursive set of strings $\{\sigma_n\}_{n\in\omega}$ such that $Q(x) = \lim_{n\to\omega}\sigma_n(x)$ for all x. Let $X = \{\sigma_n\}_{n\in\omega}$ be a recursive sequence converging to Q and closed under initial segments. Then an r.e. subsequence $Y \subset X$ is Σ_1-*dense* if no initial segment of Q belongs to Y, and if for all infinite r.e. $Z \subset X$, there is a string $\sigma \in Z$ which extends one in Y.

For our purposes we need the following generalization of this notion:

Definition. Let B be any set and P be an r.e. set of strings. We say that P is Σ_1 dense for B if the following conditions are satisfied:

(1) $\sigma \in P$ implies that σ is not an initial segment of B;

(2) for any r.e. set of strings Q, if $D(Q) = \{\sigma \mid \sigma \leqslant \tau \text{ for some } \tau \in Q\}$ (the downward closure of Q) contains arbitrarily long initial segments of B, then

$$(\exists\tau \in Q)(\exists\sigma \in P)[\tau \geqslant \sigma].$$

Observe that our definition of Σ_1 density here is stronger than that given in [1]. For Δ_2^0 sets the two definitions turn out to be equivalent, since the existence of a Σ_1 dense set for one recursive approximation to a Δ_2^0 set automatically implies its existence for every other recursive approximation, a fact which follows from the main result of [1].

In §2 we shall first prove

LEMMA B. *If B has a Σ_1 dense set of strings, then B is not recursive in any 1-generic set.*

This is mainly given for the sake of completeness, since the proof is essentially the same as in Chong[2]. In [1] it is shown that if A has no Σ_1 dense set of strings and $A <_T \varnothing'$ then A is recursive in a 1-generic set, and furthermore there exist sets of minimal degree below $\varnothing'$ with no Σ_1 dense set of strings.

Thus the property of having a Σ_1 dense set of strings is degree-theoretically invariant for degrees below $0'$. (The first author has recently shown that this property holds for all degrees.) We feel that this (and similar) notions promise further applications since they reduce the global property of being recursive in 1-generic sets to a local one of having a certain 'simple' set of strings. Furthermore the notion of Σ_1 density is useful in α recursion theory where for example in [2] it is exploited to establish that no minimal degree below $0'$ is recursive in a 1-generic degree for ordinals such as $\aleph_\omega^L$.

In view of Lemma B, Theorem A follows once we establish

THEOREM C. *There is a set M of minimal degree below $\varnothing'$ with a Σ_1 dense set of strings.*

Our terminology and notation are fairly standard. The reader is assumed to be familiar with the usual tree constructions of minimal degrees (Shoenfield[13], Soare[14], Lerman[8], Epstein[4, 5]). Sets of natural numbers are identified with their characteristic functions, and the language of strings (elements of $2^{<\omega}$) is used throughout.

2. *Proofs*

We first establish Lemma B.

LEMMA B. *If M has a Σ_1 dense set of strings P, then M is recursive in no 1-generic set.*

Proof. We follow [2]. Let G be a 1-generic set with $\Phi(G) = M$, and let M, P satisfy the hypothesis of the lemma. Let

$$R = \{\sigma \mid \Phi(\sigma) \text{ extends some string in } P\}.$$

Then R is r.e. and contains no initial segment of G, since P contains no initial segment of M. Thus, as G is 1-generic, there is an initial segment τ of G such that for all σ in R, σ does not extend τ. This follows from Posner's[10] characterization of 1-genericity (see [7], lemma 2·7). It follows from the definition of R that no extension of τ is mapped by Φ to a string extending one in P. Now let

$$Q = \{\sigma \mid \sigma \leqslant \Phi(\gamma) \text{ for some } \gamma \text{ extending } \tau\}.$$

Then Q is r.e., and $D(Q) = Q$ contains arbitrarily long initial segments of M. As P is Σ_1 dense, there are strings σ_1, σ_2 with $\sigma_1 \in Q$ and $\sigma_2 \in P$ such that $\sigma_1 \geqslant \sigma_2$. Now as $\sigma_1 \in Q$, we have $\sigma_1 \leqslant \Phi(\gamma)$ for some γ extending τ. Therefore $\sigma_2 \leqslant \Phi(\gamma)$ for some γ extending τ. As $\sigma_2 \in P$ and $\Phi(\gamma) \geqslant \sigma_2$, we see that $\gamma \in R$ by definition. But this is impossible since $\gamma \geqslant \tau$. ∎

We now return to the proof of Theorem C, namely the construction of a set M of minimal degree below $\varnothing'$ with a Σ_1 dense set of strings P. We shall simultaneously build $M = \lim_s M_s$ and $P = \bigcup_s P_s$ by a full approximation method along the lines of, for example, Epstein [4, 5]. We shall construct nested sequences of recursive trees $T_{0,s} \supseteq T_{1,s} \supseteq \ldots \supseteq T_{s,s}$ for each stage s. Each node σ with $lh(\sigma) \leqslant s$ has an e-state $\alpha \in \{0,1\}^{<\omega}$ for certain $e \leqslant lh(\sigma)$. In this case α is a string of length $e+1$.

The following requirements are to be satisfied:

N_{-1}: $\sigma \in P$ implies that σ is not an initial segment of M;
N_e: $\Phi_e(M)$ total implies $((\Phi_e(M) \equiv_T M) \vee (\Phi_e(M) \equiv_T \varnothing))$;
R_e: If $D(V_e)$ contains arbitrarily long initial segments of M, then $\sigma \leqslant \tau$ for some $\sigma \in P$ and $\tau \in V_e$.

