THE COMPUTATIONAL POWER OF MAXIMAL SETS

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Abstract. We introduce the notion of *eventually uniformly weak truth table array computable (e.u.wtt-a.c.)* sets. As our main result, we show that a computably enumerable (c.e.) set has this property iff it is weak truth table (*wtt*-) reducible to a maximal set. Moreover, in this equivalence we may replace maximal sets by quasi-maximal sets, hyperhypersimple sets or dense simple sets and we may replace *wtt*-reducibility by identity-bounded Turing reducibility (or any intermediate reducibility).

Here, a set A is e.u.wtt-a.c. if there is an effective procedure which, for any given partial *wtt*-functional $\hat{\Phi}$, yields a computable approximation $g(x, s)$ of the domain of $\hat{\Phi}^A$ together with a computable indicator function $k(x, s)$ and a computable order $h(x)$ such that, once the indicator becomes positive, i.e., $k(x, s) = 1$, the number of the mind changes of the approximation g on x after stage s is bounded by $h(x)$ where, for total $\hat{\Phi}^A$, the indicator eventually becomes positive on almost all arguments x of $\hat{\Phi}^A$.

In addition to our main result, we show several properties of the computably enumerable e.u.wtt-a.c. sets. For instance, the class of these sets is closed downwards under *wtt*-reductions and closed under join. Moreover, we relate this class to – and separate it from – well known classes in the literature. On the one hand, the class of the *wtt*-degrees of the c.e. e.u.wtt-a.c. sets is strictly contained in the class of the array computable c.e. *wtt*-degrees. On the other hand, every bounded low set is e.u.wtt-a.c. but there are e.u.wtt-a.c. c.e. sets which are not bounded low. Here a set A is bounded low if $A^\dagger \leq_{wtt} \emptyset^\dagger$, i.e., if $A^\dagger$ is ω-c.a., where $A^\dagger$ is the *wtt*-jump of A (Anderson, Csima and Lange [ACL17]).

Finally, we prove that there is a strict hierarchy within the class of the bounded low c.e. sets A depending on the order h that bounds the number of mind changes of a computable approximation of $A^\dagger$, and we show that there exists a Turing complete set A such that $A^\dagger$ is h-c.a. for any computable order h with $h(0) > 0$.

1. Introduction

The first goal of this paper is to seek to understand the computational power of a class of computably enumerable sets, the maximal sets, in terms of what kinds of sets they can compute, at least through the eyes of a strong reducibility. Our answer to this question, yields another goal of this paper. We introduce a new hierarchy classifying computably enumerable sets according to their ability to compute functions and sets, measured in terms of their "mind change" moduli. This is in the spirit of the Strong Jump Tracing [GT18] and Downey-Greenberg Hierarchies [DG20]. However, our new hierarchy is not aligned to either of these hierarchies. The new hierarchy is generated via a calibration method involving

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strong reducibilities and restricted forms of the jump, which may well have further applications. Before we turn to our results, we wish to place them in a historical context.

1.1. **Post's Programme and Maximal Sets.** Early applications of computability theory to demonstrate problems in classical mathematics were algorithmically undecidable all worked essentially in the same way. These proofs directly code the halting problem into the decision question at hand. It seemed that all *semidecidable* (i.e. computably enumerable) problems, such as the entscheidungsproblem or the word problem for groups, were simply the halting problem in disguise. This observation led to *Post's Problem* which asked if there were intermediate computably enumerable Turing degrees. That is, do there exist c.e. $\mathbf{a}$ with $\mathbf{0} <_T \mathbf{a} <_T \mathbf{0}'$?

In the quest to solve this question, *Post's Programme* [Pos44] tried to find a "thinness" property of the complement of a c.e. set which would guarantee Turing incompleteness. In [Pos44], Post gave the motivation for this programme. Whilst he could not solve his problem, he observed that it is possible to solve it using the thinness approach for reducibilities stronger than $\leq_T$, such as m- and tt-reducibilities[1]. For example, recall that a co-infinite c.e. set A is *simple* if its complement is immune: it has no infinite c.e. subsets. Also, A is called *hypersimple* if there is no computable sequence of pairwise disjoint canonical finite sets $\{D_{f(x)} \mid x \in \omega\}$ where for all x, $D_{f(x)} \cap \overline{A} \neq \emptyset$. Post showed that if A is simple then it has intermediate m-degree, and hypersimple sets have intermediate tt-degrees. One reducibility stronger than T-reducibility (but weaker than tt-reducibility) is *weak truth table* (*wtt*-) reducibility, where $A \leq_{wtt} B$ means that there is a Turing procedure Φ and a computable function φ, such that $\Phi^B(x) = A(x)$ and the use of $\Phi^B(x)$ is less than $\varphi(x)$ for all x. If φ is the identity function, then we would say $A \leq_{ibT} B$ (identity bounded Turing reducibility). Friedberg and Rogers [FHR59] showed that hypersimple sets have intermediate wtt-degrees[2].

It is worth noting that the concepts introduced by Post [Pos44] have been highly influential. The original solution to Post's Problem was by Friedberg [Fri57] and Muchnik [Muc56]. These papers famously introduced the priority method in computability theory. The concepts of immunity and hyperimmunity correlate with various domination properties whose ramifications are still being explored today, both in computability theory and in reverse mathematics. As well as being mainstays of computational complexity (in time bounded form) Post's fine-grained reducibilities have had applications especially in the theory of algorithmic randomness as they allow for transfer of measure.

Since Post was unable to show that any of his c.e. sets with thin complements were necessarily Turing incomplete, he suggested that perhaps there were c.e. sets with even thinner complements. He asked if there exists a c.e. maximal set. That is, a c.e. co-infinite set M such that for all c.e. sets W, if $M \subseteq W$ then either $M =^* W$ (finitely different) or $W =^* \omega$. Post did not know if there was a maximal c.e. set. In [Fri58], Friedberg gave a novel and intricate construction of a maximal

[1]We remind the reader that $A \leq_m B$ means that A is computable or there is a computable function f such that $x \in A$ iff $f(x) \in B$. $A \leq_{tt} B$ can be formulated as $A \leq_T B$ via a Turing procedure Φ, $\Phi^B = A$, such that Φ^X is total for all oracles X. Both of these reducibilities were clarified by Post [Pos44].

[2]In fact Downey and Jockusch [DJ87] showed that if A is hypersimple, then there is no set X with $A \not\leq_{wtt} X$ and $A \oplus X \geq_{wtt} \emptyset'$.

set, using a primitive form of the infinite injury method. Would such a set yield a realization of Post's Programme?

Alas no. We now know that Post's Programme, in its original form, has a negative solution since there is a Turing complete maximal set (Yates [Yat65]). Moreover, since Soare [Soa74] showed that all maximal sets were automorphic in the automorphism group of the lattice of c.e. sets, there are no "extra" properties we could add to maximality which would guarantee Turing incompleteness. Indeed, Cholak, Downey and Stob [CDS92] showed that no property of the lattice of supersets of a c.e. set A *alone* can guarantee incompleteness.

Ultimately Post's intuition that structural properties of a c.e. set in the lattice of c.e. sets can guarantee incompleteness does have a realization. Harrington and Soare [HS91] showed that there *is* an elementarily definable property Q of c.e. sets such that $Q(A)$ guarantees incompleteness and noncomputability, and there were c.e. sets A such that $Q(A)$ held.

1.2. **Maximal sets.** Turning the Post programme on its head, Martin [Mar66] and Tennenbaum [Ten61] showed that maximal sets are *computationally powerful*, rather than weak, as measured by Turing degree. In particular, they are all *high*. That is, if M is maximal then $M' \equiv_T \emptyset''$. Thus, computationally, they are indistinguishable from the halting problem when we use only the Turing jump to understand them.

The high c.e. degrees are an important well-understood class. Martin realized that the high c.e. degrees capture the computational complexity of a number of classes of c.e. sets. He showed that the high c.e. degrees are precisely the degrees capturing the combinatorics and computational power (in terms of $\leq_T$) of dense simple, r-maximal, hyperhypersimple, and similar sets[3]. High sets have the ability to compute a function g wich *dominates* all computable functions[4]. This highness characterization in terms of domination properties has also been used in many other contexts from computable model theory, degree theory (see Lerman [Ler85]), algorithmic learning theory (Gold [Gol67]), algorithmic randomness (see, e.g., Nies, Stephan and Terwijn [NST05]), etc. Similar jump characterizations such as low$_2$ (i.e. $X'' \equiv_T \emptyset''$) have proven very productive (Lerman [Ler85], for example).

1.3. **New initiatives.** Beginning with the work of Downey, Jockusch and Stob [DJS90, DJS96], a finer classification of c.e. sets has been initiated. This classifies sets according to the number of mind changes needed to compute approximations. Shoenfield's Limit Lemma says that $f \leq_T \emptyset'$ iff there is a computable approximation $g(\cdot,\cdot)$ such that $g(x,s+1) \neq g(x,s)$ for only finitely many s and $f(x) = \lim_s g(x,s)$. Going back to work of Ershov [Ers70], it is possible to understand how complex a Δ^0_2 set or function is by classifying the complexity of its computable approximations g according to their "mind change" functions:

$$m_g(x) = |\{s \mid g(x,s+1) \neq g(x,s)\}|.$$

For example, $f \leq_{wtt} \emptyset'$ iff there is a computable approximation g of f and a computable h where $m_g(x) \leq h(x)$. We say that f is *h-c.a. (computably approximable)* where we usually assume that the bound h is an *order*, i.e., nondecreasing and unbounded.

[3] It is not important here to define these sets, save to say that they are important classes of c.e. sets.

[4] That is, if f is computable then $g(x) > f(x)$ for almost all x.

The [DJS90] intuition is that we can classify c.e. degrees and sets according to the mind-change complexity of the functions computable from them. A degree would be computationally weak in this sense if it could only compute things with computable approximations which have few mind changes. Downey, Jockusch and Stob studied the c.e. degrees $\mathbf{a}$, such that there is a computable h such that every function $f \leq_T \mathbf{a}$ is h-c.a. These degrees are called the *array computable* degrees. They precisely capture the combinatorics of a wide class of degree classes in several parts of computability theory. (This is elaborated in Section 7.) This idea was later generalized by Downey and Greenberg [DG20] by using computable ordinals into an infinite hierarchy, where array computability was the bottom of this hierarchy. The second level is a non-uniform version of being array computable.

Definition 1.1. $\mathbf{a}$ *is called* totally ω-c.a. *iff for each* $f \leq_T \mathbf{a}$*, there is a computable* h *such that* f *is* h*-c.a.*

Again the notion of being totally ω-c.a. captures the combinatorics of a large number of constructions in computability theory, algorithmic randomness and effective model theory. We refer the reader to [DG20] for a detailed discussion.

Related here, and of great relevance to us, comes the notion of approximating *partial* $\mathbf{a}$-computable functions. Frequently this is done in terms of *tracing.* We say that a set A (and its degree $\mathbf{a}$) is *jump traceable at order* h (or h*-jump traceable* for short) if, for any partial A-computable function ψ, there are uniformly c.e. sets $\{T_n \mid n \in \omega\}$ with $|T_n| < h(n)$ for all n and $\psi(n)\downarrow$ implying $\psi(n) \in T_n$. This notion was explicitly introduced by Nies [Nie06], although the idea had been used earlier. Certainly the idea of taming the complexity of a function using tracing had arisen in set theory, and this was the inspiration for its use in algorithmic randomness where it is used to characterize lowness for Schnorr randomness and helps understand K-triviality (see Terwijn and Zambella [TZ01], Downey and Hirschfeldt [DH10]).

Jump tracing is widely used in the theory of algorithmic randomness, and is a *lowness* property, in that it implies *computational weakness.* For example, a natural refinement of the notion of a low set is called *superlowness.* A set A is superlow if $A' \equiv_{wtt} \emptyset'$ (equivalently, $A' \equiv_{tt} \emptyset'$). It is easy to see that a c.e. set A is superlow iff A is h-jump traceable for some computable order h.

1.4. **The computational power of maximal sets.** Our work was inspired by the following attractive result.

Theorem 1.2 (Barmpalias, Downey and Greenberg [BDG10])**.** *A c.e. set* A *is wtt-computable from a hypersimple c.e. set iff* A *has totally* ω*-c.a. Turing degree.*

Our motivating question, asked by Ambos-Spies, is

> "What is the analog of Theorem 1.2 if we replace hypersimple by maximal?".

Our hope was that we would get a class of sets A which fell into one of the classes totally ω-c.a., array computable, superlow, or similar classes already investigated in the literature. Unfortunately, this is not the case. It turned out that the answer to Ambos-Spies's question lay not in understanding the *Turing jump*, but a weaker notion called the *wtt-jump.*

The way that this classification came about was quite natural. Initially, we found that if A was a c.e. *superlow* set, $A \leq_{wtt} M$ for some maximal set M. We analysed the proof and realized that all of the uses of the computations in the construction had computably bounded uses, and a weaker notion being *wtt-superlow* sufficed. We discuss this concept in the next subsection.

1.5. **The *wtt*-jump.** The strong reducibilities, in particular *wtt*-reducibility and *tt*-reducibility, turned out to be a central unifying idea in algorithmic randomness (e.g., Downey-Hirschfeldt [DH10]). This fact, plus work in many other areas using reducibilities stronger than Turing, show that, rather than mere artifacts of definitions in classical computability theory, hierarchies related to strong reducibilities and bounded jump operators (such as those below) can give classification and unification of combinatorics in parts of computable mathematics. As a consequence, it seems we should better understand analogs of the core notions of classical computability for such hierarchies. This paper contributes to that program.

The earliest analog of a jump operator using only bounded reducibilities is the "mini-jump" hierarchy introduced by Ershov [Ers70] as discussed in Odifreddi [Odi99], Chapter XI.6. Ershov's hierarchy concerned a jump operator for the m-degrees involving the partial m-degrees. Also a bounded analog of the jump for *tt*-reductions was investigated by Gerla [Ger79].

For us the bounded jump for *wtt*-reductions will be of interest. As in Downey and Greenberg [DG20], from a standard listing of all pairs consisting of a partial Turing procedure and a partial computable function, we obtain a standard listing $\{\hat{\Phi}_e\}_{e\in\omega}$ of the partial *wtt*-functionals together with a computable listing $\{\hat{\varphi}_e\}_{e\in\omega}$ of the corresponding partial computable use bounds (see Section 3 below for details). Then the *wtt-jump* or *bounded jump* of a set A is defined by

$$A^{\dagger} = \{\langle e, x\rangle : \hat{\Phi}_e^A(x)\downarrow\}.$$

Clearly the usual equivalences obtained by the s-m-n theorem apply. So the *wtt*-jump of A is (up to m-degree) the same as the diagonal *wtt*-jump $\{e \mid \widehat{\Phi}_e^A(e)\downarrow\}$ (in the literature sometimes the latter is denoted by $A^{\dagger}$). Note that $\emptyset' \equiv_m \emptyset^{\dagger}$, and that for a c.e. set A, $\emptyset' \leq_{wtt} A^{\dagger} \leq_{wtt} (\emptyset')^{\dagger}$. Moreover if X is Δ^0_2, $X^{\dagger}$ is also Δ^0_2.

The analog of the idea of lowness for the bounded jump can be defined as follows. A set A is *bounded low* or *wtt-superlow* if $A^{\dagger} \leq_{wtt} \emptyset'$ (or, equivalently, $A^{\dagger} \leq_{tt} \emptyset'$). Variations of bounded lowness - all of them *wtt*-equivalent to this notion - have been studied by Coles, Downey, and LaForte [CDL98], Csima, Downey and Ng [CDN11], Anderson and Csima [AC14], Ambos-Spies, Downey and Monath [ASDMss], and Wu and Wu [WW19]. It is easy to see that all superlow sets A are bounded low (i.e., *wtt*-superlow), but below we prove that there are Turing complete bounded low c.e. sets. (This result was independently obtained by Wu and Wu [WW19].)

We discovered that, if A is *wtt*-superlow, then $A \leq_{wtt} M$ for some maximal set M. However, *wtt*-superlowness is not a necessary condition to be $\leq_{wtt} M$ for some maximal M.

1.6. **The main theorem.** In the end, we discovered that a technical variation of the idea above actually gives a necessary and sufficient condition. This variation will be defined in Section 4, and is called *eventually uniformly wtt-array computable.* Armed with this notion we prove the following.

Theorem 1.3. *For a c.e. A, $A \leq_{ibT} M$ for some maximal set M iff $A \leq_{wtt} M$ for some maximal set M iff A is eventually uniformly wtt-array computable.*

Moreover, in this theorem we may replace maximal sets by quasimaximal sets or hyperhypersimple sets or dense simple sets. The proof of Theorem 1.3 is quite technical and will be given in Section 4.

In the later sections we show that the *wtt*-degrees of the eventually uniformly *wtt*-array computable c.e. sets form an ideal (Section 5), and we relate the e.u.wtt-a.c. sets to other lowness notions thereby giving strict lower and upper bounds on the class of the c.e. sets with this property (and their *wtt*-degrees). First we show that any *wtt*-superlow set is eventually uniformly *wtt*-array computable (Section 6) and that any eventually uniformly *wtt*-array computable c.e. set is array computable (Section 7). Then, in the final Section 8, we give separations of these concepts by showing that there are maximal sets which are not *wtt*-superlow, and there are array computable c.e. sets which are not *wtt*-reducible to any maximal set. Note that, by *wtt*-invariance of the e.u.wtt-a.c. property, these separations extend to the corresponding *wtt*-degrees.

1.7. **A new hierarchy of bounded lowness.** In addition, in Section 6, we have a closer look at the *wtt*-superlow (i.e., bounded low) c.e. sets. We remark that Anderson, Csima and Lange already demonstrated in [ACL17] that the bounded jump and the Turing jump are quite different with respect to the low/high hierarchy by showing the existence of both a low set which is bounded high and a high set which is bounded low. For example we can sharpen at least on of these results by demonstrating the following.

Theorem 1.4. *There is a T-complete wtt-superlow c.e. set.*

In fact, we get more. The *wtt*-analog of (h-)jump traceability, (h-)*wtt*-jump traceability, turns out to be equivalent to *wtt*-superlowness (just as *jump*-traceability is equivalent to superlowness). This leads to a new hierarchy of the *wtt*-superlow sets based on the growth rates of the orders h. As we show, this hierarchy is proper. So we can define very strong lowness notions such as A being *strongly wtt*-superlow if A is h-*wtt*-jump traceable for all computable orders h. Theorem 1.4 can actually be improved to say that there is a Turing complete strongly *wtt*-superlow set. Whilst we have only begun exploration of this new hierarchy, we will prove this and some other results in Section 6.

2. Notation

We follow the standard notation as given in [Soa87]. In particular, $\{\Phi_e\}_{e\in\omega}$ denotes a standard enumeration of all Turing functionals, where $\varphi_e^A(x)$ denotes the use of a computation of $\Phi_e^A(x)$ with oracle A and input x. Moreover, $\{\varphi_e\}_{e\in\omega}$ denotes a standard enumeration of all unary partial computable functions and $\{W_e\}_{e\in\omega}$, where $W_e = \text{dom}(\varphi_e)$ – the domain of φ_e – denotes the induced standard enumeration of all c.e. sets. We let $\Phi_{e,s}^A(x)$, $\varphi_{e,s}^A(x)$ and $\varphi_{e,s}(x)$ denote the approximation of $\Phi_e^A(x)$, $\varphi_e^A(x)$ and $\varphi_e(x)$ within s steps, respectively, and we let $W_{e,s} = \text{dom}(\varphi_{e,s})$. We adapt the now commonly used Lachlan notation for approximations of computations, i.e., if A is a set and $\{A_s\}_{s\in\omega}$ is a sequence of sets approximating A in the limit then we let $\Phi_e^A(x)[s] = \Phi_{e,s}^{A_s}(x)$. Finally, we follow the usual convention on converging computations, i.e., for any oracle A and any numbers e, x, y, s, if $\Phi_{e,s}^A(x) \downarrow= y$ then $\max\{e, x, y, \varphi_e^A(x)\} < s$, and, similarly, if $\varphi_{e,s}(x) \downarrow= y$ then $\max\{e, x, y\} < s$; in particular, we have $W_{e,s} \subseteq \omega \upharpoonright s$.

3. Basic Definitions and Properties

Let us start by giving the definition of the bounded jump. The underlying notation is mostly adapted from [DG20].

Definition 3.1 ([DG20]). *For any set $X \subseteq \omega$ and for any numbers $e_0, e_1, y \in \omega$, let*

$$\hat{\Phi}^X_{\langle e_0,e_1\rangle}(y) = \begin{cases} \Phi^X_{e_0}(y) & \text{if } \Phi^X_{e_0}(y)\downarrow,\ \varphi_{e_1}(y)\downarrow \text{ and } \varphi^X_{e_0}(y) \le \varphi_{e_1}(y), \\ \uparrow & \text{otherwise,} \end{cases} \tag{1}$$

$$\hat{\varphi}_{\langle e_0,e_1\rangle} = \varphi_{e_1}. \tag{2}$$

Given a set A, the (diagonal) bounded jump *and the* bounded jump function *of A, denoted by $A^\dagger$ ($A^\dagger_d$) and $\hat{J}^A$, respectively, are defined as*

$$A^\dagger = \{\langle e,x\rangle : \hat{\Phi}^A_e(x)\downarrow\}, \tag{3}$$

$$A^\dagger_d = \{e : \hat{\Phi}^A_e(e)\downarrow\}, \text{ and} \tag{4}$$

$$\hat{J}^A(e) = \hat{\Phi}^A_e(e). \tag{5}$$

For notational convenience, we define the bounded jump $A^\dagger$ of a set A such that $A^\dagger$ codes all computations of partial *wtt*-functionals instead of only the diagonal computations, the latter one being denoted by $A^\dagger_d$. However, it is easy to see that $A^\dagger$ and $A^\dagger_d$ are computably isomorphic (see clause 3. of Lemma 3.4 below). Before we start examining some of the properties of $A^\dagger$ and $\hat{J}^A$ for a (c.e.) set A, let us make some general remarks on the definition of $\hat{\Phi}_e$ and introduce some terminology to be used below which is also mostly taken from [DG20]. First of all, we say that a Turing functional Φ is a *wtt-functional* if there exists a number $e \in \omega$ such that $\Phi = \hat{\Phi}_e$. Note that, for any set A and any total function g, $g \le_{wtt} A$ holds iff there exists $e \in \omega$ such that $g = \hat{\Phi}^A_e$. So $\{\hat{\Phi}_e\}_{e\in\omega}$ incorporates all *wtt*-reductions.

Using $\{\hat{\Phi}_e\}_{e\in\omega}$, we may extend the definition of being *wtt*-reducible to a set A to partial functions. We say that a partial function $\varphi : \omega \to \omega$ is *wtt-reducible* to a set A, and denote it by $\varphi \le_{wtt} A$, if there exists $e \in \omega$ such that $\varphi = \hat{\Phi}^A_e$. Furthermore, for sets A and B, we say that A is *bounded computably enumerable in B*, *bounded c.e. in B* or *bounded B-c.e.* for short, if there exists a partial function φ such that φ is *wtt*-reducible to B and $A = \mathrm{dom}(\varphi)$. In particular, $A^\dagger$ is bounded c.e. in A for all sets A.

We fix computable approximations $\hat{\Phi}^X_{\langle e_0,e_1\rangle,s}(y)$ ($s \ge 0$) of $\hat{\Phi}^X_{\langle e_0,e_1\rangle}(y)$ where $\hat{\Phi}^X_{\langle e_0,e_1\rangle,s}(y)$ is defined iff $\hat{\Phi}^X_{\langle e_0,e_1\rangle}(y)$, $\Phi^X_{e_0,s}(y)$ and $\varphi_{e_1,s}(y)$ are defined. Then, for any c.e. set A and any fixed computable enumeration $\{A_s\}_{s\in\omega}$ of A, we have a canonical approximation to $A^\dagger$, denoted by $\{A^\dagger_s\}_{s\in\omega}$, such that, for all numbers e, x, we have that $\langle e,x\rangle \in A^\dagger_s$ iff $\hat{\Phi}^A_e(x)[s]\downarrow$. We tacitly assume that this approximation to $A^\dagger$ is clear from the context whenever a c.e. set A and a computable enumeration of A is given to or constructed by us. Note that if $\hat{\Phi}^A_e(x)[s]\downarrow$ holds for infinitely many stages s then $\hat{\Phi}^A_e(x)\downarrow$ holds as the use of $\hat{\Phi}_e$ is bounded (this does not hold for Turing functionals in general).

Moreover, we will often make use of the Recursion Theorem (with Parameters) with respect to $\{\hat{\Phi}_e\}_{e\in\omega}$. For that, we need the following definition.

Definition 3.2. *A sequence of wtt-functionals $\{\Psi_e\}_{e\in\omega}$ is* uniformly computable *if $\{\Psi_e\}_{e\in\omega}$ is uniformly computable in the sense of Turing functionals and there exists* *a uniformly computable sequence of partial computable functions $\{\psi_e\}_{e\in\omega}$ such that, for any $e \in \omega$, the use of Ψ_e is bounded by ψ_e.*

Then the following lemma says that $\{\hat{\Phi}_e\}_{e\in\omega}$ is a Gödel numbering of the *wtt*-functionals whence we may argue as in the proof of the classical Recursion Theorem (with Parameters) that the Recursion Theorem also holds for uniformly computable sequences of *wtt*-functionals.

Lemma 3.3 (Recursion Theorem (with Parameters))**.** *Let $\{\Psi_e\}_{e\in\omega}$ be a sequence of wtt-functionals and $g : \omega \to \omega$ and $H : \omega^2 \to \omega$ be total computable functions. Then the following holds.*

1. *$\{\Psi_e\}_{e\in\omega}$ is uniformly computable iff there exists a computable one-one function $f : \omega \to \omega$ such that $\Psi_e^A = \hat{\Phi}_{f(e)}^A$ holds for any number e and any set A.*
2. *There exists $e \in \omega$ such that $\hat{\Phi}_{g(e)} = \hat{\Phi}_e$.*
3. *There exists a computable function $h : \omega \to \omega$ such that $\hat{\Phi}_{h(e)} = \hat{\Phi}_{H(h(e),e)}$ holds for any $e \in \omega$.*

Proof. For the "only if"-part of clause 1., note that a sequence $\{\hat{\Phi}_{f(e)}\}_{e\in\omega}$, where $f : \omega \to \omega$ is a computable function, is a uniformly computable sequence of *wtt*-functionals since the use bound $\{\hat{\varphi}_{f(e)}\}_{e\in\omega}$ is a uniformly computable sequence of partial computable functions. For the "if"-direction, by Definition 3.2, we may fix computable one-one functions $f_i : \omega \to \omega$ ($i \leq 1$) such that, for any $e \in \omega$, we have $\Psi_e = \Phi_{f_0(e)}$ and $\psi_e = \varphi_{f_1(e)}$. Then, by (1) and by assumption on Ψ_e, we have that $\Psi_e = \hat{\Phi}_{f(e)}$ for the computable one-one function $f(e) = \langle f_0(e), f_1(e)\rangle$.

For the proofs of clauses 2. and 3., it is easy to see that the proofs of the Recursion Theorem and the Recursion Theorem with Parameters can be carried out in the setting of uniformly computable *wtt*-functionals. In the following, we give a sketch of the proofs by outlining the critical parts.

For clause 2., the proof is as follows. For any numbers $e, x \in \omega$ and any set A, let

$$\Psi_e^A(x) = \begin{cases} \hat{\Phi}_{\varphi_e(e)}^A(x) & \text{if } \varphi_e(e)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then the sequence $\{\Psi_e\}_{e\in\omega}$ is a uniformly computable sequence of Turing functionals whose use if uniformly bounded by $\psi_e(x) = \hat{\varphi}_{\varphi_e(e)}(x)$. So since $\{\psi_e\}_{e\in\omega}$ is a uniformly computable sequence of partial computable functions, by clause 1., we may fix a computable function $d : \omega \to \omega$ such that $\Psi_e^A = \hat{\Phi}_{d(e)}^A$ holds for any $e \in \omega$ and any set A, and we may fix $i \in \omega$ such that $\varphi_i(x) = g(d(x))$. Then, by virtually the same argument as in the proof of the classical Recursion Theorem, it follows that $e = d(i)$ is a fixed point for g.

For clause 3., we argue analogously. Let

$$\Psi_{\langle x,y\rangle}^A(z) = \begin{cases} \hat{\Phi}_{\varphi_x(\langle x,y\rangle)}^A(z) & \text{if } \varphi_x(\langle x,y\rangle)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

Then since $\{\Psi_e\}_{e\in\omega}$ is clearly a uniformly computable sequence of Turing functionals and $\{\psi_e\}_{e\in\omega}$, where $\psi_{\langle x,y\rangle}(z) = \hat{\varphi}_{\varphi_x(\langle x,y\rangle)}(z)$ is a uniformly computable sequence of partial computable functions bounding the use of $\Psi_{\langle x,y\rangle}^A$ for any $x, y \in \omega$ and any set A, we may easily argue as in the proof of the Recursion Theorem with Parameters that $h(x) = d(i, x)$ is as desired, where, by clause 1., $d : \omega^2 \to \omega$ is chosen such that $\Psi^A_{\langle x,y\rangle} = \hat{\Phi}^A_{d(x,y)}$ holds and $i \in \omega$ is chosen such that $\varphi_i(\langle x, y\rangle) = H(d(x, y), y)$ holds for all $x, y \in \omega$. □

It is natural to ask what properties does the bounded jump operator share with the classical Turing jump operator if we replace Turing reductions by *wtt*-reductions. In the following lemma, we list some of the common properties which can be found in [DG20, p.30pp].

Lemma 3.4 ([DG20])**.** *Let A and B be any (not necessarily c.e.) sets. Then the following holds.*

1. *If $A \leq_{wtt} B$ then there exists a strictly increasing computable function $f : \omega \to \omega$ such that, for any $e \in \omega$, $\hat{\Phi}^A_e = \hat{\Phi}^B_{f(e)}$.*
2. *$A^\dagger$ is 1-complete for the class of the bounded A-c.e. sets. In particular, $\emptyset'$ is computably isomorphic to $\emptyset^\dagger$.*
3. *There exists a strictly increasing computable function $f : \omega \to \omega$ such that, for any e, x and any set A, $\hat{\Phi}^A_e(x) = \hat{J}^A(f(\langle e, x\rangle))$. Hence, $A^\dagger$ is computably isomorphic to $A^\dagger_d$.*
4. *$A <_{wtt} A^\dagger$.*
5. *$A \leq_{wtt} B$ implies $A^\dagger \leq_1 B^\dagger$.*

However, not every property of the Turing jump carries over to the bounded jump as the following lemma of [DG20] shows.

Lemma 3.5 ([DG20], Lemma 3.6)**.** *There is a c.e. set B and a set A such that $A^\dagger \leq_1 B^\dagger$ holds but $A \not\leq_{wtt} B$.*

The fact that the converse of clause 5. in Lemma 3.4 fails is due to the fact that the Complement Lemma does not carry over to bounded-c.e. sets as Downey and Greenberg also show in [DG20, Proposition 3.1(3)]. However, the proof of Lemma 3.5 (and similarly for [DG20, Proposition 3.1(3)]) builds on the fact that the set A constructed there may change its mind whether a given x is in A or not more than once. This leaves the question open whether the Complement Lemma and hence the converse of clause 5. in Lemma 3.4 hold if A is chosen to be computably enumerable. We can affirmatively answer both questions.

Lemma 3.6. *For any sets A and B such that A is c.e. or co-c.e., if A and $\overline{A}$ are bounded-c.e. in B then $A \leq_{wtt} B$. In particular, if A and B are c.e. then $A^\dagger \leq_1 B^\dagger$ implies that $A \leq_{wtt} B$ holds.*

Proof. For a proof of the first part of the lemma, fix sets A and B such that A is c.e. or co-c.e. and A and $\overline{A}$ are bounded-c.e. in B. By $\overline{A} \equiv_{wtt} A$ w.l.o.g. we may assume that A is computably enumerable. So fix a computable enumeration $\{A_s\}_{s\in\omega}$ of A and fix a number e such that $\overline{A} = \mathrm{dom}(\hat{\Phi}^B_e)$. Then we can compute A from B by a Turing reduction whose use is computably bounded as follows.

Let $f(x) = \mu s(x \in A_s \text{ or } \hat{\varphi}_{e,s}(x)\downarrow)$. Then f is a total computable function as $\hat{\varphi}_e(x)\downarrow$ holds for any number $x \notin A$. Given x, with oracle B compute the least stage $s \geq f(x)$ such that either $x \in A_s$ or $\hat{\Phi}^B_{e,s}(x)\downarrow$. Then, by our assumptions on A, stage s exists, and $x \in A$ iff $x \in A_s$. Moreover, since, by the convention on converging computations, $\hat{\varphi}_e(x) < f(x)$ if $\hat{\varphi}_e(x)\downarrow$, $B \upharpoonright f(x)$ can compute the stage s.

