ON SUPERSETS OF NON-LOW$_2$ SETS

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Abstract. We solve a longstanding question of Soare by showing that if $\mathbf{d}$ is a non-low$_2$ computably enumerable degree then $\mathbf{d}$ contains a c.e. set with no r-maximal c.e. superset.

1. Introduction

A longstanding programme in computability theory explores the relationship between the computably enumerable degrees and the lattice of computably enumerable sets. A great deal of Soare's classic text [6] is devoted to this analysis. For example, if $\mathbf{a}$ is high then $\mathbf{a}$ contains a maximal c.e. set, and no non-high c.e. degree contains a maximal set[1], as proven in the ground-breaking paper of Martin [4].

One of the fundamental themes is that low and low$_2$ sets resemble computable sets in that, for example, low$_2$ c.e. sets have maximal supersets. Here A is low if $A' \equiv_T \emptyset'$ and A is low$_2$ if $A'' \equiv_T \emptyset''$. Indeed, if A is c.e. and low, then the lattice of c.e. supersets of A is isomorphic to the lattice of c.e. sets (Soare [7]) and this is long conjectured to be true for the supersets of low$_2$ sets, but this much harder question remains open (see Soare [6], Ch. X and XVI, for example).

On the other hand, low$_2$ seems to be a demarcation point. Non-low$_2$ sets have certain domination properties which are certainly useful in the structure of the global degrees. For example, non-low$_2$ degrees bound 1-generic degrees, and are complemented below, whereas there are minimal low$_2$ degrees. See, for example, Lerman [3].

For the c.e. sets, in 1976, Shoenfield [5] proved that non-low$_2$ c.e. degrees contain c.e. sets with no maximal c.e. supersets, and Lachlan [2] showed that low$_2$ c.e. sets have maximal supersets. Thus there is an elementary difference between properties of low$_2$ c.e. sets and some non-low$_2$ c.e. sets. Indeed, Shoenfield's Theorem shows that a

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[1]Recall M is maximal if it is c.e. and co-infinite and no c.e. set splits $\overline{M}$ meaning that if W is c.e. and $W \supseteq M$ then either $M =^* W$ or $W =^* \omega$.

certain natural set called the deficiency set, has no hyperhypersimple c.e. superset.

One of the natural variations on being maximal is what is known as r-maximal. We recall that a co-infinite c.e. set D is r-maximal if for all computable sets X, $X \cup D =^* \omega$, or $\overline{X} \cup D =^* \omega$. That is no *computable* set splits $\overline{D}$.

In this paper we solve a question first stated in 1987 by Soare [6], p. 233. Soare asks to classify the class of c.e. sets with no r-maximal supersets. In this paper we show that the classification aligns with that of those with no maximal supersets.

Theorem 1.1. *Suppose that* $\mathbf{a}$ *is a c.e. non-low*$_2$ *degree. Then* $\mathbf{a}$ *contains a c.e. set* D *with no* r*-maximal superset.*

The techniques we use are quite different than the mysterious ones of Shoenfield [5], and build on ideas of Downey and Shore [1].

2. The Proof

The proof of Theorem 1.1 uses some general machinery developed by Downey and Shore [1] for working below a non-low$_2$ c.e. degree. We will need to discuss some of these ideas to make the current paper self contained.

In the global degrees, there is a well known technique of working below a non-low$_2$ degree using escape functions. A degree $\mathbf{d} \leq \mathbf{0}'$ is non-low$_2$ iff for every function $h \leq_T \emptyset'$, there is a function g computable in $\mathbf{d}$ such that g is not dominated by h. This characterization is used for constructions related to $\mathbf{d}$ as follows: Relying on specific properties of the requirements to be met, one defines "in advance" a function h which gives an appropriate "search space" inside of which one should look for witnesses to satisfy some requirements. This function will be computable from $\emptyset'$ due to the specific nature of the relevant requirements. Now if $\mathbf{d}$ is non-low$_2$, there is a strictly increasing function g computable in $\mathbf{d}$ not dominated by h. The idea then is to use g to $\mathbf{d}$-computably bound searches and hence make the construction an oracle one computable in $\mathbf{d}$. By the way g and h have been constructed, the fact that $g(x) > h(x)$ infinitely often guarantees that, by a priority argument, we get to meet all the requirements. The classical example of this is showing that all non-low$_2$ degrees bound 1-generics, where we use a function h for which $h(x)$ says: for all strings σ of length x and all $e \leq x$, compute a stage where if σ has an extension in V_e, the e-th c.e. set of strings, we can see it by stage s. Then if we take a non-low$_2$ set D there is a function $g \leq_T D$ which infinitely often escapes this function. We use $g(x)$ as our search space to figure out which requirement to pursue. Then a standard finite injury argument works (see e.g. Lerman [3], Downey and Shore [1]).