Here V_e denotes the eth r.e. collection of strings in $2^{<\omega}$ under some standard enumeration. The reader should note that the R_e requirements will automatically make M non-recursive. To see this, suppose otherwise. Let W_i be r.e. with $\overline{W}_i = M$. Let $V_e = V_{e(i)}$ be the recursive collection of strings consisting of all initial segments of $\overline{W}_i$. That is, $V_e = \{\tau \mid \tau < \overline{W}_i = M\}$. Note that V_e is recursive as $\overline{W}_i$ is recursive. Now, as R_e is met, for some $\tau \in V_e$ and some $\sigma \leqslant \tau$ in P, we have $\sigma < M$. But N_{-1} says that $\sigma \nless M$, contradicting the definition of V_e.

A string σ may or may not be *forbidden*. Once σ is forbidden at stage s, we ensure that σ is never chosen to be an initial segment of M_t for all $t \geqslant s$. Only forbidden strings are put in P. This ensures that the requirement N_{-1} is met, since, once σ is forbidden, all extensions of σ are forbidden as well. Also we adopt the following *forbiddenness condition*: if $\sigma * 0$ and $\sigma * 1$ are forbidden, then so is σ. Similarly if σ and τ are forbidden and on $T_{e,s}$ (for some e, s) with $T_{e,s}(\eta * 0) = \sigma$ and $T_{e,s}(\eta * 1) = \tau$, then we forbid $T_{e,s}(\eta)$ as well. One ramification of this condition will be that if σ is on $T_{e,s}$ and σ is non-forbidden, then there will be at least one non-forbidden path through $T_{e,s}$. (In fact, the construction will ensure that there is a non-forbidden full subtree of $T_{e,s}$, and so many non-forbidden paths.)

As usual, we shall satisfy the condition N_e by building, at each stage s, a collection of recursive trees $T_{1,s} \supseteq \ldots \supseteq T_{s,s}$ 'fully approximating' a minimal degree construction by maximizing the appropriate e-states. There are two basic problems in the attempt to implement the forbidding idea in the presence of the requirements R_e. Both of these problems stem from the fact that the requirement R_e tends to kill off vast portions of the trees T_j. This means that we lose the complete freedom we usually have to argue (in the standard construction) that if we see an e-splitting, then we automatically take it.

The first problem is that we might kill off far too much of the tree making M recursive. Perhaps M is the unique non-forbidden path on some T_e. In order to carry out our strategy for meeting R_e, we must allow many possible choices for M. Thus, to make sure that R_e is met, we make sure that there are certain *non-forbidden cones* of strings available for R_e, should it desire them. The basic problem is that when R_e forbids some string σ this is an irrevocable decision. Now it is not difficult to make sure that our approximation M_s is not forbidden at any stage (by, roughly speaking, defining M_s first, before the forbidding procedure, never forbidding M_s and never cutting off all paths on $T_{s,s}$ by the reshaping of the trees due to the 'maximizing e-state' machinery). There is a timing element involved in the definition of the high e-state $\alpha * 1$. This timing element means, roughly, e-splitting *before* forbidding. That is, once a string is forbidden we no longer consider it as a possible e-split (if it is not already an e-split). The principal conflict that occurs is that we might make all strings except M_s forbidden, making M recursive.

Thus for the sake of R_e we shall have a number $n(e,s)$ such that $\lim_s n(e,s)$ exists, and such that, once we see a string $\tau \in V_{e,s}$ with τ 'appearing like' M at stage s, and $T_{e,s}(\sigma * 0) < \tau$ for some σ with $lh(\sigma) = n(e,s)$ (of course we must argue that this happens if $D(V_e)$ contains arbitrarily long initial segments of M), we shall *e-abandon* τ in favour of some new M_{s+1} extending $T_{e,s}(\sigma * 1)$. This will be permissible since we shall argue that all extensions of $T_{e,s}(\sigma * 1)$ on $T_{e,s}$ will be non-forbidden. The extensions of $T_{e,s}(\sigma * 1)$ on $T_{e,s}$ give the 'non-forbidden cone of strings set aside for R_e'. This cone is given by a Π_2 argument and in the limit depends both on e-states and initialization by R_j for $j < e$. If τ is *e-held* (i.e. on $T_{e,s}$ and $\tau > T_{e,s}(\sigma * 1)$), then τ cannot be forbidden except by R_k for $k < e$. If for example R_k, where $k < e$, forbids $T_{e,s}(\sigma * 1)$, we initialize the entire e-held cone (and $n(e,s)$). We argue that R_k for $k < e$ act only finitely often (as does initialization by e-states) and so both $\lim_s n(e,s) = n(e)$ exists and eventually the e-cones become stable (stringwise).

All of this serves no purpose unless we eventually meet R_e by forbidding something. Note that, should $T_{e,s}(\sigma * i) = T_e(\sigma * i)$ for $i = 0, 1$ and should we ensure $M > T_e(\sigma * 1)$, then we will meet R_e if we forbid $T_{e,s}(\sigma * 0)$ (since then $T_{e,s}(\sigma * 0) < \tau$ and $\tau \in V_e$ with $T_{e,s}(\sigma * 0) \in P$). In fact this works provided that we keep M not extending $T_e(\sigma * 0)$, although we have not explicitly incorporated this in the construction.

However, the whole problem is that we may not be able to forbid τ immediately we e-abandon it, again due to timing difficulties. The point is that *forbidden strings cannot be chosen as e-splits*. Suppose that the final 'well resided' e-state is $\alpha * 0$ and $\Phi_e(M)$ is total. We aim to conclude that $\Phi_e(M)$ is recursive. In the usual argument we reason as follows. Let $\hat{\sigma} \in T_e$, with $\hat{\sigma}$ having e-state $\alpha * 0$, and let s_0 be the stage where $M_s > \hat{\sigma}$ for all $s > s_0$. Then to compute $\Phi_e(M; z)$ we simply find any stage $s > s_0$ where $\Phi_{e,s}(M_s; z)\downarrow$. Now we know that, although perhaps $M_t | M_s$ for some stage $t > s$, it must be that $\Phi_{e,s}(M_t; z) = \Phi_{e,s}(M_s; z)$ if $\Phi_{e,s}(M_t; z)\downarrow$, since otherwise we would use M_t and M_s to e-split $\hat{\sigma}$ on $T_{e,s}$ and hence $\hat{\sigma}$ must have e-state $\alpha * 1$.