For the second part of Lemma 3.6, it suffices to note that $A^\dagger \leq_1 B^\dagger$ implies that A and $\overline{A}$ are bounded-c.e. in B. So the second part follows from the first part. $\square$

Next, we formulate and prove the main result of this paper.

4. C.E. Sets Which Are Bounded Turing Reducible To Maximal Sets

For our main result, we make the following definition.

Definition 4.1. *A set A is* eventually uniformly *wtt*-array computable *(*e.u.wtt-a.c. *for short) if there exist computable functions $g, k : \omega^2 \to \{0,1\}$ and a computable order $h : \omega \to \omega$ such that, for all e, x,*

$$A^\dagger(x) = \lim_{s\to\infty} g(x,s), \tag{6}$$

$$k(x,s) \leq k(x,s+1), \tag{7}$$

$$k(x,s) = 1 \Rightarrow \ |\{t \geq s : g(x,t+1) \neq g(x,t)\}| \leq h(x), \tag{8}$$

$$\forall e\ (\hat{\Phi}_e^A \textit{ total } \Rightarrow \ \forall^\infty x\ \exists s\ (k(\langle e,x\rangle, s) = 1)). \tag{9}$$

For functions g, k and h as above, we say that A is eventually uniformly *wtt*-array computable via *g, k* and *h, and we let* EUwttAC *denote the class of all c.e. e.u.wtt-a.c. sets.*

Now the main result is as follows.

Theorem 4.2 (Characterization Theorem)**.** *For a c.e. set A the following are equivalent.*

(i) A is eventually uniformly wtt-array computable.
(ii) A is wtt-reducible to some maximal (quasi-maximal, hh-simple, dense simple) set.
(iii) A is ibT-reducible to some maximal (quasi-maximal, hh-simple, dense simple) set.

Since *ibT*-reducibility is stronger than *wtt*-reducibility, for a proof of Theorem 4.2 it suffices to prove the implications $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (i)$. In fact, since the strength of the simplicity notions considered here is ordered by

maximal $\Rightarrow$ quasi-maximal $\Rightarrow$ hh-simple $\Rightarrow$ dense simple

(see, e.g., Soare [Soa87], page 211), in the proof of the former implication it suffices to consider maximal sets, and in the proof of the latter implication it suffices to consider dense simple sets. So Theorem 4.2 follows from the following two theorems.

Theorem 4.3. *Let A be c.e. and eventually uniformly wtt-array computable. Then A is ibT-reducible to some maximal set.*

Theorem 4.4. *Let A and D be c.e. sets such that $A \leq_{wtt} D$ and D is dense simple. Then A is eventually uniformly wtt-array computable.*

In the remainder of this section we prove these two theorems.

Proof of Theorem 4.3. Let $\{A_s\}_{s\in\omega}$ be a computable enumeration of A and fix computable functions $\hat{g}$, $\hat{k}$ and $\hat{h}$ which witness that A is e.u.wtt-a.c. according to Definition 4.1. We construct a c.e. set M in stages s, where M_s denotes the finite set of numbers which are enumerated into M by stage s, such that M is maximal and $A \leq_{ibT} M$. Clearly, any such M witnesses that Theorem 4.3 holds.

Before we give the formal construction, let us discuss some of the ideas behind it and introduce some of the concepts to be used in the construction. We start with the task of making M maximal.

In order to make M maximal, it suffices to ensure that the complement of M is infinite,

$$|\overline{M}| = \omega, \tag{10}$$

and that M meets the requirements

$$\mathcal{R}_e : \ \overline{M} \subseteq^* W_e \text{ or } \overline{M} \subseteq^* \overline{W_e}. \tag{11}$$

for $e \in \omega$.

In order to achieve these goals, just as in the classical maximal set construction (as for instance in Soare [Soa87]), we use n-states and "optimize" the states of almost all elements in $\overline{M}$. Since we use a priority tree here, however, in our definition of the states the infinitary outcome (corresponding to the case that $W_e \cap \overline{M}$ is infinite) is denoted by 0 (as common on priority trees) and not by 1 as in the classical definition of states. So here the *n-state of* a number x *at stage* s is the unique binary string $\sigma(n, x, s)$ of length n such that, for $e < n$,

$$\sigma(n, x, s)(e) = 0 \ \text{ iff } \ x \in W_{e,s},$$

and the *(true) n-state of* x is the unique binary string $\sigma(n, x)$ of length n such that, for $e < n$,

$$\sigma(n, x)(e) = 0 \ \text{ iff } \ x \in W_e.$$

Note that requirements $\mathcal{R}_0, \ldots, \mathcal{R}_n$ are met if almost all elements of $\overline{M}$ have the same $(n+1)$-state. So, in order to meet the maximal set requirements, it suffices to guarantee that, for any $n \geq 0$, almost all numbers in $\overline{M}$ have the same n-state. In the construction of M we achieve this by attempting to minimize the n-states of the numbers in $\overline{M}$ (which corresponds to the classical strategy of maximizing the (classically defined) n-states).

For this sake we use the full binary tree $T = \{0, 1\}^{<\omega}$ as the priority tree. Elements of T are called *nodes*. As usual, we say for two nodes α and β that α has *higher priority than* β and denote it by $\alpha < \beta$ iff $\alpha \sqsubset \beta$ (i.e., α is a proper initial segment of β) or α is to the left of β, denoted by $\alpha <_{left} \beta$, i.e., there exists $\gamma \in T$ such that $\gamma 0 \sqsubseteq \alpha$ and $\gamma 1 \sqsubseteq \beta$. Nodes are viewed as states in the following sense. A node $\alpha \in T$ of length n codes the guess that there are infinitely many numbers in $\overline{M}$ with n-state α. Then, assuming that $\overline{M}$ is infinite, there is a leftmost path through T such that, for any node α on this path, there are infinitely many elements of $\overline{M}$ which have state α. So it suffices to guarantee that almost all elements of $\overline{M}$ have state α.

In order to approximate the true path, for any node α and any stage s, we let

$$\begin{aligned} V_{\alpha,s} &= \overline{M_s} \restriction s \cap \{y : \sigma(|\alpha|, y, s) = \alpha\} \\ &= \overline{M_s} \restriction s \cap \{y : \forall\, e < |\alpha| \ (y \in W_{e,s} \Leftrightarrow \alpha(e) = 0)\} \end{aligned}$$

and

$$V_\alpha = \overline{M} \cap \{y : \sigma(|\alpha|, y) = \alpha\} = \overline{M} \cap \{y : \forall\, e < |\alpha| \ (y \in W_e \Leftrightarrow \alpha(e) = 0)\},$$

and we use the following *length of agreement function*

$$l(\alpha, s) = |V_{\alpha,s}|. \tag{12}$$

Based on $l(\alpha, s)$, we define the set of α-stages by induction on $|\alpha|$ as follows. Every stage is a λ-stage. An α-stage s is called *α-expansionary* if $s = 0$ or $l(\alpha 0, s) > l(\alpha 0, t)$ holds for all α-stages $t < s$. Then a stage s is an $\alpha 0$-stage if it is α-expansionary and an $\alpha 1$-stage if it is an α-stage but not α-expansionary. At stage s, the current approximation δ_s of the true path is the unique node α of length s such that s is an α-stage, and we say that α is *accessible* at stage $s+1$ if α is an initial segment of δ_s, i.e., $\alpha \sqsubseteq \delta_s$. Then the *true path* TP through T is defined by $TP = \liminf_{s\to\infty} \delta_s$, i.e., $TP \upharpoonright n$ is the leftmost node of length n which is accessible infinitely often (for every n).

Next we explore under which assumptions on M the true path TP actually has the desired properties, i.e., satisfies that, for any n, $TP \upharpoonright n$ is the leftmost node α of length n such that V_α is infinite. We start with some observations. Note that

$$V_{\alpha 0,s} = V_{\alpha,s} \cap W_{|\alpha|,s} \text{ and } V_{\alpha 1,s} = V_{\alpha,s} \cap \overline{W_{|\alpha|,s}}. \tag{13}$$

So $V_{\alpha,s}$ is the disjoint union of $V_{\alpha 0,s}$ and $V_{\alpha 1,s}$,

$$V_{\alpha,s} = V_{\alpha 0,s} \,\dot\cup\, V_{\alpha 1,s}, \tag{14}$$

and

$$l(\alpha, s) = l(\alpha 0, s) + l(\alpha 1, s). \tag{15}$$

Note that the analog of (14) holds for V_α, too, and that the equation can be extended to

$$V_{\alpha,s} = \dot{\bigcup}_{|\beta|=n} V_{\alpha\beta,s} \text{ and } V_\alpha = \dot{\bigcup}_{|\beta|=n} V_{\alpha\beta} \tag{16}$$

for any $n \geq 0$. Next note that, for any node α, $\{V_{\alpha,s}\}_{s\in\omega}$ is a computable approximation of V_α, i.e., for any number y,

$$V_\alpha(y) = \lim_{s\to\infty} V_{\alpha,s}(y). \tag{17}$$

Moreover, a number $y \in V_{\alpha,s}$ is in V_α unless y is enumerated into M after stage s or the $|\alpha|$-state of y decreases after stage s. So, if we let

$$\hat{V}_\alpha = \bigcup_{\{\alpha' : |\alpha'|=|\alpha| \,\&\, \alpha' \leq_{left} \alpha\}} V_{\alpha'},$$

then

$$\hat{V}_\alpha = \overline{M} \cap \left(\bigcup_{s\in\omega} \bigcup_{\{\alpha' : |\alpha'|=|\alpha| \,\&\, \alpha' \leq_{left} \alpha\}} V_{\alpha',s} \right). \tag{18}$$

In fact, if we say that α' is *stronger* than α ($\alpha' \prec \alpha$) if $\alpha' <_{left} \alpha$ or $\alpha \sqsubset \alpha'$ (i.e., viewed as a state, either α' is less than α or α' contains more information than α) then, by the definition of $\hat{V}_\alpha$ and (16), $\hat{V}_{\alpha'} \subseteq \hat{V}_\alpha$ for any α' which is stronger than α whence

$$\hat{V}_\alpha = \bigcup_{\{\alpha' : \alpha' \preceq \alpha\}} V_{\alpha'} = \overline{M} \cap \left(\bigcup_{s\in\omega} \bigcup_{\{\alpha' : \alpha' \preceq \alpha\}} V_{\alpha',s} \right). \tag{19}$$

We can now state the two crucial facts on TP used in the proof.

Claim 1 (Infinity Lemma). Assume (10). *For any node $\alpha \sqsubset TP$, the set S_α of the α-stages is infinite and*

$$\lim_{s\to\infty,\, s\in S_\alpha} l(\alpha, s) = \omega. \tag{20}$$

Moreover, if α' is to the left of TP then $S_{\alpha'}$ and $\hat{V}_{\alpha'}$ are finite.

Proof. For a proof of the first part, fix $\alpha \sqsubset TP$. The infinity of S_α is immediate by the definition of TP. The proof of (20) is by induction on $|\alpha|$. We distinguish the following three cases. First assume that $\alpha = \lambda$. Then $S_\alpha = \omega$ and $V_\lambda = \overline{M}$. So (20) holds by the infinity of $\overline{M}$. Next assume that $\alpha = \hat{\alpha}0$ for some node $\hat{\alpha}$. Then, by $\alpha \sqsubset TP$ there are infinitely many $\hat{\alpha}$-expansionary stages. So S_α is infinite and (20) holds by definition. Finally assume that $\alpha = \hat{\alpha}1$ for some node $\hat{\alpha}$. Then, by $\hat{\alpha}1 \sqsubset TP$, there are only finitely many $\hat{\alpha}0$-stages, whence $l(\hat{\alpha}0, s)$ is bounded. By the former, $S_\alpha =^* S_{\hat{\alpha}}$ while, by the latter and by (15), there is a constant c such that $l(\alpha, s) + c \geq l(\hat{\alpha}, s)$ for all stages s. So infinity of (20) follows by the inductive hypothesis.

For a proof of the second part, fix α' to the left of TP, let $\alpha = TP \upharpoonright |\alpha'|$ and let $\hat{\alpha}$ be the longest common initial segment of α' and α. Then $\hat{\alpha}0 \sqsubseteq \alpha'$ and $\hat{\alpha}1 \sqsubseteq \alpha$ whence $\hat{\alpha}$ and $\hat{\alpha}1$ are on the true path. By the definition of TP, it follows that $S_{\hat{\alpha}0}$ is finite. Since, by $\hat{\alpha}0 \sqsubseteq \alpha'$, $S_{\alpha'} \subseteq S_{\hat{\alpha}0}$, $S_{\alpha'}$ is finite, too. Finally, in order to show that $\hat{V}_{\alpha'}$ is finite, for a contradiction assume that $\hat{V}_{\alpha'}$ is infinite. Since $\hat{V}_{\alpha'}$ is the finite union of the sets $V_{\alpha''}$ where $|\alpha''| = |\alpha'|$ and $\alpha'' \leq_{left} \alpha'$, for notational convenience, w.l.o.g. we may assume that $V_{\alpha'}$ is infinite. It follows that $V_{\hat{\alpha}0}$ is infinite since, by $\hat{\alpha}0 \sqsubseteq \alpha'$, $V_{\alpha'} \subseteq V_{\hat{\alpha}0}$. By (17) this implies that $lim_{s\to\omega} l(\hat{\alpha}0) = \omega$. Since, by $\hat{\alpha} \sqsubset TP$, $S_{\hat{\alpha}}$ is infinite, it follows that there are infinitely many $\hat{\alpha}$-expansionary stages, hence $S_{\hat{\alpha}0}$ is infinite contradicting the above observation that $S_{\hat{\alpha}0}$ is finite. □

Claim 2 (Maximal Set Lemma). Assume that M is c.e. and coinfinite. If, for any $\alpha \sqsubset TP$, $\overline{M} \subseteq^* \hat{V}_\alpha$ then M is maximal.

Proof. Since $\hat{V}_\alpha$ is the finite union of V_α and the sets $V_{\alpha'}$ such that $\alpha' <_{left} \alpha$ and $|\alpha'| = |\alpha|$, it follows by the second part of the Infinity Lemma that $\overline{M} \subseteq^* V_\alpha$ for all $\alpha \sqsubset TP$. So, for any $n \geq 0$, almost all numbers in $\overline{M}$ have $(n+1)$-state $TP \upharpoonright n+1$. As pointed out before, this implies that all requirements $\mathcal{R}_n$ are met. Since, by assumption, M is c.e. and coinfinite this implies that M is maximal. □

The Infinity Lemma shows that (assuming $\overline{M}$ is infinite), for any α on the true path and for any numbers r and k, there are infinitely many stages at which α is accessible and where we can pick k numbers greater than r of current state α which have not yet been enumerated into M. (Note that, for meeting a finitary requirement we typically need such a set of numbers, where later in the construction some of these numbers may be put into M and some of the numbers may be kept out of M.) On the other hand, the Maximal Set Lemma tells us that if we make sure that infinitely many of the numbers we pick in this way are kept out of M and that, for any $\alpha \sqsubset TP$, up to finitely many exceptions, only those numbers picked for α or a stronger node α' are kept out of M then M is maximal. These observations lead to the following strategy ensuring maximality. We pick the numbers which become associated with a given node α for ensuring any of the additional finitary tasks in such a way that one of these numbers is never needed for this task (this will ensure that $\overline{M}$ infinite). Moreover, if the task assigned to the numbers associated with a state α can be taken over by the numbers associated with a stronger state (or, as in the following, associated with a finite collection of stronger states) then the original attempt becomes superfluous and we may cancel it and enumerate the corresponding numbers into M.

Having introduced the basic technical notions needed for the maximal set strategy, we now turn to the second goal of the construction, namely, to ensure that the given computably enumerable e.u.wtt-a.c. set A is ibT-reducible to the maximal set M that we construct. We first note that this part requires the construction of a uniformly computable sequence of auxiliary *wtt*-functionals $\{\Psi_\alpha\}_{\alpha\in\{0,1\}^*}$, where we denote the partial computable use bound of Ψ_α by ψ_α. By identifying $\{0,1\}^*$ with ω in the standard way, by Lemma 3.3 (Recursion Theorem), we may assume that *in advance* we are given a computable function $f : \{0,1\}^* \to \omega$ such that

$$\Psi_\alpha = \hat{\Phi}_{f(\alpha)} \tag{21}$$

holds for all $\alpha \in \{0,1\}^*$. So, by letting $g(\langle\alpha,x\rangle,s) = \hat{g}(\langle f(\alpha),x\rangle,s)$, $k(\langle\alpha,x\rangle,s) = \hat{k}(\langle f(\alpha),x\rangle,s)$ and $h(\langle\alpha,x\rangle) = \hat{h}(\langle f(\alpha),x\rangle)$, we obtain

$$\lim_{s\to\infty} g(\langle\alpha,n\rangle,s) = \begin{cases} 0 & \text{if } \Psi_\alpha^A(n)\uparrow, \\ 1 & \text{otherwise} \end{cases} \tag{22}$$

$$k(\langle\alpha,n\rangle,s) \le k(\langle\alpha,n\rangle,s+1) \tag{23}$$

$$k(\langle\alpha,n\rangle,s) = 1 \;\Rightarrow\; |\{t \ge s : g(\langle\alpha,n\rangle,t+1) \ne g(\langle\alpha,n\rangle,t)\}| \le h(\langle\alpha,n\rangle) \tag{24}$$

$$\Psi_\alpha^A \text{ is total } \;\Rightarrow\; \forall^\infty n\, \exists\, s\; (k(\langle\alpha,n\rangle,s) = 1) \tag{25}$$

and we may use these equations in the construction.

Now, coming back to the second goal of the construction, in order to ensure that A is ibT-reducible to M, we use a variant of straight permitting: if a number x enters A at a “late” stage s then, in order to indicate that x is in A we enumerate a number $y \le x$ into M at stage s or at a later stage. Note that if we reserve a number y for such a permitting and x does not enter A then y will not enter M, too. So, in order to be compatible with the maximal set strategy, we have to ensure that the states of the permitters y are sufficiently small. In order to show that there are sufficiently many permitters of small state, we exploit that A is eventually uniformly *wtt*-array computable. The basic idea of how to obtain permitters (for almost all numbers x) of a given m-state α (on or to the right of TP) is as follows. We attempt to define a strong array $\{B_n^\alpha\}_{n\in\omega}$ of finite sets B_n^α, in the following called (α-)*blocks*. The α-blocks are defined one after the other in increasing order, and we ensure that the numbers in B_{n+1}^α are greater than the numbers in B_n^α. Moreover, when an α-block becomes defined, say, at stage $s+1$ then all of its elements are not in M_s and have m-state α or stronger than α at stage s. (Note that (assuming that $\overline{M}$ is infinite), by the Infinity Lemma, for α on or to the right of the true path, we will find such numbers no matter how large we want to make the blocks. So, for such α, all the α-blocks will become defined.) Now the idea is that the numbers y in block B_n^α serve as permitters for the numbers x in the interval $I_n^\alpha = [\max B_n^\alpha, \max B_{n+1}^\alpha]$ (note that these intervals cover all numbers $x \ge \max B_0^\alpha$). In order to guarantee that the size (i.e., cardinality) of B_n^α is large enough to provide the required numbers of permitters, we appropriately define the corresponding auxiliary *wtt*-functional Ψ_α. We let $\psi_\alpha(n) = \max B_{n+1}^\alpha$ (if the latter block becomes defined) be the use of $\Psi_\alpha^X(n)$. Moreover, if $\psi_\alpha(n)$ is defined then we ensure that $\Psi_\alpha^A(n)$ is defined, too, where – exploiting that, by (22), $g(\langle\alpha,n\rangle,s)$ approximates the domain of Ψ_α^A – we make sure that any enumeration of a number $x \in I_n^\alpha$ in A is followed by a change of $g(\langle\alpha,n\rangle,s+1) \ne g(\langle\alpha,n\rangle,s)$ at a later stage

s. Now, since Ψ_α^A is total, it follows by (25) that (for almost all n) there is a least stage s_n such that $k(\langle\alpha, n\rangle, s_n) = 1$, and, by (24), $\lambda s.g(\langle\alpha, n\rangle, s)$ will change after stage s_n at most $h(\langle\alpha, n\rangle)$ times. So if we say that a number $x \in I_n^\alpha$ enters A "late" if it does so after stage s_n then $h(\langle\alpha, n\rangle)$ permitters suffice for dealing with all numbers in I_n^α. So it suffices to let B_n^α have size $h(\langle\alpha, n\rangle)$.

The above explains how, for a single α on or to the right of the true path, we can ensure that $A \leq_{ibT} M$ and at the same time only numbers of state α or astronger state are left in $\overline{M}$ (namely, it suffices to enumerate all numbers which are not in any α-block into M). Moreover, by adding one more element to each α-block we can guarantee that no α-block becomes completely enumerated into M whence $\overline{M}$ will be infinite.

For the actual construction, however, we have to ensure that, for *any* α on the true path, almost all numbers left in $\overline{M}$ have state α or stronger state. We achieve this by (1) carrying out the above strategy for all α and by (2) suspending the permitting numbers in block B_n^α (in the actual construction we say that the block B_n^α becomes *frozen*) and enumerating them into M once we see that, for any number $x \in I_n^\alpha$, there is a node $\alpha' \prec \alpha$ and a number n' such that x is in the interval $I_{n'}^{\alpha'}$ covered by the α'-block $B_{n'}^{\alpha'}$ and x is considered to be "late" relative to this block, too (i.e., $k(\langle\alpha', n'\rangle, s) = 1$ if this happens at stage $s+1$). As we will show, this will provide the required improvements of states.

There is one technical problem left, however. We cannot achieve that, for $\alpha \neq \alpha'$, the α-blocks and α'-blocks are disjoint. So when determining the sizes of the blocks we have to consider possible overlaps. By allowing the α'-strategy to use a number in the intersection of the blocks B_n^α and $B_{n'}^{\alpha'}$ only if α' is stronger than α, we have to ensure that any block B_n^α contains a core $\hat{B}_n^\alpha$ of size $h(\langle\alpha, n\rangle) + 1$ which does not intersect any α'-block for all stronger α'. The sole purpose of the priority tree is to resolve this problem. The interval B_n^α will be defined by one of the nodes β which extends α and has length $\langle|\alpha|, n\rangle$. As long as B_n^α is not yet defined there will be (at most) one such β "eligible" to define B_n^α. The stage when this node becomes eligible gives a lower bound on $\min B_n^\alpha$ and, by initializing a node, its eligibility can be (temporarily) deleted. This will suffice to avoid overlaps between α-blocks and α'-blocks for comparable α and α' and will give an eligible node β a bound on the sizes of the potential overlaps in terms of the higher priority nodes currently admissible.

Having explained the ideas of the construction and some of its technical features, we now turn to the construction. Any stage $s+1$ consists of 5 steps (Stage 0 is vacuous).

In Step 1 the blocks are defined. We let the nodes β with $\alpha \sqsubseteq \beta$ and $|\beta| = \langle|\alpha|, n\rangle$ define the block B_n^α. We call such a node β a B_n^α-*node* and call B_n^α the *block associated with* β. Moreover, we call two nodes *equivalent* if they are associated with the same block. If B_n^α is defined by (activity of) the node β then we say that B_n^α has *priority* β. As long as B_n^α is not yet defined, there will be at most one B_n^α-node β which is *eligible*. This node attempts to define B_n^α. Once B_n^α is defined, no B_n^α-node will be eligible. A node β can become eligible only at a stage $s+1$ such that $\beta \sqsubset \delta_s$ or $\delta_s <_{left} \beta$. Once β is eligible, β stays eligible unless β becomes *initialized*. The only effect of initialization of a node is to make it non-eligible. If initialized, a node may become eligible at a later stage again. We write $B_n^\alpha[s]\downarrow$ if B_n^α is defined by the end of Step 1 of stage s, and we write $B_n^\alpha[s]\uparrow$ otherwise.

Moreover, $B^\alpha_n \downarrow$ ($B^\alpha_n \uparrow$) denotes that B^α_n is eventually defined (never defined). For any α and n such that B^α_n is defined, we let

$$\hat{B}^\alpha_n = \{y \in B^\alpha_n : \nexists \alpha' \prec \alpha \ \nexists n' \ (B^{\alpha'}_{n'} \downarrow \ \& \ y \in B^{\alpha'}_{n'}\}$$

be the *core* of B^α_n. Similarly, for s such that $B^\alpha_n[s] \downarrow$, we let

$$\hat{B}^\alpha_n[s] = \{y \in B^\alpha_n : \nexists \alpha' \prec \alpha \ \nexists n' \ (B^{\alpha'}_{n'}[s] \downarrow \ \& \ y \in B^{\alpha'}_{n'}\}$$

be the *core* of B^α_n at stage s.

In Steps 2 and 3, the partial use functions ψ_α and the *wtt*-functionals Ψ_α are defined. We write $\psi_\alpha(n)[s] \downarrow$ if $\psi_\alpha(n)$ has been defined by the end of Step 2 of stage s and write $\psi_\alpha(n)[s] \uparrow$ otherwise, and we write $\Psi^A_\alpha(n)[s] \downarrow$ if $\Psi^{A_s}_\alpha(n)$ has been defined by the end of Step 3 of stage s and $\Psi^A_\alpha(n)[s] \uparrow$ otherwise. We say that the α-block B^α_n is *realized* at stage s if $\psi_\alpha(n)[s] \downarrow$ and we say that B^α_n is *truly realized* at stage s if B^α_n is realized at stage s and $k(\langle \alpha, n \rangle, s) = 1$; and B^α_n is *realized* (*truly realized*) if it is realized (truly realized) at some stage. Finally, we say that x is *(truly) covered by B^α_n (at stage s)* – or (truly) *$\langle \alpha, n \rangle$-covered (at stage s)* for short – if $\langle \alpha, n \rangle$ is (truly) realized (at stage s) and $x \in [\max B^\alpha_n, \psi_\alpha(n)]$; and we say that x is *α-covered (at stage s)* if x is $\langle \alpha, n \rangle$-covered (at stage s) for some n.

In Step 4 blocks become frozen. We say that a block B^α_n is *admissible* at stage s, if it is truly realized at stage s and has not been frozen by the end of Step 4 of stage s.

In Step 5 numbers are enumerated into M, i.e., M_{s+1} becomes defined.

Now, using the notation introduced above, the steps of stage $s+1$ are as follows.

Step 1 (Defining the blocks B^α_n). A B^α_n-node β requires attention at stage $s+1$ if one of the following holds.

(a) (i) $B^\alpha_n[s] \uparrow$
  (ii) $\beta \sqsubset \delta_s$ or $\delta_s <_{left} \beta$ and $|\beta| < s$.
  (iii) Neither β nor any equivalent node β' such that $\beta' <_{left} \beta$ is eligible at stage s.
  (iv) For any node β' such that $\beta' \sqsubset \beta$, the block associated with β' is defined at stage s.
  (v) For any node β' such that $\beta <_{left} \beta'$, $|\beta'| = |\beta|$ and β' is not equivalent to β, the block associated with β' is defined at stage s.

(b) β is eligible at stage s, and there is a block B which is suitable for the definition of B^α_n by β at stage $s+1$. Here a block B is *suitable* for the definition of B^α_n by the B^α_n-node β at stage $s+1$ if B has the following properties.
  (i) $r(\beta, s) < \min B$,
  (ii) $B \subseteq \bigcup_{\{\alpha' : |\alpha'| = |\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha', s}$,
  (iii) The block B has cardinality $|B| = F(\beta, s)$ where $F(\gamma, s)$ is defined (by induction on γ) by

$$F(\gamma, s) = 2 + H(\gamma) + \sum_{\{\gamma' : \gamma' <_{left} \gamma \text{ and } \gamma' \text{ is eligible at stage } s\}} F(\gamma', s)$$

where, for a $B^{\alpha'}_{n'}$-node γ, $H(\gamma) = h(\langle \alpha', n' \rangle)$ (and where $\sum_\emptyset = 0$). (Note that, at any given stage s, there are only finitely many eligible nodes, hence $F(\gamma, s)$ is well-defined.)

Fix β minimal such that β requires attention.

If (a) holds then declare that β becomes eligible, set $r(\beta', s+1) = s$ for all $\beta' \geq \beta$, and initialize all nodes β' with $\beta < \beta'$ (i.e., no such β' is eligible at stage $s+1$).

If (b) holds then let $B^\alpha_n = B$ for the least (w.r.t. the canonical index) block B which is suitable for the definition B^α_n by β at stage $s+1$, let β be the *priority* of B^α_n, set $r(\beta', s+1) = s$ for all $\beta' > \beta$, and initialize all nodes β' such that $\beta \leq \beta'$.

If no node requires attention then Step 1 of stage $s+1$ is vacuous.

Step 2 (Defining the partial computable use functions ψ_α). For any α and any n such that either $n = 0$ or $\psi_\alpha(n-1)[s]\downarrow$, $\psi^\alpha(n)[s]\uparrow$ and $B^\alpha_{n+1}[s]\downarrow$, let $\psi_\alpha(n) = \max B^\alpha_{n+1}$.

Step 3 (Defining the wtt-functionals Ψ_α). For any α and any n such that $\psi_\alpha(n)[s]\downarrow$ let

$$\Psi^A_\alpha(n)[s+1]\downarrow \text{ if } \Psi^A_\alpha(n)[s]\uparrow \text{ and } g(\langle\alpha, n\rangle, s) = 0, \tag{26}$$

and let

$$\Psi^A_\alpha(n)[s+1]\uparrow \text{ if } \Psi^A_\alpha(n)[s]\downarrow,\ g(\langle\alpha, n\rangle, s) = 1 \text{ and } A_{s+1} \upharpoonright \psi_\alpha(n) \neq A_s \upharpoonright \psi_\alpha(n). \tag{27}$$

In any other case let $\Psi^A_\alpha(n)[s+1]\downarrow$ if and only if $\Psi^A_\alpha(n)[s]\downarrow$.

Step 4 (Freezing blocks). A block B^α_n is *freezable* at stage $s+1$ if the following hold.

(i) $\langle|\alpha|, n\rangle < s$.

(ii) B^α_n is not frozen at stage s.

(iii) For any x covered by B^α_n there is a block $B^{\alpha_x}_{n_x}$ such that $\alpha_x \prec \alpha$, $B^{\alpha_x}_{n_x}$ is admissible at stage s, and $B^{\alpha_x}_{n_x}$ covers x.

If there is a freezable block then choose $q = \langle m, n\rangle$ minimal such that there is a freezable block B^α_n with $|\alpha| = m$ and fix the rightmost α such that $|\alpha| = m$ and B^α_n is freezable. Declare that B^α_n *becomes frozen* at stage $s+1$.

Step 5 (Enumerating M). A number $y \notin M_s$ is enumerated into M at stage $s+1$ if (at least) one of the following hold.

(i) *(Freezing)* There is a block B^α_n which becomes frozen in Step 4 of stage $s+1$ and y is in the core $\hat{B}^\alpha_n[s+1]$ of B^α_n at stage $s+1$.

(ii) *(Enumerating nonblock numbers)* y is not in any block defined at stage $s+1$ and y is less than the maximum of a block defined at stage $s+1$.

(iii) *(Coding A into M)* There is a node α and a number n such that the block B^α_n is admissible at stage s and

$$\Psi^A_\alpha(n)[s]\downarrow \quad\text{and}\quad \Psi^A_\alpha(n)[s+1]\uparrow \tag{28}$$

or

$$g(\langle\alpha, n\rangle, s) = 1 \quad\text{and}\quad g(\langle\alpha, n\rangle, s+1) = 0, \tag{29}$$

holds, and y is the least element of the core $\hat{B}^\alpha_n[s+1]$ of B^α_n at stage $s+1$ which is not in M_s. In this case, call y an *$\langle\alpha, n\rangle$-coding number.*

This completes the construction. In the remainder of the proof we show that M has the required properties.

We first summarize the properties of the blocks we will need.

Claim 3. The definition of the blocks satisfies the following conditions.

(B_0) *If B^α_n becomes defined at stage $s+1$ (i.e., $B^\alpha_n[s+1]\downarrow$ and $B^\alpha_n[s]\uparrow$) then $B^\alpha_n \cap M_s = \emptyset$.*

(B_1) *If B^α_n is defined then*

$$B^\alpha_n \cap \overline{M} \subseteq \bigcup_{\{\alpha' : |\alpha'|=|\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha'}.$$

(B_2) *If B^α_n is defined then $\langle |\alpha|, n\rangle \leq \min B^\alpha_n$.*

(B_3) *If B^α_{n+1} is defined then B^α_n is defined and $\max B^\alpha_n < \min B^\alpha_{n+1}$.*

(B_4) *If B^α_n is defined then, for the core*

$$\hat{B}^\alpha_n = B^\alpha_n \setminus \bigcup_{\{(\alpha',n') :\ \alpha' \prec \alpha,\ n' \geq 0 \text{ and } B^{\alpha'}_{n'}\downarrow\}} B^{\alpha'}_{n'}$$

of B^α_n, $|\hat{B}^\alpha_n| > h(\langle \alpha, n\rangle) + 1$.