We consider this method in the context of c.e. degrees. The natural idea we pursue is to use the global characterization of non-low$_2$ and then approximate the functions g and h via the Limit Lemma. Now we will be given a c.e. set $D = \bigcup_s D_s$ of non-low$_2$ degree and a "witness" function h for the satisfaction of some requirements. Again $h \leq_T \emptyset'$. We apply the Limit Lemma to h so that $h(x) = \lim_s h(x, s)$ with $h(x, s)$ a computable approximation to h. Again, since D is not low$_2$, there will be a function $g \leq_T D$ not dominated by h which we can also approximate via the Limit Lemma. Since $g \leq_T D$, there is a reduction $\Gamma(D) = g$. The problem is that, as $h(x)$ only equals $\lim_s h(x, s)$ and $g(x)$ only equals $\lim_s g(x, s)$, we must be able to "correct" our mistakes. This is a serious problem since the objects we need to construct must not only be computable from D *but also c.e.*

The main idea in the constructions from Downey and Shore [1] is that we must be able to correct the mistakes that occur when $g(x, s)$ does not have its final value by *dumping* elements into the set we are constructing whenever $g(x, s)$ or even $D \upharpoonright g(x, s)$ changes where $g(x, s)$ is the standard approximation of $\Gamma(D)$. (In fact, below we will choose a c.e. member D of the given non-low$_2$ degree $\mathbf{a}$ such that D has the desired dumping property.) Moreover, as in [1], for our construction we will require that our computable approximations $h(x, s)$ and $g(x, s)$ have certain nice properties like monotonicity (see the construction below for details).

Having explained some of its basic features we now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Fix any c.e. set A in the given non-low$_2$ c.e. degree $\mathbf{a}$, let f be a computable 1-1 function enumerating A, and let

$$D = \{t : \exists\, s > t(f(s) < f(t))\}$$

be the deficiency set of A w.r.t. f. Since $D \equiv_T A$ (see Soare [6], p.81), it suffices to show that no superset of D is r-maximal. So, for the remainder of the proof, fix a coinfinite c.e. superset W of D. It suffices to define a c.e. splitting (X_0, X_1) of ω, i.e., c.e. sets X_0 and X_1 such that

$$X_0 \cup X_1 = \omega \text{ and } X_0 \cap X_1 = \emptyset \qquad (1)$$

(hence X_0 is computable and $X_1 = \overline{X_0}$), which meet the requirements

$$R_{2e+i} : \exists\, x \geq e\ (x \in X_i \cap \overline{W})$$

for $e \geq 0$ and $i \leq 1$. Note that this ensures that $X_0 \cap \overline{W}$ and $\overline{X_0} \cap \overline{W}$ are infinite. So the computable set X_0 witnesses that W is not r-maximal. In the remainder of the proof we construct the desired c.e. sets X_0 and X_1 by a finite injury argument.

Let $\{D_s\}_{s\geq 0}$ be the natural enumeration of D defined by $D_0 = \emptyset$ and

$$D_{s+1} = D_s \cup \{x \leq s : f(s+1) < f(x)\}.$$

As in [1]'s proof that D has no maximal superset, the crucial property of this enumeration is that if x is enumerated into D at stage $s+1$ then all elements of the interval $(x, s]$ not yet in D also enter D at stage $s+1$:

$$x \in D_{s+1} \setminus D_s \;\Rightarrow\; \big(x \leq s \;\;\&\; [x,s] \subseteq D_{s+1}\big) \tag{2}$$

(namely, if $r \in (x, s]$ then $f(r) > f(x)$ as $x \notin D_r$ and so $r \in D_{s+1}$). In the following we refer to this property as the *dump property* of $\{D_s\}_{s\geq 0}$.

The function h is defined by recursion as follows. Given a computable enumeration $\{W_s\}_{s\geq 0}$ of W such that $D_s \subseteq W_s \subset \omega \upharpoonright s$, let

$$h(0) = 0$$

and

$$\begin{gathered} h(x+1) = \mu s > t_x\ (\overline{W} \cap [t_x, s) \neq \emptyset) \\ \text{where } t_x = \mu t > h(x)\ (W_t \upharpoonright h(x) = W \upharpoonright h(x)). \end{gathered} \tag{3}$$

Then, for any number x,

$$x \leq h(x) < t_x < h(x+1), \tag{4}$$

$$W_{t_x} \upharpoonright h(x) = W \upharpoonright h(x), \tag{5}$$

and

$$\overline{W} \cap [t_x, h(x+1)) \neq \emptyset. \tag{6}$$

We will work with computable approximations $h(x,s)$ and t^s_x of $h(x)$ and t_x, respectively, recursively defined as follows. Let $h(0,s) = h(0) = 0$ for $s \geq 0$ and, let

$$\begin{gathered} h(x+1,s) = \mu s' > t^s_x\ (\overline{W_s} \cap [t^s_x, s') \neq \emptyset) \\ \text{where } t^s_x = \mu t > h(x,s)(W_s \upharpoonright h(x,s) \subseteq W_t \upharpoonright h(x,s)). \end{gathered} \tag{7}$$

for $x, s \geq 0$. Note that

$$x \leq h(x,s) < t^s_x < h(x+1,s) < t^s_{x+1}, \tag{8}$$

$$h(x,s) \leq h(x,s+1) \;\&\; t^s_x \leq t^{s+1}_x, \tag{9}$$

$$h(x,s) \neq h(x,s+1) \;\Rightarrow\; W_{s+1} \upharpoonright h(x,s) \neq W_s \upharpoonright h(x,s), \tag{10}$$

