My Mathematical Encounters with Anil Nerode-Updated!

Rod Downey
Victoria University
Wellington
New Zealand

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Plan

- In 2013, I was asked to give a talk for Anil's 80-th, I was not only honoured, but also had occasion to reflect on how many times his work has influenced mine.
- In 2013, it seemed to me that this would be a nice basis for a talk.
- In 2022, I was asked to give a talk at Anil's 90-th. I am even more honoured! The talk from San Diego reflects a lot of my thinking about Anil, so I will repeat and update it.
- Hopefully I will be around for his 100th!

The beginning

- I think I met Anil in 1979 (certainly 79 or 80) in Monash University in Australia. I was beginning my PhD.
- My supervisor, Crossley, said I should talk to Anil Nerode, to which I said "who?"
- My first impression was "this guy is really old". Of course he was not yet 50!
- Denis Hirschfeldt once told me that when I met him in Cornell when I was around 40 he had the same thought about me!
- Now I would have said "this guy is really young to be so smart".

My thesis

- Was in Effective Alegebra.
- This considers algebraic structures and endows them with some kind of computational structure and seeks to see what kind of algorithms come with this.
- For example. A computable group is one where the group operations are computable and the universe is too.
- Nerode was there way before me.

Background

- Begins implicitly with work of Kronecker, etc in the late 19th century.
- Explicitly with the work of Max Dehn in 1911 asking about the word, conjugacy and isomorphism problems in finitely presented groups. (That is, groups of the form $F(x_1, \ldots, x_n)/G(y_1, \ldots, y_m)$ with y_i words in x_i, F and G free and G normal.)
- Before the language of computability theory.
- Arguably going back to Kronecker.
- Van ver Waerden (based on Emmy Noether's lectures), Grete Hermann (1926) for ideal theory, Post and Turing in the 1930's for semigroups.
- Discussion: Metakides (a student of Anil) and Nerode: The introduction of nonrecursive methods into mathematics. The L. E. J. Brouwer Centenary Symposium (Noordwijkerhout, 1981), 319-335.
- Modern incarnation: Fröhlich and Shepherdson 1956, *Effective procedures in field theory*,
- Rabin, *Computable algebra, general theory and theory of computable fields*, 1960

Nerode Patras April 1980 I PROLOGUE

CRUDE HISTORY OF FIELD THEORY 1771-1980

LAGRANGE (1771) [Algebra as string manipulations. Solvability by Radicals. Galois theory.

GAUSS (1801). QUADRATIC FIELDS, cyclotomy.

ABEL, GALOIS (1820's) COMPUTATION OF GALOIS Groups

KUMMER (1840's) [ideals as systems of higher congruences in cyclotomic fields

THE SEPARATION OF Methods

Highly constructive

L. KRONECKER (1882)

[STUDENT OF KUMMER
CONSTRUCTIVE Algebraic NUMBER THEORY AND GEOMETRY VIA KUMMER ideals

M. Noether - COMPUTATIONS IN Algebraic GEOMETRY

HENZELT (1915) ELIMINATION theory UNPUBLISHED

E. Noether - HENZELT (1923) Elimination Theory of Polynomial Ideals.

(NOT HIGHLY CONSTRUCTIVE)

R. Dedekind (1879?)

[LAST STUDENT OF GAUSS
SUBSTITUTION OF SET DEFINITION IN ideals AND Reals

Weber (1890's) abstract fields

Hilbert

STEINITZ (1909) General Theory of Fields

E. ARTIN-Schreier (1927) Real Fields

W. Krull (1928) ∞ Galois THEORY

- Maltsev constructed a computable abelian group without a computable basis.
- Frölich and Shepherdson showed that there are computable fields without computably unique algebraic closures (meaning no computable isomorphism between the algebraic closures) (but see van der Waerden early editions).
- Rabin showed that a computable field had a computable algebraic closure.
- When does a computable field have a computably unique computably algebraic closure?
- What about the rest of classical field theory?
- For example, does a computable algebraically closed field have a computable transcendence base?

III MODERN ACT I

The theorems above do not involve non-trivial recursion theory. They leave open what happens when F is recursive but does NOT have a splitting Algorithm: is its recursive Algebraic closure recursively unique, or can it have two recursive Algebraic closures which do NOT differ by a recursive isomorphism over F?

This is what we mean by Analyzing the effective content of Steinitz's famous theorem that: Every field has an algebraic closure; Any two such differ by an automorphism of the base field.

$\boxed{\text{Metakides-Nerode (1977)}}$ Let F be a recursive field. Any two Recursive Algebraic closures of F differ by a recursive F-isomorphism if and only if F has a splitting Algorithm.

