The Geometry of Computable Banach Spaces

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This Talk

- There has been an increasing work in computable aspects of computable uncountable algebraic structures i.e. computable analytic algebra.
- For example, topological structures, metric spaces, totally disconnected locally compact groups.
- There is a very nice book (Computable Structure Theory: A Unified Approach: D. and Melnikov)
- In this talk I will look at, and update, a number of open question and recent results about computable Banach spaces, particularly geometric aspects.
- Recall that a normed vector space has associated with it a distance $d(x, y) = ||x - y||$. If this is a complete metric space, it is called a Banach space.
- Banach spaces are fundamental to the field of functional analysis, and have extensive applications.

History

- The modern history began with Pour-El and Richards.
- Also Metakide-Nerode-Shore (Hahn Banach Theorem)
- A long paper by Brattka ("Computability of Banach Space Principles") covering basic results such as Open Mapping, Closed Graph, etc. (Available online)
- Many, many of the results are consequences of computable compactness, and with Melnikov I have a survey of many in the BSL.

Theorem

For a computably, completely metrized Polish space M, (i.e. separable computable metrized topological space, and hence including computable Banach spaces) the following are equivalent:

- *Given n, we can effectively compute (the finite set of parameters describing) a finite 2^n-cover K_n of M by basic open balls.*
- *We can effectively enumerate all finite basic open covers (each given at once as a finite set of parameters) of the space.*
- *Same as 2, but additionally in each finite cover n we can uniformly decide whether two basic open balls $B_1, B_2 \in K_n$ intersect.*
- *In the notation of 3, we can additionally uniformly decide (non)emptiness of intersection for any $B_1, B_2 \in \cup_n K_n$, but balls may have merely computable radii (and, thus, are not necessarily basic).*

Theorem (continued)

- *There is a computable sequence of computable reals $\{\epsilon_n : n \in \omega\}$ such that $\epsilon_n \leq 2^{-n}$ so that, for every n, we can compute the maximal number of points in the space that are at least ϵ_n-far from each other.*
- *M is computably homeomorphic to a computable closed subset of the Hilbert cube.*
- *M is a computable surjective image of 2^ω.*
- *The full continuous diagram of M is decidable.*

- The following result is more or less implicit in the literature, and certainly well-known. The reference is the first occurrence to my knowledge and only for $C([0,1])$.

Lemma (Pour-El and Richards)

Let X be a computable normed space, further let K be some computably compact subset of X. Let $f : X \to \mathbb{R}$ be any computable function. Then $max_K(f)$ is computable. (Therefore $min_K(f)$ is also computable)

- our goal is to give a Cauchy name for e.g. $\max_K(f)$. In essence build a collection of balls whose limit is $\max_K(f)$. We use the fact that as K is computably compact then f is computably uniformly continuous on K, which is easily proven via an effectivization of the classical proof that a continuous function on a compact set is uniformly continuous (since $f(K)$ will also be computably compact).
- At step n, we take a 2^{-2n} cover of K At step n, we take a 2^{-2n} cover of K and then we can compute, say, $f(e)$ for each e with precision 2^{-2n} in in the centre of the balls (K_p) generating the cover.
- we look at the maximum of all such values, this will generate a value for $\max_K(f)$ of precision at least 2^{-n}.

Computable Banach Spaces

Definition

A *computable (separable) Banach space B* is a computable vector space (over $\mathbb{R}$ or $\mathbb{C}$) with a computable dense sequence $\{e_i : i \in \mathbb{N}\}$ making the operations $||\cdot||$, $+$ computable in the sense described for computable real functions.

- Here we work in the "type 2" framework, so that elements are given as fast Cauchy sequences $x = \lim_s e_{i_s}$ with $||x - e_{i_j}|| < 2^{-j}$.
- $+$ and $\cdot$ are computable functions in the sense that they are computable operators taking fast Cauchy sequences to fast Cauchy sequences, so acting on all members of the space.
- "Most" normal Banach spaces have such representations.

Computable Banach spaces and compactness

- A subset of K a computable Banach space X is computable compact iff it is closed, located, and totally computably bounded.
- Here K is located iff for all $x \in X$, $\inf_{y \in K} d(x, y)$ is computable and K satisfies any of the conditions in Theorem 1.
- Note that the unit ball in a computable Banach space is compact iff the space has finite dimension.