(11) $$W_{s+1} \restriction h(x,s) \neq W_s \restriction h(x,s) \ \Rightarrow\ t_x^{s+1} \geq s+1,$$

and

(12) $$\lim_{s\to\infty} h(x,s) = \sup_{s\to\infty} h(x,s) = h(x) \text{ and } \lim_{s\to\infty} t_x^s = \sup_{s\to\infty} t_x^s = t_x$$

hold for all $x, s \geq 0$. For the sake of completeness, the rather straightforward proofs of (8) - (12) are given in the appendix.

Obviously, $h \leq_T W$ hence $h \leq_T \emptyset'$. So, since D is not low$_2$, there is a function $\hat{g} \leq_T D$ not dominated by h. For the construction we replace $\hat{g}(x)$ by a D-computable function $g(x)$ majorizing $\hat{g}(x)$ which has a more amenable computable approximation $g(x,s)$. The properties of the function $g(x)$ and its approximation $g(x,s)$ which are crucial for the construction of the desired c.e. sets X_0 and X_1 are given in the following claim.

Claim 1. There are total functions $g(x)$ and $g(x,s)$ such that the unary function g is not dominated by h, i.e.,

(13) $$\exists^\infty x\ (h(x) < g(x)),$$

the binary function $g(x,s)$ is computable and approximates $g(x)$ from below, i.e.,

(14) $$g(x) = \sup_{s\to\infty} g(x,s) = \lim_{s\to\infty} g(x,s),$$

and, for any numbers x and s,

(15) $$g(0,s) = g(0) = 0,$$

(16) $$x \leq g(x,s) \leq g(x+1,s) \ \&\ g(x,s) \leq g(x,s+1),$$

and

(17)
$$g(x,s) < g(x,s{+}1) \ \Rightarrow\ (D_{s+1} \restriction g(x,s) \neq D_s \restriction g(x,s) \ \&\ g(x,s{+}1) > s).$$

Proof of Claim 1. Since $h \leq_T \emptyset'$ and D is not low$_2$, we may fix a function $\hat{g} \leq_T D$ not dominated by h where w.l.o.g. $\hat{g}$ is nondecreasing, $\hat{g}(0) = 0$ and $\hat{g}(x) \geq x$ for $x \geq 0$. Then there is a Turing functional Γ with use function γ such that $\hat{g} = \Gamma^D$ and such that, for any set X, $\Gamma^X(0)\downarrow$, $\Gamma^X(0) = \gamma^X(0) = 0$ and

(18)
$$\Gamma^X(x)\downarrow \Rightarrow\ \forall x' \leq x(\Gamma^X(x')\downarrow \ \&\ x' \leq \Gamma^X(x') \leq \gamma^X(x') \leq \gamma^X(x))$$

for any number $x \geq 0$. Moreover, we may choose a computable enumeration $\{\Gamma_s\}_{s\geq 0}$ of Γ such that, for any set X, $\Gamma_0^X(0)\downarrow$ and

$$\Gamma_s^X(x)\downarrow \Rightarrow \forall x' \leq x(\Gamma_s^X(x')\downarrow \;\&\; \gamma^X(x') \leq s) \tag{19}$$

for any number x and any stage s.

It follows that, for the function $u(x,s)$ defined by

$$u(x,s) = \mu u \geq s\ (\Gamma_u^{D_u}(x)\downarrow),$$

u is total and computable, $u(0,s) = s$ for all numbers s, and

$$x \leq u(x,s) \leq u(x+1,s) \;\&\; u(x,s) \leq u(x,s+1) \tag{20}$$

and

$$\forall x' \leq x(\Gamma_{u(x,s)}^{D_{u(x,s)}}(x')\downarrow). \tag{21}$$

hold for all numbers x and s.

Now, for the definition of the functions $g(x)$ and $g(x,s)$, we recursively define the binary function $g(x,s)$ by letting

$$g(x,s) = \begin{cases} u(x,s) & \text{if } s = 0 \text{ or } s > 0 \text{ and} \\ & \exists x' \leq x\ (D_s \restriction \gamma^{D_{g(x',s-1)}}(x') \not\subseteq D_{g(x',s-1)}), \\ g(x,s-1) & \text{otherwise} \end{cases}$$

and we let

$$g(x) = \lim_{s\to\infty} g(x,s).$$

It remains to show that these functions have the required properties. Obviously the binary function $g(x,s)$ is total and computable. Moreover, for any x and s, $g(x,s) = u(x,t)$ for some $t \leq s$. In the following we let t_s^x be the least such number t.

Next we show that (15), (16) and (17) hold.

For a proof of (15) it suffices to note that $g(0,s) = g(0,0) = u(0,0) = 0$ since $\gamma^{D_s}(0) = 0$ for all numbers s.