Note: R. Smith carried out the characteristic

IV

ANOTHER TWO UNIVERSALLY COMPREHENDIBLE THEOREMS FROM THE SAME WORK.

Steinitz: Every field has a transcenden[ce] base

Metakides-Nerode: LET F be a recursive field. Then there is A recursive Algebrai[c] closed field G OVER F of transcendence degree $\aleph_0$ such that every recursively enumerable independent set IN G OVER F IS FINITE.

(thus No Trace of a decent transcendence base Persists.)

Artin-Schreier: Every formally real field has a real closure.

Metakides-Nerode. The SPACE OF ORDERING[S] OF A RECURSIVE FORMALLY Real Field is a bounded Π^0_1 class, AND EVERY SUCH CLAS[S] SO ARISES (UP TO EFFECTIVE HOMEOMORPHISM

Classic papers

- Metakides and Nerode :
- Recursion theory and algebra, in *Algebra and Logic* (ed. J. N. Crossley), Lecture notes in Math., vol. 450, New York (1975), 209–219.
- Recursively enumerable vector spaces, Ann. Math. Logic, Vol. 11 (1977), 141-171.
- Effective content of field theory, Ann. Math. Logic, vol. 17 (1979), 289–320.

- The last one I found particularly inspiring. I assign this to students:
- It shows that a computable field has a computably unique algebraic closure iff it has a (separable) splitting algorithm.
- That is, an algorithm to decompose polynomials and hence use the usual method of adjoining roots.
- Hidden message: There must be some other way to construct algebraic closures by Rabin's Theorem.
- Also shows how to classify the orderings of computable formally real fields in terms of effective approximations called Π^0_1 classes.
- These are the infinite paths through computable trees.
- Uses the priority method in algebra.
- Remains an area of great interest, e.g. Russell Miller, Reed Slomon, Steffen Lempp, Julia Knight, Valentina Harizanov, Sasha Melnikov, Nikolay Bazhenov, etc.

- One recent use is to show that certain algebraic objects cannot have decent invariants using computation.
- "decent"= should make the problem less complex than the invariant "the isomorphism type"
- Example. (Downey and Montalbán) The isomorphism problem for torsion-free abelian groups is Σ_1^1-complete.
- This work developed also into feasible algebra (polynomial time presented structures-notably Nerode and Remmel), automatic structures (notably Khoussainov-Nerode) more later.
- (Bazhenov, et. al.) $\{e \mid M_e$ has an automatic, polynomial time, primitive recursive presentation$\}$ is Σ_1^1-complete.
- Also recycled as reverse mathematics (Harvey Friedman, Steve Simpson etc)
- Personally, also I was highly influenced by Anil, and his students who visited Monash, particularly Rick Smith (profinite groups) and the late Jeff Remmel (boolean algebras, and lots of other things).

Isols

- The Metakides-Nerode work developed from 1960's and 1970's interest in effective maths of a different type.
- $A \sim B$ means that there is a partial computable 1-1 function φ such that $\mathrm{dom}(\varphi) \supseteq A$ and $\mathrm{ra}(\varphi) \supseteq B$, and $\varphi(A) = B$.
- A and B are called recursively equivalent. $[A]$ is the RET (recursive equivalence type).
- This is an effective notion of cardinality. What does "finite" mean? (Dedekind) A cannot be mapped to a proper subset of itself.
- $[A]$ is an isol if it is not equivalent to a proper subset.
- For example, all finite sets plus immune sets.
- McCarty proved that models of the isols are models of choice free mathematics in the sense of Kleene realizability.

- A big project in the 60's and 70's was to develop arithmetic for the isols.
- For example $[A] + [B] =_{\text{def}} [A \oplus B]$, $[A] \cdot [B] = [\{\langle a, b\rangle \mid a \in A, b \in B\}]$.
- Many authors gave ad hoc development but Anil showed with a very general construction how to do a wide class simultaneously.
  1966: Diophantine correct non-standard models in the isols. Ann. of Math. (2) 84 421-432.
  1962: Extensions to isolic integers. Ann. of Math. (2) 75 419-448.
  1961: Extensions to isols. Ann. of Math. (2) 73 362-403.
- Essentially forcing arguments before Cohen.