(B_5) *If B^α_n is defined and $\alpha \prec \alpha'$ and $|\alpha'| \leq |\alpha|$ then $B^{\alpha'}_n$ is defined, too.*

(B_6) *Assume that $\overline{M}$ is infinite. Then, for any α on or to the right of the true path, the blocks B^α_n are defined for all n.*

(B_7) *There is an infinite path p through $T = \{0,1\}^*$ such that, for any α, all blocks B^α_n ($n \geq 0$) are defined if and only if α is on or to the right of p.*

Proof. With the exception of property (B_7) the proof only depends on the definition of the blocks and not on the other parts of the construction. In case of (B_7) we use that at any stage $s+1$ any number y which is enumerated into M at stage $s+1$ is bounded by the maximum of some block existing at this stage.

We tacitly use that the restraint function is nondecreasing in both arguments, i.e., $r(\beta, s) \leq r(\beta', s')$ for $\beta \leq \beta'$ and $s \leq s'$, and that for any pair $\langle \alpha, n\rangle$ and any stage s there is at most one eligible B^α_n-node at stage s and there is no such node if $B^\alpha_n[s]\downarrow$.

(B_0). This is immediate by clause (ii) in the definition of suitability since, for any node α and any stage s, $V_{\alpha,s} \subseteq \overline{M_s}$.

(B_1). This is immediate by clause (ii) in the definition of suitability since, for any node α and any stage s,

$$\overline{M} \cap \bigcup_{\{\alpha' : |\alpha'|=|\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha',s} \subseteq \bigcup_{\{\alpha' : |\alpha'|=|\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha'}.$$

(B_2). If β is the priority of the block B^α_n and B^α_n becomes defined at stage $s+1$ then $r(\beta, s) < \min B^\alpha_n$ by clause (i) in the definition of suitability. Moreover, there is a stage $s'+1 \leq s$ such that β receives attention via (a) and becomes eligible at stage $s'+1$. By clause (ii) in (a), this implies that $|\beta| < s' = r(\beta, s'+1) \leq r(\beta, s)$. Finally, since β is a B^α_n-node, $|\beta| = \langle |\alpha|, n\rangle$.

(B_3). Assume that B^α_{n+1} becomes defined by β at stage $s+1$. Then, for the greatest stage $t < s+1$ such that β is not eligible at stage t, $t+1 \leq s$ and β receives attention via clause (a) at stage $t+1$. So, since there is B^α_n-node β' such that $\beta' \sqsubset \beta$, $B^\alpha_n[t+1]\downarrow$ whence $\max B^\alpha_n \leq t$. Moreover, $r(\beta, t+1) = t$ hence, by $t+1 \leq s$, $r(\beta, s) \geq t$. By the latter and by clause (i) in the definition of suitability of a block B, it follows that $t < \min B^\alpha_{n+1}$, which completes the proof of (B_3).

(B_4). Assume that B^α_n is defined. Fix the node β and the stage $s+1$ such that B^α_n has priority β and B^α_n becomes defined by activity of β at stage $s+1$. Then, given any state α' and any number n' such that

$$\alpha' \prec \alpha,\ B^{\alpha'}_{n'} \text{ is defined and } B^{\alpha'}_{n'} \cap B^\alpha_n \neq \emptyset, \tag{30}$$

it suffices to show that, for the priority β' of $B^{\alpha'}_{n'}$,

$$\beta' <_{left} \beta, \tag{31}$$

$$\beta' \text{ is eligible at stage } s, \tag{32}$$

and

$$|B^{\alpha'}_{n'}| = F(\beta', s) \tag{33}$$

hold. Namely, since different blocks have different priorities, it follows that

$$\begin{aligned}
|\hat{B}^\alpha_n| &= |B^\alpha_n \setminus \textstyle\bigcup_{\{(\alpha',n'):\ \alpha' \prec \alpha,\ n' \geq 0 \text{ and } B^{\alpha'}_{n'}\downarrow\}} B^{\alpha'}_{n'}| \\
&\geq |B^\alpha_n| - \textstyle\sum_{\{(\alpha',n'):\ \alpha' \prec \alpha,\ n' \geq 0,\ B^{\alpha'}_{n'}\downarrow \text{ and } B^{\alpha'}_{n'} \cap B^\alpha_n \neq \emptyset\}} |B^{\alpha'}_{n'}| \\
&\geq F(\beta,s) - \textstyle\sum_{\{\beta':\beta' <_{left} \beta \text{ and } \beta' \text{ is eligible at stage } s\}} F(\beta',s) \\
&\quad \text{(by the definition of } B^\alpha_n \text{ and by (30) implying (31) - (33))} \\
&\geq H(\beta) + 2 \\
&\quad \text{(by the definition of } F(\beta,s)) \\
&= h(\langle \alpha, n\rangle) + 2 \\
&\quad \text{(by the definition of } H(\beta))
\end{aligned}$$

So, assuming that (30) implies (31) - (33), (B_4) holds.

Hence, for the remainder of the proof of (B_4), fix α' and n' such that (30) holds, and let β' be the priority of $B^{\alpha'}_{n'}$. We have to show that (31) - (33) hold. Fix $t < s$ maximal such that β is not eligible at stage t and fix $t'+1 < s'+1$ such that $B^{\alpha'}_{n'}$ becomes defined via β' at stage $s'+1$ and t' is maximal such that $t' < s'$ and β' is not eligible at stage t'. Note that β becomes eligible at stage $t+1$, β is not initialized (hence eligible) at any stage u such that $t+1 \leq u < s+1$, and B^α_n is defined by β at stage $s+1$. Hence

$$t = r(\beta, t+1) = r(\beta, s) < \min B^\alpha_n \leq \max B^\alpha_n \leq s. \tag{34}$$

Similarly, β' becomes eligible at stage $t'+1$, β' is not initialized (hence eligible) at any stage u' such that $t'+1 \leq u' < s'+1$, and $B^{\alpha'}_{n'}$ is defined by β' at stage $s'+1$. Hence

$$t' = r(\beta', t'+1) = r(\beta', s') < \min B^{\alpha'}_{n'} \leq \max B^{\alpha'}_{n'} \leq s'. \tag{35}$$

Moreover, any node γ with $\beta < \gamma$ is initialized at stages $t+1$ and $s+1$, and any node γ' with $\beta' < \gamma'$ is initialized at stages $t'+1$ and $s'+1$. Finally note that, by $(\alpha, n) \neq (\alpha', n')$, $\beta \neq \beta'$ and the stages t, s, t', s' are pairwise distinct.

Now, the proof of (31) - (33) is in two steps: before we prove (31), we show that (31) implies (32) and (33). So assume (31). Now, if $s'+1 < s+1$ then (by (31)) β is initialized at stage $s'+1$ hence $s'+1 < t+1$. So, by (34) and (35), $\max B^{\alpha'}_{n'} < \min B^\alpha_n$ contradicting (30). Similarly, if $s+1 < t'+1$, then by (34) and

(35), $\max B_n^\alpha < \min B_{n'}^{\alpha'}$, again contradicting (30). So $t'+1 < s+1 < s'+1$ must hold. Now (35) is immediate by the choice of t'. Moreover, again by the choice of t', no node $\gamma \leq \beta'$ is initialized at any stage $u' \in [t'+1, s'+1)$, whence no node $\gamma' < \beta'$ becomes active at any such stage. So a node $\gamma' < \beta'$ is eligible at stage s if and only if it is eligible at stage s'. By the definition of F, this implies that $F(\beta', s) = F(\beta', s')$. Equation (33) follows since $|B_{n'}^{\alpha'}| = F(\beta', s')$.

It remains to establish (31). By assumption $\alpha' \prec \alpha$, hence $\alpha' <_{left} \alpha$ or $\alpha \sqsubset \alpha'$. In the former case, (31) is immediate since $\alpha' \sqsubseteq \beta'$ and $\alpha \sqsubseteq \beta$. So, for the remainder of the argument assume that $\alpha \sqsubset \alpha'$ and, for a contradiction, assume that (31) fails, i.e. that $\beta' \sqsubset \beta$ or $\beta \sqsubset \beta'$ or $\beta <_{left} \beta'$. If $\beta' \sqsubset \beta$ then, by construction, $B_{n'}^{\alpha'}$ has to be defined before β can become eligible, i.e., $s'+1 < t+1$ whence $\max B_{n'}^{\alpha'} < \min B_n^\alpha$ contrary to (30). Similarly, if $\beta \sqsubset \beta'$ then $s+1 < t'+1$ hence $\max B_n^\alpha < \min B_{n'}^{\alpha'}$ contrary to (30).

This leaves the case that $\beta <_{left} \beta'$. If $s'+1 < t+1$ or $s+1 < s'+1$ then, as above, we may conclude from (34) and (35) that (30) fails (note that in the latter case, $s+1 < t'+1$ by $\beta < \beta'$). So w.l.o.g. $t+1 < s'+1 < s+1$. But since $\alpha \sqsubset \alpha'$ and since $\beta < \beta'$, $V_{\alpha', s'} \subseteq V_{\alpha, s'}$, $F(\beta, s') \leq F(\beta', s')$ and $r(\beta, s') \leq r(\beta', s')$. So since the block $B_{n'}^{\alpha'}$ is suitable for β' at stage $s'+1$, $B_{n'}^{\alpha'}$ or a subblock B of it will be suitable for β at stage $s'+1$, too. So, since β is eligible at stage $s'+1$, β will require attention at stage $s'+1$. Since $\beta < \beta'$, this contradicts the fact that β' receives attention. This completes the proof of (31) and the proof of (B_4).

(B_5). Fix α, α' and n such that $B_n^\alpha \downarrow$ and either $\alpha' \sqsubset \alpha$ or $\alpha <_{left} \alpha'$ and $|\alpha| = |\alpha'|$. It suffices to show $B_n^{\alpha'} \downarrow$. (Then the claim follows by induction on $|\alpha|$.) Let β be the priority of B_n^α and fix the least stage $s+1$ at which β becomes eligible hence requires attention via clause (a). Then the subclauses (iv) and (v) of (a) guarantee that, for any node β' such that either $\beta' \sqsubset \beta$ or $\beta <_{left} \beta'$, $|\beta| = |\beta'|$ and β and β' are not equivalent, the block associated with β' is defined at stage s. But if $\alpha' \sqsubset \alpha$ then $B_n^{\alpha'}$ is associated with the proper initial segment $\beta' = \beta \restriction \langle |\alpha'|, n \rangle$ of β and if $\alpha <_{left} \alpha'$ and $|\alpha| = |\alpha'|$ then $B_n^{\alpha'}$ is associated with the node $\beta' = \alpha' 1^{|\beta| - |\alpha|}$ and $\beta <_{left} \beta'$, $|\beta| = |\beta'|$ and β and β' are not equivalent. So in either case $B_n^{\alpha'} \downarrow$.

(B_6). The proof is indirect. Assume that $\overline{M}$ is infinite and that there is a node α and a number n such that $TP \restriction |\alpha| \leq_{left} \alpha$ and B_n^α is not defined. Fix $q = \langle m, n \rangle$ minimal such that there is a node α of length m such that $TP \restriction m \leq_{left} \alpha$ and B_n^α is not defined and fix the rightmost corresponding α. Moreover, let β be the rightmost B_n^α-node. (Note that $\beta = \alpha 1^{q-m}$. In particular, $\alpha \sqsubseteq \beta$, $|\beta| = q$ and, by $TP \restriction |\alpha| \leq_{left} \alpha$ and by the definition of β, $TP \restriction q \leq_{left} \beta$.)

We claim that there is a stage s^* such that no node β' with $\beta' < \beta$ which is not equivalent to β requires attention after stage s^*. This is shown as follows. Note that any node β' with $\beta' < \beta$ which is not equivalent to β is element of one of the following sets.

$$
\begin{aligned}
N_0 &= \{\beta' : |\beta'| \leq |\beta| \;\&\; \beta' <_{left} TP \restriction |\beta'|\} \\
N_1 &= \{\beta' : |\beta'| < |\beta| \;\&\; TP \restriction |\beta'| \leq_{left} \beta'\} \\
N_2 &= \{\beta' : |\beta'| = |\beta| \;\&\; TP \restriction |\beta| \leq_{left} \beta' <_{left} \beta \;\&\; \beta' \text{ is not a } B_n^\alpha\text{-node}\} \\
N_3 &= \{\beta' : |\beta| < |\beta'| \;\&\; \beta' <_{left} \beta\}
\end{aligned}
$$

So it suffices to show that for $i \leq 4$ there is a stage s_i such that no node in N_i requires attention after stage s_i.

$i = 0$. Fix t_0 minimal such that $TP \restriction q < \delta_s$ for all stages $s \geq t_0$. Then no $\beta' \in N_0$ can become eligible after stage t_0. So whenever a node $\beta' \in N_0$ requires attention at a stage $s+1 > t_0$, either the block associated with β' becomes defined (namely, if β' acts at stage $s+1$) or β' becomes initialized (namely, if a higher priority node $\beta'' < \beta'$ acts at stage $s+1$). In either case β' will not require attention after stage $s+1$. Since N_0 is finite, this gives the existence of the desired stage s_0.

$i = 1$. Note that by the minimality of q, any node $\beta' \in N_1$ is associated with a block which eventually becomes defined. Since N_1 is finite, this gives the existence of the desired stage s_1.

$i = 2$. Note that, for any $\beta' \in N_2$, $\beta' <_{left} \beta$, $|\beta'| = |\beta|$ and β' and β are not equivalent. Since the block B^α_n associated with β is never defined, it follows, by clause (v) in the definition of requiring attention via (a), that no node in N_1 will ever require attention via (a). So $s_2 = 0$ will do.

$i = 3$. If $\beta' \in N_3$ then, for the proper initial segment $\beta'' = \beta' \restriction |\beta|$ of β' of length $|\beta|$, either $\beta'' \in N_2$ or β'' is a B^α_n-node. In either case the block associated with β'' is never defined. So, by clause (iv) in the definition of requiring attention via (a), β' does not require attention via (a), hence does not require attention. So $s_3 = 0$ will do.

Having established the existence of s^*, we next claim that there is a stage $t^* > s^*$ and a B^α_n node $\hat{\beta}$ such that $\hat{\beta}$ is eligible at all stages $s \geq t^*$. Since, by the choice of s^*, a B^α_n-node β' can be initialized at a stage $s+1 > s^*$ only if a B^α_n-node β'' to the left of it becomes active at stage $s+1$, and since by $B^\alpha_n \uparrow$ this implies that β'' acts via clause (a) hence becomes eligible at stage $s+1$, it suffices to show that some B^α_n-node β' will be eligible at some stage $s+1 > s^*$. For a contradiction assume that such β' and $s+1$ do not exist. By the minimality of q and maximality of α, we may fix a stage $s^{**} \geq s^*$ such that for any $q' = \langle m', n' \rangle < n$ the block $B^{\beta \restriction m'}_{n'}$ is defined at stage s^{**} and, for any α' with $|\alpha'| = |\alpha|$ and $\alpha <_{left} \alpha'$, the block $B^{\alpha'}_n$ is defined at stage s^{**}, too. Then, for the rightmost B^α_n-node β and any $s \geq s^{**}$, the subclauses (i) (by $B^\alpha_n \uparrow$), (iii) (by assumption) and (iv) and (v) (by the choice of s^{**}) in the definition of requiring attention (a) hold at stage s. So if we let s be the least stage $\geq s^{**}$ such that $TP \restriction q \sqsubset \delta_s$ then β requires attention via (a), at stage $s+1$ hence becomes eligible (since by assumption and by the choice of s^* no higher priority node requires attention). Contradiction.

So, for the remainder of the argument we may fix the B^α_n-node $\hat{\beta}$ which is permanently eligible after stage t^*. In order to get the final contradiction, we show that, eventually, there is a stage $s+1 > t^*$ such that $\hat{\beta}$ requires attention via (b) at stage $s+1$. Since no higher priority node requires attention after stage t^*, it follows that the block B^α_n becomes defined at stage $s+1$ contrary to choice of α and n. Now, by the choice of t^*, for any node $\beta' \leq \hat{\beta}$, eligibility of β' does not change after stage t^*. So $r(\hat{\beta}, s) = r(\hat{\beta}, t^*)$ and $F(\hat{\beta}, s) = F(\hat{\beta}, t^*)$.

So in order to show that $\hat{\beta}$ eventually requires attention via (b), it suffices to show that there is a stage $s \geq t^*$ such that

$$|\bigcup_{\{\alpha' : |\alpha'| = |\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha', s}| > r(\hat{\beta}, t^*) + F(\hat{\beta}, t^*).$$

But, since $TP \restriction m \leq_{left} \alpha$, this follows from the fact, that, by the assumption that $\overline{M}$ is infinite and by the True Path Lemma, $V_{TP \restriction m}$ is infinite.

This completes the proof of (B_6).

(B_7). First note that infinitely many blocks become defined. (Namely, otherwise, it follows that M is finite since any number y which is enumerated into M at stage $s+1$ is less than or equal to the maximum of a block B^α_n defined at stage s. So, by (B_6), infinitely many blocks will be defined contrary to assumption.) Now, call a node β a *block node* if for all $\beta' \sqsubseteq \beta$ the block associated with β' is defined, and let B be the set of all block nodes. Note that any initial segment of a block node is a block node again, and any priority of a block which becomes defined is a block node. Moreover, for $\langle \alpha, n\rangle \neq \langle \alpha', n'\rangle$, the blocks B^α_n and $B^{\alpha'}_{n'}$ (if defined) have different priorities. Since infinitely many blocks become defined, we may conclude that the set B of block nodes is an infinite subtree of the priority tree $T = \{0,1\}^*$.

Now, by König's Lemma, let p be the leftmost infinite path through B. To show that p has the required properties, first fix α on p and $n \geq 0$. Then $\beta = p \upharpoonright \langle |\alpha|, n\rangle$ is a block node and B^α_n is associated with β. So B^α_n is defined. By (B_4) we may conclude that, for α to the right of the path p, the blocks B^α_n ($n \geq 0$) are defined, too. Finally, fix α to the left of p and, for a contradiction, assume that B^α_n is defined for all $n \geq 0$. Then, the set of priorities β_n of the blocks B^α_n, $n \geq 0$, is an infinite subset of nodes in B all extending the node α. By $\alpha <_{left} p$ and by König's Lemma this contradicts the fact that p is the leftmost infinite path through B.

This completes the proof of (B_7) and the proof of Claim 3. □

Next we summarize relevant properties of the use functions ψ_α and the *wtt*-functionals Ψ_α.

Claim 4. The partial functions ψ_α, $\alpha \in \{0,1\}^*$, are uniformly computable. Moreover, for any α, the domain of ψ_α (at stage s) is an initial segment of ω, and ψ_α is strictly increasing on its domain. Finally, ψ_α is total if and only if the blocks B^α_n are defined for all $n \geq 0$.

Proof. Uniform computability follows by the effectivity of (part 1 of) the construction. The second part of the claim is immediate by definition and by (B_3). The third part is immediate by definition. □

Claim 5. The functionals Ψ_α are uniformly computable and, for any X, α and n such that $\Psi^X_\alpha(n)$ is defined, $\psi_\alpha(n)$ is defined and the use of $\Psi^X_\alpha(n)$ is bounded by $\psi_\alpha(n)$. Moreover, for any α and n such that $\psi_\alpha(n)$ is defined, $\Psi^A_\alpha(n)$ is defined, too.

Proof. The proof of the first part is straightforward. For a proof of the second part, for a contradiction assume $\psi_\alpha(n)\downarrow$ and $\Psi^A_\alpha(n)\downarrow$. Since Ψ is a *wtt*-functional, it follows that $\Psi^A_\alpha(n)[s]\uparrow$ for almost all s. Moreover, by (22), $g(\langle\alpha, n\rangle, s) = 0$ for almost all s. So there is a least stage s_0 such that $\psi_\alpha(n)[s_0]\downarrow$ and $\Psi^A_\alpha(n)\uparrow$ and $g(\langle\alpha, n\rangle, s) = 0$ for all stages $s \geq s_0$. By clause (26) in the definition of Ψ, this implies $\Psi^A_\alpha(n)[s_0+1]\downarrow$. Contradiction. □

Note that the first part of Claim 5 justifies that in advance we have fixed a computable function f satisfying (21).

Claim 6. Assume that ψ_α is total. Then the following hold.

(i) Ψ^A_α is total.

(ii) There is a number n_α such that, for any $n \geq n_\alpha$, there is a stage s such that $k(\langle\alpha, n\rangle, s) = 1$.

Proof. Part (i) is immediate by the second part of Claim 5. Part (ii) follows from part (i) by (23) and (25). □

For the remaining claims we need some more notation. Let p be the unique path through T defined in (B_7). Then, for any node α such that $\alpha \sqsubset p$ or $p <_{left} \alpha$, all blocks B^α_n are defined. So, by Claims 4 and 6, ψ_α and Ψ^A_α are total and we may fix n_α such that $\lim_{s\to\infty} k(\langle \alpha, n\rangle, s) = 1$ for all $n \geq n_\alpha$. It follows that, for $n \geq n_\alpha$, the block B^α_n will eventually become truly realized. So, if we let $x_\alpha = \max B^\alpha_{n_\alpha}$, then all numbers $x \geq x_\alpha$ are eventually truly α-covered. Hence, for such x we may fix n^α_x and s^α_x such that n^α_x is the unique n such that x becomes covered by B^α_n and s^α_x is the least stage s such that x is truly covered by $B^\alpha_{n^\alpha_x}$ at stage s.

Claim 7. Let $\alpha \sqsubset p$ and let $x \geq x_\alpha$. There is a node $\alpha' \preceq \alpha$, a number $n \geq 0$ and a stage t such that the block $B^{\alpha'}_n$ covers x and is admissible at all stages $s \geq t$ (hence is never frozen).

Proof. Note that, by (B_2), there are only finitely many blocks which may cover x. So there is a stage t_0 such that any block which covers x and becomes frozen is frozen by stage t_0. So it suffices to show that, for almost all stages s, there is a block $B^{\alpha'}_n$ such that $\alpha' \preceq \alpha$, $B^{\alpha'}_n$ covers x and $B^{\alpha'}_n$ is admissible at stage s. This is established by proving the following two facts. (a) There is a stage s such that x is covered by a block $B^{\alpha'}_n$ where $\alpha' \preceq \alpha$ and $B^{\alpha'}_n$ is admissible at stage s. (b) If x is covered by a block $B^{\alpha'}_n$ which is admissible at stage s then, at any stage $s' > s$, x is covered by a block $B^{\alpha''}_{n'}$ such that $\alpha'' \preceq \alpha'$ and $B^{\alpha''}_{n'}$ is admissible at stage s'.

For a proof of (a) recall that x will be truly α-covered eventually. So there is a stage s and a number n such that the block B^α_n truly covers x at stage s. If B^α_n is not frozen at stage s then B^α_n is admissible at stage s and we are done. Otherwise, there is a stage $\hat{s} \leq s$ such that B^α_n becomes frozen at stage $\hat{s}$. But, by construction, this implies that there is a block $B^{\alpha'}_{n'}$ such that $\alpha' \preceq \alpha$, $B^{\alpha'}_{n'}$ covers x and $B^{\alpha'}_{n'}$ is admissible at stage $\hat{s}$. So (a) holds in this case, too.

For a proof of (b), it suffices to consider the case of $s' = s+1$. (Then the general case follows by induction.) So assume that $B^{\alpha'}_n$ covers x and is admissible at stage s. If $B^{\alpha'}_n$ does not become frozen at stage $s+1$ then we are done. Otherwise, it follows by construction that there is a block $B^{\alpha''}_{n'}$ such that $\alpha'' \preceq \alpha'$, $B^{\alpha''}_{n'}$ covers x and $B^{\alpha''}_{n'}$ is admissible at stage $s+1$. This completes the proof of (b) and the proof of the claim. □

Claim 8. Assume that B^α_n becomes defined and is never frozen. Then, for the core $\hat{B}^\alpha_n$ of B^α_n, $\hat{B}^\alpha_n \cap \overline{M} \neq \emptyset$. Similarly, if B^α_n is defined but not frozen at stage s then $\hat{B}^\alpha_n[s+1] \cap \overline{M_s} \neq \emptyset$.

Proof. We prove the first part of the claim. The second part is obtained by straightforward modifications of the proof.

We first show that a number $y \in \hat{B}^\alpha_n$ can be enumerated into M only if it is an $\langle \alpha, n\rangle$-coding number. For a contradiction assume that $y \in \hat{B}^\alpha_n$ is enumerated into M at stage $s+1$ and y is not an $\langle \alpha, n\rangle$-coding number. Then y cannot be enumerated into M as a nonblock number according to clause (ii). (Namely, if so, $B^\alpha_n[s+1]\uparrow$. Hence B^α_n becomes defined at a stage $t+1 > s+1$. But, by (B_0) this implies that $B^\alpha_n \cap M_{s+1} = \emptyset$ hence $y \notin B^\alpha_n$. The claim follows since $\hat{B}^\alpha_n \subseteq B^\alpha_n$.) Since B^α_n is never frozen, this leaves the case that $y \in \hat{B}^{\alpha'}_{n'}[s+1]$ for some $\langle \alpha', n'\rangle \neq \langle \alpha, n\rangle$

and y becomes enumerated into M since $B^{\alpha'}$ becomes frozen at stage $s+1$ or y is an $\langle \alpha', n' \rangle$-coding number. Since, by (B_0), y can't be in a block which is not yet defined at stage $s+1$, it follows by $y \in \hat{B}^{\alpha'}_{n'}[s+1]$ and by definition of the core $\hat{B}^{\alpha'}_{n'}$ that $y \in \hat{B}^{\alpha'}_{n'}$. So it suffices to show that $\hat{B}^{\alpha'}_{n'} \cap \hat{B}^{\alpha}_{n} = \emptyset$. Since the core of a block is contained in the block this is done as follows. If $\alpha' = \alpha$ the claim is immediate. So, by symmetry, w.l.o.g. $\alpha' \prec \alpha$. But then, by the definition of $\hat{B}^{\alpha}_{n}$, $B^{\alpha'}_{n'} \cap \hat{B}^{\alpha}_{n} = \emptyset$.

Now, by the above and by construction, a number $y \in \hat{B}^{\alpha}_{n}$ is enumerated into M at stage $s+1$ only if B^{α}_{n} is admissible at stage s (hence $k(\langle \alpha, n \rangle, s) = 1$) and (28) or (29) holds. Moreover, at any such stage $s+1$ at most one number $y \in \hat{B}^{\alpha}_{n}$ is enumerated into M. So, by (B_0) and (B_4), it suffices to show that

$$|\{s \geq s_0 : \text{ (28) or (29) holds}\}| < h(\langle \alpha, n \rangle) + 2 \tag{36}$$

where s_0 is minimal such that $k(\langle \alpha, n \rangle, s_0) = 1$.

Since between any two stages $s < s'$ for which (29) holds there must be a stage t such that $g(\langle \alpha, n \rangle, t) = 0$ and $g(\langle \alpha, n \rangle, t+1) = 1$,

(37)
$$\begin{array}{rcl} 2 \cdot |\{s \geq s_0 : \text{ (29) holds}\}| & \leq & |\{s \geq s_0 : g(\langle \alpha, n \rangle, s+1) \neq g(\langle \alpha, n \rangle, s)\}| + 1 \\ & \leq & h(\langle \alpha, n \rangle) + 1 \end{array}$$

where the second inequality holds by (24). Moreover, since $\Psi^A_\alpha(n) \downarrow$ by Claim 5, any stage s at which (28) holds has to be followed by a stage $t > s$ such that $\Psi^A_\alpha(n)[t] \uparrow$ and $\Psi^A_\alpha(n)[t+1] \downarrow$, where $t < s'$ for the least stage $s' > s$ such that (28) holds (if there is such a stage s'). Since, by construction, $g(\langle \alpha, n \rangle, t) = 0$ and $g(\langle \alpha, n \rangle, s') = 1$ for any such stage t, it follows that

$$|\{s \geq s_0 : \text{ (28) holds}\}| \leq |\{s \geq s_0 : \text{ (29) holds}\}|$$

holds. So, by (37), (36) holds. □

Claim 9. $A \leq_{ibT} M$.

Proof. It suffices to give an effective procedure which computes $A(x)$ from $M \upharpoonright x+1$ for all sufficiently large x.

Let $x \geq x_\lambda$ and let s be the least stage such that there is a node α and a number n such that

(I) B^{α}_{n} covers x at stage s,
(II) B^{α}_{n} is admissible at stage s,
(III) $\Psi^A_\alpha(n)[s] \downarrow$ and $g(\langle \alpha, n \rangle, s) = 1$, and
(IV) $M \upharpoonright x+1 = M_s \upharpoonright x+1$.

Note that such a stage s exists. (Namely, by Claim 7, there is a block B^{α}_{n} which covers x and which is admissible at all sufficiently large stages. So (I) and (II) hold for all sufficiently large s. Moreover, since $\Psi^A_\alpha(n) \downarrow$ (by the second part of Claim 5), it follows that (III) holds for all sufficiently large s, too (by (22)). Finally, (IV) obviously holds for all sufficiently large s.) Moreover, for any stage s, we can effectively check whether, among the finitely many blocks defined at stage s, there is a block B^{α}_{n} satisfying (I) - (III). So we can find the above stage s by using $M \upharpoonright x+1$ as an oracle.

We claim that $A(x) = A_s(x)$. For a proof, first note that B^{α}_{n} does not become frozen after stage s hence is admissible at all later stages. (To wit, if B^{α}_{n} becomes frozen at stage $s'+1 > s$ then $\hat{B}^{\alpha}_{n}[s']$ is completely enumerated into M at stage $s'+1$ whence, by the second part of Claim 8, there is a number $y \in \hat{B}^{\alpha}_{n}[s'+1]$

such that $y \in M_{s'+1} \setminus M_{s'}$. Since $\hat{B}^{\alpha}_n[s'+1]$ is contained in B^{α}_n and $\max B^{\alpha}_n \leq x$ it follows that $y \leq x$ hence $M \upharpoonright x+1 \neq M_s \upharpoonright x+1$ contrary to (iv).) Now, for a contradiction, assume that $A(x) \neq A_s(x)$. Fix $s' \geq s$ minimal such that a number $x' \leq x$ is enumerated into A at stage $s'+1$. Then, assuming that $\Psi^A_\alpha(n)[s']\downarrow$ and $g(\langle \alpha, n\rangle, s') = 1$, $\Psi^A_\alpha(n)[s'+1]\uparrow$ by construction. So, in any case, there is a least stage s'' such that $s \leq s'' \leq s'$ and such that (28) or (29) holds for s' (in place of s). It follows, by construction and by Claim 8, that there is a number $y \in \hat{B}^{\alpha}_n[s'+1]$ which is newly enumerated into M at stage $s'+1$. But, as observed before, this contradicts (iv). □

Claim 10. $\overline{M}$ is infinite.

Proof. By (B_2) and by Claim 7 there are infinitely many blocks which are never frozen. So the claim follows by Claim 8. □

Claim 11. For any node $\alpha \sqsubset p$ there are only finitely many blocks $B^{\alpha'}_n$ such that $\alpha \prec \alpha'$, $n \geq 0$ and $B^{\alpha'}_n$ is never frozen.

Proof. Fix $\alpha \sqsubset p$. By (B_2) it suffices to show that any block $B^{\alpha'}_n$ such that $\alpha \prec \alpha'$ and $x_\alpha \leq \max B^{\alpha'}_n$ becomes frozen eventually. So fix such a block $B^{\alpha'}_n$ and, for a contradiction, assume that $B^{\alpha'}_n$ is never frozen. Note that, by $\alpha \sqsubset p$ and $\alpha \prec \alpha'$, α' is on p or to the right of p whence $B^{\alpha'}_n$ becomes defined, say, at stage $s+1$. By Claim 7 we may fix a stage $t \geq s+1$ such that, for any of the finitely many numbers x covered by $B^{\alpha'}_n$ there is a block $B^{\alpha_x}_{n_x}$ such that $\alpha_x \preceq \alpha$ (hence $\alpha_x \prec \alpha'$), $B^{\alpha_x}_{n_x}$ covers x and $B^{\alpha_x}_{n_x}$ is admissible at all stages $t' \geq t$. So $B^{\alpha'}_n$ is freezable at all stages $t' \geq t$. Since there are only finitely many blocks $B^{\hat{\alpha}}_{\hat{n}}$ such that $\langle |\hat{\alpha}|, \hat{n}\rangle < \langle |\alpha'|, n\rangle$ or $\langle |\hat{\alpha}|, \hat{n}\rangle = \langle |\alpha'|, n\rangle$ and $\alpha' <_{left} \hat{\alpha}$, it follows that $B^{\alpha'}_n$ becomes frozen eventually. Contradiction. □

Claim 12. The true path TP coincides with the path p.