In our construction we do not look at all strings for e-splits, but only at *non-forbidden* ones. The difficulty is that perhaps $\Phi_{e,s}(M_s; z)\downarrow$ and that at stage $t > s$, again $M_t | M_s$. But perhaps $\Phi_e(M_t; z)$ converges very slowly, and $\Phi_{e,t}(M_t; z)\uparrow$. Then M_s might now get forbidden if we are careless. Indeed, there may be a situation where M_s is e-abandoned and $M_t > T_{e,s}(\sigma * 1)$, and some $\hat{\sigma} \leqslant T_{e,s}(\sigma)$ has e-state $\alpha * 0$. The point is that perhaps $\Phi_{e,\hat{t}}(M_{\hat{t}}; z)\downarrow \neq \Phi_{e,s}(M_s; z)$ at stage $\hat{t} > t$. But now M_s is forbidden so we cannot use $M_{\hat{t}}$ and M_s to e-split $\hat{\sigma}$.

Our solution is to *squeeze* Φ_e. After all, we really do not need to do anything for N_e if $\Phi_e(M)$ is not total. Thus we put an *e-delay* in the construction. Roughly speaking, in the situation outlined above, we shall declare M_s as *e-frozen* and keep it e-frozen until a stage $t > s$ occurs where for all z and for all $\hat{z} \leqslant z$, if $\Phi_{e,s}(M_s; \hat{z})\downarrow$ then so too does $\Phi_{e,t}(M_t; z)\downarrow$. Strictly speaking this is not quite correct, in the sense that M_t must be available for the relevant e-splitting and so must have the appropriate $(e-1)$-state. Furthermore, in the actual construction M_s will be σ-frozen for various $\hat{e}$-states σ rather than simply e-frozen. Thus if τ is on T_e and e-frozen then τ will *blame* some node $\hat{\sigma} \leqslant \tau$. This will necessitate both τ and $\hat{\sigma}$ having the same e-state $\alpha * 0$. That is, this node $\hat{\sigma}$ will be the shortest one with e-state $\alpha * 0$ for which, as above, we are waiting for proof that $\Phi_e(M)$ is total. Now we do not e-thaw (i.e. unfreeze) τ until we see $\hat{\tau} \leqslant M_t$ so that for all z, if $\Phi_{e,s}(\tau; z)\downarrow$ then $\Phi_{e,t}(\hat{\tau}; z)\downarrow$ and furthermore $\hat{\tau}$ has e-state $\alpha * 0$.

This strategy, in turn, creates problems with the satisfaction of the R_e, since perhaps some e-abandoned τ is never e-thawed. Our solution is the obvious one: we start doing work on R_e anew 'higher up' on the tree. The point is that whilst some potential witness for R_j is e-frozen it is a temporary witness to either the well-resided $(e-1)$-state not being α or $\Phi_e(M; z)\uparrow$ for some z. The next such version (if permanent) higher up on the e-tree is similarly a witness either for $\Phi_e(M)$ not being total or for the final $(e-1)$-state not being $\hat{\alpha}$ for some lower $(e-1)$-state $\hat{\alpha}$. Thus eventually some version of R_e must get a string τ that, at worst, becomes e-thawed. If τ blamed $\hat{\sigma}$ (as above) and if when τ was thawed, the e-state of $\hat{\sigma}$ remained $\alpha * 0$, then we can attack R_e by forbidding τ and win, since we know $\Phi_e(M_t; z)$ agrees with $\Phi_e(\tau; z) = \Phi_e(M_s; z)$. The final conflict we must resolve is that if the e-state of $\hat{\sigma}$ improves to $\alpha * 1$ then perhaps it uses splittings (for example) $\tau_1, \tau_2 > \tau$, and we cannot forbid τ without killing both τ_1 and τ_2. Now the driving force behind our construction is to try to be very conservative as to when to forbid. In particular if we forbid say $T_{e,s}(\sigma * 0)$ in the situation above, we would obviously try not to forbid $T_{e,s}(\sigma * 1)$, since we want $M > T_{e,s}(\sigma * 1)$. This would happen if $\hat{\sigma} = T_{e,s}(\sigma)$ and τ_1, $\tau_2 > \tau$. The situation might be even worse. If $\hat{\sigma} = T_{e,s}(\varnothing) = T_{0,s}(\varnothing)$ and $\tau_1, \tau_2 > \tau$ (so that τ was 0-frozen) then if we were to forbid τ we would kill all extensions of $\varnothing$ on all the T_j. This might injure R_0 since perhaps $n(e, s) = 1$ (in combination, R_j for $j > 0$ could ensure R_0 is never met).

Our solution is simply to *initialize* (nor forbid τ) and reset $n(e, t)$ back to $n(e, s)$. Roughly speaking, since we argue that $\lim_s n(e, s) = n(e)$ exists, we ensure that such initialization occurs finitely often. For the cognoscenti, we are guaranteeing that R_e is met by the appropriate '$\alpha * 1$-strategy'. This waits for $T_{e,s}(\sigma)$ to achieve the high e-state $\alpha * 1$ and then attacks R_e. The aim, of course, is to ensure the existence of $\lim_s n(e, s) = n(e)$. The point is that $n(e, s+1) > n(e, s)$ at the stage s when we e-abandoned τ. Of course the reset version of R_e working above $n(e, s+1)$ is essentially guessing that either $\alpha * 0$ is too high an e-state or $\Phi_e(M)$ is not total. Should we find out at a stage $t > s$ that in fact the blamed $\hat{\sigma}$ e-splits, then these versions of R_e must be wrong. It makes sense then to reset $n(e, t+1) = n(e, s)$. What we have gained by doing this is the knowledge that $\hat{\sigma}$ has now the high e-state, and we need only worry about $\sigma' > \hat{\sigma}$. Hence this all can happen only finitely often.