- Given $f : \omega \to \omega$, we can write f as $\sum_{i=0}^{\infty} c_i \binom{n}{i}$ (uniquely) with the c_i called Stirling coefficients. There's a k-ary version of this. If all the $c_i \geq 0$, then f is called combinatorial. Note that we see all f can be expressed as the difference of two combinatorial functions; and similarly for computable functions.
- Nerode showed via extensions of ideas of Myhill how to extend every computable f in this way to the isols using "frames" where are esentially forcing techniques, as later demonstrated by Ellentuck.
- I should remark that problems in this area can be very hard. One of my own most complicated argument was with Slaman, it was in the isols and needed a nonuniform-$\mathbf{0}'''$ priority argument.

- This is essentially the view that what is extendable is that which has a *uniform* combinatorial proof, something I have recently pursued in studying online algorithms.
- The idea that computability theorists look at uniform activities acting on finite strings in a continiuous way leads to a new view of online computation is a branch of computable analysis.
- Tractability comes via parametrizations.
- I am currently pursuing this with Melnikov, Ng and others.

Encounter II-Analysis

- There is a tradition of computable analysis going back to Turing 1936, although the ideas can clearly be seen in work of Borel.
- Reals are fast growing Cauchy sequences.
- Effective functions are effective maps taking such approximations to similar approximations.
- Turing only looked at the field of computable reals, the Type II view only came later through the work of people like and Kleene in the west and Grzegorczyk and Mazur in the east.
- Note Turing computed Bessel functions in 1936.
- There is a tradition of computable analysis-Abeth, Markov, Myhill, Pour-El Richards, more recently Weihrauch, Brattka, Hertling, Joe Miller, Bravermann, Yampolsky, Westrick, Pauly, etc.
- Mekides-Nerode-Shore The effective content of the Hahn-Banach Theorem.
- Recently, it has been shown that this is hand-in-hand with the theory of algorithmic randomness (Demuth's program). Differentiability$\approx$randomness.

Encounter IIb-Analysis, again

- Quite recently I got interested in computable analysis again, and began several projects with several people, one of which concerned Schauder bases for computable Banch spaces. As usual I asked something of Anil. Copied from e-mails with spelling mistakes from both sides.
- (Me to Anil) "I am looking at the theorem that every infinite dimensional Banach space has a infinite dimensional subspace with a Schauder basis.

  All the treatments I have seen use the dual space. Do you know of any historiacl ones which had another more direct approach" (April, 2021)
- (Reply:) He pointed me at a classic text (Lindenstrauss) and... "....So the key notion was to generate a sequence which each element is outside the bclosed linear sphere generated by the previous elements. I am not sure this works or is relevant. "
- Anil is a fountain of historical knowledge....

Encounter III-Automata

- Mike Fellows and I were developing parameterized complexity, and I ran into Anil's work here.
- It related to Anil's famous:
- Myhill-Nerode Theorem L is finite state iff $\sim_L$ has a finite number of equivalence classes.
- $\sim_L$: $x \sim_L y$ iff for all z, $xz \in L$ iff $yz \in L$.
- So being regular (an apparently computational property) is in effect simply saying that something is finite.

Treewidth and Courcelle's Theorem

Definition

[Tree decomposition and Treewidth] Let $G = (V, E)$ be a graph. A tree decomposition, TD, of G is a pair $(T, \mathcal{X})$ where
1. $T = (I, F)$ is a tree, and
2. $\mathcal{X} = \{X_i \mid i \in I\}$ is a family of subsets of V, one for each node of T, such that

(i) $\bigcup_{i \in I} X_i = V$,

(ii) for every edge $\{v, w\} \in E$, *there is an* $i \in I$ *with* $v \in X_i$ *and* $w \in X_i$, *and*

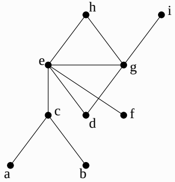
(iii) for all $i, j, k \in I$, *if* j *is on the path from* i *to* k *in* T, *then* $X_i \cap X_k \subseteq X_j$.

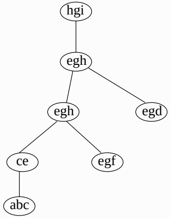
The width of a tree decomposition $((I, F), \{X_i \mid i \in I\})$ is $\max_{i \in I} |X_i| - 1$. The treewidth of a graph G, denoted by $tw(G)$, is the minimum width over all possible tree decompositions of G.

The canonical method

- The following refers to any of these inductively defined graphs families. Notes that many commercial constructions of, for example chips are inductively defined.
  1. Find a bounded-width tree (path) decomposition of the input graph that exhibits the underlying tree (path) structure.
  2. Perform dynamic programming on this decomposition to solve the problem.