The individual parts of (16) are shown as follows. The relation $x \leq g(x,s)$ is immediate since $x \leq u(x,t_s^x)$ by (20). For a proof of $g(x,s) \leq g(x,s+1)$ w.l.o.g. assume that $g(x,s) \neq g(x,s+1)$. Then $t_x^s \leq s$ and $t_x^{s+1} = s+1$, hence $g(x,s) \leq g(x,s+1)$ by (20). The proof of $g(x,s) \leq g(x+1,s)$ is by induction on s. If $s = 0$ then $g(x,s) = u(x,s)$ and $g(x+1,s) = u(x+1,s)$, and the claim is immediate by (20). So fix $s > 0$. Since, by inductive hypothesis, $g(x,s-1) \leq g(x+1,s-1)$, w.l.o.g. we may assume that $g(x,s) \neq g(x,s-1)$ or $g(x+1,s) \neq g(x+1,s-1)$. If the former holds then $g(x,s)$ is defined according to the first case of the definition. But, by definition, this implies that $g(x+1,s+1)$ is defined according to this case too, hence $t_s^x = t_s^{x+1} = s$.

If the latter holds, then $t_s^{x+1} = s$ while $t_s^x \leq s$. So, in either case, the claim follows by (20).

Finally, for a proof of (17), fix x and s such that $g(x,s) < g(x,s+1)$. Then, by definition, there is a number $x' \leq x$ such that

$$D_{s+1} \upharpoonright \gamma^{D_{g(x',s)}}(x') \not\subseteq D_{g(x',s)}. \tag{22}$$

Then, in particular, $g(x',s) \leq s$. Moreover, by $g(x',s) = u(x', t_s^{x'})$, and by (21), $\Gamma_{g(x',s)}^{D_{g(x',s)}}(x') \downarrow$. Hence, by (19) and (16), $\gamma^{D_{g(x',s)}}(x') \leq g(x',s) \leq g(x,s)$. So it suffices to show

$$D_{s+1} \upharpoonright \gamma^{D_{g(x',s)}}(x') \not\subseteq D_s. \tag{23}$$

For a contradiction assume that (23) fails. Then, by (22), $g(x',s) < s$, hence $g(x',s) = g(x',s-1)$, and

$$D_s \upharpoonright \gamma^{D_{g(x',s)}}(x') \not\subseteq D_{g(x',s)}.$$

But, by definition of g, this implies that $g(x',s) = u(x',s) \geq s$, a contradiction.

It remains to show that the unary function $g(x)$ is total and that (14) and (13) hold. For this sake, by choice of $\hat{g}$ and by (17), it suffices to show that, for given x, there is a stage s_x such that $g(x,s) = g(x,s_x+1)$ for all $s > s_x$ and $\hat{g}(x) \leq g(x, s_x+1)$.

Fix s'_x such that, for any $x' \leq x$, $\Gamma^{D_{s'_x}}(x')$ is defined and $D_{s'_x} \upharpoonright \gamma^D(x') = D \upharpoonright \gamma^D(x')$. Then $\Gamma^{D_s}(x') = \Gamma^D(x')$ and $\gamma^{D_s}(x') = \gamma^D(x') \leq s'_x$ for all numbers $x' \leq x$ and $s \geq s'_x$. So we may fix $s_x \geq s'_x$ such that $D_{s_x} \upharpoonright m = D \upharpoonright m$ for $m = \max\{\gamma^{D_s}(x') : x' \leq x \;\&\; s \geq 0 \;\&\; \Gamma^{D_s}(x') \downarrow\}$. To show that $g(x,s) = g(x,s_x+1)$ for $s > s_x$, for a contradiction fix $s \geq s_x$ such that $g(x,s+1) \neq g(x,s)$. Then there is some $x' \leq x$ such that $D_{s+1} \upharpoonright \gamma^{D_{g(x',s)}}(x')$ is not contained in $D_{g(x',s)}$. By choice of s_x this implies that $g(x',s) < s_x$. So, just as in the proof of (17), we may argue that $g(x',s_x) = u(x',s_x) \geq s_x$ contrary to (17).

Finally, for a proof of $\hat{g}(x) \leq g(x,s_x+1)$, for a contradiction assume that $g(x,s_x+1) < \hat{g}(x)$. Note that, by definition of Γ and by choice of s_x, $\hat{g}(x) = \Gamma^D(x) = \Gamma_s^{D_s}(x) \leq s$ for all $s \geq s_x+1$ (where the last inequality holds by (18) and (19)). On the other hand, by definition of g, $\Gamma_{g(x,s_x+1)}^{D_{g(x,s_x+1)}}(x) \downarrow$, hence (by (18) and (19)) $\Gamma^{D_{g(x,s_x+1)}}(x) \leq \gamma^{D_{g(x,s_x+1)}}(x) \leq g(x,s_x+1)$. By assumption we may conclude that $\Gamma^{D_{g(x,s_x+1)}}(x) < \Gamma^{D_{s_x+1}}(x)$, $g(x,s_x+1) < s_x+1$ and $D_{s_x+1} \upharpoonright \gamma^{D_{g(x,s_x+1)}}(x) \not\subseteq D_{g(x,s_x+1)}$ (hence $D_{s_x+2} \upharpoonright \gamma^{D_{g(x,s_x+1)}}(x) \not\subseteq D_{g(x,s_x+1)}$). It follows by definition of g that $g(x,s_x+2) = u(x,s_x+2) \geq s_x+2 > g(x,s_x+1)$, a contradiction.