We now turn to the formal details of the argument. We say that R_e is *satisfied at stage* s if

$$(\exists \sigma, \tau)[\sigma \in P_s \ \&\ \tau \in V_{e,s} \ \&\ \sigma \leqslant \tau].$$

We say that R_e *requires attention* at substage e of stage $s+1$ if R_e is unsatisfied at stage s and that there exists $\tau \in V_{e,s}$ with $\tau > T_{e,s+1}(\sigma * 0)$ where $lh(\sigma) = \hat{n}(e, s+1)$ and

$T_{e,s}(\sigma * 0) < M(e, s+1)$. ($M(e, s+1)$ is an approximation to M_s defined in the construction; similarly $\hat{n}(e, s+1)$ is an approximation to $n(e, s)$.)

We use the convention that $\tau \in V_{s,s}$ implies $lh(\tau) < s$. In the construction to follow, it is convenient to treat the $T_{j,s}$, for $j \leqslant s$, as being infinite trees, and M_s as an infinite branch on $T_{s,s}$. This does no harm since we use the convention that above some level (say s) $T_{s,s}$ is simply the identity tree. This will be done automatically.

Construction at stage $s+1$.

Substage $e \leqslant s+1$.

Step 1 (improving e-states). In order of j and then in order of $\sigma \in 2^{<\omega}$ (for $j \leqslant s+1$ and $lh(\sigma) = j$) we shall define $T_{e,s+1}(\sigma)$ and simultaneously the e-state of $\tau \in T_{e,s+1}$ (i.e. τ on $T_{e,s+1}$). For convenience set $T_{-1,s} = 2^{<\omega}$ for all s and assign to all $\sigma \in 2^{<\omega}$ the (-1)-state $\varnothing$ (the empty string). (The reader should note that changing e-states in this step will 'probably' mean cancellation of attacks on R_k for $k \geqslant e$ in Step 2.)

Let $T_{e,0} = 2^{<\omega}$ and assign to all $\sigma \in T_{e,0}$ the e-state $0^{(e+1)}$. Set $n(e, 0) = e+1$ for all e. We use $n(e, s)$ to define the relevant cone of e-held strings. If we denote this by $C(e, s)$, it will be

$$C(e,s) = \{\tau \mid \tau \in T_{e,s} \;\&\; \tau \geqslant T_{e,s}(\sigma * 1)\},$$

where $lh(\sigma) = n(e, s)$ and $T_{e,s}(\sigma)$ is an initial segment of the left-most non-forbidden path of $T_{e,s}$. Thus at stage 0, $C(e, 0)$ is simply $\{\tau \mid \tau \in 0^{<\omega} \;\&\; \tau \geqslant 0^{e+1} * 1\}$.

For stages $s \geqslant 0$, implement one of the following which pertains to σ:

Case 1. $lh(\sigma) < n(e, s)+1$. Let $T_{e,s+1}(\sigma) = T_{e-1,s+1}(\sigma)$. Such $\hat{\sigma} = T_{e,s}(\sigma)$ only has j-states for $j < e$.

Case 2. $lh(\sigma) \geqslant n(e, s)+1$. If $lh(\sigma) = n(e, s)+1$, set $T_{e,s+1}(\sigma) = T_{e-1,s+1}(\sigma)$ and define $\sigma(e, s+1) = \sigma$. In general $\sigma(e, s+1)$ is the node in the domain of $T_{e-1,s+1}$ corresponding to $T_{e,s+1}(\sigma)$ (i.e. $\sigma(e, s+1) = T^{-1}_{e-1,s+1}(T_{e,s+1}(\sigma))$. Let α denote the $(e-1)$-state of $T_{e-1,s+1}(\sigma(e, s+1))$ on $T_{e-1,s+1}$.

We shall define the e-state of $\hat{\sigma} = T_{e,s+1}(\sigma)$ together with $T_{e,s+1}(\sigma * i)$ and $\sigma * i(e, s+1)$ for $i = 0, 1$. In general, we assume that we are given $\sigma(e, s+1)$ and the $(e-1)$-state of $\hat{\sigma}$ on $T_{e-1,s+1}$. We adopt the first case below which applies to σ.

Case A. For all $\sigma' \leqslant \sigma$, $\sigma'(e, s+1) = \sigma'(e, s)$ (if defined), the e-state of $T_{e,s+1}(\sigma)$ at the end of stage s was $\alpha * 1$ and $T_{e-1,s+1}(\sigma * i(e, s)) = T_{e-1,s}(\sigma * i(e, s))$ for $i = 0, 1$. In this case we claim that nothing has changed since stage s and furthermore since $T_{e,s+1}(\sigma)$ has already the high e-state $\alpha * 1$ (for $(e-1)$-state α via the e-splittings $T_{e-1,s+1}(\sigma * i(e, s))$, the obvious action is to change nothing. Thus, for $i = 0, 1$, we set

$$\sigma * i(e, s+1) = \sigma * i(e, s) \quad \text{and} \quad T_{e,s+1}(\sigma * i) = T_{e,s}(\sigma * i).$$

Hence $T_{e,s+1}$ remains locally unchanged and keeps the e-state $\alpha * 1$.

Case B. Case A does not apply, and there exist extensions τ_1, τ_2 of $T_{e-1,s+1}(\sigma(e, s+1))$ on $T_{e-1,s+1}$ such that

(3) both τ_1 and τ_2 have $(e-1)$-state α;

(4) neither τ_1 nor τ_2 is forbidden;

(5) τ_1 is 'left' of τ_2 and the two strings e-split.