Proof. Claim 10 and (B_6) immediately imply that $p \leq_{left} TP$. For a proof of the converse, i.e., $TP \leq_{left} p$, it suffices to show that, for any given node $\alpha' <_{left} TP$, only finitely many α'-blocks become defined. Now, by Claim 10 and by the second part of the Infinity Lemma (Claim 1), there are only finitely many stages s such that $\delta_s <_{left} \alpha'$ or $\alpha' \sqsubset \delta_s$. So only finitely many α'-nodes can become eligible, hence only finitely many α'-blocks can be defined. □

Claim 13. For any α on TP, $\overline{M} \subseteq^* \hat{V}_\alpha$.

Proof. Fix $\alpha \sqsubset TP$. Since (by (B_1) and (19)) $\overline{M} \cap B^{\alpha'}_n \subseteq \hat{V}_\alpha$ for any block $B^{\alpha'}_n$ such $\alpha' \preceq \alpha$, it suffices to show

$$\overline{M} \subseteq^* \bigcup_{\{(\alpha',n)\colon \alpha' \preceq \alpha, n \geq 0 \text{ and } B^{\alpha'}_n \downarrow\}} B^{\alpha'}_n. \tag{38}$$

For a proof of (38), first recall that (by Claim 12) the true path TP coincides with the path p. So, by Claim 11, we may let B be the finite union of the blocks $B^{\alpha'}_n$ such that $\alpha \prec \alpha'$, $n \geq 0$ and $B^{\alpha'}_n$ is never frozen. Now, call a number y a *block number* if y is element of some block, and call a block number y an *α'-number* if α' is $\prec$-minimal such that y is in an α'-block. (Note that any number is element of at most finitely many blocks. So α' is well-defined.) Then it suffices to show that any number $y \in \overline{M}$ which is not an α'-number for some $\alpha' \preceq \alpha$ is an element of

B. So fix such y. We first observe that y is a block number. Namely, since there are infinitely many blocks, it follows by (B_2) that there is a stage s such that y is less than the maximum of a block defined at stage s. So if y is not a block number then y is enumerated into M at stage $s+1$ for the least such s contrary to choice of y. So we may fix α' and the corresponding unique n such that y is an α'-number and $y \in B_n^{\alpha'}$. It suffices to show that $B_n^{\alpha'}$ is contained in B. For a contradiction, assume that this is not the case. Since, by the choice of y, $\alpha \preceq \alpha'$, this implies that there is a stage $s+1$ at which $B_n^{\alpha'}$ becomes frozen. So $\hat{B}_n^{\alpha}[s+1] \subseteq M_{s+1}$ by construction. But since y is an α'-number, y is in the core $\hat{B}_n^{\alpha'}$ of $B_n^{\alpha'}$. Since, obviously, $\hat{B}_n^{\alpha'} \subseteq \hat{B}_n^{\alpha'}[s+1]$, it follows that $y \in M$ contrary to assumption.

This completes the proof of Claim 13. □

Claim 14. M is maximal.

Proof. By the effectivity of the construction and by Claims 10 and 13, the hypotheses of the Maximal Set Lemma (Claim 2) are satisfied. □

By Claims 9 and 14, M has the required properties. This completes the proof the Theorem 4.3. □

In order to complete the proof of the Characterization Theorem it remains to prove Theorem 4.4. For this sake, we use the following characterization of the dense simple sets given in Robinson [Rob67]: a c.e. set D is dense simple if and only if D is coinfinite and, for every strong array $\{F_n\}_{n\in\omega}$ of pairwise disjoint sets, there is a number m such that

$$\forall\, n \geq m \; (|F_n \cap \overline{D}| < n). \tag{39}$$

Proof of Theorem 4.4. Fix c.e. sets A and D such that $A \leq_{wtt} D$ and D is dense simple. It suffices to define computable functions g, h and k witnessing that A is eventually uniformly wtt-array computable.

Fix a wtt-functional Γ such that $A = \Gamma^D$ and fix a computable function γ such that the use of Γ^D is bounded by γ where w.l.o.g. γ is strictly increasing. Moreover, fix computable enumerations $\{A_s\}_{s\in\omega}$, $\{D_s\}_{s\in\omega}$ and $\{\Gamma_s\}_{s\in\omega}$ of A, D and Γ, respectively, such that the length of agreement function

$$l(s) = \max\{y : A_s \restriction y = \Gamma_s^{D_s} \restriction y\}$$

is strictly increasing in s. (Such enumerations can be obtained by speeding up any given computable enumerations of A, D and Γ.) Note that this ensures

$$(x < l(s) \;\&\; A_{s+1}(x) \neq A_s(x)) \;\Rightarrow\; D_{s+1} \restriction \gamma(x) \neq D_s \restriction \gamma(x) \tag{40}$$

for all numbers x and stages s.

Now the computable functions $g, k : \omega^2 \to \{0,1\}$ and $h : \omega \to \omega$ are defined as follows. Define g by letting

$$g(\langle e, y\rangle, s) = \begin{cases} 1 & \text{if } \hat{\Phi}_{e,s}^{A_s}(y)\downarrow, \\ 0 & \text{otherwise,} \end{cases}$$

and let h be the order defined by

$$h(x) = (x+1)^2.$$

Finally, for the definition of k, define the auxiliary uniformly partial computable functions $\tilde{\varphi}_e$ by $\tilde{\varphi}_e(y) = \lim_{s\to\infty} \tilde{\varphi}_{e,s}(y)$ where

$$\tilde{\varphi}_{e,s}(y) = \begin{cases} y + \max\{\hat{\varphi}_{e,s}(y') : y' \leq y\} & \text{if } \forall\, y' \leq y\ (\hat{\varphi}_{e,s}(y') \downarrow), \\ \uparrow & \text{otherwise.} \end{cases}$$

Note that $\tilde{\varphi}_e$ is defined on an initial segment of ω, $\tilde{\varphi}_e$ is strictly increasing on its domain, $\tilde{\varphi}_e$ majorizes $\hat{\varphi}_e$ on its domain, and $\tilde{\varphi}_e$ is total if $\hat{\varphi}_e$ is total. So, for total $\hat{\Phi}^A_e$, $\tilde{\varphi}_e$ is total, strictly increasing and bounds the use of $\hat{\Phi}^A_e$. Now, the 0-1-valued function k is defined by letting $k(\langle e, y\rangle, s) = 1$ iff

$$\tilde{\varphi}_{e,s}(y) \downarrow\ \&\ l(s) > \tilde{\varphi}_e(y)\ \&\ |\overline{D_s} \upharpoonright \gamma(\tilde{\varphi}_{e,s}(y))| < \frac{(\langle e, y\rangle + 1)^2}{2}. \tag{41}$$

Obviously, the functions g, h and k are computable, and h is an order. Moreover, g is the canonical approximation of $A^\dagger$ whence (6) holds. So it only remains to show that the functions g, h and k satisfy conditions (7) - (9) in Definition 4.1, too.

For a proof of (7) it suffices to note that k is 0-1-valued and that the three clauses in equation (41) that characterize the stages s such that $k(\langle e, y\rangle, s) = 1$ persist if we replace s by a stage $t \geq s$. (For the second clause, recall that the length function $l(s)$ is nondecreasing in s.)

For a proof of (8) fix $x = \langle e, y\rangle$ and s such that $k(x, s) = 1$. By the definition of g and h, it suffices to show that

$$|\{t \geq s : \hat{\Phi}^{A_t}_{e,t}(y) \downarrow\ \&\ \hat{\Phi}^{A_{t+1}}_{e,t+1}(y) \uparrow\}| < \frac{(\langle e, y\rangle + 1)^2}{2}. \tag{42}$$

(Namely, (42) guarantees that $g(x, t)$ switches from 1 to 0 less than $(x + 1)^2 \cdot 2^{-1}$ times after stage s. So, since g is 0-1-valued, g may change on x after stage s at most $2((x + 1)^2 \cdot 2^{-1})$ $(= h(x))$ times.)

So fix t as in (42). Then $A_{t+1} \upharpoonright \hat{\varphi}_e(y) \neq A_t \upharpoonright \hat{\varphi}_e(y)$. Note that, by $k(x, s) = 1$, (41) holds. So, by $\hat{\varphi}_e(y) \leq \tilde{\varphi}_e(y)$ (if defined), by (40) and by the first two clauses in (41), there is a number $\leq \gamma(\tilde{\varphi}_{e,s}(y))$ that is enumerated into D at stage $t + 1$. But, by the third clause in (41), the latter can happen for at most $\frac{(\langle e,y\rangle+1)^2}{2} - 1$ stages $t \geq s$. So (8) holds.

Finally, for a proof of (9), fix e such that $\hat{\Phi}^A_e$ is total. Then $\tilde{\varphi}_e$ is total, computable and strictly increasing (and so is γ). So we can define a computable partition of ω into nonempty intervals $\{F_n\}_{n\in\omega}$ by letting $F_0 = [0, \gamma(\tilde{\varphi}_e(0)))$ and $F_{n+1} = [\gamma(\tilde{\varphi}_e(n)), \gamma(\tilde{\varphi}_e(n + 1)))$. Now, since D is dense simple, it follows, by Robinson's characterization of the dense simple sets given above, that there is a number m such that (39) holds. So there is a constant c such that

$$|\overline{D} \upharpoonright \gamma(\tilde{\varphi}_e(n))| = |\overline{D} \upharpoonright 1 + \max F_n| = \sum_{n'\leq n} |\overline{D} \cap F_{n'}| \leq (\sum_{n'\leq n} n') + c = \frac{n(n + 1)}{2} + c$$

for all $n \geq 0$. Since, by $y \leq \langle e, y\rangle$, $y(y + 1) \cdot 2^{-1} + c < (\langle e, y\rangle + 1)^2 \cdot 2^{-1}$ for all sufficiently large y, it follows that, for almost all y, there is a stage s_y such that (41) holds for all stages $s \geq s_y$. So, by the definition of $k(\langle e, y\rangle, s)$, $\lim_{s\to\infty} k(\langle e, y\rangle, s) = 1$ for all sufficiently large numbers y, whence (9) holds.

This completes the proof of Theorem 4.4. □

5. Closure Properties of EUwttAC

In this section, we prove that EUwttAC is closed downwards under $\leq_{wtt}$ and closed under join. The former holds by the following slightly more general result where we do not require that the sets are computably enumerable.

Lemma 5.1. *Let A and B be any (not necessarily c.e.) sets such that $A \leq_{wtt} B$ and such that B is e.u.wtt-a.c. Then A is e.u.wtt-a.c., too.*

Proof. Fix computable functions g, k and h such that B is e.u.wtt-a.c. via g, k and h, and, by clause 1. of Lemma 3.4, fix a computable function f such that $\hat{\Phi}^A_e = \hat{\Phi}^B_{f(e)}$ for $e \geq 0$. Then A is e.u.wtt-a.c. via $\tilde{g}, \tilde{k}$ and $\tilde{h}$ where $\tilde{g}(\langle e, x\rangle, s) = g(\langle f(e), x\rangle, s)$, $\tilde{k}(\langle e, x\rangle, s) = k(\langle f(e), x\rangle, s)$ and $\tilde{h}(\langle e, x\rangle) = h(\langle f(e), x\rangle)$ (for $e, x, s \in \omega$). □

For the closure under the join operation (and for some later applications), we need the following technical lemma.

Lemma 5.2. *Let A_0 and A_1 be c.e. sets. There exist strictly increasing computable functions $f_0, f_1 : \omega \to \omega$ such that, for all $e, x \in \omega$,*

$$\hat{\Phi}^{A_0 \oplus A_1}_e(x)\downarrow \Leftrightarrow (\hat{\Phi}^{A_0}_{f_0(e)}(x)\downarrow \;\&\; \hat{\Phi}^{A_1}_{f_1(e)}(x)\downarrow) \tag{43}$$

and

$$\hat{\Phi}^{A_0 \oplus A_1}_e(x)\downarrow \Rightarrow \exists\, i \leq 1\, (\hat{\Phi}^{A_i}_{f_i(e)}(x) = \hat{\Phi}^{A_0 \oplus A_1}_e(x)). \tag{44}$$

Proof. Given computable enumerations $\{A_{i,s}\}_{s\in\omega}$ of A_i $(i \leq 1)$, for each $i \leq 1$ and $e \geq 0$ define the functional $\Psi_{i,e}$ by letting, for any set Z and any number x,

$$\Psi^Z_{i,e}(x)\downarrow \Leftrightarrow \exists s\, (\hat{\Phi}^{A_0 \oplus A_1}_e(x)[s]\downarrow \;\&\; A_{i,s} \upharpoonright \hat{\varphi}_e(x) + 1 = Z \upharpoonright \hat{\varphi}_e(x) + 1)$$

and by setting

$$\Psi^Z_{i,e}(x) = \hat{\Phi}^{A_0 \oplus A_1}_e(x)[s]$$

for the least such s if $\Psi^Z_{i,e}(x)$ is defined. Note that the use of $\Psi^Z_{i,e}(x)$ is bounded by $\hat{\varphi}_e(x)$, and, for $i \leq 1$, $\{\Psi_{i,e}\}_{e\in\omega}$ is a uniformly computable sequence of *wtt*-functionals. So, by clause 1. of Lemma 3.3, there is a strictly increasing computable function f_i such that $\Psi_{i,e} = \hat{\Phi}_{f_i(e)}$. We claim that f_0 and f_1 are as desired.

Note that $\hat{\Phi}^{A_0\oplus A_1}_e(x)\downarrow$ trivially implies that $\hat{\Phi}^{A_0}_{f_0(e)}(x)$ and $\hat{\Phi}^{A_1}_{f_1(e)}(x)$ are defined. So, assuming that $\hat{\Phi}^{A_0}_{f_0(e)}(x)$ and $\hat{\Phi}^{A_1}_{f_1(e)}(x)$ are defined, it suffices to show that $\hat{\Phi}^{A_i}_{f_i(e)}(x) = \hat{\Phi}^{A_0\oplus A_1}_e(x)$ for some $i \leq 1$. By assumption, for $i \leq 1$ fix the least stage s_i such that $\hat{\Phi}^{A_0\oplus A_1}_e(x)[s_i]\downarrow$ and $A_{i,s_i} \upharpoonright \hat{\varphi}_e(x){+}1 = A_i \upharpoonright \hat{\varphi}_e(x){+}1$ holds. Then, for $s = \max\{s_0, s_1\}$, it follows by the use-principle that $\hat{\Phi}^{A_0\oplus A_1}_e(x) = \hat{\Phi}^{A_0\oplus A_1}_e(x)[s]$. So, for the least $i \leq 1$ such that $s = s_i$, we may deduce that $\hat{\Phi}^{A_i}_{f_i(e)}(x) = \hat{\Phi}^{A_0\oplus A_1}_e(x)$. □

By applying Lemma 5.2, now we can prove that EUwttAC is closed under join.

Lemma 5.3. *Let A_0 and A_1 be c.e. e.u.wtt-a.c. sets. Then $A_0 \oplus A_1$ is e.u.wtt-a.c., too.*

Proof. Fix computable functions g_i, k_i and h_i such that A_i is e.u.wtt-a.c. via g_i, k_i and h_i $(i \leq 1)$. By Lemma 5.2, fix computable functions $f_i : \omega \to \omega$ $(i \leq 1)$ such that (43) holds. Define the functions g, k and h by letting

$$g(\langle e,x\rangle,s) = g_0(\langle f_0(e),x\rangle,s)\cdot g_1(\langle f_1(e),x\rangle,s),$$
$$k(\langle e,x\rangle,s) = k_0(\langle f_0(e),x\rangle,s)\cdot k_1(\langle f_1(e),x\rangle,s), \text{ and}$$
$$h(\langle e,x\rangle) = h_0(\langle f_0(e),x\rangle)+h_1(\langle f_1(e),x\rangle)$$

(for all $e,x,s\in\omega$). We claim that $A_0\oplus A_1$ is e.u.wtt-a.c. via g,k and h. Obviously, the functions g,k and h are computable. So it suffices to show (6) – (9) for $A=A_0\oplus A_1$. Now, by the choice of g_i and k_i, (6) is immediate by (43) and (7) is immediate. For a proof of (8), note that, for any $e,x,s\in\omega$, $g(\langle e,x\rangle,s+1)\neq g(\langle e,x\rangle,s)$ implies that $g_0(\langle f_0(e),x\rangle,s+1)\neq g_0(\langle f_0(e),x\rangle,s)$ or $g_1(\langle f_1(e),x\rangle,s+1)\neq g_1(\langle f_1(e),x\rangle,s)$ and $k(\langle e,x\rangle,s)=1$ implies $k_0(\langle f_0(e),x\rangle,s)=k_1(\langle f_1(e),x\rangle,s)=1$. So, by the choice of g_i, k_i and h_i, $k(\langle e,x\rangle,s)=1$ implies

$$\begin{aligned}
& |\{t\geq s: g(\langle e,x\rangle,t+1)\neq g(\langle e,x\rangle,t)\}| \\
\leq\ & |\{t\geq s: g_0(\langle f_0(e),x\rangle,t+1)\neq g_0(\langle f_0(e),x\rangle,t)\}|+ \\
& |\{t\geq s: g_1(\langle f_1(e),x\rangle,t+1)\neq g_1(\langle f_1(e),x\rangle,t)\}| \\
\leq\ & h_0(\langle f_0(e),x\rangle)+h_1(\langle f_1(e),x\rangle) \\
=\ & h(\langle e,x\rangle).
\end{aligned}$$

Finally, for a proof of (9), fix e such that $\hat{\Phi}_e^{A_0\oplus A_1}$ is total. Then, by (43), $\hat{\Phi}_{f_0(e)}^{A_0}$ and $\hat{\Phi}_{f_1(e)}^{A_1}$ are total, too. So, by the choice of k_0 and k_1, there is a number x_0 such that $\lim_{s\to\infty}k_0(\langle f_0(e),x\rangle,s)=1$ and $\lim_{s\to\infty}k_1(\langle f_1(e),x\rangle,s)=1$ for all $x\geq x_0$. By the definition of k, this implies that $\lim_{s\to\infty}k(\langle e,x\rangle,s)=1$ $x\geq x_0$. □

The above closure properties of EUwttAC show that the *wtt*-degrees of the c.e. e.u.wtt-a.c. sets are an ideal in the upper semilattice of the c.e. *wtt*-degrees. Moreover, by the Characterization Theorem, this ideal intersects all high c.e. Turing degrees.

Theorem 5.4. *The class* $\mathrm{EUwttAC}_{wtt}$ *of the wtt-degrees of c.e. e.u.wtt-a.c. sets is an ideal in the upper semilattice of the c.e. wtt-degrees. Moreover, for any high c.e. Turing degree* $\mathbf{a}$*, there is a c.e. set* $A\in\mathbf{a}$ *such that* $deg_{wtt}(A)\in\mathrm{EUwttAC}_{wtt}$*.*

Proof. The first part of the theorem is immediate by Lemmas 5.1 and 5.3. For the second part of the theorem, note that, by Theorem 4.2, any maximal set is e.u.wtt-a.c. So the claim follows by Martin's Theorem [Mar66] which asserts that any high c.e. Turing degree contains a maximal set. □

In the remainder of the paper we relate the eventually uniformly *wtt*-array computable sets to the *wtt*-superlow sets and to the array computable sets. As we will show this provides strict lower bounds and upper bounds, respectively. We start with the *wtt*-superlow sets which are of great interest for themselves.

6. *Wtt*-Superlow Sets

In this section, we will study notions of lowness for the bounded jump, i.e., we will look at the *wtt*-superlow (i.e., bounded low) sets introduced in the introduction already. After showing that *wtt*-superlow sets are eventually uniformly *wtt*-array computable (Subsection 6.1), we have a closer look at the class of this low sets.

So we observe that the *wtt*-degrees of the c.e. *wtt*-superlow sets form an ideal, and we give an analog of the equivalence of superlowness and jump traceability by introducing a corresponding notion of *wtt*-jump traceability (Subsection 6.2). We use this equivalence in order to give a strict hierarchy of the *wtt*-superlow sets depending on the order of the mind changes needed in computable approximations of the bounded jump (Subsection 6.3). Finally, we look at the lowest level of this hierarchy, the class of the strongly *wtt*-superlow sets, and we show that there are Turing complete sets in this class (Subsection 6.4).

6.1. ***Wtt*-superlow sets are eventually uniformly *wtt*-array computable.** We recall the following definition from the introduction.

Definition 6.1. *A (not necessarily c.e.) set A is wtt-*superlow *if $A^\dagger \leq_{wtt} \emptyset'$.*

In order to show that any (not necessarily c.e.) *wtt*-superlow set is eventually uniformly *wtt*-array computable, we characterize the *wtt*-superlow sets in terms of approximability of their bounded jumps. We first recall the relevant notions needed. A total function $f : \omega \to \omega$ is called *h-computably approximable via g* or *h-c.a. via g* for short, if $g : \omega^2 \to \omega$ is a computable function and $h : \omega \to \omega$ is a computable order such that $f(x) = \lim_{s\to\infty} g(x, s)$ and $|\{s : g(x, s + 1) \neq g(x, s)\}| \leq h(x)$ (for any x), i.e., g is a computable approximation of f where the number of mind changes of g is computably bounded by h; f is called *h-computably approximable (h-c.a.)* if f is h-computably approximable (h-c-a.) via some computable function $g : \omega^2 \to \omega$; and f is *ω-computably approximable* or *ω-c.a.* for short if f is h-c.a. for some computable order h. (Note that if the range of f is bounded, say, $f(x) \leq k$ for all x, then we may assume that the approximating function g is also bounded by k. So if A is an ω-c.a. set and g approximates A in the limit then in the following we tacitly assume that g is 0-1 valued.)

Lemma 6.2. *Let A be any (not necessarily c.e.) set. The following are equivalent.*

1. *A is wtt-superlow, i.e., $A^\dagger \leq_{wtt} \emptyset'$.*
2. *$A^\dagger \leq_{tt} \emptyset'$.*
3. *$A^\dagger$ is ω-c.a.*
4. *There exists a computable order h such that any set B which is bounded-c.e. in A is h-c.a.*

Proof. The equivalence of the first three clauses 1., 2. and 3. is immediate by the general fact that, for *any* set B, $B \leq_{wtt} \emptyset'$ iff $B \leq_{tt} \emptyset'$ iff B is ω-c.a., see, e.g., [Odi89, III.8.14] and [DH10, Corollary 2.6.2]. Moreover, the implication "4. $\Rightarrow$ 3." is immediate, too, since $A^\dagger$ is bounded-c.e. in A. This leaves the implication "3. $\Rightarrow$ 4.".

So suppose that $A^\dagger$ is ω-c.a. Fix a computable function $g : \omega^2 \to \{0, 1\}$ and a computable order $\hat{h}$ such that $A^\dagger(x) = \lim_{s\to\infty} g(x, s)$ and $|\{s : g(x, s + 1) \neq g(x, s)\}| \leq \hat{h}(x)$ hold for all x. We claim that any bounded A-c.e. set is h-c.a. for the order $h(x) = \hat{h}(\langle x, x\rangle)$. So let B be a bounded A-c.e. set. Fix $e \in \omega$ such that $B = \mathrm{dom}(\hat{\Phi}^A_e)$. Then $x \in B$ iff $\langle e, x\rangle \in A^\dagger$. Define the computable function $\tilde{g} : \omega^2 \to \{0, 1\}$ by letting

$$\tilde{g}(x, s) = \begin{cases} B(x) & \text{if } x < e, \\ g(\langle e, x\rangle, s) & \text{otherwise.} \end{cases}$$

By definition, $B(x) = \lim_{s\to\infty} \tilde{g}(x,s)$ holds for all x. So it suffices to show that the number of mind changes of $\tilde{g}(x,\cdot)$ is bounded by $h(x)$ for any x. The latter clearly holds if $x < e$. On the other hand, for $x \geq e$ we may argue that

$$|\{s : \tilde{g}(x,s+1) \neq \tilde{g}(x,s)\}| = |\{s : g(\langle e,x\rangle, s+1) \neq g(\langle e,x\rangle, s)\}| \leq \hat{h}(\langle e,x\rangle) \leq h(x),$$

where the latter inequality holds since $\hat{h}$ is a computable order. □

Corollary 6.3. *Any (not necessarily c.e.) wtt-superlow set is eventually uniformly wtt-array computable.*

Proof. Assume that A is *wtt*-superlow. Then, by Lemma 6.2, $A^\dagger$ is ω-c.a. So we may fix a computable order h and a computable function g such that $A^\dagger$ is h-c.a. via g. It follows that A is eventually uniformly *wtt*-array computable via g, k and h where we may let k be the constant function $k(x,s) = 1$. □

From Lemma 6.2 we can further deduce that the class of the *wtt*-superlow sets is closed downwards under *wtt*-reducibility and that the class of the c.e. *wtt*-superlow sets is closed under join. So the class of the *wtt*-superlow c.e. *wtt*-degrees is an ideal in EUwttAC.

Corollary 6.4. *(a) Let A and B be any (not necessarily c.e.) sets such that $A \leq_{wtt} B$ and B is wtt-superlow. Then A is wtt-superlow, too.*

(b) Let A_0 and A_1 be wtt-superlow c.e. sets. Then $A_0 \oplus A_1$ is wtt-superlow, too.

Proof. (a). By *wtt*-superlowness of B, $B^\dagger \leq_{wtt} \emptyset'$ while, by $A \leq_{wtt} B$ and by part 5. of Lemma 3.4, $A^\dagger \leq_{wtt} B^\dagger$. Hence $A^\dagger \leq_{wtt} \emptyset'$. By Lemma 6.2 this implies that A is *wtt*-superlow.

(b). By Lemma 5.2 fix computable functions f_i $(i \leq 1)$ satisfying (43). Then, for all $e, x \in \omega$, we have

$$\langle e,x\rangle \in (A_0 \oplus A_1)^\dagger \;\Leftrightarrow\; \forall i \leq 1\ (2\langle f_i(e), x\rangle + i \in A_0^\dagger \oplus A_1^\dagger).$$

Hence, $(A_0 \oplus A_1)^\dagger \leq_{tt} A_0^\dagger \oplus A_1^\dagger \leq_{tt} \emptyset'$. □

6.2. ***Wtt*-superlowness and *wtt*-jump traceability.** For computably enumerable sets the equivalent characterizations of *wtt*-superlowness given in Lemma 6.2 can be expanded. In particular, a computably enumerable set A is *wtt*-superlow iff A is *wtt*-jump traceable where the latter is defined as follows.

Definition 6.5. *A set A is h-wtt-*jump traceable *via $\{V_e\}_{e\in\omega}$ if h is a computable order and $\{V_e\}_{e\in\omega}$ is a uniformly c.e. sequence of finite sets such that, for all $e \geq 0$, $|V_e| \leq h(e)$ and $\hat{J}^A(e)\downarrow$ implies $\hat{J}^A(e) \in V_e$; A is h-wtt-*jump traceable *if there exists a uniformly c.e. sequence $\{V_e\}_{e\in\omega}$ such that A is h-wtt-jump traceable via $\{V_e\}_{e\in\omega}$; and A is wtt-*jump traceable *if there exists a computable order h such that A is h-wtt-jump traceable. If A h-wtt-jump traceable via $\{V_e\}_{e\in\omega}$ then we say that $\{V_e\}_{e\in\omega}$ is an h-*trace *for $\hat{J}^A$.*

Theorem 6.6. *For a c.e. set A, A is wtt-superlow if and only if A is wtt-jump traceable.*

By Lemma 6.2, Theorem 6.6 is immediate by the following two lemmas. In these lemmas, in addition we analyze how the relevant orders are affected if we go from one notion to the other. (This analysis will be used below in the proof of Lemma 6.12).

Lemma 6.7. *Let A be a c.e. set, let h be a computable order, and suppose that $A^\dagger$ is h-c.a. Then A is $\hat{h}$-wtt-jump traceable for the computable order $\hat{h}(x) = \lceil \frac{h(\langle x,x\rangle)}{2} \rceil$*

Proof. We adapt some of the techniques from [Nie06, Theorem 4.1] where it is shown that the c.e. superlow sets coincide with the c.e. jump traceable sets.

Fix a computable function $g : \omega^2 \to \{0,1\}$ such that $A^\dagger$ is h-c.a. via g and fix a computable enumeration $\{A_s\}_{s\in\omega}$ of A. We show that there exists a number $d \in \omega$ and a uniformly c.e. sequence $\{V_e\}_{e\in\omega}$ such that A is h'-*wtt*-jump traceable via $\{V_e\}_{e\in\omega}$ for the computable order $h'(x) = \lceil \frac{h(\langle d,x\rangle)}{2} \rceil + 1$. Then, obviously, A is $\hat{h}$-*wtt*-jump traceable via $\{\hat{V}_e\}_{e\in\omega}$ via the uniformly c.e. sequence

$$\hat{V}_e = \begin{cases} \emptyset & \text{if } e < d \text{ and } \hat{J}^A(e)\uparrow \\ \{\hat{J}^A(e)\} & \text{if } e < d \text{ and } \hat{J}^A(e)\downarrow \\ V_e & \text{otherwise.} \end{cases}$$

Now, along with $\{V_e\}_{e\in\omega}$, we define an auxiliary *wtt*-functional Ψ in stages s where, by the Recursion Theorem, we may assume that in advance we know an index $d \in \omega$ such that $\Psi = \hat{\Phi}_d$ holds (the intuition behind $\Psi^A(x)$ is that its computation is a delayed version of the computation of $\hat{J}^A(x)$). In more detail, we define a uniformly computable sequence of *wtt*-functionals $\{\tilde{\Psi}_e\}_{e\in\omega}$ (intuitively, for any $e \in \omega$, we have a version for the definition of Ψ, where e is a guess for an index of Ψ). Then, in the construction, we make $\tilde{\Psi}_e^A(x)$ defined (undefined) at a certain stage $s+1$ only if $g(\langle e,x\rangle, s)$ correctly approximates the status of definedness of $\tilde{\Psi}_e^A(x)[s]$. Then, by the Recursion Theorem, there exists a number d such that $\tilde{\Psi}_d = \hat{\Phi}_d$. So $\Psi = \tilde{\Psi}_d$ is as desired. Now the definition of V_e and $\Psi^A(e)$ for given $e \in \omega$ is as follows.

> *Stage 0.* Let $V_{e,0} = \emptyset$ and $\Psi^A(e)[0]\uparrow$.
>
> *Stage $s+1$.* Let $V_{e,s}$ and $\Psi^A(e)[s]$ be given. If $\hat{\varphi}_e(e)[s]\uparrow$ or if $A_{s+1} \upharpoonright \hat{\varphi}_e(e)+1 \neq A_s \upharpoonright \hat{\varphi}_e(e)+1$ holds then let $\Psi^A(e)[s+1]\uparrow$ and $V_{e,s+1} = V_{e,s}$. Otherwise, distinguish between the following cases.
>
> (i) If $\Psi^A(e)[s]\uparrow$, $\hat{J}^A(e)[s]\downarrow$ and $g(\langle d,e\rangle, s) = 0$ then let $\Psi^A(e)[s+1]\downarrow = \hat{J}^A(e)[s]$ with use $\hat{\varphi}_e(e)$ and let $V_{e,s+1} = V_{e,s}$.
>
> (ii) If $\Psi^A(e)[s]\downarrow$ and $g(\langle d,e\rangle, s) = 1$ then let $\Psi^A(e)[s+1] = \Psi^A(e)[s]$ and $V_{e,s+1} = V_{e,s} \cup \{\Psi^A(e)[s]\}$.
>
> If neither of the previous cases applies then let $\Psi^A(e)[s+1] = \Psi^A(e)[s]$ and $V_{e,s+1} = V_{e,s}$.