This completes the proof of Claim 1. □

For the remainder of the proof fix $g(x)$ and $g(x,s)$ as in Claim 1. We call a number x *(truly) good* if $h(x) < g(x)$, and we call x *good at stage* s if $h(x,s) < g(x,s)$. Note that (by (13)) there are infinitely many truly good numbers x and, by (12) and (14), a number x is good iff it is good at infinitely many stages iff it is good at all sufficiently large stages. By the former we may inductively define good numbers x_n such that x_0 is the least good number and x_{n+1} is the least good number $> g(x_n)$. (Note that, by $h(0) = g(0) = 0$ and by (8), $1 \leq x_n \leq h(x_n) < g(x_n) < x_{n+1}$ for any number $n \geq 0$.) We will ensure that, for $n = 2e + i$,

$$\overline{W} \cap [t_{x_n-1}, g(x_n)) \subseteq X_i \tag{24}$$

holds (where t_{x_n-1} is defined according to (3)). By trueness of x_n and (6) this implies that R_n is met.

In order to achieve this, in the course of the construction we assign approximations of x_n to requirement R_n - called followers. At any stage $s+1$ of the construction at which a new approximation x_n^{s+1} of x_n is assigned to R_n as follower we enumerate the numbers $< g(x_n^{s+1}, s+1)$ which haven't been put into X_0 or X_1 at previous stages into X_i (where $n = 2e+i$). We will argue that if a number in the interval $[t_{x_n-1}, g(x_n))$ has been previously put into X_{1-i} by some other requirement then, by the dump property, a later change of D will cause this number to enter D (hence W) whence this will not affect satisfaction of (24).

If it seems that an approximation is not correct or the corresponding g-value is not yet final then we cancel the follower as well as the existing lower priority followers (we say that we initialize R_n and the lower priority requirements) and we later assign a new follower. If defined, we let x_n^s denote the follower of R_n at the end of stage s (and we write $x_n^s \uparrow$ otherwise). At any stage s there is a number $m \geq 0$ such that the requirements R_n with $n < m$ are the ones which have a follower, and $1 \leq x_0^s < x_1^s < \cdots < x_{m-1}^s$ (if $m > 0$). For convenience, we let $x_{-1}^s = 0$. $X_{i,s}$ denotes the finite part of X_i enumerated by the end of stage s. The construction will ensure that $X_{i,s} \subseteq \omega \upharpoonright s$.

Using the above introduced notation, the formal construction is as follows.

> *Stage* 0 is vacuous, i.e., $X_{0,s} = X_{1,s} = \emptyset$ and no requirement has a follower (i.e., $x_n^0 \uparrow$ for $n \geq 0$).
>
> *Stage* $s+1$. The stage consists of two steps.

Step 1. For any requirement R_n which has a follower x_n^s at the end of stage s, R_n is *initialized* and x_n^s is *cancelled* if, for some number $n' \leq n$, $x_{n'}^s$ is not good at stage $s+1$ or there is a number $x \leq x_{n'}^s$ such that $g(x,s) < g(x,s+1)$.
Step 2. Requirement R_n *requires attention via* x if the following hold.

(i) R_n is the highest priority requirement which does not have a follower after Step 1.

(ii) $x \geq 1$ and x is good at stage $s+1$, i.e., $h(x,s+1) < g(x,s+1)$.

(iii) $x > g(x_{n-1}^s, s+1)$.

(iv) $g(x,s+1) \leq s+1$.

If R_n requires attention then declare that R_n becomes *active* at stage $s+1$; for the least x such that R_n requires attention via x, appoint $x_n^{s+1} = x$ as R_n-follower; and let

$$\begin{array}{lcl} X_{i,s+1} & = & X_{i,s} \cup \big([0, g(x,s+1)) \setminus X_{1-i,s}\big) \\ X_{1-i,s+1} & = & X_{1-i,s} \end{array}$$

where $i = 0$ if n is even and $i = 1$ if n is odd.

If no requirement requires attention, let $X_{i,s+1} = X_{i,s}$ for $i \leq 1$. In any case, for any follower $x_{n'}^s$ defined at the end of stage s and not cancelled in Step 1, $x_{n'}^{s+1} = x_{n'}^s$.

In the remainder of the proof we will show that the sets X_0 and X_1 have the required properties. Obviously, the sets X_0 and X_1 are c.e. and disjoint. So it suffices to show that $X_0 \cup X_1 = \omega$ and that the requirements R_n are met. In order to show this we prove a series of claims.

