Yet More on Q- and Z−reducibilities

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Q-degrees

Definition (Tennenbaum)

$A \leq_Q B$ iff there is a computable f such that $x \in A$ iff $W_{f(x)} \subseteq B$.

- First came to prominence via Post's programme.
- Post suggested that if A has a "very thin" complement it would not be T-complete.
- Inspired by the fact that if A is simple then A is not m-complete.
- And hypersimple sets are not tt-complete.

- On the c.e. sets this really is a reducibility, since $W_{f(x)}$ can be taken as finite. (Odifreddi attributes this observation to someone called Soloviev.)
- Moreover, we can always have a unique $m(x, s)$ such that, at any stage s, if $m(x, s)$ enters $B_{s+1} - B_s$, then either $x \in A_{s+1}$ or a new $m(x, s+1) \notin B_{s+1}$ is picked, and such that $\lim_s m(x, s)$ exists.
- Kind looks like a "fickle" m-reduction, and we say x can enter A only if $m(x, s)$ enters B.
- On the c.e. sets $A \leq_Q B$ means $A \leq_T B$.
- Henceforth, everything is c.e., so $\leq_Q$ is a strong reducibility.

Post's Problem and Programme

- Post's Problem was whether there is an intermediate c.e. Turing degree, or are all c.e. problems simply the halting problem in disguise.

Theorem (Friedberg, Muchnik)

There are incomplete c.e. Turing degrees.

- Friedberg and Muchnik solved this not using Post's programme.
- Why would Post think a thinness poperty of $\overline{A}$ guarantee incompleteness?
- $\emptyset' = \{\langle x, y\rangle : \varphi_x(y)\downarrow\}$ so there are lots of elements of $\overline{\emptyset'[s]}$ waiting to halt.
- So if $\emptyset' \leq_m A$, and A is c.e. then this must also be true of $\overline{A}$. So simple sets solve Post's problem for $\leq_m$ (Post, 1945).
- Similar argument for hypersimple (in fact hypersimple c.e. sets are not even *wtt*-cuppable (Downey-Jockusch)).

A Programme Fails

- Post's programme, in its original form fails.

Theorem (Yates)

There is a T-complete maximal set. Here coinfinite A is maximal if for all c.e. $W \supset A$, either $W =^* A$ or $W =^* \omega$.

- Soare in a BSL announcement "conjectured" that every c.e. noncomputable set is automorphic to a complete set.

Theorem (Cholak, D, Stob)

No property of the lattice of supersets of A alone can guarantee Turing incompleteness.

However:

Theorem (Harrington and Soare)

There is a property P such that there are c.e. sets satisfying $P(A)$, and for any c.e. set A with $P(A)$ A cannot be complete.

- $P(A)$ is a 4 quantifier statement obtained from analysing how the "automorphism machinery" fails when you try to prove all c.e. noncomputable sets are automorphic to complete ones.

Degtev-Marchenkov Theorem

- Marchenkov (1976) gave a resurrection of Post's programme in the original thinness spirit.

Definition (Jockusch)

A is called semirecursive if there is a computable function $f(x, y)$ such that for all x, y, $f(x, y) \in \{x, y\}$ and

$$x \in A \vee y \in A \text{ iff } f(x, y) \in A.$$

Lemma

If c.e. B is semirecursive and c.e. $A \leq_T B$, then $A \leq_Q B$.

- Proof. Let $\Phi^B = A$, and accelerate things so that every stage is expansionary. Suppose $x \notin A_s$. Consider $D_s = \{z : z \leq \varphi(x,s) \wedge z \notin B_s\} = \{b_1, \ldots b_{n(s)}\}$. Consider the pairs of elements of D_s. Compute $f(b_1, b_2) = b_{i_1}$. Note if either b_1 or b_2 enter $B - B_s$, then b_{i_1} does. Now consider $f(b_{i_1}, b_3) = b_{i_2}$. The if any of $b_1, b_2, b_3 \in B - B_s$, then so is b_{i_2}, etc. So we can compute an elements $m(x,s) = b_{i_{n(s)-1}}$, where enters $B - B_s$ if B changes on the use $\varphi(x,s)$ after stage s. This is a Q-reduction.