Markov computable Banach spaces

- To my knowledge, nobody has pursued the theory of Markov computable Banach spaces.
- By this I mean that, in the spirit of Turing's original 1936 paper we assume that we are looking at functionals only on the computable objects of the domains.
- In Turing's case, he looked at $[0, 1]$ and functions on the computable reals.
- I think this would be an interesting project and ties in with work in RCA_0.

Examples

- $\mathbb{R}^n$ with the Euclidean norm.
- Hilbert spaces.
- ℓ^p (sometimes written ℓ_p); infinite sequences of reals, and such that $||\overline{x}|| = \sqrt[p]{\sum_{k=0}^{\infty} |x_k|^p}$;
- c_0 infinite sequences of reals with the sup norm, but which converge to 0.
- $\ell_p(X)$ all sequences in X with $\sum_{n=0}^{\infty} ||x||^2 < \infty$ and norm as $||\overline{x}|| = \sqrt[p]{\sum_{k=0}^{\infty} ||x_k||^p}$
- $C[0,1]$ continuous functions with sup norm. Dense set rational polynomials.
- $L_p(0,1)$ for $1 < p < \infty$.

Generalized Computable Banach spaces

- The above is all fine provided that the space is separable.
- There are lots of spaces from classical mathematics which are not.
- For example, ℓ^∞, infinite sequences of reals with the sup norm (which needs to exist).
- For example dual spaces.
- Brattka defined a *general computable normed space*, which is a represented space in which the operation taking (names of) fast converging Cauchy sequences to (a name of) the limit of the sequence, is a computable function on names.
- Duals of computable Banach spaces also have this property.
- Additionally, they always can be represented with the norm left or right c.e., uniformly as a function.

Question

Do something with these spaces. Add the left c.e.-ness of the norm. When is the dual computable? More specific question later.

- Brattka and Dillhage have given some conditions on effective bases (e.g. “shrinking”) where the dual is computable.

Classification

- The general theory seems hard. We don't have a good way to turn graphs into Banach spaces.
- The isomorphism problem is hard:

Theorem (Downey and Melnikov)

The isomorphism problem for computable Banach spaces is Σ^1_1-complete.

- Ferenczi, Louveau and Rosendal showed a similar result in the context of Borel equivalence relations.
- The DM result is from a paper called computably compact spaces, which shows the central role of computable compactness and effective duality.

- The upper bound is Σ^1_1 since it is sufficient to state that there is an isometry that works for special points, maps zero to zero, and is, furthermore, surjective (these properties are closed).
- The well-known Mazur-Ulam theorem asserts that every isometry with these properties has to be linear.
- Completeness follows from the Σ^1_1-completeness for Boolean algebras, as follows. First, uniformly produce the computably compact Stone space $\widehat{B}$ of a given Boolean algebra B, and then consider $C(\widehat{B}; \mathbb{R})$ whose computable Banach space structure can be produced uniformly effectively from the compact presentation of the space.
- It is well-known that the homeomorphism type of the compact domain determines the linear isomorphism type of the resulting space, and vice versa (this is Banach-Stone duality). This gives the Σ^1_1-completeness.

Kadets' Theorem

- Kadets' Theorem: Any two infinite dimensional separable Banach spaces are homeomorphic as *topological* spaces, and hence homeomorphic to $\mathbb{R}^{\mathbb{N}}$.
- This result answered a question from Fréchet (1928) and was only proven in 1966.
- In terms of effective content the best we might hope for is that it is effectively true.

Question

Is it true that any two infinite dimensional computable Banach spaces are computably homeomorphic as topological spaces, and hence homeomorphic to a computable copy of $\mathbb{R}^{\mathbb{N}}$?

- Anderson (1966) extended Kadets' result to Fréchet spaces. (A Fréchet space is a complete locally convex metrizable topological vector space. We have not yet looked at this generalization.)

Kadets' continued

- We remark that most results in effective topology are either straightforward and computably true or head into in to the outer space of set existence axioms like Π^1_2-CA.
- Thus the following was a surprise (to us!)

Theorem (Downey, Franklin, Melnikov)

Given any two infinite dimensional computable Banach spaces the halting problem can compute an isomorphism bwtween them.

Corollary

(RCA_0) ACA_0 proves Kadets' Theorem.

Question

Are they equivalent?

- We'll look at the proof shortly.

Geometry

- The fundamental tool in finite dimensional vector spaces is the use of (Hamel) bases. Independent sets where every element is a finite linear combination of the basis elements.
- For infinite dimensional Banach spaces every Hamel basis must be uncountable.
- Yet spaces like ℓ^p are coded by countable information.