Claim 2. If $x_n^s \downarrow$ then $x_{n-1}^s \downarrow$, x_n^s is the least number $> g(x_{n-1}^s, s)$ which is good at stage s, and $g(x_n^s, s) \leq s$. Hence, by (16) *and* (8),

$$x_{n-1}^s \leq g(x_{n-1}^s, s) < x_n^s \leq h(x_n^s, s) < g(x_n^s, s) \leq s. \tag{25}$$

Proof. The straightforward proof is by main induction on s and side induction on n. □

Next we show that any requirement R_n eventually obtains the good number x_n defined above as *permanent follower* (i.e., there is a stage s such that x_n is appointed as R_n-follower at stage s and R_n is not initialized at any greater stage, hence $x_n = x_n^{s'}$ for all $s' \geq s$).

Claim 3. x_n *is the permanent follower of requirement* R_n.

Proof. The proof is by induction on n. Fix n and assume the claim to be correct for $n' < n$. Fix s_0 minimal such that the following hold.

(a) For any $n' < n$ and any stage $s \geq s_0$, $x_{n'}$ is the follower of $R_{n'}$ at the end of stage s, i.e., $x_{n'} = x^s_{n'}$.

(b) For any $x \leq x_n$, $h(x, s_0) = h(x)$ and $g(x, s_0) = g(x) \leq s_0$.

Then, for $s \geq s_0$, x_n is the least number $> g(x_{n-1}, s)$ which is good at stage s. So, if R_n has no follower at the end of stage s_0, then R_n requires attention at stage $s_0 + 1$ and x_n becomes appointed. If R_n has a follower x at the end of stage s_0 then $x = x_n$ by Claim 2. In either case, it follows with (b) that R_n will not be initialized later. So the follower x_n is permanent. □

Note that, by Claim 3 and by (16), all numbers $\leq x_n$ have entered X_0 or X_1 by the end of the first stage at which x_n becomes appointed as R_n-follower. So, by $\lim_{n\to\infty} x_n = \omega$, $X_0 \cup X_1 = \omega$. It remains to show that the requirements R_n are met. For this sake we first prove another auxiliary claim.

Claim 4. Let $n \geq 1$ *and let* s_n *be minimal such that no requirement* $R_{n'}$ *with* $n' < n$ *becomes active or is initialized after stage* s_n. *Then the following hold.*

(A) *For* $n' < n$ *and* $s \geq s_n$, $x^s_{n'} = x_{n'}$, $x_{n'}$ *is good at stage* s *and* $g(x_{n'}, s) = g(x_{n'}) \leq s_n$.

(B) x_{n-1} *is appointed as* R_{n-1}*-follower at stage* s_n.

(C) $x^{s_n}_n \uparrow$.

(D) *For* $s \geq s_n$, $g(x', s) = g(x')$ *for all* $x' \leq x_{n-1}$ *and* $h(x', s) = h(x')$ *and* $t^s_{x'} = t_{x'}$ *for all* $x' < x_{n-1}$ *(where* $t^s_{x'}$ *and* $t_{x'}$ *are defined in* (7) *and* (3), *respectively).*

(E) $X_{0,s_n} \cup X_{1,s_n} \subseteq \omega \upharpoonright g(x_{n-1})$.

Proof. For a proof of part (A), fix $n' < n$. By choice of s_n and by Claims 1 and 3, $x^s_{n'} = x_{n'}$ and $g(x_{n'}, s) = g(x_{n'})$ for all stages $s \geq s_n$. The other items of (A) follow by Claim 2. Part (B) follows by minimality of s_n and by construction, and part (C) is immediate by (B). The first part of (D) is immediate by choice of s_n. For a proof of the second part, for a contradiction assume that there are numbers $x' < x_{n-1}$ and $s \geq s_n$ such that $h(x', s) < h(x', s+1)$ or $t^s_{x'} < t^{s+1}_{x'}$. By (10) and by definition of $t^s_{x'}$ this implies that $W_{s+1} \upharpoonright h(x', s) \neq W_s \upharpoonright h(x', s)$. (Namely, if $h(x', s) < h(x', s+1)$ then this is immediate by (10); and if $h(x', s) = h(x', s+1)$ and $t^s_{x'} < t^{s+1}_{x'}$ then this follows by definition of $t^s_{x'}$.) So, by (11), we may conclude that $t^{s+1}_{x'} \geq s+1$, hence

$h(x'+1, s+1) > s+1$ by definition. Since $x'+1 \leq x_{n-1}$ and since x_{n-1} is good this implies that $g(x_{n-1}) > s+1 > s_n$ contrary to (A).

This leaves (E). For a contradiction, assume that there is a number $y \geq g(x_{n-1})$ such that $y \in X_{0,s_n} \cup X_{1,s_n}$. By (B), only numbers $<$ $g(x_{n-1})$ are enumerated into X_0 and X_1 at stage s_n. So there must be a stage $s+1 < s_n$, a requirement R_m, and a number x such that x becomes appointed as R_m-follower at stage $s+1$ and $y < g(x, s+1) \leq s+1$.