By the effectivity of the construction, $\{V_e\}_{e\in\omega}$ is uniformly c.e. and Ψ is a *wtt*-functional. We claim that $\{V_e\}_{e\in\omega}$ and the number d obtained from the Recursion Theorem are as desired. We first prove that $\{V_e\}_{e\in\omega}$ is a trace for $\hat{J}^A$. So let $e \in \omega$ be given such that $\hat{J}^A(e)\downarrow$. Then we may fix the least stage s such that $\hat{J}^A(e)[s]\downarrow$ and $A \upharpoonright \hat{\varphi}_e(e)+1 = A_s \upharpoonright \hat{\varphi}_e(e)+1$. Since $\lim_{s\to\infty} g(\langle d,e\rangle, s) = \mathrm{dom}(\Psi^A)(e)$, it follows that there exists a stage s_0 such that (i) applies at stage s_0+1. So $\Psi^A(e)[s_0+1]\downarrow = \hat{J}^A(e)$ holds by construction; hence, for the least stage $s_1 > s_0$ such that (ii) applies at stage s_1+1, it follows that $\hat{J}^A(e) \in V_{e,s_1+1}$; hence, $\hat{J}^A(e) \in V_e$.

It remains to show that $\{V_e\}_{e\in\omega}$ is an h'-trace. For that, we observe that, by construction, a number x may be enumerated into V_e at stage $s+1$ only if $x = \Psi^A(e)[s]\downarrow$. So if $s_0 < s_1$ are stages such that $\Psi^A(e)[s_0]\downarrow \neq \Psi^A(e)[s_1]\downarrow$ and such that $\Psi^A(e)[s_i]$ enter V_e at stage $s_i + 1$ ($i \leq 1$) then, by construction, there must be a stage s such that $s \in (s_0, s_1)$ and such that $\Psi^A(e)[s+1]\uparrow$. Thus, by (i), there exists a stage $t \in (s, s_1)$ such that $\Psi^A(e)[t+1]\downarrow$. So, by (ii), we can argue that each new element that enters V_e corresponds to a change of $g(\langle d, e\rangle, \cdot)$ from 1 to 0 and back to 1. Since there are at most $\lceil \frac{h(\langle d,e\rangle)}{2} \rceil$ many such stages, this completes the proof. $\square$

Lemma 6.8. *Let A be a c.e. set. There exists a strictly increasing computable function $f : \omega \to \omega$ such that, for any computable order h such that A is h-wtt-jump traceable, $A^\dagger$ is $\tilde{h}$-c.a. via the computable order $\tilde{h}(x) = 2h(f(x)) + 1$.*

Proof. Given a computable enumeration $\{A_s\}_{s\in\omega}$ of A, consider the wtt-functional Ψ such that, for any oracle X and any input $e, x \in \omega$, we have

$$\Psi^X(\langle e, x\rangle) = \mu s(\hat{\Phi}^A_e(x)[s]\downarrow \ \&\ X \upharpoonright \hat{\varphi}_e(x) + 1 = A_s \upharpoonright \hat{\varphi}_e(x) + 1) \tag{45}$$

and, by 3. of Lemma 3.4, let $f : \omega \to \omega$ be a computable function such that $\Psi^X(n) = \hat{J}^X(f(n))$ holds for all oracles X and all numbers n.

Now fix a computable order h and suppose that A is h-wtt-jump traceable. By the latter, fix a uniformly c.e. sequence $\{V_e\}_{e\in\omega}$ which is an h-trace for $\hat{J}^A$. Then, for all $n, e, x, s \in \omega$, let

$$t(n, s) = \max(V_{f(n),s}), \text{ and} \tag{46}$$

$$g(\langle e, x\rangle, s) = \begin{cases} 1 & \text{if } \hat{\Phi}^A_e(x)[t(\langle e, x\rangle, s)]\downarrow \text{ and} \\ & A_s \upharpoonright \hat{\varphi}_e(x) + 1 = A_{t(\langle e,x\rangle,s)} \upharpoonright \hat{\varphi}_e(x) + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

We claim that $A^\dagger$ is $\tilde{h}$-c.a. via g for the computable order $\tilde{h}$ as given by the lemma. First of all, we show that $\lim_{s\to\infty} g(\langle e, x\rangle, s) = A^\dagger(\langle e, x\rangle)$ holds for all $e, x \in \omega$. First, suppose that $\hat{\Phi}^A_e(x)\uparrow$. Then $\hat{\Phi}^A_e(x)[s]\uparrow$ holds for almost all stages s; hence, $\lim_{s\to\infty} g(\langle e, x\rangle, s) = 0$, as desired. Otherwise, $\Psi^A(\langle e, x\rangle)\downarrow$; hence, $\hat{J}^A(f(\langle e, x\rangle)) \leq t(\langle e, x\rangle, s)$ holds for almost all s by the definition of f and by (47) which in turn implies that $\lim_{s\to\infty} g(\langle e, x\rangle, s) = 1$, as desired.

In order to show that the number of mind changes of $g(\langle e, x\rangle, \cdot)$ is bounded by $2h(f(\langle e, x\rangle)) + 1$, by the fact that $g(\langle e, x\rangle, 0) = 0$, it suffices to show that the number of stages $s_0 < s_1$ such that $g(\langle e, x\rangle, s_0) = 1$, $g(\langle e, x\rangle, s_0 + 1) = 0$ and such that s_1 is the least stage greater than s_0 such that $g(\langle e, x\rangle, s_1) = 1$ is bounded by $h(f(\langle e, x\rangle))$. For the latter, let $e, x \in \omega$ be given and suppose that $s_0 < s_1$ are as above. We claim that $t(\langle e, x\rangle, s_0) < t(\langle e, x\rangle, s_1)$ holds. Otherwise, since $t(\langle e, x\rangle, s)$ is nondecreasing in s and by (47), it follows that

$$\hat{\Phi}^A_e(x)[t(\langle e, x\rangle, s_0)]\downarrow$$

and

$$A_{s_1} \upharpoonright \hat{\varphi}_e(x) + 1 = A_{t(\langle e,x\rangle,s_0)} \upharpoonright \hat{\varphi}_e(x) + 1.$$

Hence, $g(\langle e, x\rangle, s) = 1$ holds for all $s \in [s_0, s_1)$, contrary to choice of stage s_0. So for any such two stages $s_0 < s_1$ there exists a number which is enumerated into $V_{f\langle e,x\rangle}$. As $\{V_e\}_{e\in\omega}$ is an h-trace, this completes the proof. $\square$

6.3. **A hierarchy of *wtt*-superlow sets.** We conclude the section by looking at strong variants of *wtt*-superlowness and by introducing a hierarchy of *wtt*-superlow sets. By Lemma 6.2 a set A is *wtt*-superlow if there is a computable order h such that $A^\dagger$ is h-c.a. So we may ask whether the function h depends on A or not. In this subsection we show that in general this is the case. In fact, we show that, for any computable order h_1, there is a (faster growing) computable order h_2 such that there is a c.e. set A such that the bounded jump $A^\dagger$ of A is h_2-c.a. but not h_1-c.a., and there is a (slower growing) computable order h_0 such that there is a c.e. set B such that the bounded jump $B^\dagger$ of B is h_1-c.a. but not h_0-c.a. On the other hand, in the next subsection we will show that there are noncomputable – in fact, Turing complete – c.e. sets A such that $A^\dagger$ is h-c.a. for all computable orders.

The key to the hierarchy results in this subsection is the following technical lemma.

Lemma 6.9. *Let h, $\hat{h}$, H and $\hat{H}$ be computable orders such that, for $n \geq 0$,*

$$\hat{h}(n) = h(\langle n, n\rangle) \quad \text{and} \quad H(n) = 2\hat{H}(n) + 1 \tag{48}$$

and such that there are a computable order $neg(n)$ and a strong array $\{F_n\}_{n\in\omega}$ of mutually disjoint finite sets satisfying

$$\forall\, n\ (|F_n| = neg(n) + 1) \tag{49}$$

and

$$\forall\, m\ (\sum_{\{n : neg(n) \leq m\}} (\hat{h}(\max F_n) + 1) \leq \hat{H}(m)). \tag{50}$$

Then there is a c.e. set A such that $A^\dagger$ is H-c.a. but not h-c.a.

Proof. By a finite injury argument, we give a computable enumeration $\{A_s\}_{s\in\omega}$ of a c.e. set A with the required properties.

We make $A^\dagger$ H-c.a. via the canonical computable approximation $g : \omega^2 \to \{0, 1\}$ of $A^\dagger$ induced by $\{A_s\}_{s\in\omega}$ where (for $e, x \geq 0$)

$$g(\langle e, x\rangle, s) = 1 \ \Leftrightarrow\ \hat{\Phi}^A_e(x)[s]\downarrow. \tag{51}$$

For this sake it suffices to ensure that

$$m_g(\langle e, x\rangle) \leq H(\langle e, x\rangle)$$

for all $e, x \geq 0$ where

$$m_g(\langle e, x\rangle) = |\{s : g(\langle e, x\rangle, s + 1) \neq g(\langle e, x\rangle, s)\}|$$

is the number of mind changes of g on $\langle e, x\rangle$. In order to achieve this, it suffices to meet the (negative) requirements

$$\mathcal{N}_{\langle e,x\rangle} :\ \hat{\varphi}_e(x)\downarrow \Rightarrow\ |(A \setminus A_{s_{\langle e,x\rangle}}) \upharpoonright \hat{\varphi}_e(x)| \leq \hat{H}(\langle e, x\rangle)$$

for $e, x \geq 0$ where $s_{\langle e,x\rangle}$ is the least stage s such that $\hat{\varphi}_{e,s}(x)\downarrow$. Namely, for any stage s such that $g(\langle e, x\rangle, s) = 1$ and $g(\langle e, x\rangle, s + 1) = 0$, the definition of g implies that $s \geq s_{\langle e,x\rangle}$ and $A_{s+1} \upharpoonright \hat{\varphi}_e(x) \neq A_s \upharpoonright \hat{\varphi}_e(x)$. Since $g(\langle e, x\rangle, s) = 1$ for any other stage s such that $g(\langle e,x\rangle, s+1) \neq g(\langle e,x\rangle, s)$, it follows that

$$\begin{aligned} m_g(\langle e,x\rangle) &\leq 2\cdot|\{s : g(\langle e,x\rangle,s)=1 \;\&\; g(\langle e,x\rangle,s+1)=0\}|+1 \\ &\leq 2\cdot|\{s \geq s_{\langle e,x\rangle} : A_{s+1}\restriction\hat{\varphi}_e(x) \neq A_s\restriction\hat{\varphi}_e(x)|\}|+1 \\ &\leq 2\cdot|(A\setminus A_{s_{\langle e,x\rangle}})\restriction\hat{\varphi}_e(x)|+1 \\ &\leq 2\cdot\hat{H}(\langle e,x\rangle)+1 \\ &= H(\langle e,x\rangle) \end{aligned}$$

where the last inequality holds by $\mathcal{N}_{\langle e,x\rangle}$.

In order to guarantee that $A^{\dagger}$ is not h-c.a., we define an auxiliary *wtt*-functional Ψ together with a corresponding partial computable use bound ψ such that

(52) $$dom(\Psi) \text{ is not } \hat{h}\text{-c.a.}$$

The proof that this guarantees that $A^{\dagger}$ is not h-c.a. is indirect. For a contradiction assume that $A^{\dagger}$ is h-c.a. Fix $\hat{g}$ such that $A^{\dagger}$ is h-c.a. via $\hat{g}$ and fix e such that $\Psi = \hat{\Phi}_e$. Then, for $x \geq 0$, $\lambda s.\hat{g}(\langle e,x\rangle, s)$ converges to $dom(\Psi)(x)$ with $\leq h(\langle e,x\rangle)$ mind changes. Since $h(\langle e,x\rangle) \leq \hat{h}(x)$ for all numbers $x \geq e$, this implies that $dom(\Psi)$ is $\hat{h}$-c.a. contrary to (52).

Since, for any order h, any h-c.a. set B is h-c.a. via a primitive recursive function, condition (52) can be broken up into the (positive) requirements

$$\mathcal{P}_n :\; dom(\Psi) \text{ is not } \hat{h}\text{-c.a. via } g_n.$$

($n \geq 0$) where $\{g_n\}_{n\in\omega}$ is a computable numbering of the primitive recursive functions of type $\omega^2 \to \{0,1\}$.

The basic strategy for meeting requirement $\mathcal{P}_n$ is as follows. We pick a number y, called ($\mathcal{P}_n$-)*follower*, such that $\mathcal{P}_n$ may define Ψ and ψ on y. Then we ensure that the follower y witnesses that $\mathcal{P}_n$ is met by guaranteeing

(53) $$dom(\Psi^A)(y) = \lim_{s\to\infty} g_n(y,s) \;\Rightarrow\; |\{s : g_n(y,s+1) \neq g_n(y,s)\}| > \hat{h}(y).$$

For this sake we pick $\hat{h}(y)+1$ numbers $z_0 < z_1 < \dots < z_{\hat{h}(y)}$, called ($\mathcal{P}_n$-)*attackers*, which are not used as attackers by other strategies, let $\psi(y) = z_{\hat{h}(y)}+1$ (note that this allows us to make a convergent computation $\Psi^A(y)[s]\downarrow$ divergent at stage $s+1$ by enumerating one of the attackers into A at this stage), and define Ψ on y as follows (where initially $\Psi^A(y)[0]\uparrow$). For any stage s such that $\Psi^A(y)[s]\uparrow$ and $g_n(y,s) = 0$ we let $\Psi^A(y)[s+1]\downarrow$ (note that this does not require to change the oracle A_s) and for any stage s such that $\Psi^A(y)[s]\downarrow$, $g_n(y,s) = 1$ and there is at least one attacker z_i left which is not yet in A, we put the least such z_i into A at stage $s+1$ and let $\Psi^A(y)[s+1]\uparrow$. Obviously, if the hypothesis of (53) holds, this guarantees that there are at least $\hat{h}(y)+1$ stages s such that $g_n(y,s) = 1$ and $g_n(y,s+1) = 0$. So, in particular, (53) holds. Moreover, the functional Ψ defined in this way is a *wtt*-functional with partial computable bound ψ on the use.

Now, in order to make the $\mathcal{P}_n$-strategies compatible with the goal of meeting the negative requirements $\mathcal{N}_{\langle e,x\rangle}$, we have to adjust the basic strategy. In particular, it may happen that the $\mathcal{P}_n$-follower may be cancelled by a negative requirement, and the basic strategy for meeting $\mathcal{P}_n$ has to be started all over again with a new follower (and new attackers).

We say that $\mathcal{P}_n$ *injures* $\mathcal{N}_{\langle e,x\rangle}$ via follower y and corresponding attacker z at stage $s+1$ if $\hat{\varphi}_{e,s}(x)\downarrow$ (i.e., $s_{\langle e,x\rangle} \leq s$), $z < \hat{\varphi}_e(x)$ and $\mathcal{P}_n$ enumerates z into A at stage $s+1$. So, since attackers are the only numbers which may enter A, in order to ensure that $\mathcal{N}_{\langle e,x\rangle}$ is met it suffices to guarantee that there are at most $\hat{H}(\langle e,x\rangle)$ stages at which the requirement $\mathcal{N}_{\langle e,x\rangle}$ is injured. In order to achieve this, first we ensure that if a $\mathcal{P}_n$-follower y is appointed at stage $s+1$ then the corresponding attackers z_i are chosen to be $\geq s+1$ (in the actual construction we achieve this by letting $z_i = \langle y, s+1, i\rangle$ which, in addition, ensures that the sets of attackers associated with different followers are disjoint) whence, for any requirement $\mathcal{N}_{\langle e,x\rangle}$ such that $\hat{\varphi}_{e,s}(x)\downarrow$, $\mathcal{P}_n$ will not injure $\mathcal{N}_{\langle e,x\rangle}$ after stage s since $\hat{\varphi}_e(x) \leq s_{\langle e,x\rangle} \leq s \leq z_i$ for any attacker z_i associated with y (or with any $\mathcal{P}_n$-follower appointed later). Next we assign priorities to the requirements, and we ensure that a negative requirement $\mathcal{N}_{\langle e,x\rangle}$ cannot be injured by any lower priority positive requirement $\mathcal{P}_n$ as follows. If $\hat{\varphi}_e(x)$ becomes defined at stage s (i.e., if $s = s_{\langle e,x\rangle}$) then $\mathcal{N}_{\langle e,x\rangle}$ *initializes* the lower priority positive requirements $\mathcal{P}_n$ at stage s by cancelling the current follower y of $\mathcal{P}_n$ (if any) and the corresponding attackers. So the strategy for meeting $\mathcal{P}_n$ has to be restarted with a new follower and new attackers after stage s, thereby guaranteeing that the new attackers are too large to injure $\mathcal{N}_{\langle e,x\rangle}$.

Note that $\mathcal{P}_n$ can be injured by any higher priority negative requirement at most once. So in order to guarantee that there will be a follower y of $\mathcal{P}_n$ left which is never cancelled (whence the basic strategy using follower y will succeed to meet $\mathcal{P}_n$) it suffices to assign a *reservoir* of followers to $\mathcal{P}_n$ which is greater than the number of the negative requriements that have higher priority than $\mathcal{P}_n$.

Here we achieve this by letting $\mathcal{N}_m$ have *higher priority* than $\mathcal{P}_n$ iff $m < neg(n)$ (and by letting $\mathcal{P}_n$ have higher priority than $\mathcal{N}_m$ otherwise) and by letting the finite set F_n be the reservoir of $\mathcal{P}_n$-followers. Then there are $neg(n)$ negative requirements of higher priority than $\mathcal{P}_n$ and, by (49) there are $neg(n)+1$ potential $\mathcal{P}_n$-followers. So the positive requirements $\mathcal{P}_n$ are met.

It remains to show that the negative requirements $\mathcal{N}_{\langle e,x\rangle}$ are met, too. By initialization, $\mathcal{N}_{\langle e,x\rangle}$ can be injured only by the higher priority positive requirements, i.e., by the requirements $\mathcal{P}_n$ where $neg(n) \leq \langle e,x\rangle$. Moreover, for any such requirement $\mathcal{P}_n$, $\mathcal{N}_{\langle e,x\rangle}$ can be injured via one $\mathcal{P}_n$-follower only. Namely, if $\mathcal{N}_{\langle e,x\rangle}$ becomes injured by $\mathcal{P}_n$ via y at stage $s+1$ then $s_{\langle e,x\rangle} \leq s$. So the attackers of any $\mathcal{P}_n$-followers which may be appointed later are greater than $s_{\langle e,x\rangle}$ and hence cannot injure $\mathcal{N}_{\langle e,x\rangle}$. So $\mathcal{N}_{\langle e,x\rangle}$ can be injured by a single higher priority positive requirement $\mathcal{P}_n$ at most $\hat{h}(\max F_n)+1$ times, since any $\mathcal{P}_n$-follower y is picked from the reservoir F_n and since y is associated with $\hat{h}(y)+1$ attackers.

So, if we let $\mathcal{P}_n > \mathcal{N}_m$ denote that $\mathcal{P}_n$ has higher priority than $\mathcal{N}_m$, then, for any $\langle e,x\rangle$ such that $\hat{\varphi}_e(x)\downarrow$,

$$\begin{array}{rcl} |(A \setminus A_{s_{\langle e,x\rangle}}) \restriction \hat{\varphi}_e(x)| & \leq & \sum_{\{n:\mathcal{P}_n > \mathcal{N}_{\langle e,x\rangle}\}} (\hat{h}(\max F_n)+1) \\ & = & \sum_{\{n:neg(n) \leq \langle e,x\rangle\}} (\hat{h}(\max F_n)+1) \\ & \leq & \hat{H}(\langle e,x\rangle) \end{array}$$

where the last inequality holds by assumption (50). So the negative requirements $\mathcal{N}_{\langle e,x\rangle}$ are met, too.

Having outlined the construction, we conclude the proof by giving the formal construction. We start with some additional notation. Let $y_n[s]$ be the follower of $\mathcal{P}_n$ at stage s (if any); if $y_n[s]\downarrow$ let $z_{n,i}[s]$ $(i \le \hat{h}(y_n[s]))$ be the attackers associated with $y_n[s]$; let $y^n_0 < y^n_1 < \cdots < y^n_{neg(n)}$ be the elements of F_n in order of magnitude; call a negative requirement *critical* at stage s if $\hat{\varphi}_{e,s}(x)\downarrow$ (i.e., $s_{\langle e,x\rangle} \le s$); and let

$$l(n,s) = |\{\langle e,x\rangle < neg(n) : \hat{\varphi}_{e,s}(x)\downarrow\}| = |\{\langle e,x\rangle < neg(n) : s_{\langle e,x\rangle}\downarrow \le s\}|$$

be the number of the negative requirements of higher priority than $\mathcal{P}_n$ which are critical at stage s. (Note that $\lambda s.l(n,s)$ is nondecreasing in s, $l(n,0) = 0$ and $l(n,s) \le neg(n)$ whence $y^n_{l(n,s)}$ is a well-defined element of F_n.) In the construction all parameters persist unless explicitly stated otherwise.

Stage 0 is vacuous, i.e., $A_0 = \emptyset$, Ψ and ψ are nowhere defined, and no followers and attackers are defined.

Stage $s+1$. Requirement $\mathcal{P}_n$ requires attention if
(a) either $n = s$, or $n < s$ and $l(n,s) < l(n,s+1)$ or
(b) $n < s$ and $l(n,s+1) = l(n,s)$ and
  (b1) $\Psi^A(y_n[s])[s]\uparrow$ & $g_n(y_n[s],s) = 0$ or
  (b2) $\Psi^A(y_n[s])[s]\downarrow$ & $g_n(y_n[s],s) = 1$ and there is an attacker $z_{n,i}[s]$ which is not in A_s.

For any requirement $\mathcal{P}_n$ which requires attention act as follows according to the case via which the requirement requires attention.
(a) If $n < s$ and $l(n,s) < l(n,s+1)$ declare that $\mathcal{P}_n$ is *initialized* at stage $s+1$ and cancel the follower and attackers of $\mathcal{P}_n$ existing at stage s. In any case appoint $y_n[s+1] = y^n_{l(n,s+1)}$ as (new) $\mathcal{P}_n$-follower, assign $z_{n,i}[s+1] = \langle y_n[s+1], s+1, i\rangle$ as the corresponding attackers $(i \le \hat{h}(y_n[s+1]))$, and let $\psi(y_n[s+1]) = z_{n,\hat{h}(y_n[s+1])}[s+1]+1$.
(b) Distinguish the following subcases. If (b1) holds then let $\Psi^A(y_n[s])[s+1]\downarrow$. If (b2) holds then let $\Psi^A(y_n[s])[s+1]\uparrow$ and, for the least i such that $z_{n,i}[s] \notin A_s$, enumerate $z_{n,i}[s]$ into A.

This completes the construction. The correctness of the construction follows from the preceding discussion. A formal verification is left to the reader. □

Theorem 6.10. *Let h_1 be any computable order. There are computable orders h_0 and h_2 such that the following hold.*

(a) There is a c.e. set A such that $A^\dagger$ is h_2-c.a. but not h_1-c.a.
(b) There is a c.e. set A such that $A^\dagger$ is h_1-c.a. but not h_0-c.a.

Proof. (a). Let h, $\hat{h}$, H, $\hat{H}$ be the computable orders defined by $h = h_1$, $\hat{h}(n) = \langle n,n\rangle$,

$$\hat{H}(n) = n \cdot (\hat{h}(\langle n,n\rangle) + 1), \tag{54}$$

and $H(n) = 2\hat{H}(n) + 1$ $(n \ge 0)$, let neg be the computable order $neg(n) = n+1$, and let $\{F_n\}_{n\in\omega}$ be the strong array of mutually disjoint finite sets given by

$$F_n = |\{\langle n,k\rangle : k \le n\}|.$$

We claim that (a) holds for $h_2 = H$. By Lemma 6.9 it suffices to show that (49) and (50) hold. The former is immediate. The latter holds by

$$\begin{array}{rcl} \sum_{\{n:neg(n)\leq m\}}(\hat{h}(\max F_n)+1) & = & \sum_{\{n:n+1\leq m\}}(\hat{h}(\langle n,n\rangle)+1) \\ & \leq & m\cdot(\hat{h}(\langle m,m\rangle)+1) \\ & = & \hat{H}(m) \end{array}$$

where the first equality holds by the definition of $neg(n)$ and F_n while the last equality holds by the definition of $\hat{H}$.

(b). Note that, for any computable orders $\tilde{h}$ and $\tilde{\tilde{h}}$ such that $\tilde{h}$ dominates $\tilde{\tilde{h}}$, any $\tilde{\tilde{h}}$-c.a. set is $\tilde{h}$-c.a. Moreover, for any computable order $\tilde{h}$ there is a computable order $\hat{H}$ such that $\tilde{h}$ dominates the computable order $H(n) = 2\hat{H}(n)+1$. So w.l.o.g. we may assume that there is a computable order $\hat{H}$ such that h_1 is the corresponding computable order H, i.e., $h_1(n) = H(n) = 2\hat{H}(n)+1$ for $n \geq 0$. It suffices to define computable orders h, $\hat{h}$, neg and a strong array $\{F_n\}_{n\in\omega}$ of disjoint finite sets such that h, $\hat{h}$, H, $\hat{H}$, neg and $\{F_n\}_{n\in\omega}$ satisfy the hypotheses of Lemma 6.9. Then (b) holds for $h_0 = h$.

Let neg be a strictly increasing computable function such that $\hat{H}(neg(n)) \geq s(n+1)$ where $s(n) = 0+1+\cdots+n$, and let $\{F_n\}_{n\in\omega}$ be the computable partition of ω into intervals such that $\max F_n + 1 = \min F_{n+1}$ and

$$|F_n| = neg(n)+1.$$

Finally, let h be any computable order such that

$$h(\langle n,n\rangle) = m \text{ iff } n \in F_m$$

and let $\hat{h}(n) = h(\langle n,n\rangle)$.

It remains to show that (49) and (50) hold. The former is immediate by the definition of F_n. For a proof of (50) fix m. W.l.o.g. we may assume that there is a number n such that $neg(n) \leq m$ (otherwise, (50) trivially holds since $\sum_\emptyset(\dots) = 0$). So, since neg is an order, there is a greatest such n, say, n_0. It follows that

$$\begin{array}{rcl} \sum_{\{n:neg(n)\leq m\}}(\hat{h}(\max F_n)+1) & = & \sum_{\{n:neg(n)\leq m\}}(n+1) \\ & & \text{(by the definition of } h \text{ and } \hat{h}) \\ & \leq & \sum_{\{n:n\leq n_0\}}(n+1) \\ & & \text{(by the maximality of } n_0) \\ & = & s(n_0+1) \\ & \leq & \hat{H}(neg(n_0)) \\ & & \text{(by the definition of } neg) \\ & \leq & \hat{H}(m) \\ & & \text{(by } neg(n_0)\leq m) \end{array}$$

which completes the proof of (50) and the proof of the theorem. □

6.4. **Strongly *wtt*-superlow sets.** We now show that there is a noncomputable c.e. set – in fact, a Turing complete set – A such that $A^\dagger$ is h-c.a. for any order h.

Definition 6.11. *A set A is* strongly *wtt*-superlow *if $A^\dagger$ is h-computably approximable for any computable order h; and A is* strongly *wtt*-jump traceable *if A is h-wtt-jump traceable for any order h such that $h(0) > 0$.*

We first observe that the equivalence of *wtt*-superlowness and *wtt*-jump traceability for c.e. sets extends to strong *wtt*-superlowness and strong *wtt*-jump traceability.

Lemma 6.12. *Let A be a c.e. set. A is strongly wtt-superlow if and only if A is strongly wtt-jump traceable.*

Proof. First assume that A is strongly *wtt*-superlow. Then, given a computable order h such that $h(0) > 0$, we have to show that A is h-*wtt*-jump traceable. Let h' be a computable order such that $\lceil \frac{h'(\langle x,x\rangle)}{2} \rceil \leq h(x)$ for all $x \geq 0$. Then, by assumption, $A^\dagger$, is h'-c.a. But, by Lemma 6.7, this implies that A is h-*wtt*-jump traceable.

Now assume that A is strongly *wtt*-jump traceable. Then, given a computable order h, we have to show that $A^\dagger$ is h-c.a. Since any set which is h-c.a. is h'-c.a. for any finite variant h' of h, w.l.o.g. we may assume that $h(0) \geq 3$. Fix a strictly increasing computable function f as in Lemma 6.8 and let h' be a computable order such that $h'(0) = 1$ and $2h'(f(x)) + 1 \leq h(x)$ for $x \geq 0$ (note that, by $h(0) \geq 3$ such h' exists). Then, by assumption, A is h'-*wtt*-jump traceable. But, by Lemma 6.8, this implies that $A^\dagger$ is h-c.a. □

Theorem 6.13. *There exists a Turing complete set A which is strongly wtt-superlow.*

Proof. We give a computable enumeration $\{A_s\}_{s\in\omega}$ of a c.e. set A with the required properties. In order to make A Turing complete we use marker permitting. Fix a Turing complete set K, let $k : \omega \to \omega$ be a computable one-to-one function enumerating K, and let $K_s = \{k(t) : t < s\}$. We inductively define the computable marker function $\gamma : \omega^2 \to \omega$ by letting

$$\gamma(x,0) = \langle x, 0\rangle$$

(55)
$$\gamma(x,s+1) = \begin{cases} \gamma(x,s) & \text{if } x < x_s, \\ \langle x, s+1\rangle & \text{otherwise,} \end{cases}$$

where the number x_s is determined at stage $s+1$ of the construction. Moreover, we let

(56)
$$A_s = \{\gamma(x_t,t) : t < s\}$$

(for $s \geq 0$). Note that $\gamma(x,s)$ is strictly increasing in x and nondecreasing in s. Moreover, if γ is moved on x at stage $s+1$ then $\gamma(x,s) < \gamma(x,s+1)$, $\gamma(x,s+1) \geq s+1$, and $\gamma(x,s+1) \neq \gamma(y,t)$ for all numbers y and all stages $t \leq s$. It follows that $\gamma(x,s) \notin A_s$ for all numbers x and stages s. So the marker $\gamma(x_s,s)$ is enumerated into A at stage $s+1$, and the markers $\gamma(x_s,s)$ ($s \geq 0$) are the only numbers enumerated into A. Now in order to ensure that K is Turing reducible to A it suffices to choose the numbers x_s such that

(57)
$$\forall\, s\ (x_s \leq k(s))$$

and

(58)
$$\forall\, x\ (\{s : x_s \leq x\} \text{ is finite}).$$

Namely, the latter ensures that, on any x, the marker γ reaches a final position $\gamma^*(x)$, i.e., $\lim_{s\to\infty} \gamma(x,s) = \gamma^*(x) \in \omega$ exists. Moreover, $\gamma(x)$ reaches its final position at the least stage s such that $x < x_t$ for all $t \geq s$, i.e., at the least stage s such that $A \upharpoonright \gamma(x,s)+1 = A_s \upharpoonright \gamma(x,s)+1$. So $\gamma^* \leq_T A$. By (57) this implies $K \leq_T A$ since, for any x, $x \in K$ iff $x \in K_s$ for the least stage s such that $\gamma(x,s) = \gamma^*(x)$.

Note that, for any stage s,

$$|(A \setminus A_s) \upharpoonright s+1| \leq x_s + 1. \tag{59}$$

(Namely, $\gamma(x_t,t)$ is the unique number which enters A at stage $t+1$ and $\gamma(x,t') \geq t+1$ for $x \geq x_t$ and $t' \geq t+1$. So, for $t' > t \geq s$ such that $\gamma(x_t,t) \leq s$ and $\gamma(x_{t'},t') \leq s$, we have $x_{t'} < x_t \leq x_s$.) So, for any number $n > 0$ and any stage s, we can ensure that A changes below $s+1$ after stage s at most n times by letting x_s be less than n. This will be crucial for achieving our second goal, namely the goal to make A strongly wtt-superlow.

By Lemma 6.12, it suffices to make A strongly wtt-jump traceable, i.e., to meet the requirements

$\mathcal{R}_e$: If φ_e is an order such that $\varphi_e(0) > 0$ then A is φ_e-wtt-jump traceable.

for $e \geq 0$. The strategy for meeting these requirements is based on the following observation.

Claim 1. Assume that φ_e is an order, $\varphi_e(0) > 0$, and

$$\forall^\infty n\ (\hat{\varphi}_n(n)\downarrow \Rightarrow\ \gamma^*(\varphi_e(n)-1) \geq \hat{\varphi}_n(n))) \tag{60}$$

holds. Then $\mathcal{R}_e$ is met.