Definition (Schauder 1927)

Let X be a Banach space. A sequence $(x_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}}$ is a *Schauder basis* of X if for all $x \in X$, there is a unique sequence of coefficients $(a_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that

$$\sum_{i=1}^{\infty} a_i x_i = x$$

A sequence that is the Schauder basis of the closure of its linear span is called a basic sequence.

- We emphasise that order counts.

- There are situations where the order does not count and this is called an and such a basis is called unconditional.
- Not every (computable) Banach space has an unconditional basis, e.g. $C([0,1])$.
- Although (as we see) every infinite dimensional Banach space has an infinite dimensional subspace with a Schauder basis, this is not true in general for unconditional bases, a result of Gowers and Maurey (1992).
- Brattka dn Dillhage have some results showing that nice Schauder bases have real computability-theoretical consequences.

Question

Explore the relationship of these concepts. Can the the Gowers and Maurey result can be made effective? Classify when a computable Banach space has an unconditional (computable) basis, or when it has a infinite dimensional (computable) subspace with an unconditional basis.

- Sometimes Schauder bases are hard to construct.
- Haar 1910 gave a Schauder basis for $L^p(0,1)$ for $1 \leq p < \infty$. Define a sequence of functions $\{x_n(t)\}_{n\geq 1}$ by $x_1(t) \equiv 1$ (i.e. $(\forall t)(x_1(t) = 1)$.) and, for $k = 0, 1, 2, \ldots, l = 1, 2, \ldots, 2^k$,

$$x_{2^k+l}(t) = \begin{cases} 1 & t \in [(2l-2)2^{-k-1}, (2l-1)2^{-k-1}] \\ -1 & t \in ((2l-1)2^{-k-1}, 2l \cdot 2^{-k-1}] \\ 0 & \text{otherwise} \end{cases}$$

  this is known as the *Haar system* (1910), which forms a basis of $L_p(0,1)$ for every $1 \leq p < \infty$.
- If you have a Schauder basis you must be separable.

Banach's fundamental problem

Question

Suppose that B is a separable banach space. Must B have a Schauder basis?

- Many, many properties were analysed related to this question.
- 40 years (!) later Enflo NO.

Theorem (Bosserhoff)

There is a computable copy of Enflo's example, and hence there is a computable Banach space without the approximation property and hence without a Schauder basis.

Question

What is the complexity of having a basis? Specifically, what is the complexity of the index set of computable Banach spaces that have a basis?

Upper bound

Lemma (Banach)

Let X be a Banach space and $(x_i)_{i\in\mathbb{N}} \subseteq X$ a sequence of nonzero elements. Then (x_i) is a basis of X if and only if:

1. *There is a constant $K \in \mathbb{R}$ such that for all $n, m \in \mathbb{N}$ with $m < n$, for all sequences of scalars $(a_i)_{i\in\mathbb{N}}$, we have*

$$\left\|\sum_{i=1}^{m} a_i x_i\right\| \leqslant K \left\|\sum_{i=1}^{n} a_i x_i\right\|$$

2. *The finite linear span of $(x_i)_{i\in\mathbb{N}}$ is dense in X.*

The proof of the harder direction of the Lemma consists of considering the projections $\{S_i\}_{i\in\mathbb{N}}$ associated with the basis (x_i), defined by $S_k\left(\sum_{i=0}^{\infty} \alpha_i x_i\right) = \sum_{i=0}^{k} \alpha_i x_i$. Then (1) is equivalent to requiring the value $\sup_i \|S_i\|$ to be finite.

Basis Constants

Definition

Let X be a Banach space and $(x_i)_{i\in\mathbb{N}}$ be a basis of X, and $\{S_i\}_{i\in\mathbb{N}}$ its associated sequence of projections. The *basis constant* of (x_i), denoted as $\mathrm{bc}\left((x_i)\right)$, is the value $\sup_i \|S_i\|$.

- Note that $\mathrm{bc}\left((x_i)\right)$ is equivalent to the infimum of all K that satisfies the requirements of the Lemma.
- The basis constant of the space X, denoted $\mathrm{bc}(X)$, is the infimum of basis constants across all of its bases. We set $\mathrm{bc}(X) = \infty$ if X has no basis.

Intuition Fails

- A basis is called monotone if its basis constant is 1.
- Hilbert spaces always have monotone bases, BUT
- In 1983, Szarek proved that there are finite dimensional Banach spaces which do not have a basis with basis constant 1!