Since $g(x_{n-1}, s+1) \leq g(x_{n-1}) \leq y$, it follows that $x_{n-1} < x$. Moreover, by $g(x_{n-1}) \leq y < s+1$ and by the second part of (17) (and by (16) and (14)),

$$\forall x' \leq x_{n-1}(g(x', s+1) = g(x') < s+1). \tag{26}$$

By goodness of $x_{n'}$ this implies

$$\forall n' \leq n-1(x_{n'} \text{ is good at all stages} \geq s+1). \tag{27}$$

So, by Claim 2 (and by induction on n'), $x_{n'}^{s+1} \leq x_{n'}$ if $x_{n'}^{s+1}$ is defined ($n' \leq n-1$). Since $x_{n-1} < x = x_m^{s+1}$, we may conclude that $m \geq n$ and

$$\forall n' \leq n-1(x_{n'}^{s+1} \downarrow \text{ and } x_{n'}^{s+1} \leq x_{n'}).$$

In fact,

$$\forall n' \leq n-1(x_{n'}^{s+1} = x_{n'}). \tag{28}$$

Namely, otherwise, fix $n' \leq n-1$ minimal such that $x_{n'}^{s+1} < x_{n'}$. Then $x_{n'}^{s+1}$ is good at stage $s+1$ (since it is a follower) but not truly good (by (26) and by definition of $x_{n'}$). So $h(x_{n'}^{s+1}, s+1) < h(x_{n'}^{s+1})$, and - since $x_{n'}^{s+1} < x_{n-1}$ - as in the proof of the second part of (D) we may argue that $g(x_{n-1}) > s+1$ contrary to (26).

Now, by (26), (27) and (28), no requirement $R_{n'}$ with $n' < n$ becomes active or is initialized after stage $s+1$. By $s+1 < s_n$, this contradicts minimality of s_n. □

Claim 5. Requirement R_n is met.

Proof. Fix $n = 2e+i$ ($e \geq 0, i \leq 1$). Since R_{n+2} is stronger than R_n, w.l.o.g. we may assume that $n \geq 1$. We will show that (24) holds. Since $e \leq n \leq x_n - 1$ and since x_n is good, by (4) and (6) this guarantees that R_n is met.

Fix s_n and s_{n+1} (correspondingly defined for $n+1$ in place of n) as in Claim 4. Then $x_n^{s_n} \uparrow$, x_n becomes assigned to R_n permanently at stage s_{n+1} (hence $x_n = x_n^s$ for $s \geq s_{n+1}$), and

$$[t_{x_n-1}^{s_{n+1}}, g(x_n, s_{n+1})) = [t_{x_n-1}, g(x_n)).$$

By construction, this implies that all elements of $[t_{x_n-1}, g(x_n))$ which have not been put into X_0 or X_1 at a previous stage are enumerated into X_i at stage s_{n+1}. So it suffices to show that, for any number y,

$$y \in X_{1-i,s_{n+1}-1} \cap [t_{x_n-1}, g(x_n)) \;\Rightarrow\; y \in D. \tag{29}$$

Fix y as in the premise of (29). Note that $g(x_{n-1}) < t_{x_n-1}$ (since $g(x_{n-1}) < x_n$ by definition of x_n and $x_n - 1 < t_{x_n-1}$ by (4)). Since R_n does not enumerate numbers into X_{1-i}, and since, by choice of s_n, no requirement $R_{n'}$ where $n' < n$ becomes active after stage s_n, it follows, by Claim 4 (E), that there is a requirement R_m where $m = 2e'+(1-i) > n$, a number x, and a stage $s+1$ such that $s_n < s+1 < s_{n+1}$, x becomes appointed as R_m-follower at stage $s+1$, hence $x = x_m^{s+1}$, and

$$y < g(x_m^{s+1}, s+1) \leq s+1 \;\&\; y \notin X_{0,s} \cup X_{1,s}. \tag{30}$$

Moreover, x_m^{s+1} is good at stage $s+1$, hence $h(x_m^{s+1}, s+1) < g(x_m^{s+1}, s+1)$.

By $n < m$ and by construction, R_n has a follower $x' < x$ at stage s and is not initialized at stage $s+1$. So $x_n^{s+1} = x_n^s < x$ and $g(x_n^s, s+1) = g(x_n^s, s)$ where, by $s \geq s_n$, $x_{n-1}^s = x_{n-1}$ and $g(x_{n-1}) = g(x_{n-1}, s) < x_n^s$. Now distinguish between the following cases.

Case 1: $x_n^s < x_n$. Then, by $g(x_{n-1}) < x_n^s < x_n$ and by choice of x_n, x_n^s is not good but (by Claim 2) x_n^s is good at stage s. So there is a stage $t \geq s+1$ such that $h(x_n^s, t+1) \neq h(x_n^s, t)$. By (10) and by (12) this implies $W_t \restriction h(x_n^s) \neq W \restriction h(x_n^s)$ which in turn (by definition of t_{x_n-1}) implies that $t_{x_n-1} > t$. Since $y \leq s+1 \leq t$, this contradicts the choice of y. So this case cannot apply.