Proof. Fix n_0 such that the inner clause of (60) holds for $n \geq n_0$. We have to show that there is a φ_e-trace $\{V_n\}_{n\in\omega}$ for $\hat{J}^A$. Let $V_n = \emptyset$ if $n < n_0$ and $\hat{J}^A_n(n)\uparrow$, let $V_n = \{\hat{J}^A_n(n)\}$ if $n < n_0$ and $\hat{J}^A_n(n)\downarrow$, and let

$$V_n = \{\hat{J}^A_n(n)[s] : s \geq 0\ \&\ \gamma(\varphi_e(n)-1,s) \geq \hat{\varphi}_{n,s}(n)\downarrow\ \&\ \hat{J}^A_n(n)[s]\downarrow\}$$

if $n \geq n_0$. Obviously, $\{V_n\}_{n\in\omega}$ is a c.e. sequence of finite sets. Moreover, by the choice of n_0, $\hat{J}^A_n(n) \in V_n$ if $\hat{J}^A_n(n)$ is defined. So it suffices to show that $|V_n| \leq \varphi_e(n)$. Since $\varphi_e(n) \geq 1$ for all n by assumption, this is immediate for $n < n_0$. So fix $n \geq n_0$ and, for a contradiction, assume that $|V_n| > \varphi_e(n)$. Then there are stages $s_0 \leq s_1 < \cdots < s_{\varphi_e(n)}$ such that $\hat{\varphi}_{n,s_0}(n)\downarrow \leq s_0$, $\gamma(\varphi_e(n)-1,s_0) \geq \hat{\varphi}_n(n)$ and, for $m < \varphi_e(n)$, $\hat{J}^A_n(n)[s_{m+1}]\downarrow \neq \hat{J}^A_n(n)[s_m]\downarrow$. By the latter,

$$|(A \setminus A_{s_0}) \upharpoonright \hat{\varphi}_n(n)| \geq \varphi_e(n). \tag{61}$$

On the other hand, if y is the first number $< \hat{\varphi}_n(n)$ which enters A after stage s_0, say, at stage $t+1 > s_0$, then

$$y = \gamma(x_t,t) < \hat{\varphi}_n(n) \leq \gamma(\varphi_e(n)-1,s_0) \leq \gamma(\varphi_e(n)-1,t).$$

So $x_t < \varphi_e(n)-1$. It follows that

$$\begin{array}{rcll} |(A \setminus A_{s_0}) \upharpoonright \hat{\varphi}_n(n)| & = & |(A \setminus A_t) \upharpoonright \hat{\varphi}_n(n)| & \text{(by the choice of } t) \\ & \leq & |(A \setminus A_t) \upharpoonright t+1| & \text{(by } \hat{\varphi}_n(n) \leq s_0 \leq t+1) \\ & \leq & x_t + 1 & \text{(by (59))} \\ & < & \varphi_e(n) & \end{array}$$

contrary to (61). This completes the proof of Claim 1. $\square$

Now, by the above discussion, it suffices to choose the numbers x_s so that conditions (57) and (58) as well as, for $e \geq 0$ such that φ_e is an order and $\varphi_e(0) > 0$, condition (60) are satisfied. Unless the strategies for satisfying (60) assign a number $< k(s)$ to x_s we let $x_s = k(s)$. Obviously this guarantees (57) and, since k is one-to-one, this is consistent with (58).

The strategy for meeting (60) (if necessary) is as follows. Given e and n such that $\varphi_e(n) \downarrow> 0$ and $\hat{\varphi}_n(n) \downarrow$, $\varphi_{e,s}(n) \downarrow$ and $\hat{\varphi}_{n,s}(n) \downarrow$ for almost all stages s and, in order to guarantee that the inner clause of (60) is satisfied, it suffices that $x_s \leq \varphi_e(n) - 1$ for at least one of these stages s, since this ensures that

$$\gamma^*(\varphi_e(n) - 1) \geq \gamma(\varphi_e(n) - 1, s + 1) \geq \gamma(x_s, s + 1) = \langle x_s, s + 1\rangle \geq s + 1 > \hat{\varphi}_n(n).$$

(Also note that if e or n is not as above then we do not have to satisfy (60) or the inner clause is trivially satisfied.) So the following (preliminary) definition of the numbers x_s will guarantee that (57) and (60) are satisfied.

> Given s, fix e, n, t such that $s = \langle e, n, t\rangle$. If $\hat{\varphi}_{n,s}(n) \downarrow$ (hence $\hat{\varphi}_n(n) \leq s$), $\varphi_{e,s}(n) \downarrow\geq 1$, $\gamma(\varphi_e(n) - 1, s) < \hat{\varphi}_n(n)$, and $\varphi_e(n) - 1 < k(s)$ then let $x_s = \varphi_e(n) - 1$. Otherwise, let $x_s = k(s)$.

Unfortunately, however, this definition does not satisfy (58). Still, for fixed e, such that φ_e is an order and $\varphi_e(0) > 0$, the strategy for satisfying (60) will let $x_s \leq x$ for fixed x only finitely often, since this happens only for n such that $\varphi_e(n) \leq x + 1$ and, for each such n, this may happen at most once. So the claim follows since, by φ_e being an order, there are only finitely many n such that $\varphi_e(n) \leq x + 1$. Moreover, since it suffices to meet the inner clause of (58) for almost all n and since, for an order φ_e, $\varphi_e(n) > e + 1$ for almost all n, we may restrict the action for φ_e to such numbers n. So, for given x, there are only finitely many e which may let $x_s \leq x$. Hence, the above modification will suffice to make the action of all orders φ_e together compatible with (58). So we have only to ensure that the action for functions φ_e which are not an order (and for which we do not have to satisfy (60)) does not affect (58) more seriously than the action for an order. For this sake, for any e, we let $x_s = \varphi_e(n) - 1$ only if $\varphi_e(n) > e + 1$ and if we can be sure that we will do so for only finitely many n' with $\varphi_e(n') = \varphi_e(n)$. Note that the latter can be guaranteed, by letting $x_s = \varphi_e(n) - 1$ only if there is a number $n' > n$ such that $\varphi_{e,s}(m)$ is defined and nondecreasing on $\omega \upharpoonright n' + 1$ and $\varphi_e(n) < \varphi_e(n')$. Also note that this may delay the necessary action for an order φ_e only for finitely many stages. So the following definition of x_s ($s \geq 0$) will have the required properties.

> Given s, fix e, n, t such that $s = \langle e, n, t\rangle$. If
> (i) $\hat{\varphi}_{n,s}(n) \downarrow$ (hence $\hat{\varphi}_n(n) \leq s$),
> (ii) there is a number $n' > n$ such that $\varphi_{e,s}(m) \downarrow$ for $m \leq n'$, φ_e is nondecreasing on $\omega \upharpoonright n' + 1$, $\varphi_e(0) > 0$, and $\varphi_e(n') > \varphi_e(n) > e + 1$,
> (iii) $\gamma(\varphi_e(n) - 1, s) < \hat{\varphi}_n(n)$, and
> (iv) $\varphi_e(n) - 1 < k(s)$
> then let $x_s = \varphi_e(n) - 1$. Otherwise, let $x_s = k(s)$.

We complete the proof by arguing more formally that condition (57), condition (58), and, for orders φ_e where $\varphi_e(0) > 0$, condition (60) are satisfied. Condition (57) is immediate by the definition of x_s.

For a proof of (58) fix x. Let u be minimal such that $k(s) > x$ for $s \geq u$. Call s an *(e, n)-stage* if $s = \langle e, n, t\rangle \geq u$ for some number t and $x_s \leq x$, call s an *e-stage* if s is an (e, n)-stage for some number n, and call s *critical* if s is an e-stage for some number e. It suffices to show that there are only finitely many critical stages. We do this by showing that (a) any critical stage s is an e-stage for some $e \leq x$ and (b) for fixed e there are only finitely many e-stages. For a proof of (a) let s be critical. Fix the unique e, n, t such that $s = \langle e, n, t \rangle$. Then s is an (e, n)-stage. Hence (i) - (iv) in the definition of x_s hold and $x_s = \varphi_e(n) - 1 \leq x$. By the latter and by (ii), $e \leq \varphi_e(n) - 1 = x_s \leq x$. So (a) holds. For a proof of (b) fix e, and, for a contradiction, assume that there are infinitely many e-stages. Obviously, for any n, there is at most one (e, n)-stage. So, for any number m, there is an (e, n)-stage such that $n > m$. On the other hand, if there is an (e, n)-stage s, then φ_e is defined and nondecreasing on $\omega \upharpoonright n+1$ and $\varphi_e(n)$ is not the maximum of $range(\varphi_e)$. So φ_e is an order. It follows that there is a number n_0 such that $\varphi_e(n) > x+1$ for $n \geq n_0$. So there is no (e, n)-stage with $n \geq n_0$ which gives the desired contradiction.

Finally, fix e such that φ_e is an order and $\varphi_e(0) > 0$. We have to show that (60) holds, i.e., that there is a number n_0 such that $\gamma^*(\varphi_e(n) - 1) \geq \hat{\varphi}_n(n)$ for any $n \geq n_0$ such that $\hat{\varphi}_n(n) \downarrow$. Let n_0 be the least number n such that $\varphi_e(n) > e + 1$. Then, for any $n \geq n_0$ such that $\hat{\varphi}_n(n) \downarrow$, fix t minimal such that, for $s = \langle e, n, t \rangle$, clauses (i) and (ii) in the definition of x_s hold. (Note that such a stage must exist since φ_e is an order.) It suffices to show that $\gamma(\varphi_e(n)-1, s+1) \geq \hat{\varphi}_n(n)$. If (iii) fails then this is immediate. Otherwise, $x_s \leq \varphi_e(n) - 1$. So $\gamma(\varphi_e(n) - 1, s + 1) \geq \hat{\varphi}_n(n)$ in this case too.

This completes the proof of Theorem 6.13. □

We conclude this section by showing that the class of the strongly *wtt*-superlow sets is downward closed under *wtt*-reducibility and that the class of the c.e. strongly *wtt*-superlow sets is closed under join. Compare this with the corresponding results for the eventually uniformly *wtt*-array computable sets (Lemmas 5.1 and 5.3) and the *wtt*-superlow sets (Corollary 6.4).

Theorem 6.14. *(a) Let A and B be any (not necessarily c.e.) sets such that $A \leq_{wtt} B$ and B is strongly wtt-superlow. Then A is strongly wtt-superlow, too.*

(b) Let A_0 and A_1 be strongly wtt-superlow c.e. sets. Then $A_0 \oplus A_1$ is strongly wtt-superlow, too.

Proof. (a). Given a computable order h, it suffices to show that $A^\dagger$ is h-c.a. By clause 1. of Lemma 3.4, fix a strictly increasing computable function f such that, for $e \geq 0$, $\hat{\Phi}^A_e = \hat{\Phi}^B_{f(e)}$, hence

$$A^\dagger(\langle e, x \rangle) = B^\dagger(\langle f(e), x \rangle)$$

for $e, x \geq 0$. Now, since f is strictly increasing and so is $\langle \cdot, \cdot \rangle$ (in either argument), $\langle f(e), x \rangle \leq f(\langle e, x \rangle)$. So, for any order h' and any h'-bounded computable approximation g' of $B^\dagger$, g defined by $g(\langle e, x \rangle) = g'(\langle f(e), x \rangle)$ is a computable approximation of $A^\dagger$ and g is $h'(f(n))$-bounded. Since, for any computable order h there is a computable order h' such that $h'(f(n)) \leq h(n)$ for $n \geq 0$, and since, by assumption, $B^\dagger$ is h'-c.a. for any computable order h', this shows that $A^\dagger$ is h-c.a.

(b) Given a computable order h, it suffices to show that $(A_0 \oplus A_1)^\dagger$ is h-c.a. Fix strictly increasing computable functions $f_0, f_1 : \omega \to \omega$ as given by Lemma 5.2, let $f(n) = f_0(n) + f_1(n)$ and let h' be a computable order such that $h'(f(n)) \leq h(n)$ for $n \geq 0$. Then, since A_0 and A_1 are strongly *wtt*-superlow, we may fix h'-bounded computable approximations g_i of $A_i^\dagger$ $(i \leq 1)$. Now define g by

$$g(\langle e, x \rangle, s) = \min\{g_0(\langle f_0(e), x \rangle, s), g_1(\langle f_1(e), x \rangle, s)\}.$$

By (43), g is a computable approximation of $(A_0 \oplus A_1)^\dagger$. Moreover, by the definition of g and by the choice of g_0 and g_1, g is $\hat{h}$-bounded by for the computable order $\hat{h}$ defined by

$$\hat{h}(\langle e, x\rangle) = h'(\langle f_0(e), x\rangle) + h'(\langle f_1(e), x\rangle) \leq 2h'(f(\langle e, x\rangle)).$$

But, since f majorizes f_0 and f_1 and f and $\langle \cdot, \cdot \rangle$ are strictly increasing, it follows by the choice of h' that $\hat{h}(n) \leq h'(f(n)) \leq n$ for $n \geq 0$. So $(A_0 \oplus A_1)^\dagger$ is h-c.a. $\square$

7. EUwttAC and Array Computable Sets

In the preceding section we have shown that the class of the c.e. *wtt*-superlow sets is a subclass of EUwttAC. Here we show that the class of c.e. sets having array computable (a.c.) *wtt*-degree is a superclass of EUwttAC, i.e., there is no e.u.wtt-a.c. c.e. set which is *wtt*-equivalent to an array noncomputable (a.n.c.) set.

Before we do so, we give some background on the array (non)computable sets and degrees. We mentioned in the introduction, already, that the *array computable* degrees, introduced by Downey, Jockusch and Stob [DJS90], have proven a highly successful unifying tool in the study of the computational power of c.e. (and general) sets and Turing degrees. We recall from [DJS90] that a degree **a** is *array noncomputable (a.n.c.)* iff for all functions $f \leq_{wtt} \emptyset'$ there is a function g computable from **a** such that

$$\exists^\infty x(g(x) > f(x)).$$

So array noncomputability is a kind of non-lowness property, closely resembling – but more general than – non-low$_2$-ness since the latter property is obtained if in the above definition we consider all functions f which are Turing reducible to $\emptyset'$ and not only the ones which are *wtt*-reducible to $\emptyset'$. It turned out that many constructions which were originally proven using non-low$_2$-ness, could be adapted to work with the weaker assumption that **a** is array noncomputable. For example, Downey, Jockusch and Stob [DJS96] showed that every array noncomputable degree bounds a 1-generic degree. The unifying power of such degrees can be seen in the following summary theorem.

Theorem 7.1. *The c.e. a.n.c. degrees are those that:*

- *(Kummer* [Kum96]*) contain c.e. sets of infinitely often maximal Kolmogorov complexity*[5]*.*
- *(Barmpalias, Downey and McInerney* [BDM15]*) have integer valued randoms.*
- *(Downey and Greenberg* [DG08]*) have reals of effective packing dimension 1.*

Moreover, (Cholak et al. [CCDH01]*) the array noncomputable c.e. degrees form an invariant class for the lattice of Π^0_1 classes via the thin perfect classes.*

Having illustrated the importance of the array noncomputable sets and degrees, we now come back to our goal. For this purpose, we have to consider array noncomputable sets and their *wtt-degrees* (not their Turing degrees as in the examples above). We use a characterization of the a.n.c. *wtt*-degrees in terms of multiple permitting which is closer to the original definition of the computably enumerable

[5]Again, these classes are mentioned only to demonstrate the amazing unifying power of this class, and hence we won't formally define them, as it would interrupt the narrative flow.

a.n.c. set in [DJS90] than the non-domination characterization given above. Multiply permitting sets have been introduced by Ambos-Spies in [AS18], and there it is shown that the array noncomputable c.e. *wtt*-degrees, i.e., the *wtt*-degrees which contain a computably enumerable a.n.c. set, can be characterized as those c.e. *wtt*-degrees whose c.e. members are multiply permitting. For the definition of a multiply permitting sets, recall that a *very strong array* (*v.s.a.* for short) is a sequence $\mathcal{F} = \{F_n\}_{n\in\omega}$ of finite sets such that there exists a computable function $f : \omega \to \omega$ such that for all n, $F_n = D_{f(n)}$, i.e., $f(n)$ is the canonical index of F_n, $0 < |F_n| < |F_{n+1}|$ and $F_n \cap F_m = \emptyset$ hold for all $m \neq n$. Then multiply permitting c.e. sets are defined as follows.

Definition 7.2 ([AS18])**.** *Let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a v.s.a., let f be a computable function, let A be a c.e. set, and let $\{A_s\}_{s\in\omega}$ be a computable enumeration of A. Then A is* $\mathcal{F}$-permitting via f and $\{A_s\}_{s\in\omega}$ *if, for any partial computable function ψ,*

$$\exists^\infty n\ \forall x \in F_n\ (\psi(x)\downarrow\ \Rightarrow\ A \upharpoonright f(x)+1 \neq A_{\psi(x)} \upharpoonright f(x)+1) \tag{62}$$

holds. A is $\mathcal{F}$-permitting via f *if there is a computable enumeration $\{A_s\}_{s\in\omega}$ of A such that A is $\mathcal{F}$-permitting via f and $\{A_s\}_{s\in\omega}$; A is* $\mathcal{F}$-permitting *if A is $\mathcal{F}$-permitting via some computable f; and A is* multiply permitting *if A is $\mathcal{F}$-permitting for some v.s.a. $\mathcal{F}$. Finally, a c.e. wtt-degree $\boldsymbol{a}$ is* multiply permitting *if there is a multiply permitting set $A \in \boldsymbol{a}$.*

By [AS18, Lemma 1], the property of being multiply permitting for a c.e. set does not depend on the choice of the very strong array.

Lemma 7.3 ([AS18])**.** *Let A be multiply permitting and let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be a v.s.a. Then A is $\mathcal{F}$-permitting.*

Moreover, as shown in [AS18], too, the multiple-permitting property is *wtt*-invariant and the multiply permitting *wtt*-degrees coincide with the c.e. array noncomputable *wtt*-degrees.

Lemma 7.4 ([AS18])**.** *For a c.e. wtt-degree $\boldsymbol{a}$, the following are equivalent.*

(1) $\boldsymbol{a}$ is a.n.c.
(2) $\boldsymbol{a}$ is multiply permitting.
(3) Every c.e. set $A \in \boldsymbol{a}$ is multiply permitting.

Using Lemma 7.3, we can show that the following holds.

Theorem 7.5. *Let A be multiply permitting. Then A is not e.u.wtt-a.c.*

Proof. Suppose that A is multiply permitting. It suffices to show that, for any given computable functions $g, k : \omega^2 \to \{0,1\}$ and any given computable order h such that (6), (7) and (9) hold, (8) fails. For that, let $\mathcal{F} = \{F_n\}_{n\in\omega}$ be the unique very strong array such that each F_n is an interval such that $|F_n| = \hat{h}(n)$, where $\hat{h}(n) = \lfloor \frac{h(\langle n,n\rangle)+1}{2} \rfloor$ (note that $\hat{h}$ is a computable order) and such that $\min(F_{n+1}) = \max(F_n) + 1$ holds for all n. By Lemma 7.3, we may fix a computable function f and a computable enumeration $\{A_s\}_{s\in\omega}$ of A such that A is $\mathcal{F}$-permitting via f and $\{A_s\}_{s\in\omega}$, where, w.l.o.g., we may assume that f is strictly increasing.

Then we define a *wtt*-functional Γ in stages s where, by Lemma 3.3, we may assume that in advance we know a number d such that $\Gamma = \hat{\Phi}_d$ holds. In particular, by (6), $\lim_{s\to\infty} g(\langle d, n\rangle, s) = 1$ holds iff $n \in \mathrm{dom}(\Gamma^A)$. In more detail, we define a uniformly computable sequence of wtt-functionals $\{\tilde{\Gamma}_e\}_{e\in\omega}$ and we declare $\tilde{\Gamma}^A_e(n)$ to be defined (undefined) at a stage $s+1$ only if $g(\langle e,n\rangle, s)$ correctly approximates whether or not $\tilde{\Gamma}^A_e(n)[s]$ is defined (so below, the reader may replace Γ by $\tilde{\Gamma}_e$ and any occurence of d in any of the functions g and k by e). Then, by clauses 1. and 2. of Lemma 3.3 there exists $d \in \omega$ such that $\tilde{\Gamma}_d = \hat{\Phi}_d$. So d and $\Gamma = \tilde{\Gamma}_d$ are as desired.

Then the definition of Γ is as follows, where we stick to the convention that $\Gamma^A(n)[s+1] = \Gamma^A(n)[s]$ holds for any n and any stage s unless otherwise stated. Fix n in the following.

Definition of $\Gamma^A(n)$.

Stage 0. Let $\Gamma^A(n)[0]\uparrow$.

Stage $s+1$. Let $\Gamma^A(n)[s]$ be given. Then we destinguish between the following two cases.

(1) If $\Gamma^A(n)[s]\uparrow$ and $g(\langle d,n\rangle, s) = 0$ hold then declare $\Gamma^A(n)[s+1]\downarrow$.

(2) If $\Gamma^A(n)[s]\downarrow$, $g(\langle d,n\rangle, s) = 1$, $k(\langle d,n\rangle, s) = 1$ and we have that $A_{s+1} \upharpoonright f(\max(F_n)) + 1 \neq A_{s^-} \upharpoonright f(\max(F_n)) + 1$, where s^- is the largest stage $\leq s$ such that $\Gamma^A(n)[t]\downarrow$ holds for all $t \in [s^-, s]$, then declare $\Gamma^A(n)[s+1]\uparrow$.

By definition, Γ is a Turing functional and since, by clause (2), the use of Γ on input n is bounded by $f(\max(F_n))$, it follows that Γ is indeed a wtt-functional. Moreover, by clause (1), we may argue that Γ^A is total as we keep $\Gamma^A(n)[s]\uparrow$ for any stage s unless (1) holds. However, as $g(\langle d,n\rangle, s)$ correctly approximates the question as to whether or not $x \in \mathrm{dom}(\Gamma^A)$ holds, it follows that, for any stage s such that $\Gamma^A(n)[s]\uparrow$, there exists a least stage $t \geq s$ such that $\Gamma^A(n)[t]\uparrow$ and $g(\langle d,n\rangle, t) = 0$. So for the least s such that $A_s \upharpoonright f(\max(F_n)) + 1 = A \upharpoonright f(\max(F_n)) + 1$ and $\Gamma^A(n)[s]\downarrow$, it follows that $\Gamma^A(n)[t]\downarrow$ for all $t > s$. Hence, by (9), we may fix $n_0 \in \omega$ such that $\lim_{s\to\infty} k(\langle d,n\rangle, s) = 1$ holds for all $n \geq n_0$. Likewise, we can argue that for any stage s such that $\Gamma^A(n)[s]\downarrow$ there exists a least stage $t \geq s$ such that $\Gamma^A(n)[t]\downarrow$ and $g(\langle d,n\rangle, t) = 1$. In particular, the clauses (1) and (2) always apply alternatingly to $\Gamma^A(n)$.

Now consider the partial computable function $\psi : \omega \to \omega$ which is defined as follows. Given n, let $x^n_0 < \cdots < x^n_{\hat{h}(n)-1}$ be the elements of F_n. Then $\psi(x^n_i)$ is defined inductively such that, for all $i < \hat{h}(n) - 1$, we have

$$\psi(x^n_0) = \mu s(P(n,s)),$$

$$\psi(x^n_{i+1}) = \mu s(s > \psi(x^n_i) \ \&\ P(n,s) \ \&\ \exists t \in (\psi(x^n_i), s)\ (\Gamma^A(n)[t]\uparrow)),$$

where $P(n,s)$ holds iff $\Gamma^A(n)[s]\downarrow$, $g(\langle d,n\rangle, s) = 1$ and $k(\langle d,n\rangle, s) = 1$ holds. Note that, for all n, we have that either $\mathrm{dom}(\psi) \cap F_n = \emptyset$ or $F_n \subset \mathrm{dom}(\psi)$. Namely, by definition, $\psi(x^n_i)\downarrow$ can only hold if $\psi(x^n_j)\downarrow$ holds for all $j < i$ and, if $\psi(x^n_i)\downarrow$ holds for some $i < \hat{h}(n)$ then, by (62) and since $\lim_{s\to\infty} g(\langle d,n\rangle, s) = 1$ holds, there exists a stage $t > \psi(x^n_i)$ such that (2) applies at stage t in the definition of $\Gamma^A(n)$; hence, by (7), by the definition of $P(n,s)$ and by the totality of Γ^A, we may infer that

$\psi(x^n_{i+1})\downarrow$ holds. So since $\lim_{s\to\infty} k(\langle d,n\rangle, s) = 1$ holds for all $n \geq n_0$, it follows that there exist infinitely many n such that $\psi(x^n_0)\downarrow$; hence, $F_n \subset \mathrm{dom}(\psi)$ holds. However, for any such n, by the definition of Γ, it follows that, for any $i \leq \hat{h}(n)$, there exist two stages $\psi(x^n_i) \leq s_0 < s_1$ such that $g(\langle d,n\rangle, s_i+1) \neq g(\langle d,n\rangle, s_i)$ holds for all $i \leq 1$; and, if $i < \hat{h}(n)$ then $s_1 < \psi(x^n_{i+1})$ holds. So for any $n \geq d$ such that $\psi(x^n_0)\downarrow$ holds the number of mind changes of $g(\langle d,n\rangle, \cdot)$ after stage $\psi(x^n_0)$ is at least

$$2\hat{h}(n) > h(\langle n,n\rangle) > h(\langle d,n\rangle),$$

so (8) fails for any such n. However, as there are infinitely many $n \geq d$ such that $\psi(x^n_0)\downarrow$, we conclude that (8) fails, conrary to choice of A. This completes the proof. □

Corollary 7.6. *Let A be c.e. and e.u.wtt-a.c. Then any c.e. set B which is wtt-equivalent to A is array computable.*

Proof. By Lemma 7.4 and Theorem 7.5. □

8. Separations

In the preceding sections we have given lower and upper bounds for the class of the c.e. e.u.wtt-a.c. sets in terms of *wtt*-superlowness and array computability, respectively: any *wtt*-superlow set is e.u.wtt-a.c. (Corollary 6.3) and any c.e. e.u.wtt-a.c. set is array computable (Corollary 7.6). We conclude our investigations of the e.u.wtt-a.c. sets by showing that these inclusions are proper. In fact, in the case of the second inclusion, we get a slightly stronger result by showing that there is an array computable c.e. Turing degree which contains a c.e. set which is not e.u.wtt-a.c.

We start with the separation of *wtt*-superlowness and eventually uniform *wtt*-array computability on the c.e. sets.

8.1. **A c.e. e.u.wtt-a.c. set which is not *wtt*-superlow.** In order to separate the c.e. *wtt*-superlow sets from the c.e. e.u.wtt-a.c. sets, by the Characterization Theorem 4.2, it suffices to show the following.

Theorem 8.1. *There is a maximal set M which is not wtt-superlow.*

In the proof of the theorem we use the following sufficient condition for a c.e. set M to be not *wtt*-superlow.

Lemma 8.2. *Assume that M is c.e. and there is a partial computable function ψ and a Turing functional Ψ such that the following hold.*

(63) *If $\Psi^M(x)\downarrow$ then $\psi(x)\downarrow$ and $\Psi^M(x) = \Psi^{M\restriction\psi(x)}(x)$ (for $x \geq 0$).*

(64) *The domain of Ψ^M is not ω-c.a.*

Then M is not wtt-superlow.

Proof of Lemma 8.2 (sketch). By (63) there is an index e such that the domain of Ψ^M coincides with the domain of $\hat{\Phi}^M_e$. So, for any x, $x \in dom(\Psi^M)$ iff $\langle e,x\rangle \in M^\dagger$. By (64) this implies that $M^\dagger$ is not ω-c.a. So M is not *wtt*-superlow by Lemma 6.2. □

Proof of Theorem 8.1. We construct a c.e. set M, an auxiliary partial computable function ψ, and an auxiliary Turing functional Ψ such that M is maximal and (63) and (64) hold. Then, by Lemma 8.2, the set M has the required properties. The construction is in stages, and we let M_s, ψ_s and Ψ_s denote the finite parts of M, ψ and Ψ, respectively, enumerated by the end of stage s. Moreover, as in other places too, we abbreviate $\Psi_s^{M_s}(x)$ by $\Psi^M(x)[s]$.

The proof is similar to the proof of Theorem 4.3 though less involved. In particular, in order to make M maximal, we use the maximal set technique based on a priority tree introduced there. We use the notation introduced there as well as the basic observations made there, hence assume the reader to be familiar with the first part of the proof of Theorem 4.3 discussing the maximal set strategy (up to the Maximal Set Lemma).

The strategy to make M not *wtt*-superlow, i.e., the strategy to ensure that the functional Ψ and its use function ψ satisfy conditions (63) and (64) locally resembles the strategy used in the proof of Theorem 4.3 in order to ensure that $A \leq_{ibT} M$. So Ψ and ψ here and the functionals and functions Ψ_α and ψ_α defined there show some fundamental similarities. Condition (64) is split into the requirements

$\hat{\mathcal{R}}_e$: If $dom(\Psi^M) = \lambda x. \lim_s g_{e_0}(x,s)$ and φ_{e_1} is total then there is a number x such that $|\{s : g_{e_0}(x,s+1) \neq g_{e_0}(x,s+1)\}| > \varphi_{e_1}(x)$.

for $e \geq 0$ where $\{g_e\}_{e\in\omega}$ is a computable numbering of the primitive recursive functions of type $\omega^2 \to \{0,1\}$ and where (here and in the following) we assume that $e = \langle e_0, e_1\rangle$.

The basic strategy for meeting requirement $\hat{\mathcal{R}}_e$ is as follows. We fix a number x, called the *target*, which we make to witness that requirement $\hat{\mathcal{R}}_e$ is met. So we leave the definition of $\psi(x)$ and $\Psi^X(x)$ to the $\hat{\mathcal{R}}_e$-strategy. We wait for a stage s_0 such that $\varphi_{e_1,s_0}(x)$ is defined. (Note that if there is no such stage then x witnesses that φ_{e_1} is not total whence $\hat{\mathcal{R}}_e$ is trivially met.) Once we see stage s_0, we pick $\varphi_{e_1}(x)+1$ many numbers $y_{\varphi_{e_1}(x)} < y_{\varphi_{e_1}(x)-1} < \cdots < y_0$ not yet in M, called *followers*, and let the $\hat{\mathcal{R}}_e$-strategy decide which of these numbers are enumerated into M. Moreover, we let the use $\psi(x)$ of Ψ on x be a strict upper bound on the followers, say, $\psi(x) = y_0+1$, declare the strategy to be *saturated* and let the attack reach its final phase where we guarantee that either g_{e_0} does not approximate Ψ^M on x or the number of mind changes of the approximation exceeds the allowed bound $\varphi_{e_1}(x)$. Note that when we start this phase, say, at stage s_1, then $\Psi^{M_{s_1}}(x)$ is still undefined and none of the followers is in M_{s_1}. Now at stage $s+1 > s_1$ act as follows. If $\Psi^{M_s}(x)\uparrow$ and $g_{e_0}(x,s+1) = 0$ then make $\Psi^{M_{s+1}}(x)$ be defined thereby making the approximation incorrect at stage $s+1$. Note that this does not require to change the oracle. If $\Psi^{M_s}(x)\downarrow$ and $g_{e_0}(x,s+1) = 1$ and

$$|\{t \leq s : g_{e_0}(x,t+1) \neq g_{e_0}(x,t)\}| \leq \varphi_{e_1}(x) \tag{65}$$

then enumerate the greatest follower into M at stage $s+1$ that has not been enumerated into M previously. This allows to make $\Psi^{M_{s+1}}(x)$ to be undefined (thereby making the approximation incorrect at stage $s+1$).

Note that this procedure ensures that the approximation g_{e_0} of Ψ^M is incorrect on x unless g_{e_0} changes its mind on x after stage s_1 more than $\varphi_{e_1}(x)$ times, whence $\hat{\mathcal{R}}_e$ is met. Namely, if the approximation is correct, then, by using $\varphi_{e_1}(x)$ of the followers we may force the approximation to change $1+2\cdot\varphi_{e_1}(x)$ times by making the computation of Ψ on x alternatingly defined and undefined when the current approximation is correct where the first switch is from undefined to defined and where only a switch from defined to undefined requires to change the current oracle below its use by enumerating a follower into M. So, in fact, the least follower will never be enumerated into M, a fact which will be utilized in the maximal set part of the construction (in particular, it will allow us to argue that $\overline{M}$ is infinite).

In order to make this strategy compatible with the maximal set strategy, for any node α of length e there will be a strategy $\hat{\mathcal{R}}_\alpha$ for meeting requirement $\hat{\mathcal{R}}_e$. This strategy, which is based on the guess that α is on the true path, may act only if α is accessible and it picks only followers which have current e-state $\leq \alpha$. Moreover, it picks followers one-by-one. We will argue that, for the node α of length e on the true path, these modifications will not undermine the basic strategy. In particular for such α where φ_{e_0} is defined on the target x, the strategy eventually will become saturated.