Theorem (Xie 2024)

For each k, for sufficiently large n, there is a computable norm on $\mathbb{R}^n$ whose associated basis constant is greater than k.

- Moreover, we are used to building bases sequentially. You have a subspace $V \leq B$ of dimension n with basis $\{b_1, \ldots, b_n\}$ and adding b_{n+1} to make this bigger.
- There are examples where $V \leq W$, the dimension of W is finite, W has a basis with basis constant d, V has a basis with basis constant d, but that basis cannot be extended to one for W with basis constant d!

- The proof of Xie's Theorem can be found in a paper by Downey, Greenberg, Qian and Xie in the references at the end of this talk and involves effectivising the complex arguments of Szarek.
- We remark that, on the face of them, they were not effective, and also the proofs involved a deep dive into some complex historical papers in analysis.
- This is all too long and difficult to talk about today.

Finite Dimension

Lemma (Bosserhoff)

Let X be a computable Banach space, and $\{x_0, \ldots, x_n\}$ be a computable sequence of independent points. Then $bc(x_0, \ldots, x_n)$ is computable, uniformly in $\{x_0, \ldots, x_n\}$.

Proof: Let $[x_0, \ldots, x_n]$ denote the space spanned by the points. Since the basis constant is the maximum of the norms of the associated projections, it suffices to observe that given an operator on a finite-dimensional computable Banach space, we can compute its norm. To do this, we use the fact that the unit ball of a finite-dimensional Banach space is compact, and if the space is computable, then the unit ball is computably compact. The maximum of a real-valued function on a computably compact set is computable, uniformly. (Again, note the role of computable compactness.)

- We improved this for the basis constant of the whole space:

Lemma (Downey, Greenberg, Qian)

Let X be a computable Banach space, and $\{x_0, \ldots, x_n\}$ be a computable sequence of linearly independent points. Then $bc([x_0, \ldots, x_n])$ is computable. Furthermore, this is uniform in $\{x_0, \ldots, x_n\}$.

- The proof is a slightly more elaborate use of computable compactness.
- For the infinite dimensional case, since we have sequence of (computable) projections:

Lemma

Let X be a computable Banach space and $(x_i)_{i\in\mathbb{N}}$ a computable basis of X; then $bc((x_i))$ is a left-c.e. real.

Theorem

For any $\alpha \in \mathbb{R}$ that is left-c.e and $\alpha \geq 1$, there is Banach space X with basis $(e_i)_{i \in \mathbb{N}}$ such that $bc((e_i)_{i \in \mathbb{N}}) = \alpha$.

The proof is not hard. It also shows that degrees of bases are closed upwards. The following question seems completely open.

Question

Let X be a computable Banach space with basis. What is the complexity of $bc(X)$? What if X has a computable basis?

Bosserhoff's Theorem

Theorem (Bosserhoff)

There is a computable Banach space with a Schauder basis, but no computable Schauder basis.

Question

Suppose that computable X has no computable Schauder basis but does have a basis. What complexity basis does it have?

Question (Bosserhoff)

Suppose that a computable Banach space X has a monotone Schauder basis. Must X have a computable Schauder basis?

Bosserhoff's construction gives a computable presentation of a Banach space with a basis and no computable one. The following seems open:

Question

Is there a computable Banach space X with a basis such that no computable presentation of X has a computable basis? Is having a computable basis presentation dependent amongst computable presentations?

- What about subspaces?

Theorem (Mazur)

Every infinite dimensional Banach space (separable or otherwise) has a an infinite dimensional subspace with a Schauder basis.

- Metakides and Nerode showed that there is an infinite dimensional computable Vector Space over $\mathbb{Q}$ with no infinite computable independent sets. (Simpson showed that they could all code $\emptyset'$.)

- However, using computable compactness we get:

Theorem (Downey, Greenberg, Qian)

Let X be an infinite dimensional computable Banach space, then there is a computable basic sequence in X.

Question

What about this for generally computable Banach spaces? More generally, how complicated are the basic sequences in X? We know that they are closed upwards in the Turing degrees.

Index Sets

- The characterization of having a basis iff you have a basis constant means that it is possible to observe that the index set of computable Banach spaces with bases is in Σ^1_1, when on the face of it it was only Σ^1_2.

Theorem (Downey, Greenberg, Qian)

The index set of computable Banach spaces with computable Schauder bases is Σ^0_3-complete.

This is proven with a variation of Bosserhoff's construction.