Case 2: $x_n^s > x_n$. Then, by Claim 2, x_n is not good at stage $s+1$. Since x_n is good, it follows that there is a stage $t \geq s+1$ such that $g(x_n, s+1) = g(x_n, t) < g(x_n, t+1)$. So, by (17) and by the dump property (2),

$$[g(x_n, s+1), s+1] \subseteq D. \tag{31}$$

On the other hand, since $x_n < x_n^s < x$, x_n^s is follower of R_n at stage $s+1$ and x is follower of R_m at stage $s+1$, it holds that $g(x_n, s+1) < g(x_n^s, s+1) = g(x_n^s, s)$ and all numbers $< g(x_n^s, s)$ are in $X_{0,s} \cup X_{1,s}$. So, by the second part of (30), $g(x_n, s+1) \leq y < s+1$ whence $y \in D$ by (31).

Case 3: $x_n^s = x_n$. Since $s+1 < s_{n+1}$, R_n becomes initialized after stage $s+1$. Since x_n is good, this implies that there is a stage $t \geq s+1$ such that $g(x_n, s+1) = g(x_n, t) < g(x_n, t+1)$. But then we may argue as in Case 2 that $y \in D$.

This completes the proof Claim 5 and the proof of the theorem. □

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3. Appendix: Proofs of (8) - (12)

Proof of (8). The strict inequalities are immediate by definition. So, in particular, $h(x,s)$ is strictly increasing in the first argument, hence $x \leq h(x,s)$. □

Proof of (9). The proof is by induction on x. Fix x and s and assume the claim to be true for all numbers less than x. For the proof of the first part, we may assume that $x > 0$ since $h(0,s) = 0$ for all stages s. Note that $t^s_{x-1} \leq t^{s+1}_{x-1}$ by inductive hypothesis and $\overline{W_{s+1}} \subseteq \overline{W_s}$. So, for any number $s' > t^{s+1}_{x-1}$, it holds that $s' > t^s_{x-1}$ and

$$\overline{W_{s+1}} \cap [t^{s+1}_{x-1}, s') \subseteq \overline{W_s} \cap [t^s_{x-1}, s').$$

But this implies $h(x,s) \leq h(x,s+1)$ by definition. For a proof of the second part, note that, by the first part and by $W_s \subseteq W_{s+1}$, $h(x,s) \leq h(x,s+1)$ and $W_s \upharpoonright h(x,s) \subseteq W_{s+1} \upharpoonright h(x,s+1)$. So $t^s_x \leq t^{s+1}_x$ by definition. □

Proof of (10). The proof is by induction on x. Fix x and assume that $h(x,s) \neq h(x,s+1)$ hence $h(x,s) < h(x,s+1)$ by (8). Then $x > 0$ and, by (8) and inductive hypothesis, we may assume that $W_{s+1} \upharpoonright h(x-1,s) = W_s \upharpoonright h(x-1,s)$ and $h(x-1,s) = h(x-1,s+1)$. It follows that $t^s_{x-1} = t^{s+1}_{x-1}$ hence $[t^s_{x-1}, h(x,s)) = [t^{s+1}_{x-1}, h(x,s))$. By $h(x,s) < h(x,s+1)$ and by definition of $h(x,s+1)$ this implies that $\overline{W_{s+1}} \cap [t^s_{x-1}, h(x,s)) = \emptyset$. On the other hand, by definition of $h(x,s)$, $\overline{W_s} \cap [t^s_{x-1}, h(x,s)) \neq \emptyset$. So $W_{s+1} \upharpoonright h(x,s) \neq W_s \upharpoonright h(x,s)$. □

Proof of (11). Assume that $W_{s+1} \restriction h(x,s) \neq W_s \restriction h(x,s)$. Then, by $h(x,s) \leq h(x,s+1)$, $W_{s+1} \restriction h(x,s+1) \not\subseteq W_t \restriction h(x,s+1)$ for all $t \leq s$. So $t_x^{s+1} \geq s+1$ by definition. □

Proof of (12). The proof is by induction on x. Given x, by (9) it suffices to show that $h(x,s) = h(x)$ and $t_x^s = t_x$ for all sufficiently large s. If $x = 0$ then this is immediate since $h(0,s) = 0 = h(0)$ and $t_0^s = 1 = t_0$ for all stages s. So we may assume that $x > 0$ and, by inductive hypothesis, we may fix a stage s_0 such that $h(x-1,s) = h(x-1)$, $t_{x-1}^s = t_{x-1}$ and $W_s \restriction u+1 = W \restriction u+1$ for all $s \geq s_0$ where u is minimal such that $u \geq t_{x-1}$ and $u \notin W$. Then, for $s \geq s_0$, $h(x,s) = u+1 = h(x)$. Moreover, for the least stage $t > h(x)$ such that $W_t \restriction h(x) = W \restriction h(x)$, $t_x^s = t = t_x$ for all stages $s \geq \max\{s_0, t\}$. □

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