In this case, define $T_{e,s+1}(\sigma * i) = \tau_{i+1}$ for $i = 0, 1$ and define $T_{e,s+1}(\sigma)$ to have e-state $\alpha * 1$. Now define

$$\sigma * 0(e, s+1) = T^{-1}_{e-1,s+1}(\tau_1) \quad \text{and} \quad \sigma * 1(e, s+1) = T^{-1}_{e-1,s+1}(\tau_2).$$

The reader should note the 'time element' involved in (4). To achieve the high e-state we must see the relevant splitting *before* the string is forbidden. Of course, later we might forbid such a string, but we never select such strings due to improving e-states. This also means (cf. Case D) that once some γ is forbidden its state is 'fixed'. Observe also that R_j for $j > e$ cannot forbid such strings τ at this stage since R_j becomes 'completely initialized'. (In Lemma 1 below, we shall see that since both τ_1 and τ_2 are non-forbidden, they have a perfect non-forbidden subtree of extensions on $T_{e-1,s+1}$.) It will follow that we will choose these extensions when defining the rest of $T_{e,s+1}$ because either this case or Case D will apply. Thus in fact the full subtree of $T_{e,s+1}$ above $T_{e,s+1}(\sigma)$ will be non-forbidden at the end of Step 1.)

On the other hand, if this case applies and there was some string ρ which was e-frozen and blamed $T_{e,s+1}(\sigma)$, we declare this string e-thawed. If R_j was attached to ρ (i.e. R_j is waiting for ρ to be unfrozen) we *completely initialize* R_j. This means that if t is the stage where R_j became attached to ρ, we set $\hat{n}(j, s+1) = n(j, t)$ and cancel all attacks on R_j using strings ρ' with $lh(\rho') > lh(\rho)$ (i.e. begun after stage t). We also initialize all R_k for $k > j$ but set $n(k, s+1) = n(k, s)+s+1$.

Case C. Neither Case A nor Case B applies, for all $\sigma' \leqslant \sigma$ we have $\sigma'(e, s+1) = \sigma'(e, s)$, the $(e-1)$-state of $T_{e,s}(\sigma)$ at stage s on $T_{e-1,s}$ was α, the e-state of $T_{e,s+1}(\sigma')$ was the same at stage s, and $T_{e-1,s+1}(\sigma * i(e, s)) = T_{e-1,s}(\sigma * i(e, s))$. As in Case A, we set

$$\sigma * i(e, s+1) = \sigma * i(e, s)$$

and define

$$T_{e,s+1}(\sigma * i) = T_{e,s}(\sigma * i).$$

Now $T_{e,s+1}(\sigma)$ has e-state $\alpha * 0$ by necessity.

Case D. No previous case applies, so that some higher priority activity has disturbed $\sigma(e, s+1)$. Also since Case B does not apply we cannot choose non-forbidden e-splitting extensions of $T_{e,s+1}(\sigma)$ on $T_{e-1,s+1}$.

First see if there exist distinct non-forbidden extensions τ_1, τ_2 of $T_{e,s+1}(\sigma)$ on $T_{e-1,s+1}$ (we claim that these exist if and only if $T_{e,s+1}(\sigma)$ is non-forbidden; see Lemma 1). If there exist such strings choose τ_1 and τ_2 of highest $(e-1)$-state and set

$$T_{e,s+1}(\sigma * 0) = \tau_1 \quad \text{and} \quad T_{e,s+1}(\sigma * 1) = \tau_2$$

where τ_1 is left of τ_2. Now define

$$\sigma * i(e, s+1) = T^{-1}_{e-1,s+1}(\tau_{i+1})$$

for $i = 0, 1$. Declare $T_{e,s+1}(\sigma)$ to have e-state $\alpha * 0$ (Case B does not apply).

If there do not exist such extensions (this will mean that $T_{e,s+1}(\sigma)$ is forbidden), we simply define

$$T_{e,s+1}(\sigma * i) = T_{e-1,s+1}(\sigma(e, s+1) * i) \quad \text{and} \quad \sigma * i(e, s+1) = \sigma(e, s+1) * i$$

for $i = 0, 1$. Let $T_{e,s+1}(\sigma)$ have e-state $\alpha * 0$ unless it already has e-state $\alpha * 1$, in which case leave it as $\alpha * 1$. The timing element means that this makes sense.

Step 2. Having defined $T_{e,s+1}$ by defining $T_{e,s+1}(\sigma)$ for all σ with $lh(\sigma) \leqslant s$ and extending accordingly, we define $M(e, s+1)$ to be the left-most non-forbidden non-k-abandoned ($k \leqslant e$) path on $T_{e,s+1}$. We define $\hat{n}(e, s+1) = n(e, s)$ unless it is already defined by Case B. We claim (see Lemma 1) that this $M(e, s+1)$ exists and furthermore $M(e, s+1) > T_{e,s+1}(\sigma * 0)$ where $lh(\sigma) = \hat{n}(e, s+1)$, unless Case B applies, in which case $lh(\sigma) = n(e, s+1)$ as given in that case.

For each ρ currently attached to R_e, see if we can now win with ρ (these ρ's are, of course, potential witnesses for R_e). We enumerate the least ρ into P winning R_e if for all $\gamma \geqslant \rho$ ($lh(\gamma) < s+1$) and $lh(\alpha) < e$, if γ is $\alpha * 0$-frozen for the sake of (i.e. blaming) some $\hat{\sigma} < \rho$ then there exists $\eta < M(e, s+1)$ such that

(6) $\eta \in T_{e,s}$;

(7) for all $z \leqslant z(\gamma, \hat{\sigma}, t)$, we have $\Phi_j(\eta; z)\downarrow$,

where t was the stage at which γ was $\alpha * 0$-frozen and blamed $\hat{\sigma}$, and $lh(\alpha) = j$. (Of course $z(\gamma, \hat{\sigma}, t)$ is the relevant 'length' of computation number for the γ, j computations at stage t, to be explicitly defined later when strings become frozen (see equation (11).)