An example for Independent Set





$\emptyset$	a	b	c	ab	ac	bc	abc
0	1	1	1	2	-	-	-

Monadic Second Order Logic

- Two sorted structure with variables for sets of objects.
  1. **Additional atomic formulas:** For all set variables X and individual variables y, Xy is an MSO-formula.
  2. **Set quantification:** If ϕ is an MSO-formula and X is a set variable, then $\exists X\ \phi$ is an MSO -formula, and $\forall X\ \phi$ is an MSO-formula.
- Eg k-col

$$\exists X_1, , , \exists X_k \left(\forall x \bigvee_{i=1}^{k} X_i x \ \wedge \ \forall x \forall y \Big(E(x,y) \rightarrow \bigwedge_{i=1}^{k} \neg (X_i x \wedge X_i y) \Big) \right)$$

Model Checking

- Instance: A structure $\mathcal{A} \in \mathcal{D}$, and a sentence (no free variables) $\phi \in \Phi$.
  Question: Does $\mathcal{A}$ satisfy ϕ?
- PSPACE-complete for FO and MSO. Classical proofs have the size of ϕ more or less the same as $\mathcal{A}$.
- Parameterize in various ways to induce tractability. E.g. bounded variables e.g. LTL, SQL etc.
- Or parameterize the structure of $\mathcal{A}$.

Courcelle's and Seese's Theorems

Theorem (Courcelle 1990)

The model-checking problem for (counting) MSO restricted to graphs of bounded treewidth is linear-time fixed-parameter tractable.

Theorem (Frick and Grohe)

First order model checking is FPT for families of graphs of bounded "local" treewidth.

Seese showed that if C is a class of graphs with of unbounded treewidth then the monadic theory of graphs in C is undecidable. This happens as you can interpret grids and then Turing machines.

Can also be applied in e.g. low dimensional topology:

Theorem (Burton and Downey)

For fixed dimension $d \in \mathbb{N}$, let K be any class of d-dimensional triangulations whose dual graphs have universally bounded treewidth. Then from a tree decomposition T from such a dual graph of this bounded treewidth: For any fixed MSO sentence φ, it is possible to test whether $T \models \varphi$ for triangulations $T \in K$ in time $O(|T|)$.

Lots of similar extensions to other width metrics: clique width, local treewidth, d-inductive, nowhere density, etc. Large industry in algorithmic graph theory.

- A proof: we work in the language of boundaried graphs, with boundaries of size $t+1$. Then define $H \oplus G$ to be the graph obtained by gluing H to G on the boundary.
- Treewidth $t-1$: can actually work with parsing operators $_t$, push_i, $\text{join}_{i,j}$, $\oplus$.
- Then algorithms can be automata running on parse sequences for broundried graphs.
- $G_1 \sim_L G_2$ iff for all H, $G_1 \oplus H \in L$ iff $G_2 \oplus H \in L$. This is has a finite number of equivalence classes iff L is "finite state" (in a certain parse language for graphs of bounded treewidth).
- Abrahamson-Langston-Fellows prove Courcelle's Theorem using structural induction. (Think about e.g. 3-colouring) This uses Myhill-Nerode by constructing the relevant test sets for the formulae.
- As Mike Fellows points out: Myhill-Nerode is often a fist step in proving hardness. If something is not finite state it is likely hard.

- One of the first talks on parameterized complexity was at Nerode's 60th.
- Immediately, Anil recognized the value of the area and was exceptionally encouraging both as a mathematician and as an editor.
- The parameterized complexity recognition of Nerode's support is replected in the annual Nerode Prize, by the EATCS.
- Winners of this prize can be found in `https://eatcs.org/index.php/nerode-prize`, and include some of the best people in theoretical computer science.

Nerode helping

- Nerode's immediate response to me was also was a reflection of his kindness to young people.
- Personally, I have always tried to live up to this model, being encouraging and positive.
- It is amazing how this can help beginning scientists.
- Some other advice has been dubious. e.g. When you are speaking your have slides, blackboard and your rhetoric. They don't have to be about the same thing.
- e.g. You must come to the US else you won't be able to do good math.
- e.g. How to write e-mails/letters (something I have used though).

Cornell University
DEPARTMENT OF MATHEMATICS
WHITE HALL
ITHACA, NEW YORK 14853-0403 U.S.A.

ANIL NERODE, *Chairman*
TEL. 607 256-4013
607 256-3577
TELEX 937478

JAMES H. BRAMBLE, *Associate Chairman*
TEL. 607 256-4185

February 23, 1985

Dear Rod,

Your paper on undecidability is nice. I will shortly send you several things, including part one of the generic structures paper, whic was done last week and sent to the Obervolfach meeting. I will get a referee's report shortly , on your paper.