Corollary (Marchenkov)

If a semirecursive set is T-complete, it is also Q-complete.

- Fortunately, no (hyperhypersimple) maximal set) is Q-complete. (Soloviev, Gill and Morris)
- Unfortunately, no (hhs-) maximal set is semirecursive. (Martin)

Definition (Malcev)

An equivalence relation $\equiv$ is called positive if $x \equiv y$ is c.e.. A is called closed if $x \equiv y$ implies $x \in A$ iff $y \in A$.

- Now we can re-do the lattice of c.e. sets replacing $=$ by $\equiv$ for some positive $\equiv$.
- A $\equiv$-finite means that it has only a finite number of cells.

Lemma (Marchenkov)

If A is $\equiv$-maximal (even hhs), then A is not Q-complete.

- Proof. For a contradiction assume it is, so that the halting problem is $\leq_Q M$. Then we use the Recursion Theorem to build an infinite collection of sets in $\overline{K}$ and these will point at a weak array hitting $\overline{M}$.

Theorem (Degtev)

There is a positive $\equiv$ and an $\equiv$-maximal semirecursive c.e. set.

- Proof. First to make M semirecursive use a "dump" construction a enters $M - M_s$ then also put $a+1, \ldots, s$ into M. Then use an e-state constrauction a al Friedberg, but note that for Friedberg if we have $m_{i,s}, \ldots, m_{j,s}$ in $\overline{M}_s$ and want to make $m_{i+1,s+1} = m_{j,s}$ we need to put the middle elements into M, whereas here we simply declare them $\equiv$ to $m_{i,s}$.

Q- IN ALGEBRA

- Dobritsa noticed that Q-reducibility was relevant to combinatorial group theory.
- For all c.e. X there is a finitely generated c.e. presented group with word problem the same Q-degree as X.

DEFINITION (W SCOTT)

G is existentially closed (algebraically closed) if all $W_i(x_j, g_k) = e$ and $W(x_j, g_k) \neq e$ have solutions in G or there is no solution in any group extension of G. This is the group version of being algebraically closed.

THEOREM (BELEGRADEK)

Suppose that H_1, H_2 are finitely generated c.e. groups, with H_2 nontrivial. Then H_1 is embeddable in any EC group that H_2 is embeddable iff $H_1 \leq_Q H_2$. Cosequently, H_1 is embeddable into every AC group iff H_1 has a solvable word problem.

- Earlier, Macintyre proved the only if for $\leq_T$ and Belegradek proved that the iff is not true for $\leq_T$.
- No a.c. group can have a c.e. presentation, which is vaguely strange to me as the construction looks like a Henkin construction.
- The proof is by C. Miller, reported in Macintyre (1972). It relies on the fact that EC groups are simple, and if it has c.e. presentation it would then have a solvable word problem and thus so too would finitely generated subgroups, contradicting a result of C. Miller.

Ziegler Reducibility

- For non-c.e. sets, $\leq_Q$ is not so well behaved. For example $\{e : \varphi_e \text{ total}\} \leq_Q \emptyset'$.
- However, Ziegler extended Belegradek's result to all finitely generated groups using an extension of $\leq_Q$ to all sets, now called Ziegler Reducibility (once $\leq^*$ but hopefully henceforth $\leq_Z$).
- $A \leq_Z B$ means $\overline{A} \leq_Q \overline{B}$ and $A \leq_e B$, where $\leq_e$ is enumeration reduciblilty.
- That is $A \leq_e B$ means there is a W such that $x \in A$ iff $\exists D(\langle x, D\rangle \in W$ and $D \subseteq B$.
- $\leq_Z$ is strictly stronger than $\leq_T$ on arbitrary sets and is useful for things like AC groups applied outside of finitely generated c.e. presented groups.
- Largely unexplored to my knowledge.

Some Results of I. Scott and others

- Isabella Scott has begun a large and interesting project towards understanding existentially closed groups.

Theorem (I. Scott)

$\emptyset'$ can build existentially closed groups. This is tight.

- Existentially closed groups are determined by their finitely generated subgroups.

Theorem (I. Scott)

Let **a** *be a PA degree. There is a EC group all of whose finitely presented subgroups are computable from* **a**.