Case 1. If such a ρ exists, we set $M_{s+1} = M(e, s+1)$ and declare all such $\gamma > \rho$ as no longer $\alpha * 0$-frozen. The reader should note that since this will mean $\hat{\sigma}$ had a k-state $\alpha * 0$ for some $k \leqslant e$, and since Case B did not apply to $\hat{\sigma}$ in substage k, it must be that the γ and η computations (which both involve those $z \leqslant z(\gamma, \hat{\sigma}, t)$, as we will see later) must be the same. Any frozen node $\beta > \rho$ is now no longer frozen but is instead forbidden. (Forbidden nodes are never frozen, and conversely.) Now initialize all the $T_{k,s+1} = T_{e,s+1}$ for all $k \geqslant e$ and initialize their k-states in the obvious way. Reset

$$n(k, s+1) = n(k, s) + s + k + 1.$$

The reader should note that, if $\tau = T_{k,s+1}(\sigma) < M_{s+1}$ and $lh(\sigma) = n(k, s+1)$, then above τ, $T_{k,s+1}$ is the full subtree of $2^{<\omega}$ and in particular $\tau * 0$ and $\tau * 1$ are non-forbidden. Furthermore $\tau * 0 < M_{s+1}$. This is because of the way we forbid strings and the $lh(\hat{\tau}) \leqslant s$ convention for $\hat{\tau} \in V_{e,s}$. Of course such initialization includes cancellation of all $\delta * 0$-freezings for all δ with $lh(\delta) > e$. We now proceed to Step 3.

Case 2. No such ρ exists, but R_e requires attention via $T_{e,s}(\sigma * 0) < \tau$. Declare $T_{e,s+1}(\sigma * 0)$ as e-abandoned. Define M_{s+1} to be the left-most non-forbidden non-k-abandoned ($k \leqslant e$) path through $T_{e,s+1}$. Note that $M_{s+1} > T_{e,s+1}(\sigma * 1)$. Initialize all $\hat{k} > e$. Reset

$$n(\hat{k}, s+1) = n(k, s) + s + \hat{k} + 1.$$

For each α with $lh(\alpha) < e$ find the longest string γ on $T_{e,s+1}$ and the correspondingly shortest string $\hat{\sigma}$ such that

(8) both $\hat{\sigma}$ and γ have $lh(\alpha)$-state $\alpha * 0$;

(9) $M_s > \gamma$;

(10) $\hat{\sigma} \leqslant T_{e,s}(\sigma)$ and $T_{e,s+1}(\sigma * 0) < \gamma$;

(11) there exists a longest $\hat{z} = z(\gamma, \hat{\sigma}, s+1)$ such that for $j = lh(\alpha)$,

   (*a*) $(\forall z < \hat{z})\,(\Phi_{j,s}(\gamma; z)\downarrow)$;

   (*b*) $(\exists z < \hat{z})\,(\forall \tau)\,[\tau \in T_{e,s+1} \ \&\ \tau < \gamma \rightarrow \Phi_{j,s}(\tau; z)\uparrow]$;

(12) there is no $\hat{\gamma}$ such that $\hat{\gamma}$ is $\alpha * 0$-frozen for the sake of $\hat{\sigma}$.

Clause (*b*) says that γ is the necessary use of the computation.

For all such $\hat{\sigma}$, γ declare γ as $\alpha * 0$-frozen with γ blaming $\hat{\sigma}$. Attach σ to R_e. Adopt the first subcase below which applies to the situation.

Subcase 1. There do not exist $\gamma, \hat{\sigma}, \alpha$ such that $\gamma > T_{e,s+1}(\sigma) > \hat{\sigma}$ and γ is $\alpha * 0$-frozen and blames $\hat{\sigma}$. In this case we forbid $T_{e,s+1}(\sigma * 0)$ by enumerating it into P, and go to Step 3. Of course we forbid all strings so demanded by the forbiddenness condition before the construction.

Subcase 2. Subcase 1 does not apply. In this case we begin a new version of R_e higher up. Thus we redefine $n(e, s+1) = \hat{n}(e, s) + e + s + 1$. Now go to Step 3.

Step 3 (at the completion of substage $e = s+1$). Define M_{s+1} to be the left-most non-forbidden non-abandoned path on $T_{s+1,s+1}$ (this may have already been done). Now we attend to any freezing/unfreezing commitments in much the same way as in Step 2.

For each $e \leqslant s$ and α with $lh(\alpha) = e$ find the longest γ and correspondingly shortest $\hat{\sigma}$ such that

(13) γ and $\hat{\sigma}$ are both on $T_{e,s+1}$ and have e-state $\alpha * 0$;

(14) $M_s > \gamma > \hat{\sigma}$;

(15) there exists a longest $\hat{z} = z(\gamma, \hat{\sigma}, s+1)$ such that

(*a*) $(\forall z < \hat{z})(\Phi_{e,s}(\gamma; z)\downarrow)$;

(*b*) $(\exists z < \hat{z})(\forall \tau)[\tau \in T_{e,s+1} \;\&\; \tau < \gamma \rightarrow \Phi_{e,s}(\tau; z)\uparrow]$.

(16) there is no $\tau \in T_{e,s+1}$ with e-state $\alpha * 0$ such that $\hat{\sigma} < \tau < M_{s+1}$ and $\Phi_{e,s}(\tau; z)\downarrow$ for all $z < \hat{z}$;

(17) there is no $\hat{\gamma}$ such that $\hat{\gamma}$ is $\alpha * 0$-frozen for the sake of $\hat{\sigma}$.

In this case, declare γ as $\alpha * 0$-frozen and blaming $\hat{\sigma}$.