Best

Anil

Cornell University
DEPARTMENT OF MATHEMATICS
WHITE HALL
ITHACA, NEW YORK 14853-0403 U.S.A.

ANIL NERODE, *Chairman*
TEL. 607 256-4013
607 256-3577
TELEX 937478

JAMES H. BRAMBLE, *Associate Chairman*
TEL. 607 256-4185

May 30 1985

Dear Rob,

I can't look at maximal theories until you formally tell H.R you have withdrawn the submission to him.

Best

Anil

Cornell University
DEPARTMENT OF MATHEMATICS
WHITE HALL
ITHACA, NEW YORK 14853-7901 U.S.A.

ANIL NERODE, *Chairman*
TEL. 607 256-4013
TELEX 937478

MICHAEL D. MORLEY, *Associate Chairman*
TEL. 607 256-4105

Dear Rod,

Enclosed is Downey-Slaman. It was read by a very patient person. You still do not write coherently. But the paper is acceptable, and I will not get it refereed again. Please revise and send back. It will be sent then directly to APAL.

Best

Anil Nerode
Editor

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DEPARTMENT OF MATHEMATICS   LA JOLLA, CALIFORNIA 92093

Dear Rob,   June 28

Bring Maximal theories

with you: I'll accept for JPAL.

Do tell North Holland where to send

Proofs.

Annoying Young People

(1)

Victoria University of Wellington
Private Bag Wellington New Zealand *Telephone* (04) 721-000

19/10/87

Dear Anil,

1) Greetings. Enclosed is a paper (with Jeff Remmel) for possible publication in APAL. Its history is as follows :- a couple of years ago I submitted to Advances. Finally after much badgering a report came through which recommended publication, but only after many revisions (~~quite a few~~ It was obvious the referee wasn't totally familiar with a lot of effective algebra, I think) Anyhow, I've enclosed Sacks' letter. I rather like the results & during revision found a new simplification/extension of one result.
Anyway ---.

2) Second did you recieve the revised Downey - Slaman that I sent on 21/8 ?

3) Not much new recently. Mostly I'm been revising old work & catching up. I (finally!) wrote up some work I did in January last that $\exists \underset{\sim}{a}$ (in $\underset{\sim}{R}$) ($\underset{\sim}{a} \neq \underset{\sim}{0}$ & $\underset{\sim}{a}$ bounds no critical triple). A critical triple is $\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3$ such that $\underset{\sim}{a}_1 \cup \underset{\sim}{a}_2 = \underset{\sim}{a}_2 \cup \underset{\sim}{a}_3$ & $\underset{\sim}{a}_1 \not\leq \underset{\sim}{a}_2$ & $\forall \underset{\sim}{c} (\underset{\sim}{c} \leq \underset{\sim}{a}_1, \underset{\sim}{a}_3 \rightarrow \underset{\sim}{c} \leq \underset{\sim}{a}_2)$. Hence the following lattices are not (downwards) densely embeddable into $\underset{\sim}{R}$:

Encounter IV-computable model theory

- Logic is the only area of mathematics that takes language seriously, and a major theme in logic (and complexity) relates definability with computation.

Theorem (Ash-Nerode)

Subject to certain decidability conditions, a relation R on a computanle structure $\mathcal{A}$ is intrinsically computably enumerable (that is computably enumerable in all computable copies of $\mathcal{A}$) iff it is formally c.e. (meaning that it is a formal c.e. disjunction of existential formulae with parameters).

- Highly influential- in the spirit of Goncharov and classifying computably categorical structures with extra decidability in terms of effective Scott families. (Ash-Knight Monograph)
- Recent result shows the limit of this : For each $\alpha < \omega_1^{CK}$, there's a structure which is is computably categorical but no $\emptyset^{\alpha}$-Scott family, so that the index set is Σ^1_1-complete. (Kach et. al. Advances in Math 2015)

- Particularly with Khoussainov, Anil developed automatic model theory.
- Plus well-known texts Automata Theory (with Khoussainov) and Logic for Computer Science (with Shore), and one with Greenberg on Elliptic Curves in preparation, possibly even just submitted.
- Anil has, of course, worked in hybrid control, nonmonotonic logic, polynomially graded logic, and quantum related computing, of which I have no clue.

What I have learned

- Quite a lot of interesting maths.
- Quite a lot of history.
- Lessons on doing your duty for mathematics.
- Being positive and a force for good whenever you can. Promote others.
- Being interested and reading as much as you can-be broad.
- Caring.....

Thank You and Happy Birthday Anil