Having explained the underlying ideas we can now give the formal construction of M and the auxiliary functional Ψ and function ψ. If a strategy $\hat{\mathcal{R}}_\alpha$ is *initialized* at stage $s+1$ then its target (if any) and followers (if any) are cancelled and the strategy is declared to be not saturated. Stage 0 is vacuous, i.e., $M_0 = \emptyset$, $\Psi^X_0(x)\uparrow$ and $\psi_0(x)\uparrow$ for all numbers x, and all strategies $\hat{\mathcal{R}}_\alpha$ are initialized.

Stage $s+1$. A strategy $\hat{\mathcal{R}}_\alpha$ *requires attention* at stage $s+1$ if $\alpha \sqsubseteq \delta_s$ and one of the following holds where $e = |\alpha|$.

(a) No target is assigned to $\hat{\mathcal{R}}_\alpha$ at the end of stage s.

(b) Target x is assigned to $\hat{\mathcal{R}}_\alpha$ at the end of stage s, $\varphi_{e_1,s}(x)\downarrow$, and $\hat{\mathcal{R}}_\alpha$ is not saturated at the end of stage s. Moreover, for the greatest number y such that $y = x$ or y is a follower of $\hat{\mathcal{R}}_\alpha$ at the end of stage s, there is a number y' such that $y < y' \leq s$, y' is greater than any follower of any higher priority strategy $\hat{\mathcal{R}}_{\alpha'}$ at the end of stage s, and $y' \in \bigcup_{\{\alpha' : |\alpha'|=|\alpha| \text{ and } \alpha' \leq_{left} \alpha\}} V_{\alpha',s}$ (i.e., $y' \notin M_s$ and $\sigma(|\alpha|, y', s) \leq \alpha$).

(c) $\hat{\mathcal{R}}_\alpha$ is saturated at the end of stage s, x is the target of $\hat{\mathcal{R}}_\alpha$ at the end of stage s and one of the following holds.
   (A) $\Psi^{M_s}_s(x)\uparrow$ and $g_{e_0}(x, s+1) = 0$.
   (B) $\Psi^{M_s}_s(x)\downarrow$, $g_{e_0}(x, s+1) = 1$, (65) holds, and there is a follower y of $\hat{\mathcal{R}}_\alpha$ at the end of stage s such that $y \notin M_s$.

Fix the least α (if any) such that $\hat{\mathcal{R}}_\alpha$ requires attention. Declare that $\hat{\mathcal{R}}_\alpha$ *receives attention* and is *active* at stage $s+1$, and perform the following action according to the case via which $\hat{\mathcal{R}}_\alpha$ requires attention.

(a) Assign $s+1$ as target to $\hat{\mathcal{R}}_\alpha$.

(b) Appoint the least y' as in (b) as follower of $\hat{\mathcal{R}}_\alpha$. Moreover, if there are $\varphi_{e_1}(x)+1$ followers of $\hat{\mathcal{R}}_\alpha$ then let $\psi_s(x) = y'+1$ and declare $\hat{\mathcal{R}}_\alpha$ to be *saturated*.

(c) If (A) holds then let $\Psi^{M_s}_{s+1}(x) = 0$. If (B) holds then enumerate the greatest follower y of $\hat{\mathcal{R}}_\alpha$ such that $y \notin M_s$ into M and let $\Psi^{M_{s+1}}_{s+1}(x)\uparrow$.

In case of (b) or (c), initialize all lower priority strategies $\hat{\mathcal{R}}_\beta$ ($\alpha < \beta$) and enumerate all numbers $z \leq s$ such that $z \notin M_s$ and z is not a follower of any strategy $\hat{\mathcal{R}}_{\beta'}$ with $\beta' \leq \alpha$ at stage $s+1$ into M.

If no strategy $\hat{\mathcal{R}}_\alpha$ requires attention then do nothing.

This completes the construction. In the remainder of the proof we show that the set M has the required properties. This proof uses that the Infinity Lemma and the Maximal Set Lemma hold which were established in the proof of Theorem 4.3 already.

Now, first note that the construction is effective and $\{M_s\}_{s\in\omega}$ is a computable enumeration of M. So M is c.e. Similarly, ψ is a partial computable function and Ψ is a Turing functional with computable enumerations $\{\psi_s\}_{s\in\omega}$ and $\{\Psi_s\}_{s\in\omega}$, respectively. Moreover, (63) holds. (Namely, assume that $\Psi_{s+1}^{M_{s+1}}(x) \neq \Psi_s^{M_s}(x)\downarrow$ for some number x and stage s. Then there is a saturated strategy $\hat{\mathcal{R}}_\alpha$ such that $\hat{\mathcal{R}}_\alpha$ has target x and an $\hat{\mathcal{R}}_\alpha$-follower y is enumerated into M at stage $s+1$. Since, by construction, $y < \psi(x)\downarrow$, it follows that $M_{s+1} \upharpoonright \psi(x) \neq M_s \upharpoonright \psi(x)$. Obviously this implies (63).)

So it only remains to show that $\overline{M}$ is infinite and, for any $\alpha \sqsubset TP$, $\overline{M} \subseteq^* \hat{V}_\alpha$ (by the Maximal Set Lemma this implies that M is maximal) and that the requirements $\hat{\mathcal{R}}_e$ are met. For this sake we prove a series of claims.

Claim 1. For any number y and any stage s there is at most one node α such that y is an $\hat{\mathcal{R}}_\alpha$-follower at the end of stage s. Moreover, if y is an $\hat{\mathcal{R}}_\alpha$-follower at the end of stage s then $|\alpha| < y \leq s$, $\sigma(|\alpha|, y) \leq \sigma(|\alpha|, y, s) \leq |\alpha|$, and y is greater than any follower of any higher priority strategy $\hat{\mathcal{R}}_\beta$ ($\beta < \alpha$) at the end of stage s. Finally if y is $\hat{\mathcal{R}}_\alpha$-follower at stages $s < s'$ then $\hat{\mathcal{R}}_\alpha$ is not initialized at any stage s'' such that $s' \leq s'' \leq s'$ (hence y is $\hat{\mathcal{R}}_\alpha$-follower at any such stage s'').

Proof. By a straightforward induction on s. For the proof of the final part note that if $\hat{\mathcal{R}}_\alpha$ is initialized at a stage s'' then any follower y appointed after stage s'' will correspond to a target appointed after this stage whence $y > s''$. □

Claim 2. Assume that y is the least follower of $\hat{\mathcal{R}}_\alpha$ at stage s. Then $y \notin M_s$.

Proof. For a contradiction assume that $y \in M_s$ and let $t_y + 1 \leq s$ be the stage at which y is enumerated into M. Let $s_y + 1 \leq s$ be the stage at which y is appointed and let x be the target of $\hat{\mathcal{R}}_\alpha$ at stage s_y. Then $\hat{\mathcal{R}}_\alpha$ is neither initialized at stage s_y nor at any stage $s'+1$ with $s_y + 1 \leq s' + 1 \leq s$. Hence, for any such stage s', y follows $\hat{\mathcal{R}}_\alpha$ at stage $s'+1$ and x is the target of $\hat{\mathcal{R}}_\alpha$ at stage s'. It follows that y can be enumerated into M at such a stage $s'+1$ only by action of $\hat{\mathcal{R}}_\alpha$ whence $\hat{\mathcal{R}}_\alpha$ has to be saturated at stage t_y. So we may pick the unique stage s'_y such that $s_y + 1 \leq s'_y + 1 < t_y + 1$ and $\hat{\mathcal{R}}_\alpha$ becomes saturated at stage $s'_y + 1$. Then $\varphi_{e_1, s'}(x)\downarrow$ and there are $\varphi_{e_1}(x) + 1$ followers of $\hat{\mathcal{R}}_\alpha$ at stage $s'_y + 1$, say, $y_0 > y_1 > \cdots > y_{\varphi_{e_1}(x)}$, where $y = y_{\varphi_{e_1}(x)}$ and none of theses followers is in $M_{s'_y+1}$. Now, after stage $s'_y + 1$ followers are enumerated into M in decreasing orders, and this happens only if $\hat{\mathcal{R}}_\alpha$ becomes active via clause (c) in the definition of requiring attention. As observed before this implies that $g_{e_0}(x, s)$ has to change at least once before the first follower is enumerated into M and between the enumeration of two followers, $g_{e_0}(x, s)$ has to change at least twice. So, for $0 \leq k \leq \varphi_{e_1}(x)$, if y_k is enumerated into M at stage $s' + 1 \leq t_y + 1$ then

$$|\{t \leq s' : g_{e_0}(x, t+1) \neq g_{e_0}(x, t)\}| \geq 1 + 2k$$

holds.It follows by the choice of t_y that

$$|\{t \leq t_y : g_{e_0}(x, t+1) \neq g_{e_0}(x, t)\}| \geq 1 + 2\varphi_{e_1}(y) > \varphi_{e_1}(y).$$

But this implies that (65) fails for $s = t_y$. So $\hat{\mathcal{R}}_\alpha$ does not require attention via clause (c) at stage $t_y + 1$ whence $y = y_{\varphi_{e_1}(x)}$ is not enumerated into M at stage M_{t_y} contrary to choice of t_y. □

Claim 3. Any strategy $\hat{\mathcal{R}}_\alpha$ on the true path ($\alpha \sqsubset TP$) is initialized at most finitely often, requires attention at most finitely often and has a permanent target.

Proof. Note that the strategies $\hat{\mathcal{R}}_\beta$ are finitary, i.e., if a strategy $\hat{\mathcal{R}}_\beta$ is not initialized after some stage s then it will act after stage s only finitely often. So, since strategies require attention only if they are accessible and since there are only finitely many stages at which strategies to the left of $\alpha \sqsubset TP$ are accessible, by a straightforward induction on $|\alpha|$, there is a stage s_0 such that no strategy $\hat{\mathcal{R}}_\beta$ with $\beta \leq \alpha$ will be initialized or will require attention after stage s_0. Moreover, $\hat{\mathcal{R}}_\alpha$ has a target at stage s_0 (since otherwise, for the first α-stage $s \geq s_0$, $\hat{\mathcal{R}}_\alpha$ will require attention via clause (a) at stage $s+1$) and the target is permanent since $\hat{\mathcal{R}}_\alpha$ is not initialized later. □

Claim 4. Assume that $\overline{M}$ is infinite and that $\alpha \sqsubset TP$. Then the following hold where $|\alpha| = e = \langle e_0, e_1 \rangle$.

- *(i) If φ_{e_1} is total then $\hat{\mathcal{R}}_\alpha$ is permanently saturated, i.e., $\hat{\mathcal{R}}_\alpha$ becomes saturated at some stage and is not initialized later.*
- *(ii) Requirement $\hat{\mathcal{R}}_e$ is met.*
- *(iii) $\overline{M} \subseteq^* \hat{V}_\alpha$.*

Proof. (i). Assume that φ_{e_1} is total. By Claim 3 fix a stage s_0 such that $\hat{\mathcal{R}}_\alpha$ is not initialized and does not require attention after stage s_0 and such that the permanent target x of $\hat{\mathcal{R}}_\alpha$ is defined at stage s_0 (whence $\varphi_{e_1,s_0}(x)$ is defined, too). Then any follower of any strategy $\hat{\mathcal{R}}_\beta$ with $\beta \leq \alpha$ which is defined at any stage $s \geq s_0$ is defined at stage s_0 hence is less than or equal to s_0. It follows that $\hat{\mathcal{R}}_\alpha$ is permanently saturated at stage s_0. Otherwise, by the Infinity Lemma, there is an α-stage $s > s_0$ such that $V_{\alpha,s} \not\subseteq \omega \upharpoonright s_0 + 1$ hence $\hat{\mathcal{R}}_\alpha$ will require attention via clause (b) at stage $s+1$ contrary to choice of s_0.

(ii). For a contradiction assume that requirement $\hat{\mathcal{R}}_e$ is not met. Then the hypotheses of the requirement are satisfied, i.e., $dom(\Psi^M) = \lambda x. \lim_s g_{e_0}(x, s)$ and φ_{e_1} is total, but the conclusion fails, whence for all numbers x and all stages s, (65) holds. It follows that, for any sufficiently large stage s, the strategy $\hat{\mathcal{R}}_\alpha$ requires attention via clause (b) at stage $s+1$ provided that s is an α-stage and $\hat{\mathcal{R}}_\alpha$ has a target x and a follower y at stage s where $y \notin M_s$. Since, by $\alpha \sqsubset TP$, there are infinitely many α-stages it follows by part (i) of the claim and by Claim 2 that there are infinitely many such stages s. So $\hat{\mathcal{R}}_\alpha$ requires attention infinitely often contrary to Claim 3.

(iii). Obviously, there are infinitely many numbers $e' = \langle e'_0, e'_1 \rangle$ such that $\varphi_{e'_1}$ is total. So, by part (i) of the claim, there are infinitely many $\alpha' \sqsubset TP$ such that $\hat{\mathcal{R}}_{\alpha'}$ has a permanent follower. So, by Claim 3, there are infinitely many stages s at which a strategy $\hat{\mathcal{R}}_{\alpha'}$ with $\alpha' \preceq \alpha$ acts via clause (b) whence any number y such that $y \leq s$ and y is not a follower of a strategy $\hat{\mathcal{R}}_{\alpha''}$ such that $\alpha'' \preceq \alpha$ or $\alpha'' \sqsubset \alpha$ at the end of stage $s+1$ will be enumerated into M at stage $s+1$ (unless y is in M_s already). Since, by Claim 3, the strategies $\hat{\mathcal{R}}_{\alpha'}$ with $\alpha' \sqsubset \alpha$ have only finitely many followers during the course of the construction, it follows that almost all numbers y in $\overline{M}$ become a follower of a strategy $\hat{\mathcal{R}}_{\alpha'}$ with $\alpha' \preceq \alpha$ at some stage s. But, by construction, this implies that y has e-state $\leq \alpha$ at stage s whence $y \in \hat{V}_\alpha$. □

Claim 5. There are infinitely many stages s at which some strategy becomes active via clause (b).

Proof. For a contradiction fix a stage s_0 such that no strategy becomes active via clause (b) after stage s_0. Then no follower is appointed after stage s_0 whence any follower is $\leq s_0$ and there is a stage $s_1 \geq s_0$ such that no strategy acts via clause (b) or (c) after stage s_1. So, by construction, no number $\geq s_1$ is enumerated into M, hence $\overline{M}$ is infinite. But, by Claim 4 (i), this implies that, for infinitely many $\alpha \sqsubset TP$ the strategy $\hat{\mathcal{R}}_\alpha$ acts via (b). Contradiction. □

Claim 6. $\overline{M}$ is infinite.

Proof. By Claim 2 it suffices to show that there are infinitely many strategies $\hat{\mathcal{R}}_\alpha$ which have a permanent follower. For a contradiction assume not. Fix the node α of lowest priority such that $\hat{\mathcal{R}}_\alpha$ has a permanent follower. Then $\hat{\mathcal{R}}_\alpha$ is initialized only finitely often, hence requires attention only finitely often. So we may fix a stage s_0 such that no strategy $\hat{\mathcal{R}}_\beta$ with $\beta \leq \alpha$ becomes active via (b) or (c) after stage s_0. On the other hand, by Claim 5, there is a strategy $\hat{\mathcal{R}}_\beta$ which becomes active via clause (b) after stage s_0. So we may fix β of highest priority such that $\hat{\mathcal{R}}_\beta$ acts via (b) or (c) after stage s_2, say, at stage $s+1$. Then $\alpha < \beta$, $\hat{\mathcal{R}}_\beta$ has a follower at stage $s+1$ and, by the minimality of β, $\hat{\mathcal{R}}_\beta$ is not initialized after stage s. So the followers of $\hat{\mathcal{R}}_\beta$ at stage $s+1$ are permanent. Contradiction. □

Claim 7. M is maximal and not wtt-superlow.

Proof. As observed before, M is c.e. and (63) holds. So, in order to show that M is maximal, by the Maximal Set Lemma it suffices to show that $\overline{M}$ is infinite and, for any e, $\overline{M} \subseteq^* \hat{V}_{TP\restriction e}$, and, in order to show that M is not *wtt*-superlow, it suffices to show that, for $e \geq 0$, requirement $\hat{\mathcal{R}}_e$ is met. But, by Claim 6 and by Claim 4 (iii) and (ii), these properties hold. □

This completes the proof of Theorem 8.1. □

Corollary 8.3. *There is an e.u.wtt-a.c. c.e. set which is not wtt-superlow.*

Proof. By Theorems 4.2 and 8.1. □

8.2. **Separating eventually uniform *wtt*-array computability from array computability.** In this subsection we show that there is an array computable Turing degree which contains a c.e. set which is not e.u.wtt-a.c. By the Characterization Theorem 4.2 it suffices to prove the following theorem.

Theorem 8.4. *There is a c.e. set A such that the Turing degree of A is array computable and such that A is not wtt-reducible to any maximal set.*

Before proving the theorem, let us first describe the basic strategy for building a c.e. set which is not *wtt*-reducible to any maximal set.

A c.e. set A such that A is not *wtt*-reducible to any maximal set can be defined in stages s as follows (where A_s denotes the finite part of A enumerated by the end of stage s and where $A_0 = \emptyset$). It suffices to meet the requirements

$$\mathcal{P}_e : \text{ If } A = \hat{\Phi}_{e_1}^{W_{e_0}} \text{ then } W_{e_0} \text{ is not maximal.}$$

for all numbers $e = \langle e_0, e_1 \rangle$.

The strategy for meeting $\mathcal{P}_e$ is based on the following observation. If $\mathcal{F} = \{F_n\}_{n\in\omega}$ is a complete disjoint strong array of intervals (i.e., the effectively given finite sets F_n are intervals partitioning ω where $\min F_{n+1} = (\max F_n) + 1$) such that

$$\exists^\infty n\, (|\overline{W_{e_0}} \cap F_n| \geq 2) \tag{66}$$

holds then W_{e_0} is not maximal. Namely, assuming (66), the c.e. set Q defined by

$$Q \cap F_n = \begin{cases} (W_{e_0} \cap F_n) \cup \{\mu x \in F_n \cap \overline{W_{e_0}}\} & \text{if } F_n \not\subseteq W_{e_0}, \\ F_n & \text{otherwise} \end{cases}$$

is a c.e. super set of W_{e_0} satisfying $W_{e_0} \subset^\infty Q \subset^\infty \omega$ hence witnesses that W_{e_0} is not maximal. On the other hand, if, for a complete disjoint strong array of intervals $\{F_n\}_{n\in\omega}$, (66) fails then

$$\forall^\infty n\, (|\overline{W_{e_0}} \restriction (\max F_n) + 1| < 2n). \tag{67}$$

This leads to the following idea. During the course of the construction we attempt to define a complete disjoint strong array of intervals $\{F_n\}_{n\in\omega}$ such that, for any n, there are $2n+1$ numbers $x_{n,0} < x_{n,1} < \cdots < x_{n,2n}$ in F_n such that $\hat{\varphi}_{e_1}(x_{n,2n}) \leq \max F_n$ (and where these numbers $x_{n,k}$ are reserved for the strategy to meet $\mathcal{P}_e$). Now, assuming $A = \hat{\Phi}_{e_1}^{W_{e_0}}$ we can define such a strong array (since the assumption implies that $\hat{\varphi}_{e_1}$ is total). Moreover, if (66) fails, hence (67) holds then, for any n which satisfies the inner clause of (67), we can guarantee $A \neq \hat{\Phi}_{e_1}^{W_{e_0}}$ by enumerating (some of) the numbers $x_{n,0} < x_{n,1} < \cdots < x_{n,2n}$ into A. Namely, assuming that $A = \hat{\Phi}_{e_1}^{W_{e_0}}$, for almost all stages s we have that $|\overline{W_{e_0,s}} \restriction (\max F_n) + 1| < 2n$ and $A_s \restriction (\max F_n) + 1 = \hat{\Phi}_{e_1,s}^{W_{e_0,s}} \restriction (\max F_n) + 1$. But, since $\hat{\varphi}_{e_0}(x_{n,k}) \leq \max F_n$, at any such stage s we may force an additional number $y \leq \max F_n$ to enter W_{e_0} after stage s by enumerating x into A. Since there are less than $2n$ numbers $y \leq \max F_n$ which are not yet in W_{e_0} at stage s whereas there are $2n+1$ numbers $x_{n,k}$, $A \neq \hat{\Phi}_{e_1}^{W_{e_0}}$ must hold.

Now we are ready to prove Theorem 8.4.

Proof of Theorem 8.4. By a tree argument, we construct a c.e. set A with the required properties. The finite part of A enumerated by the end of stage s is denoted by A_s. $A_0 = \emptyset$.

In order to ensure that A is not *wtt*-reducible to any maximal set and that $deg(A)$ is a.c., it suffices to meet the requirements

$$\mathcal{P}_e : \text{ If } A = \hat{\Phi}_{e_1}^{W_{e_0}} \text{ then } W_{e_0} \text{ is not maximal.}$$

(where $e = \langle e_0, e_1\rangle$) and

$$\mathcal{N}_e : \text{ If } \Phi_e^A \text{ is total then } \Phi_e^A \text{ is } h\text{-c.a. for } h(n) = n+1.$$

respectively (for $e \geq 0$).

We call a requirement infinitary if its hypothesis is true. We need guesses which $\mathcal{N}$-requirements are infinitary. So we use the full binary tree $T = \{0,1\}^*$ as the priority tree. Then, for a node α of length $> e$, $\alpha(e) = 0$ codes the guess that requirement $\mathcal{N}_e$ is infinitary.

Define the computable length function l by

$$l(e,s) = \max\{y : \forall\, x < y\ (\Phi_{e,s}^{A_s}(x)\downarrow)\}.$$

Then the guess δ_s at which of the first s $\mathcal{N}$-requirements are infinitary made at stage $s+1$ is defined as follows. Inductively define *α-stages* for each node α as follows. Each stage $s \geq 0$ is a *λ-stage*. If s is an α-stage, then we call s *α-expansionary* if $l(|\alpha|, s) > l(|\alpha|, t)$ for all α-stages $t < s$, and we let s be an *$\alpha 0$-stage* if s is α-expansionary and we let s be an *$\alpha 1$-stage* if s is an α-stage but not an $\alpha 0$-stage. Then $\delta_s \in T$ is the unique string α of length s such that s is an α-stage. Moreover, we say that α is *accessible* at stage $s+1$ if $\alpha \sqsubset \delta_s$, i.e., if s is an α-stage and $|\alpha| \leq s$.

The *true path* $f : \omega \to \{0,1\}$ of the construction is defined by

$$f(n) = \begin{cases} 0 & \text{if there are infinitely many } (f \restriction n)\text{-expansionary stages} \\ 1 & \text{otherwise.} \end{cases}$$

Note that f is the leftmost path through T visited infinitely often, i.e., for any n,

$$\forall^\infty s\ (f \restriction n \leq \delta_s) \text{ and } \exists^\infty s\ (\delta_s \restriction n \sqsubset f). \tag{68}$$

Moreover, since

$$\Phi_e^A \text{ total } \Rightarrow \lim_{s\to\infty} l(e,s) = \omega \tag{69}$$

it follows that, for infinitary $\mathcal{N}_e$, $f(e) = 0$.

For each node α of length e there is a strategy $\mathcal{P}_\alpha$ for $\mathcal{P}_e$ which is based on the guess α.

At stage s any strategy $\mathcal{P}_\alpha$ is in one of the following states: n-expanding or n-diagonalizing for some n. If $\mathcal{P}_\alpha$ is n-expanding (n-diagonalizing) for some n then we say $\mathcal{P}_\alpha$ is expanding (diagonalizing). The *rank* of *$\mathcal{P}_\alpha$ at stage s*, denoted by r_s^α, is the coded pair $\langle |\alpha|, m\rangle$ where m is the number of unfrozen intervals associated with $\mathcal{P}_\alpha$ at the end of stage s. If $\mathcal{P}_\alpha$ is n-expanding at stage s then the intervals $F_{n'}^\alpha$ with $n' < n$ are defined and $\mathcal{P}_\alpha$ works on defining F_n^α by first appointing the followers $x_{n,0}^\alpha < x_{n,1}^\alpha < \cdots < x_{n,2n}^\alpha$ one after the other and then by waiting for $\hat{\varphi}_{e_1,s}(x_{n,2n}^\alpha)$ to be defined in order to complete the definition of F_n^α.

If a strategy $\mathcal{P}_\alpha$ is initialized then all intervals and followers associated with it are cancelled and the state of $\mathcal{P}_\alpha$ is reset to "0-expanding". At any stage s such that $\delta_s \leq \alpha$, $\mathcal{P}_\alpha$ is initialized (in particular all $\mathcal{P}$-strategies are initialized at stage 0). In addition $\mathcal{P}_\alpha$ may be initialized at stage $s+1$ by the action of the acting strategy $\mathcal{P}_\beta$. The latter can happen only if $\beta \sqsubset \alpha$ (note that if $\beta <_L \alpha$ then $\mathcal{P}_\alpha$ is initialized automatically since $\beta \sqsubset \delta_s$). If $\mathcal{P}_\beta$ acts in order to diagonalize (i.e., according to clause (i) or (ii) below) then all $\mathcal{P}_\alpha$ with $\beta \sqsubset \alpha$ are initialized. Otherwise, i.e., if

$\mathcal{P}_\beta$ acts in order to expand, then only those $\mathcal{P}_\alpha$ with $\beta \sqsubset \alpha$ are initialized where $|\alpha|$ is greater than the rank of $\mathcal{P}_\beta$.

If a strategy is initialized then it has to start all over again. In addition to initialization there will be freezing and partial cancellation. This affects only some of the intervals and the work on the current interval to be defined, respectively. If an interval is frozen then it cannot be used for diagonalization later (hence its followers cannot be enumerated into A later). Similarly all of the followers of an interval under construction may be cancelled. In this case the construction of this interval has to be started all over again with new followers greater than the current stage. There are two events which may lead to freezing of an α-interval $\mathcal{F}^\alpha_n$ or of the followers $x^\alpha_{n,k}$ of $\mathcal{P}_\alpha$: first if a lower priority strategy $\mathcal{P}_\beta$ with $\alpha \sqsubset \beta$ acts by diagonalization and enumerates a number into A which is less than one of the followers of $\mathcal{F}^\alpha_n$ or one of the followers $x^\alpha_{n,k}$, respectively; second if the current approximation δ_s of the true path moves to the left of the guess at the true path based on which the interval $\mathcal{F}^\alpha_n$ was (or is being) built. For the latter case we associate each interval with such a guess. If $\mathcal{P}_\alpha$ is diagonalizing then it is protected against freezing.

If a strategy $\mathcal{P}_\alpha$ is initialized then all intervals and followers associated with it are cancelled and the state of $\mathcal{P}_\alpha$ is reset to "0-expanding". At any stage s such that $\delta_s \leq \alpha$, $\mathcal{P}_\alpha$ is initialized (in particular all $\mathcal{P}$-strategies are initialized at stage 0). In addition $\mathcal{P}_\alpha$ may be initialized at stage $s+1$ by the action of the acting strategy. Finally, any interval $\mathcal{F}^\alpha_n$ is associated with a guess γ. If $\alpha \sqsubset \delta_s$ and $\delta_s < \gamma$ then $\mathcal{F}^\alpha_n$ becomes frozen at the end of stage s unless $\mathcal{P}_\alpha$ is n-diagonalizing (or $\mathcal{F}^\alpha_n$ is frozen already). Freezing may also be caused by the acting strategies (see below).

At the end of any stage s, initialize all strategies $\mathcal{P}_\alpha$ such that $\delta_s \leq \mathcal{P}_\alpha$. Moreover if $\mathcal{P}_\alpha$ is n-expanding and there is an unfrozen interval $\mathcal{F}^\alpha_{n'}$ of $\mathcal{P}_\alpha$ with guess $\gamma^\alpha_{n'}$ such that $\delta_s \restriction r^\alpha_s + 1 < \gamma^\alpha_{n'}$ then, for the least such n', freeze all intervals $\mathcal{F}^\alpha_{n''}$ with $n' \leq n'' < n$ and cancel any follower $x^\alpha_{n,k}$ which is defined. Similarly, if $\mathcal{P}_\alpha$ is n-expanding, $x^\alpha_{n,0}, \dots, x^\alpha_{n,k}$ $(k \geq 0)$ are the current followers of $\mathcal{P}_\alpha$ of order n and $\delta_s \restriction r^\alpha_s + 1 < \delta_{t_{k'}} \restriction r^\alpha_{t_{k'}} + 1$ for all $k' \leq k$ where $t_{k'}+1$ is the stage at which $x^\alpha_{n,k'}$ became appointed then cancel $x^\alpha_{n,0}, \dots, x^\alpha_{n,k}$.

Then stage $s+1$ is as follows.

Requiring attention and the corresponding potential action. $\mathcal{P}_\alpha$ $(|\alpha| = e)$ requires attention at stage $s+1$ if $\alpha \sqsubset \delta_s$ and one of the following holds.

(i) $\mathcal{P}_\alpha$ is not diagonalizing and there is an unfrozen interval F^α_n such that

$$|\overline{W_{e_0,s}} \restriction (\max F^\alpha_n) + 1| < 2n. \tag{70}$$

Corresponding action. For the least such n, declare that $\mathcal{P}_\alpha$ is n-diagonali-zing.

Initialize all strategies $\mathcal{P}_\beta$ such that $\alpha \sqsubset \beta$. Moreover, for any strategy $\mathcal{P}_\beta$ such that $\beta \sqsubset \alpha$, $\mathcal{P}_\beta$ is expanding, say, n_β-expanding, and there is an $n' \leq n_\beta$ such that there is a follower $x^\beta_{n',k'} > x^\alpha_{n,0}$, fix the least such n', freeze all intervals $F^\alpha_{n''}$ with $n' \leq n'' < n_\beta$ (if not frozen already) and cancel all β-followers $x^\beta_{n_\beta,k}$ of order n_β which are defined.

(ii) There is an n such that $\mathcal{P}_\alpha$ is n-diagonalizing,

$$A_s \restriction x^\alpha_{n,2n} + 1 = \hat{\Phi}^{W_{e_0,s}}_{e_1,s} \restriction x^\alpha_{n,2n} + 1, \tag{71}$$

and there is a follower $x^\alpha_{n,k}$ in $F^\alpha_n \setminus A_s$.

Corresponding action. Put the *greatest* follower $x^\alpha_{n,k} \in F^\alpha_n \setminus A_s$ into A.

Initialize all strategies $\mathcal{P}_\beta$ such that $\alpha \sqsubset \beta$. Moreover, for any strategy $\mathcal{P}_\beta$ such that $\beta \sqsubset \alpha$, $\mathcal{P}_\beta$ is expanding, say, n_β-expanding, and there is an $n' \leq n_\beta$ such that there is a follower $x^\beta_{n',k'} > x^\alpha_{n,0}$, fix the least such n', freeze all intervals $F^\alpha_{n''}$ with $n' \leq n'' < n_\beta$ (if not frozen already) and cancel all β-followers $x^\beta_{n_\beta,k}$ of order n_β which are defined.

(iii) (i) does not hold, there is an n such that $\mathcal{P}_\alpha$ is n-expanding and the follower $x^\alpha_{n,2n}$ is not yet defined.

Corresponding action. For the least k such that $x^\alpha_{n,k}$ is not yet defined let $x^\alpha_{n,k} = s+1$. Declare that $x^\alpha_{n,k}$ becomes associated with $\mathcal{P}_\alpha$ as (n-)follower (of order k).

Initialize all strategies $\mathcal{P}_\beta$ such that $\alpha \sqsubset \beta$ and $|\beta| > r^\alpha_s$.

(iv) (i) does not hold, there is an n such that $\mathcal{P}_\alpha$ is n-expanding, the follower $x^\alpha_{n,2n}$ is defined and $\hat{\varphi}_{e_1,s}(x^\alpha_{2n})$ is defined as well.

Corresponding action. Let $F^\alpha_n = [x_n, s]$ where $x_0 = 0$ and $x_n = 1 + \max F^\alpha_{n-1}$ for $n > 0$. Assign the guess γ^α_n to F^α_n where

$$\gamma^\alpha_n = \min\{\delta_{t_k} \restriction r^\alpha_{t_k} + 1 : k \leq 2n\}$$

where $t_k + 1$ is the stage at which $x^\alpha_{n,k}$ became appointed. Declare that $\mathcal{P}_\alpha$ is $(n+1)$-expanding.

Initialize all strategies $\mathcal{P}_\beta$ such that $\alpha \sqsubset \beta$ and $|\beta| > r^\alpha_s$.

Selecting the strategy which will act. If there is a strategy which requires attention then, for any $\alpha \sqsubset \delta_s$ such that $\mathcal{P}_\alpha$ requires attention let $p^\alpha_s = 2|\alpha|$ if α requires attention via one of the clauses (i) or (ii) and let $p^\alpha_s = 2r^\alpha_s + 1$ if α requires attention via clause (iii) or (iv). Then, from the strategies which require attention, the strategy $\mathcal{P}_\alpha$ with minimal value p^α_s *receives attention* and *becomes active* and the action corresponding to the clause according to which $\mathcal{P}_\alpha$ requires attention is performed.