Finally, we thaw strings in the following way. If there exist $\gamma, \hat{\sigma}, \alpha$ such that γ is $\alpha * 0$-frozen for the sake of $\hat{\sigma}$ and there exists $\hat{\gamma} < M_{s+1}$ with $\hat{\gamma}$ on $T_{e,s+1}$, $\hat{\gamma}$ having $lh(\alpha)$-state $\alpha * 0$, and such that

$$(\forall z < z(\gamma, \hat{\sigma}, t))[\Phi_{lh(\alpha),s}(\hat{\gamma}; z)\downarrow],$$

where t was the stage where γ was $\alpha * 0$-frozen, declare γ as $\alpha * 0$-thawed (i.e. no longer $\alpha * 0$-frozen).

This completes our construction. ▌

LEMMA 1.

(*a*) *$\lim_s T_{e,s} = T_e$ exists stringwise.*

(*b*) *R_e requires attention only finitely often, and is met.*

(*c*) *$\lim_s n(e, s) = n(e)$ exists.*

(*d*) *$M(e, s)$ exists for all e and s.*

(*e*) *$M = \lim_s M_s \leqslant_T \varnothing'$.*

(*f*) *For each string σ and for all e and s, if $lh(\sigma) = n(e,s)$ and R_e is not satisfied at stage s, and if $T_{e,s}(\sigma) < M(e, s)$, then $T_{e,s} < M_s$, and both $T_{e,s}(\sigma * 0)$ and $T_{e,s}(\sigma * 1)$ are non-forbidden.*

Proof. We verify all except (*e*) by simultaneous induction. Note that (*e*) then follows by the Limit Lemma. Now assume that the lemma holds for all $j < e$. Let

s_0 be a stage so large that for all $s > s_0$ and for all $j < e$, (i) $n(j,s) = n(j,s_0) = n(j)$, and (ii) R_j does not receive attention after stage s_0.

By the way a requirement is initialized, we may assume that when R_j $(j < e)$ receives attention, if $lh(\sigma) = n(e,s_0)$ and $T_{e,s}(\sigma) < M(e,s)$, then $T_{e,s}(\sigma) < M_s$, and both $T_{e,s}(\sigma * 0)$ and $T_{e,s}(\sigma * 1)$ are non-forbidden. Furthermore all extensions of $T_{e,s}(\sigma)$ on $T_{e,s}$ must be non-forbidden if we chose s_0 minimal.

If we suppose that R_e does not receive attention, then the only way $T_{e,s}(\sigma) \neq T_{e,s+1}(\sigma)$ where $lh(\sigma) = n(e,s)$ is due to the action of the N_j for $j \leqslant e$ (i.e. improving e-states). Now if some shortest node $\gamma < T_{e,s}(\sigma)$ on $T_{j,s}$ improves its j-state $(j \leqslant e)$ at stage $s+1$, it can only be via Case B. Thus we pick non-forbidden extensions τ_1, τ_2 of γ on $T_{j,s+1}$. This implies that we must apply either Case B or Case D to all extensions of τ_1 and τ_2 on $T_{j,s+1}$. In either case we always pick non-forbidden extensions which must exist lest τ_i be forbidden by the forbiddenness condition. This means that on tree $T_{j+1,s+1}$ (if $j < e$) the same considerations must apply, and hence, if some $\gamma < T_{e,s}(\sigma)$ changes its e-state, then $T_{e,s+1}(\sigma)$ is non-forbidden as are all its extensions on $T_{e,s+1}$. The usual e-state argument implies that T_e exists. We then see that R_e is met in this case since, once $T_{e,s}(\sigma)$, $T_{e,s}(\sigma * 0)$ and $T_{e,s}(\sigma * 1)$ get their highest e-state, then $T_{e,s}(\sigma * 0) = T_e(\sigma * 0)$ is a witness for the failure of $D(V_e)$ to contain arbitrarily long initial segments of M (since $M > T_e(\sigma * 0)$).

If R_e receives attention at least stage s, then R_e is met via $T_{e,s}(\sigma * 0)$ (since $M > T_{e,s}(\sigma * 0)$ and $T_{e,s}(\sigma * 0) \in P$) unless $T_{e,s}(\sigma * 0)$ is $\alpha * 0$-frozen for some pair $(\gamma, \hat{\sigma})$ with $\gamma > T_{e,s}(\sigma * 0) > \hat{\sigma}$. In this case we reset $n(e, s+1)$ to ensure that, once we define M_{s+1} (which extends $T_{e,s+1}(\sigma * 1)$ by construction), we have

$$T_{e,s+1}(\eta * 0) < M_{s+1} = M(e, s+1)$$

with both $T_{e,s+1}(\eta * 0)$ and $T_{e,s+1}(\eta * 1)$ non-forbidden, where $lh(\eta) = n(e,s+1)$.