- For the general question of the complexity of having a Schauder basis, we can prove it is Π^0_3-hard.
- Or, at least improve Σ^1_1.
- In passing there are known constructions of Banach spaces from trees, such as the James Tree Space, but to my knowledge all have bases.
- Now that we have Schauder bases, let's go back to Kadets' Theorem.

Back to Kadets' Theorem

- There are several proofs of Kadets' Theorem, and all are nontrivial.
- To obtain the stated result on the effective content, DFM effectivized the original proof of Kadets'.
- This proof splits into two cases: One where B has a (computable) basis, and one where it does not.
- Interestingly the original proof of Kadets' *predated* Enflo's work.
- Using $\mathbf{0}'$ we massage the norm of B to construct an equivalent norm $||\cdot||_2$. (meaning that there are constants c, C with $c||\cdot|| \leq ||\cdot||_2 \leq C||\cdot||$.) which is "locally uniformly convex".
- This is effectivised by replacing certain closed conditions by open ones and realizing that it is enough to make things work on certain "special points" which are computable.
- In any case we then find that $\mathbf{0}'$ can compute a homeomorphism from B to ℓ_2.

- The second part of the classical proof observes that if B does not have a computable basis then since there is an infinite dimensional subspace C with a Schauder Basis by Mazur's Theorem.
- Kadets then decomposes as $B \cong B/C \oplus C$, and works with B/C
- In the effective case, we know there is a subspace with a computable basis, but even so, typically a quoteint structure is pretty wild in terms of computability theory since, for example, how can we compute selections: representatives of distinct equivalence classes.
- Moreover, the norm is usually the infimum norm, which looks bad. It was surprising that we could prove the following.

Theorem (Downey, Franklin, Melnikov)

If B is a computable Banach space, then B admits a computable basic sequence $\{b_i \mid i \in \mathbb{N}\}$, such that its span C is a located c.e. closed space such that B/C is a computable Banach space with the supremum norm.

- The proof uses some old material of Metakides and Nerode, a delicate and new proof of Mazur's Theorem which gives, for example, closure and locatedness.
- The proof then filters through a surprising result.

Effective Michael's Selection

- $\phi : X \to 2^Y$ is *lower semi-continuous*, or *l.s.c.* for short, if

$$V_\phi = \{x \in X \mid \phi(x) \cap V \neq \emptyset\}$$

  is open in X for every open $V \subset Y$.
- By effective lower semi-continuity we mean that, for any open set V, the set

$$V_\phi = \{\, x \in X \mid \phi(x) \cap V \neq \emptyset \,\}.$$

  is open, and furthermore given a description of V we can effectively produce a description of V_ϕ.
- For a Banach space F, let

$$F(Y) = \{C \subseteq Y : C \text{ closed and convex}\}.$$

- The proof of the following is a nontrivial modification of a nontrivial result of Michael 1956

Theorem (Effective Michael's Selection)

Suppose X is a computable Polish space, and Y is a computable Banach space. Then every effectively lower semi-continuous (l.s.c.) $\phi : X \to F(Y)$ admits a computable selection.

Theorem

Suppose B is a computable Banach space and E is a located closed linear subs pace of B. Then B/E admits a computable selection. (That is, there is a computable function $B/E \to B$ that for each coset output s a representative.)

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- For the proof, Let $T : B \to B/E$ be the projection map. For each $v \in B/E$ (viewed as a Banach space under the norm $\inf_{e \in E} ||x + e||$) associate the respective class $T^{-1}(v)$ which is an element of the hyperspace of closed convex subsets of B. Let $\phi : v \mapsto T^{-1}(v)$ be this map. Then use: Effective Michael's Selection Theorem 33 for this ϕ with $X = B/E$ and $Y = B$. This needs a final lemma:

Lemma

In the notation above, $\phi : v \to T^{-1}(v)$ is effectively lower semi-continuous.

Back to Index Sets

- Notice that there is an enormous gap between the lower bound for having a basis Π^0_3 and the upper bound Σ^1_1.
- One reason for the enormous gap is that all constructions constructing a separable Banach space without a basis do so by constructing separable Banach spaces failing to have a property implied by having a basis.
- These seem to have come naturally in the attempts to solve Banach's question, but have become important in their own right.
- Next frames have some:

Definition (Schauder decomposition)

Let X be a Banach space. A *Schauder decomposition* (SD) of X is an infinite sequence $(Z_i)_{i\in\mathbb{N}}$ of closed subspaces of X such that for all $x \in X$, there exists an unique sequence $(z_i)_{i\in\mathbb{N}}$, $z_i \in Z_i$ such that

$$x = \sum_{i=1}^{\infty} z_i.$$

A Schauder decomposition where the spaces Z_i are all finite dimensional is called a *finite dimensional Schauder decomposition* (FDD).