- Scott showed these results relativize but only in the enumeration degrees.
- Scott has shown that these ideas transfer to other settings:

Entropies

- A d-dimensional *shift* of finite type is a collection of colouring s of $\mathbb{Z}^d$ defined by local rules and a shift action. (Basically saying certain colourings are illegal), and its entropy is the asymptotic growth in the number of legal colourings.
- More formally, consider $G = (\mathbb{N}^d, +)$ or $(\mathbb{Z}^d, +)$, and A a finite set of symbols.
- A has the discrete topology and A^G the product topology.
- The shift action of G on A^G is

$$(S^g x)(h) = x(h + g), \text{ for } g, h \in G \wedge x \in A^G.$$

- A subshift is $X \subseteq A^G$ such that $x \in X$ implies $S^g x \in X$, (i.e. shift invariant)
- Symbolic dynamics studies subshifts

- These are very important wrt Ramsey Theory, Ergodic Theory etc.
- Interestingly, Tao connects these things to things like the Lebesgue theorem on almost everywhere differentiation which in turn is connected to varieties of martingales in randomness under the view (continuity=computable relative to an oracle, non-differentiability =randomness relative to some notion of algoroithmic randomness.)
- Also the work of Avigad and his group at CMU.

THEOREM (HOCHMAN AND MEYEROVITCH, ANNALS OF MATH)

The values of entropies
of subshifts of finite type over $\mathbb{Z}^d$ for $d \geq 2$ are exactly the complements of halting probabilities.

- There is a lot to do here, and almost all we don't understand.

- For subshifts over finite alphabet, Scott and Cordero have defined what it means for one to be finitely determined over and other and prove the following:

THEOREM (I. SCOTT AND CORDERO)

For subshifts S and T over finite alphabets, S is finitely determined over T iff the language complement of S is Ziegler reducible to the language complement of T.

- Clearly there is a model-theoretical fact behind this work, and lots of open questions and areas for investigation.

- $\leq_e$ is a weak reducibility.
- But has had applications in computable algebra.
- For example, in Linda Jean Richter's PhD Thesis she used it to construct models without degree (i.e. in their isomorphism type).

THEOREM (RICHTER)

Let T be a theory of a computable language L. Suppose there is a recursive antichain $W\{V_i : i \in \omega\}$ of finite L-structures and a method of combining, for all $S \subseteq \omega$, the collection $\{V_i : i \in S\}$ into a structure V_S with the following properties:

1. *V_S is a countable model of T.*
2. *V_S is enumeration reducible to S.*
3. *V_i embeds into V_S iff $i \in S$.*

Then there is a set A such that the isomorphism type of V_A has no (least Turing) degree.

- Richter used this on Abelian groups, to construct one with no degree of its isomorphism type.
- The only other method I know is one she used for linear orderings using minimal pairs.
- Echoes Joe Miller's proof that there are continuous functions with no degree.

Higman Embedding

- I remark that Belegradek-Macintye results on Q- and EC groups probably inspired by the reletavised Higman Embedding theorem.

Theorem (Higman)

A finitely generated group is embeddable into a finitely presented group iff it has a c.e. presentation.

Theorem (C. F. Miller, Higman-Scott)

Given two finitely generated groups, H_1, H_2, then H_1 is embeddable into a group finitely presented over H_2, iff H_1 is enumeration reducible to H_2.

- Belegradek (1996) has an extension to certain algebras.

Structure of $\leq_Q$ degrees of c.e. sets

- Call this R_Q.
- Upper-semilattice with join induced by $\oplus$.
- Some things are inherited from $\leq_T$, such as there are minimal pairs.

Theorem (Ambos-Spies and Fischer)

R_Q is not a lattice and not distributive.

- Not distributive : make $B \leq A_1 \oplus A_2$, so that $B \not\equiv_Q B_1 \oplus B_2$ with uses $m_1(x, s)$, $m_2(x, s)$. Roughly bait and switch. Not a lattice, using a variation of Jockusch's non-inf construction, for example.

THEOREM (D, LAFORTE, NIES)

R_Q is a dense USL, has nonbranching degrees, and has an undecidable first order theory.