VERIFICATION.

Claim 1. Assume that $\mathcal{P}_\alpha$ is n-diagonalizing at stage $s+1$ and not initialized after stage s. Then $\mathcal{P}_\alpha$ is n-diagonalizing at all stage $s' \geq s+1$ and $\mathcal{P}_\alpha$ acts only finitely often.

Proof. By assumption and by construction, strategy $\mathcal{P}_\alpha$ is n-diagonalizing at all stage $s' \geq s+1$ and $F^\alpha_n[s'] = F^\alpha_n[s+1]$ hence $x^\alpha_{n,k}[s'] = x^\alpha_{n,k}[s+1]$ for all $k \leq 2n$. So, after stage s', $\mathcal{P}_\alpha$ can act only according to clause (ii) and, whenever it acts, one of the $2n+1$ followers $x^\alpha_{n,0}[s'], \ldots, x^\alpha_{n,2n}[s']$ is enumerated into A. So this can happen at most $2n+1$ times. □

Claim 2. Assume that $\mathcal{P}_{f\restriction e}$ becomes active infinitely often. Then

$$\lim_{s\to\infty} r^{f\restriction e}_s = \omega. \tag{72}$$

Proof. For a contradiction assume that the claim fails. Fix r minimal such that, for some number e,

$$\mathcal{P}_{f\restriction e} \text{ becomes active infinitely often and } \exists^\infty s\ (r_s^{f\restriction e} \leq r). \tag{73}$$

Fix the unique numbers e and m such that $r = \langle e, m\rangle$ (note that $e \leq r$). Then, by the minimality of r, e is the unique number satisfying (73) whence

$$\forall\, e' \neq e\ (\mathcal{P}_{f\restriction e'} \text{ becomes active infinitely often } \Rightarrow \forall^\infty s\ (r_s^{f\restriction e'} > r)). \tag{74}$$

Moreover,

$$\forall^\infty s\ (r \leq r_s^{f\restriction e}) \text{ and } \exists^\infty s\ (r = r_s^{f\restriction e}). \tag{75}$$

Since, by the definition of the true path, $f \restriction r < \delta_s$ for almost all s and since, by the definition of the rank, $r_s^{f\restriction e'} > r$ for all $e' > r$ and all s, by the above we may fix a stage $s_0 > r$ such that, for any $s \geq s_0$, the following hold.

$$f \restriction r < \delta_s \tag{76}$$

$$r \leq r_s^{f\restriction e} \tag{77}$$

$$\forall\, e' \neq e\ (\mathcal{P}_{f\restriction e'} \text{ does not become active at stage } s+1 \text{ or } r_s^{f\restriction e'} > r) \tag{78}$$

By (76), (77) and (78), any strategy $\mathcal{P}_{f\restriction e'}$ with $e' \leq r$ can be initialized after stage s_0 only if some strategy $\mathcal{P}_{f\restriction e''}$ with $e'' < e'$ becomes active via clause (i) or (ii). So, by Claim 1, it follows by a straightforward induction on e' that there is a stage $s_1 > s_0$ such that no $\mathcal{P}_{f\restriction e'}$ with $e' \leq r$ is initialized or acts according to (i) or (ii) after stage s_1. Hence, in particular, any interval assigned to $\mathcal{P}_{f\restriction e}$ after stage s_1 is permanent. (In the following let F_n^α ($n \geq 0$) denote the permanent intervals of $\mathcal{P}_{f\restriction e}$ if they exist.) Moreover, since $\mathcal{P}_{f\restriction e}$ acts infinitely often we may deduce that $\mathcal{P}_{f\restriction e}$ is expanding at all stages $s > s_1$. So we fix n_s such that $\mathcal{P}_{f\restriction e}$ is n_s-expanding at stage s ($s > s_1$). Then n_s is nondecreasing in s. Moreover, if $\{n_s : s > s_1\}$ is bounded then, for $n = \max\{n_s : s > s_1\}$, the followers of $\mathcal{P}_{f\restriction e}$ are cancelled infinitely often. So (in any case) we may fix a stage $s_2 > s_1$ such that, for $s \geq s_2$, the least $\mathcal{P}_{f\restriction e}$-follower $x_{n_s,0}^{f\restriction e}[s]$ of order n_s at the end of stage s has been appointed after stage s_1 or is undefined.

Next observe that, by the second part of (75) and by the definition of the rank, there are at most m permanent $\mathcal{P}_{f\restriction e}$-intervals which are never frozen, say, $F_{p_0}^{f\restriction e}, \dots, F_{p_{m'-1}}^{f\restriction e}$ where $p_0 < \cdots < p_{m'-1}$ and $m' \leq m+1$. Moreover, we may pick a stage $s_3 > s_2$ such that $F_{p_{m'-1}}^{f\restriction e}$ is defined at stage s_3. Then, for any $s > s_3$ and for the number n_s such that $\mathcal{P}_{f\restriction e}$ is n_s-expanding at stage s, $n_s > p_{m'-1}$. So, in order to get the desired contradiction, it suffices to show that there is a stage $s > s_3$ such that $F_{n_s}^{f\restriction e}$ becomes defined and is never frozen.

For this sake we first observe that there is a stage s such that

$$s > s_3 \text{ and } s \text{ is an } (f \restriction r)\text{-stage and } r = r_s^{f\restriction e}. \tag{79}$$

The existence of such a stage is shown as follows. Since there are infinitely many $(f \restriction r)$-stages and infinitely many stages s such that $r = r_s^{f\restriction e}$, we may pick two consecutive $(f \restriction r)$-stages s' and s'' such that $s_3 < s' < s''$ and such that there is a stage $t \in [s', s'']$ such that $r = r_t^{f\restriction e}$. Fix the least such t. We claim that $r = r_{s'}^{f\restriction e}$ or $r = r_{s''}^{f\restriction e}$ (or both). For a contradiction assume not. Then $s' < t < s''$ and

$$r_t^{f\restriction e} < r_{s'}^{f\restriction e} \text{ and } r_t^{f\restriction e} < r_{s''}^{f\restriction e}. \tag{80}$$

Now, by the second part of (80), there must be an interval $F_n^{f\restriction e}$ with $n \geq n_t$ which is defined and not frozen at stage s''. On the other hand, by the first part of (80), some $(f \restriction n)$-interval becomes frozen at stage t whence, by the freezing process, all $(f \restriction e)$-followers of order n_t are cancelled at stage t. So all of the followers $x_{n,k}^{f\restriction e}$ in $F_n^{f\restriction e}$ are appointed at stages t' with $t < t' < s''$. Since, for any such stage t', $f \restriction r <_L \delta_{t'} \restriction r$ it follows that $f \restriction r <_L \gamma_n^{f\restriction e}$ for the guess $\gamma_n^{f\restriction e}$ associated with $F_n^{f\restriction e}$. Since, by the choice of s'', $f \restriction r \sqsubseteq \delta_{s''} \restriction r_{s''}^{f\restriction e}$, it follows that $F_n^{f\restriction e}$ becomes frozen at stage s'' contrary to choice of this interval. This completes the proof of (79).

Now fix a stage s as in (79). It suffices to show that $F_{n_s}^{f\restriction e}$ becomes defined and is never frozen. Fix $k \geq 0$ maximal such that $x_{n_s,k-1}^{f\restriction e}$ is defined at stage s. Note that, by $s > s_0$, $\mathcal{P}_{f\restriction e}$ becomes active at any stage $s'+1 \geq s+1$ such that s' is an $(f \restriction e)$-stage and $r = r_{s'}^{f\restriction e}$. Now, first assume that $k < 2n_s + 1$. Then, at stage $s+1$, $x_{n_s,k}^{f\restriction e} = s$ becomes appointed and all strategies $\mathcal{P}_\alpha$ with $f \restriction r < \alpha$ are initialized. Since, by the choice of s_1, no strategy $\mathcal{P}_\alpha$ with $\alpha \leq f \restriction r$ enumerates any number into A after stage s_1, it follows that no number $\leq s$ will enter A after stage s. So, in particular, if $k+1 < 2n_s + 1$ and s' is the least $(f \restriction e)$-stage $> s$ then $r = r_{s'}^{f\restriction e}$ (note that the rank of $\mathcal{P}_{f\restriction e}$ can grow only if a new $(f \restriction e)$-interval becomes assigned and that the latter can happen only at $(f \restriction e)$-accessible stages) and the next follower $x_{n_s,k+1}^{f\restriction e} = s'$ becomes appointed at stage $s'+1$. Moreover, again, all strategies $\mathcal{P}_\alpha$ with $f \restriction r < \alpha$ are initialized whence no number $\leq x_{n_s,k+1}^{f\restriction e}$ may enter A after stage s'. It follows by induction that there is an $f \restriction r$-stage $s'' > s$ such that $x_{n_s,2n_s}^{f\restriction e} = s''$ becomes appointed at stage $s''+1$ and no number $< x_{n_s,2n_s}^{f\restriction e}$ enters A after stage s''. Since $\mathcal{P}_{f\restriction e}$ acts infinitely often, it follows that at the next stage where the strategy acts, $F_{n_s}^{f\restriction e}$ becomes defined. Moreover, since no number $< x_{n_s,2n_s}^{f\restriction e}$ will enter A later, $F_{n_s}^{f\restriction e}$ will not be frozen by the action of any strategy. Finally, since $x_{n_s,k+1}^{f\restriction e}$ became appointed at an $(f \restriction r)$-accessible stage $s+1$ where $r = r_s^{f\restriction e}$, it follows that $F_{n_s}^{f\restriction e}$ is associated with the guess $\gamma_{n_s}^{f\restriction e} = f \restriction r$ whence $F_{n_s}^{f\restriction e}$ will no be frozen at all. This completes the proof that $F_{n_s}^{f\restriction e}$ becomes defined and is never frozen in the case of $k < 2n_s + 1$.

If $k = 2n_s + 1$ then the argument is similar. Consider the stage $t+1 < s+1$ at which $x_{n_s,2n_s}^{f\restriction e}$ becomes defined. Since $x_{n_s,2n_s}^{f\restriction e}$ does not become cancelled by the end of stage s, as in the proof of (79) we may argue that t is an $(f \restriction r)$-stage and $r = r_t^{f\restriction e}$. So all strategies $\mathcal{P}_\alpha$ with $f \restriction r < \alpha$ are initialized at stage $t+1$. So, as in the first case, we may argue that no number $\leq x_{n_s,2n_s}^{f\restriction e}$ will be put into A after stage t, that $F_{n_s}^{f\restriction e}$ eventually becomes defined, and that $F_{n_s}^{f\restriction e}$ will never be frozen. □

Claim 3. $\mathcal{P}_{f\restriction e}$ is initialized at most finitely often.

Proof. The proof is by induction on e. Fix e. Since $f \restriction e < \delta_s$ for all sufficiently large s, by the inductive hypothesis we may fix a stage $s_0 > e$ such that no strategy $\mathcal{P}_{f\restriction e'}$ with $e' < e$ is initialized after stage s_0 and such $f \restriction e < \delta_s$ for all $s \geq s_0$.

Moreover, w.l.o.g. we may assume that no strategy $\mathcal{P}_{f\restriction e'}$ with $e' < e$ acts after stage s_0 unless it acts infinitely often. So, by Claim 1, $\mathcal{P}_{f\restriction e}$, will be initialized at a stage $s+1 > s_0$ only if a strategy $\mathcal{P}_{f\restriction e'}$ with $e' < e$ acts via clause (iii) or (iv) at this stage and $r_s^{f\restriction e'} < e$. Since $\mathcal{P}_{f\restriction e'}$ must be a strategy which acts infinitely often, it follows by Claim 2 that this can happen only finitely often. □

Claim 4. Assume that $\mathcal{P}_{f\restriction e}$ acts infinitely often. Then $\mathcal{P}_e$ is met.

Proof. By Claim 3 let s_0 be the greatest stage at which $\mathcal{P}_{f\restriction e}$ is initialized. Then, at the end of stage s_0, no interval is associated with $\mathcal{P}_{f\restriction e}$ and $\mathcal{P}_{f\restriction e}$ waits for 0-expansion. Moreover, any interval which becomes associated with $\mathcal{P}_{f\restriction e}$ after stage s_0 is permanent. So let $F_n^{f\restriction e}$ be the nth interval permanently associated with $\mathcal{P}_{f\restriction e}$ (if defined). Since, by assumption and by Claim 2,

$$\lim_{s\to\infty} r_s^{f\restriction e} = \omega \tag{81}$$

and since intervals become associated in order of their indices, it follows that $F_n^{f\restriction e}$ is defined for all n and, by construction, $\{F_n^{f\restriction e}\}_{n\in\omega}$ is a complete disjoint strong array of intervals.

So, as pointed out in the description of the basic strategy for building a c.e. set which is not *wtt*-reducible to any maximal set given after the statement of Theorem 8.4, in order to show that $\mathcal{P}_e$ is met it suffices to show that (67) fails for $F_n = F_n^{f\restriction e}$, i.e., that

$$\exists^\infty\, n\ (|\overline{W_{e_0}} \restriction (\max F_n^{f\restriction e}) + 1| \geq 2n) \tag{82}$$

holds. We do this by showing that there are infinitely many permanent intervals which are never frozen and that any such interval satisfies the inner clause of (82):

$$\exists^\infty\, n\ (F_n^{f\restriction e} \text{ is never frozen}) \tag{83}$$

$$\text{If } F_n^{f\restriction e} \text{ is never frozen then } |\overline{W_{e_0}} \restriction (\max F_n^{f\restriction e}) + 1| \geq 2n. \tag{84}$$

For a proof of (83), for a contradiction, assume that there are only finitely many permanent $(f \restriction e)$-intervals which are never frozen, say, $F_{n_0}^{f\restriction e}, \ldots, F_{n_{m-1}}^{f\restriction e}$ where $n_0 < \cdots < n_{m-1}$ and let $r = \langle e, m\rangle$. We will show that $\liminf_{s\to\infty} r_s^{f\restriction e} \leq r$ contrary to (81). For given s it suffices to find a stage $s'' > s$ such that $r_{s''}^{f\restriction e} = r$. Fix $s' > s$ minimal such that the intervals $F_{n_0}^{f\restriction e}, \ldots, F_{n_{m-1}}^{f\restriction e}$ are defined at stage s' and let s'' be the least stage $> s'$ such that the interval with least index n which eventually becomes frozen after stage s' becomes frozen at stage s''. Then, by construction, for any $n' > n$ such that $F_{n'}^{f\restriction e}$ is defined at stage s'', $F_{n'}^{f\restriction e}$ becomes frozen at stage s'', too (unless $F_{n'}^{f\restriction e}$ had been frozen before already). Hence the only intervals which exist at stage s'' and are not frozen are the intervals $F_{n_0}^{f\restriction e}, \ldots, F_{n_{m-1}}^{f\restriction e}$. So $r_{s''}^{f\restriction e} = \langle e, m\rangle = r$, which completes the proof of (83).

It remains to show (84). For a contradiction assume that $F_n^{f\restriction e}$ is never frozen and $|\overline{W_{e_0}} \restriction (\max F_n^{f\restriction e}) + 1| < 2n$. By Claims 1, 2 and 3 we may fix a stage t such that, for $e' \leq e$, $\mathcal{P}_{e'}$ is not initialized after stage t, $\mathcal{P}_{e'}$ does not act via clause (i) or (ii) after stage t, and - if $\mathcal{P}_{e'}$ acts infinitely often - then, for any $s \geq t$, $\mathcal{P}_{e'}$ is expanding at stage s and $r_s^{f\restriction e'} > e$. Since, by assumption, $\mathcal{P}_e$ acts infinitely often, it follows that $\mathcal{P}_e$ is expanding at all stages $\geq t$ and does not require attention via clause (i) after stage t (since, otherwise, it would become active via clause (i) contrary to choice of t). On the other hand, by the choice of n, for all sufficiently large stages s, $F_n^{f\restriction e}$ is defined and unfrozen at stage s and (70) holds. So $\mathcal{P}_e$ requires attention via clause (i) at almost all stages $s > t$. A contradiction. This completes the proof of (84) and the proof of Claim 4. □

Claim 5. $\mathcal{P}_e$ is met.

Proof. W.l.o.g. we may assume that $A = \hat{\Phi}_{e_1}^{W_{e_0}}$ (whence, in particular, $\hat{\varphi}_{e_1}$ is total). By Claim 4 it suffices to show that $\mathcal{P}_{f\restriction e}$ acts infinitely often. For a contradiction assume that this is not the case. Then we may fix a stage $s_0 > e$ such that $\mathcal{P}_{f\restriction e}$ does not act after stage s_0 and such that the intervals associated with $\mathcal{P}_{f\restriction e}$ at the end of stage s_0 - say, $F_0^{f\restriction e}, \dots, F_{n-1}^{f\restriction e}$ $(n \geq 0)$ - are permanent, and such that the intervals associated with $\mathcal{P}_{f\restriction e}$ which are not frozen at stage s_0 - say, $F_{n_0}^{f\restriction e}, \dots, F_{n_{m-1}}^{f\restriction e}$ (where $n_0 < \dots n_{m-1} \leq n-1$ and $m \leq n$) - do not become frozen later, and such that the followers of order n associated with $\mathcal{P}_{f\restriction e}$ at the end of stage s_0 - say, $x_{n,0}^{f\restriction e}, \dots, x_{n,k-1}^{f\restriction e}$ $(0 \leq k \leq 2n+1)$ - are not cancelled later. Note that this implies that $r = r_s^{f\restriction e}$ for all $s \geq s_0$ where $r = r_{s_0}^{f\restriction e}$.

Now, by Claims 1, 2 and 3, fix an $(f \restriction e)$-stage $s_1 > s_0$ such that no strategy $\mathcal{P}_{f\restriction e'}$ with $e' \leq e$ is initialized after stage $s_1 - 1$, no strategy $\mathcal{P}_{f\restriction e'}$ with $e' < e$ acts via clause (i) or (ii) after stage s_1, and, for any $e' < e$ such that $\mathcal{P}_{f\restriction e'}$ acts after stage s_1, $r_s^{f\restriction e'} > r$ for all $s \geq s_1$. Moreover, by $A = \hat{\Phi}_{e_1}^{W_{e_0}}$ and by the totality of $\hat{\varphi}_{e_1}$, w.l.o.g. we may assume that

$$A \restriction 1 + \max F_{n-1}^{f\restriction e} = A_{s_1} \restriction 1 + \max F_{n-1}^{f\restriction e} = \hat{\Phi}_{e_1,s_1}^{W_{e_0,s_1}} \restriction 1 + \max F_{n-1}^{f\restriction e}$$

(provided that $n > 0$) and $\hat{\varphi}_{e_1,s_1}(x_{n,k-1}^{f\restriction e}) \downarrow$ (provided that $k = 2n+1$).

Note that s_1 is chosen so that $\mathcal{P}_{f\restriction e}$ will become active at any stage $s+1 > s_1$ at which it requires attention. So, by $s_0 < s_1$, $\mathcal{P}_{f\restriction e}$ will not require attention after stage s_1.

Now, in order to get the desired contradiction, we distinguish the following two cases depending of the state of $\mathcal{P}_{f\restriction e}$ at stage s_1.

Case 1: $\mathcal{P}_{f\restriction e}$ is diagonalizing at stage s_1. Fix $t < s_1$ maximal such that $\mathcal{P}_{f\restriction e}$ is not diagonalizing at stage t. Then $\mathcal{P}_{f\restriction e}$ acts via clause (i) at stage $t+1$ and it becomes n'-diagonalizing for some n' at this stage. Moreover, by the maximality of t, $\mathcal{P}_{f\restriction e}$ is not initialized after stage t, hence is n'-diagonalizing at all stages $s > t$, the interval $F_{n'}^{f\restriction e}[t]$ is permanent and so are the $2n'+1$-followers $x_{n',0}^{f\restriction e}[t], \dots, x_{n',2n'}^{f\restriction e}[t]$. Moreover, by construction, none of these followers is in A_{t+1} and

$$|\overline{W_{e_0,t}} \restriction (\max F_{n'}^{f\restriction e}[t]) + 1| < 2n'. \tag{85}$$

It follows by $A = \hat{\Phi}_{e_1}^{W_{e_0}}$ that at any sufficiently large $(f \restriction e)$-stage s such that there is a follower $x_{n',k}^{f\restriction e}[t]$ left which has not yet been enumerated into A, $\mathcal{P}_{f\restriction e}$ will require attention according to clause (ii). Since, by the choice of s_1, $\mathcal{P}_{f\restriction e}$ does not require attention after stage s_1, there must be stages stages $t < t_{2n'} < t_{2n'-1} < \dots < t_0$ such that $\mathcal{P}_{f\restriction e}$ acts according to clause (ii) at stage $t_{k'}+1$ and the follower $x_{n',k'}^{f\restriction e}[t]$ is enumerated into A at stage $t_{k'}+1$. So, $A_{t_{k'}+1}(x_{n',k'}^{f\restriction e}[t]) \neq A_{t_{k'}}(x_{n',k'}^{f\restriction e}[t])$. By condition (71) in (ii) this implies

$$\hat{\Phi}_{e_1,t_{k'}}^{W_{e_0,t_{k'}}}(x_{n',k'}^{f\restriction e}[t]) \neq \hat{\Phi}_{e_1,t_{k'-1}}^{W_{e_0,t_{k'-1}}}(x_{n',k'}^{f\restriction e}[t])$$

where both sides are defined (and where t_{-1} is the least stage $s > t_0$ such that $\hat{\Phi}_{e_1,s}^{W_{e_0,s}} \restriction x^{\alpha}_{n',2n'}[t]+1 = \hat{\Phi}_{e_1}^{W_{e_0}} \restriction x^{\alpha}_{n',2n'}[t]+1)$. Since $\hat{\varphi}_{e_1}(x^{f\restriction e}_{n',k'}[t]) \leq \max F^{f\restriction e}_{n'}[t]$ for all $k' \leq 2n'$, this implies that $2n'+1$ numbers $\leq \max F^{f\restriction e}_{n'}[t]$ have to enter W_{e_0} after stage t. But this contradicts (85).

Case 2: $\mathcal{P}_{f\restriction e}$ is expanding at stage s_1. Since $\mathcal{P}_{f\restriction e}$ neither acts nor is initialized after stage s_0 and since $s_0 < s_1$, it follows that $\mathcal{P}_{f\restriction e}$ is n-expanding at all stages $s \geq s_1$. Moreover, for any such stage s, $x^{f\restriction e}_{n,0}, \dots, x^{f\restriction e}_{n,k-1}$ are the followers of order n associated with $\mathcal{P}_{f\restriction e}$ at stage s. In order to get the desired contradiction, it suffices to show that $\mathcal{P}_{f\restriction e}$ requires attention after stage s_1. This is done by distinguishing the following two cases. If $k < 2n+1$ then $\mathcal{P}_{f\restriction e}$ requires attention via clause (iii) at the first stage $s+1 > s_1$ at which $\mathcal{P}_{f\restriction e}$ is accessible. If $k = 2n+1$ then $\mathcal{P}_{f\restriction e}$ requires attention via clause (iv) at the first stage $s+1 > s_1$ at which $\mathcal{P}_{f\restriction e}$ is accessible and at which $\hat{\varphi}_{e_1,s}(x^{f\restriction e}_{n,k-1})$ is defined. Note that, by $A = \hat{\Phi}_{e_1}^{W_{e_0}}$, such a stage must exist. This completes the proof of Case 2 and the proof of Claim 5. □

Claim 6. $\mathcal{N}_e$ is met.

Proof. W.l.o.g. assume that Φ_e^A is total. It suffices to define a computable function g such that, for any $n \geq 0$,

(86) $$\lim_{k\to\infty} g(n,k) = \Phi_e^A(n) \ \& \ |\{k : g(n,k+1) \neq g(n,k)\}| \leq n+1.$$

By assumption, $f(e) = 0$. So $\alpha \sqsubset f$ for $\alpha = (f \restriction e)0$. Pick the least α-stage s_0 such that

(87) $$\forall\, s \geq s_0\ (\alpha < \delta_s)$$

and

(88) $$\forall\, \beta \sqsubseteq \alpha\ (\mathcal{P}_\beta \text{ does not act via clause (i) or (ii) after stage } s_0),$$

and let $s_0 < s_1 < s_2 < \dots$ be the α-stages $\geq s_0$. Then, for $n < m$, $l(e,s_m) > n$ hence $\Phi^{A_{s_m}}_{e,s_m}(n)\downarrow$ (hence $\varphi_e^{A_{s_m}}(n)\downarrow$). Hence if we let

$$g(n,k) = \Phi^{A_{s_{n+1+k}}}_{e,s_{n+1+k}}(n)$$

then g is total and computable and $\lim_{k\to\infty} g(n,k) = \Phi_e^A(n)$. So, for a proof of (86), it suffices to show that

(89) $$|\{k : A_{s_{n+1+k+1}} \restriction \varphi_e^{A_{s_{n+1+k}}}(n) \neq A_{s_{n+1+k}} \restriction \varphi_e^{A_{s_{n+1+k}}}(n)\}| \leq n+1.$$

For a proof of (89) we first show by induction on n that, for any n, there exist at most n followers at the end of stage s_n which may enter A later whence

(90) $$|(A \restriction s_n+1) \setminus (A_{s_n} \restriction s_n+1)| \leq n$$

(since followers appointed after stage s_n are greater than s_n). For $n = 0$ the claim is obvious if $s_0 = 0$. So w.l.o.g. assume that $s_0 > 0$. Since, by the choice of s_0 no strategy $\mathcal{P}_\beta$ with $\beta \leq \alpha$ enumerates a follower into A after stage s_0, it suffices to show that no strategy $\mathcal{P}_\gamma$ with $\alpha < \gamma$ has a follower at the end of stage s_0. If $\alpha <_L \gamma$ then this is obvious since $\mathcal{P}_\gamma$ becomes initialized at the α-stage s_0. So assume that $\alpha \sqsubset \gamma$. Fix the greatest α-stage $s < s_0$. By the minimality of s_0, (87) or (88) fails for s in place of s_0. So $\mathcal{P}_\gamma$ is initalized at stage s or stage $s+1$, respectively, and $\mathcal{P}_\gamma$ does not act at stage $s+1$. So $\mathcal{P}_\gamma$ does not have a follower at the end of stage $s+1$. Since s is the greatest α-stage $< s_0$ and $\mathcal{P}_\gamma$ may act only at stages where

α is accessible, $\mathcal{P}_\gamma$ does not have any follower at the end of stage s_0, too. This completes the proof of the case $n = 0$. For the inductive step assume that $n > 0$ and that there are at most $n-1$ followers at the end of stage s_{n-1} which may enter A later. Since at stage $s_{n-1}+1$ at most one new follower is appointed and since any follower appointed at a stage $s+1$ such that $s_{n-1}+1 < s+1 \leq s_n$ has to be appointed to a strategy $\mathcal{P}_\gamma$ with $\alpha <_L \gamma$ hence will be initalized at the end of stage s_0, the claim for n follows immediately.

Now, by (90), for a proof of (89) it suffices to show that, for any $k \geq 0$, such that

$$A_{s_{n+1+k+1}} \restriction \varphi_e^{A_{s_{n+1+k}}}(n) \neq A_{s_{n+1+k}} \restriction \varphi_e^{A_{s_{n+1+k}}}(n) \tag{91}$$

we have

$$A_{s_{n+1+k+1}} \restriction s_{n+1}+1 \neq A_{s_{n+1+k}} \restriction s_{n+1}+1. \tag{92}$$

For a contradiction assume that the latter is not true. Fix k minimal such that (91) holds but (92) fails, and fix x minimal such that $x \in A_{s_{n+1+k+1}} \setminus A_{s_{n+1+k}}$. Then

$$s_{n+1} < x < \varphi_e^{A_{s_{n+1+k}}}(n) \leq s_{n+1+k}.$$

Next fix β such that x is a $\mathcal{P}_\beta$-follower, say, $x = x^\beta_{m,p}$. Note that, by the choice of s_0, $\alpha < \beta$. In fact, since at stage s_{n+1+k} all strategies $\mathcal{P}_\gamma$ with $\alpha <_L \gamma$ are initialized, $\alpha \sqsubset \beta$. So $\mathcal{P}_\beta$ may act only at stages where α is accessible, hence x is enumerated into A at stage $s_{n+1+k}+1$ and (since $s_{n+1} < x < s_{n+1+k}$) there is some $k' < k$ such that $x = x^\beta_{m,p} = s_{n+1+k'}+1$ becomes appointed as $\mathcal{P}_\beta$-follower at stage $s_{n+1+k'}+1$. Since, by the latter, $\varphi_e^{A_{s_{n+1+k'}}}(n) \leq x$ whereas, by the choice of x, $x < \varphi_e^{A_{s_{n+1+k}}}(n)$ it follows that there must be a number k'' such that $k' \leq k'' < k$ and (91) holds for k'' in place of k whence, by the minimality of k, (92) holds for k'' in place of k, too. So fix $x' \leq s_{n+1}$ and s such that $s_{n+1+k''} \leq s < s_{n+1+k''}$ and x' is enumerated into A at stage $s+1$, and let $\mathcal{P}_{\beta'}$ be the strategy which enumerates x' into A at stage $s+1$. Now, in order to get the desired contradiction, consider the relation between β and β'. Since the $\mathcal{P}_\beta$-follower x exists at stage s and is enumerated into A after stage $s+1$, $\mathcal{P}_\beta$ is not initialized at stage $s+1$ and x neither becomes frozen nor becomes cancelled at stage $s+1$. By the former, $\beta \leq \beta'$. First assume $\beta < \beta'$. Then $\mathcal{P}_\beta$ has to be m-diagonalizing at stage s since $x' < x$ and $x = x^\beta_{m,p}$ is neither frozen nor cancelled at stage $s+1$. So we may fix the greatest stage $t+1 < s$ at which $\mathcal{P}_\beta$ acts according to clause (i) (thereby becoming m-diagonalizing at stage $t+1$) then $\mathcal{B}_{\beta'}$ becomes initialized at stage $t+1$. Since, by the maximality of $t+1$, x is a $\mathcal{P}_\beta$-follower at stage $t+1$ hence $x < t+1$. Since $x' < x$ it follows that x' becomes cancelled at stage $t+1$, a contradiction. This leaves the case that $\beta' = \beta$. Then x and x' are $\mathcal{P}_\beta$-followers at stage s_{n+1+k} hence both associated with the same interval F^β_m. But, by construction, such followers are enumerated in decreasing order. So, by $x' < x$, x had to be enumerated first contrary to choice of x and x'. So this case is impossible, too. Hence (91) implies (92), which completes the proof of (89) and the proof of Claim 6. $\square$

By Claims 5 and 6 all requirements are met. This completes the proof of the theorem. $\square$

Corollary 8.5. *There is an array computable c.e. Turing degree* **a** *which contains a computably enumerable set which is not eventually uniformly wtt-array computable.*

Proof. By the Characterization Theorem 4.2 and Theorem 8.4. □

9. Questions and comments

Having introduced some new classification tools including a hierarchy of bounded lowness notions, we have an infinite number of questions we might ask. We mention a couple.

One separation we have not yet succeeded in finding is a Turing degree containing an e.u.wtt-a.c. c.e. set which does not contain a *wtt*-superlow set. There likely should be a domination/non-domination property corresponding to this question. What is it?

Some of the notions considered in this paper – like *wtt*-superlowness and *wtt*-jump traceability – are the *wtt*-analogs of notions previously defined for Turing reducibility. The reader should note that the analog game can be played from the *wtt*-structures back to the Turing degrees. In particular, we may consider the Turing analog of the notion which turned out to be central for our investigations, namely, eventually uniform *wtt*-array computability.

Definition 9.1. *A set A is* eventually uniformly array computable *if there exist computable functions $g, k : \omega^2 \to \{0,1\}$ and a computable order h such that, for all e, x,*

$$A'(x) = \lim_{s\to\infty} g(x,s), \tag{93}$$

$$k(x,s) \leq k(x,s+1), \tag{94}$$

$$k(x,s) = 1 \Rightarrow \ |\{t \geq s : g(x,t+1) \neq g(x,t)\}| \leq h(x), \tag{95}$$

$$\forall e \ (\Phi_e^A \ \textit{total} \ \Rightarrow \ \forall^\infty x \ \exists s \ (k(\langle e,x\rangle, s) = 1)) \tag{96}$$

(where A' is the general halting set for A, i.e., $A' = \{\langle e,x\rangle : \Phi_e^A(x)\downarrow\}$).

This notion and similar ones seem to yield classes of degrees of complexity different than any seen before. They appear worth investigating.

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