Think of a Schauder basis as being 1-dimensional Z_i. In 1987 Szarek proved that these properties are strictly weaker than having a Schauder basis.

Definition (Local basis structure)

Let X be a Banach space. X is said to have the *local basis structure* (LBS) if there is some constant $K \in \mathbb{R}$ such that for any finite dimensional subspace $B \subset X$, there exists a finite dimensional space $L \subset X$ such that $B \subseteq L$ and $\mathrm{bc}(L) \leq K$.

Enflo's original construction constructed a separable Banach space with LBS, and yet has not basis.

Theorem (Xie 2024)

The index-set of computable Banach spaces with the local basis structure is Σ_3^0-complete.

Definition (Approximation property)

1. Let X be a Banach space. X is said to have the *approximation property* (AP) if for all compact sets K, for all $\epsilon > 0$, there is a finite rank operator T on X such that $(\forall x \in K)\,(\|Tx - x\| < \epsilon)$.
2. Let X be a Banach space. X is said to have the *bounded approximation property* (BAP) if there is a $\lambda \geq 1$ such that for all compact sets K, for all $\epsilon > 0$, there is a finite rank operator T on X such that $(\forall x \in K)\,(\|Tx - x\| < \epsilon)$ and $\|T\| \leq \lambda$.

Szarek constructed a BAP space without LBS

Theorem

1. $\Pi^0_3 \leq BASIS_I \leq \Sigma^1_1$.
2. $\Pi^0_3 \leq BAP_I \leq \Sigma^0_4$.
3. $\Pi^0_3 \leq AP_I \leq \Pi^1_1$.
4. $\Pi^0_3 \leq FDD_I \leq \Sigma^1_1$.
5. $\Pi^0_3 \leq SD_I \leq \Sigma^1_1$.

Question

Close the gaps.

Coda: Other bases

- There are many other kinds of bases for Banach spaces.
- For Schauder bases: Shrinking, monotone, absolute, etc, and aside from 2-3 papers by Brattka and his students, very little is known.
- For example, they showed that having a shrinking basis of the right kind allows for duality.
- There is a basis notion that every separable Banach space has. A (strong) Markushevich Basis. (1943)
- (A biorthogonal system (x_α, f_α) such that closure of the span of the $x_\alpha's$ is B and the f_α separate the points of B. This this factors through duality.)

Question

Say something about them.

- As you can see, Banach spaces provide a plethora of fascinating demanding questions, and deserve greater exploration.

Some References

- Volker Bosserhoff, Computable functional analysis and probabilistic computability, Thesis, Universität München, 2008.
- Vasco Brattka, Computability of Banach space principles, FernUniversität in Hagen, 2001.
- Vasco Brattka, Ruth Dillhage, Computability of compact operators on computable Banach spaces with bases, Mathematical Logic Quarterly 53 (4–5) (2007) 345–364.
- Rod Downey, Johanna Franklin, Alexander Melnikov, The effective content of Kadets' theorem. in preparation.
- Rod Downey, Noam Greenberg and Long Qian, Some Open Questions and Recent Results on Computable Banach Spaces, in *Twenty Years of Theoretical and Practical Synergies (CiE 2024)*, Springer-Verlag, 10-26.
- Rod Downey, Noam Greenberg, Long Qian and Ruofei Xie, The geometry of computable Banach spaces, *Information and Computation*. Gold Open Access, Vol. 310 (2026) 105431 `https://doi.org/10.1016/j.ic.2026.105431`

- Rod Downey and Alexander Melnikov, Computably Compact Spaces, *Bulletin Symbolic Logic,* Vol 29 (2) (2023), 170 - 263
- Rod Downey and Alexander Melnikov, Computable Structure Theory: A Unified Approach, Springer, 2025.
- Marion Pour-El, Ian Richards, Computability and noncomputability in classical analysis, Trans. Am. Math. Soc. 275 (2) (1983) 539–560.
- Marion Pour-El, Ian Richards, Computability in analysis and physics, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, Berlin, 1989.
- Long Qian, Computability-theortic complexity of effective Banach spaces, MSc. Thesis, Victoria University, Wellington, 2021.

Thank You