If R_e fails to be met, then some $\gamma > T_{e,s}(\sigma * 0)$ is $\alpha * 0$-frozen for the sake of some $\hat{\sigma} < T_{e,s}(\sigma)$. Also, if R_e fails, then either there is a permanently $\alpha * 0$-frozen such γ (with $T_{e,s}(\sigma * 0) = T_e(\sigma * 0)$) or some node $\rho \leqslant T_{e,s}(\sigma)$ changes its j-state (for $j \leqslant e$). Note that if the latter occurs, say at stage $t > s+1$, then Case B must hold and we completely initialize R_e and reset $n(e,t)$ to $n(e,s)$ and begin anew. Arguing in this manner, assuming R_e fails to be met, we can pretend s to be after a stage $s_1 > s_0$ when $T_{e,s_1}(\sigma * 0) = T_e(\sigma * 0)$, and now some $\gamma > T_e(\sigma * 0)$ is permanently $\alpha * 0$-frozen for the sake of some $\hat{\sigma} < T_e(\sigma * 0)$. Now we have reset $T_{e,s+1}(\eta)$ where $lh(\eta) = n(e,s+1)$. In a similar fashion, eventually $\hat{\gamma} \geqslant T_e(\eta * 0)$ must be permanently $\beta * 0$-frozen for the sake of some $\rho \leqslant T_e(\eta * 0)$. The whole point is that $\beta \neq \alpha$. Once γ is permanently $\alpha * 0$-frozen for the sake of $\hat{\sigma}$, if any string extending $T_e(\eta)$ is to be $\alpha * 0$-frozen, it can only be because (since $\hat{\sigma} \leqslant T_e(\sigma) < T_e(\eta)$) it blames the shortest node below it with the same $lh(\alpha)$-state, which by assumption is $\hat{\sigma}$. But γ is already $\alpha * 0$-frozen for the sake of $\hat{\sigma}$, and so would need to be unfrozen before $\hat{\gamma}$ would be frozen. Since $\beta \neq \alpha$ we must choose a new e-state. There can thus be only finitely many permanently frozen attacks on R_e. Thus eventually R_e is met and so $\lim_s n(e,s) = n(e)$ exists and the usual e-state argument gives $\lim_s T_{e,s} = T_e$ stringwise. This proves Lemma 1.

LEMMA 2. *If $\Phi_e(M)$ is total and if α (the well resided e-state) is $\beta * 0$ then $\Phi_e(M)$ is recursive.*

Proof. Let σ be the shortest string with σ on T_e and $\sigma < M$, and such that σ has e-state $\beta * 0$. Let s_0 be a stage after which, for all $j \leqslant e$, R_j does not receive attention and all $\tau \leqslant \sigma$ do not act (i.e. have reached their final e-states and so on), and furthermore $M_s > T_{e,s}(\sigma)$ for all $s > s_0$. We show how to compute $\Phi_e(M)$. Let z be given with $z > lh(\sigma)$. To compute $\Phi_e(M; z)$, find the least stage $s_1 > s_0$ such that there exists a (least) string $\gamma < M_{s_1}$ on T_{e,s_1} with $\gamma > \sigma$ and such that (i) γ has e-state $\beta * 0$, (ii) $\Phi_{e,s_1}(\gamma; \hat{z})\downarrow$ for all $\hat{z} \leqslant z$, and (iii) $lh(\gamma) > z$ (we assume here without loss of generality that for all x, the *use function* u gives $u(\Phi_e(M; x)) > x$).

Now there are two possibilities: either there is a string η already $\beta * 0$-frozen for the sake of σ, or there is not. In the former case, if η is not $\alpha * 0$-thawed, it can only be that $z(\eta, \sigma, s) > z$. In this case set $\hat{\gamma} = \eta$. Otherwise set $\hat{\gamma} = \gamma$. We claim that for all stages $s > s_1$, it must be that if $\Phi_{e,s}(M_s; z)\downarrow$ then

$$\Phi_{e,s}(M_s; z) = \Phi_{e,s_1}(\gamma; z) = \Phi_{e,s}(\hat{\gamma}; z).$$

The reader should note that the last equality follows, since if $\Phi_{e,s}(\gamma; z) \neq \Phi_{e,s}(\hat{\gamma}; z)$ then we could use γ and $\hat{\gamma} = \eta$ as e-splitting extensions of σ on T_e (with $(e-1)$-states β). Let $s_2 > s_1$ be the least stage where, if $\hat{\gamma} \neq \gamma$, we have some $\tau < M_{s_2}$ on T_{e,s_2} with τ having $(e-1)$-state β and $\Phi_{e,s_2}(\tau; \hat{z})\downarrow$ for all $\hat{z} \leqslant z(\hat{\gamma}, \sigma, s_1)$. Then, by the choice of s_0 and the e-state of σ being $\beta * 0$, it must be that (by freezing) $\Phi_{e,s_2}(\tau; \hat{z}) = \Phi_{e,s_2}(\hat{\gamma}; \hat{z})$.

In a similar fashion whenever a stage s_3 occurs with $s_3 > s_1$ and s_2 and M_{s_3} not extending γ, it must be that some string $\delta \geqslant \gamma$ is or has been already frozen blaming σ. Obviously the same reasoning shows that eventually some ϵ with $(e-1)$-state β occurs as an initial segment of M_{s_4} $(s_4 > s_3)$ unfreezing δ (and therefore γ). This in turn means that $\Phi_e(M; z) = \Phi_{e,s_1}(\gamma; z)$ as required.

LEMMA 3. *If $\Phi_e(M)$ is total and if the well resided e-state is $\alpha = \beta * 1$, then $\Phi_e(M) \equiv_T M$.*

Proof. Let s_0, σ be chosen as in Lemma 2, but with $\alpha = \beta * 1$. Now use the standard properties of e-splittings for extensions of σ.

This completes the proof of Theorem C, and hence establishes the existence of a minimal degree below $0'$ not recursive in any 1-generic degree.

We remark that all the usual full approximation variations will apply to this construction. For example, each r.e. degree and each high Δ_2^0 degree will bound a minimal degree not recursive in a 1-generic degree. We refer the reader to Lerman [**8**] for details of such variations. There remain several interesting questions regarding the relationship between Sacks forcing and Cohen forcing. Some of these are mentioned in Chong [**1**]. By the results of Haught [**6**], there exist 1-generic degrees below $0'$ that are minimal covers (of 1-generic degrees). It would appear to be an interesting question to decide what initial segment-type results are possible. In particular, are there 1-generic strong minimal covers? A related question is the following: can a hyper-immune free minimal degree be recursive in a 1-generic degree?

It seems conceivable that the ideas of this paper may be useful in answering this question. Finally we remark that Jockusch has pointed out that another method of constructing a minimal degree not recursive in a 1-generic degree would be to construct a set M of such minimal degree separating a recursively inseparable pair of r.e. sets. However it is unclear as to how such a set may be constructed.

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