- The proofs are genuinely complicated.
- One interesting feature of R_Q is that it has join irreducible elements

- Define the dump set as follows, given a computable enumeration $f(\omega) = A$. Let $\overline{D}_s = \{d_{0,s}, \ldots, d_{n,s}, \ldots\}$, and let $D_{s+1} = D_s \cup \{d_{f(s),s}, \ldots, d_{s,s}\}$.
- Then $D \equiv_T A$ and D is semirecursive (and hypersimple).

THEOREM (D, LAFORTE, NIES)

$D(A)$ has maximal Q-degree in the T-degree of A. The Q-degree of A is join irreducible.

- Proof. It is maximal as it is semirecursive. The second part follows from the following more general result.

THEOREM (D, LAFORTE, NIES)

Suppose that semirecursive $D \leq_Q A \oplus B$. Then either $D \leq_Q A$ or $D \leq_Q B$.

- Proof: First try to make $D \leq_Q B$. Suppose that $D \leq A \oplus B$ via m, and f witness the semirecursiveness of D. Given some x find some $y > x$ which will enter D if x enters D, and $m(y, s)$ is targeted at B. Then let x point at $m(y, s)$. If this purported Q-reduction fails, then it can only be that some least $x \notin D$ yet $m(y, s) \to \infty$ and all of the $m(y, s)$ relevant to x enter B. Choose some $d \notin A$. Then for $y > x$, if y points at B, as above we will map y to d. Else it must point at A.

COROLLARY (AMBOS-SPIES AND FISCHER)

The Q-degree of $\emptyset'$ is join irreducible.

Some recent results

- When you have join irreducible elements you can have what are called Ahmad Pairs. A, B such that $A \not\leq B$, and such that for all $X \leq A$, either $A \leq X$ or $X \leq B$.

Theorem (Ben-Sahar, D, Soskova)

Neither R_Q nor R_{sQ} below have these pairs.

Definition (Omanadze)

$A \leq_{sQ} B$ iff $A \leq_Q B$ via some Q-procedure Γ for which there is a computable function f with $\gamma(x) < f(x)$.

- $\leq_{sQ}$ implies $\leq_{wtt}$ on the c.e. sets.

Theorem (Omanadze)

R_{sQ} is a dense USL, which is not a lattice. It has join irreducible elements.

- Question How do $\leq_Q$ and $\leq_{sQ}$ interact?

Definition (BDS)

We say that a Q-degree $[A]$ is sQ-contiguous iff for all $B \equiv_Q A$, $B \equiv_{sQ} A$.

Theorem (BDS)

Every c.e. Turing degree contains a sQ-contiguous Q-degree.

- There are even stranger Q-degrees.

Definition (Batyrshin)

We say that a Q-degree $[A]$ is sQ-singular iff for all $B \equiv_Q A$, $B \equiv_m A$.

Theorem (Batyrshin)

There exist non-zero sQ-singular Q-degrees.

Theorem (BDS)

They cannot have low Turing degree. Can be high.

- We had hoped that since $\leq_{sQ}$ refines $\leq_{wtt}$ it would also be distributive.

Lemma (BDS)

R_{sQ} is not distributive.

- The problem is that permitting does not work here. e.g.
- Recall that A is called nowhere simple if $\overline{A}$ is infinite and for all infinite W with $W \setminus A$ infinite, there is an infinite c.e. $\hat{W} \subseteq W$ with $\hat{W} \cap A = \emptyset$.

Theorem (Omanadze, BDS)

The Q degrees of c.e. nowhere simple sets form an ideal.

- Also $\leq_{sQ}$ and $\leq_Q$ interct badly.

THEOREM (BDS)

There exist $A \equiv_Q B$ such that the sQ-degrees of A and B form a minimal pair.

THEOREM (D, LAFORTE, NIES)

There exist $A \equiv_T B$ such that the Q-degrees of A and B form a minimal pair.

- More lack of permitting:

THEOREM (D, KO KAI JUN)

For all simple Q-degrees **a**, *there is a c.e.* $\mathbf{b} < \mathbf{a}$ *such that* **b** *bounds no join irreducible degrees.*

- However,

THEOREM (D, KO KAI JUN)

There is a c.e. nowhere simple **a** *which is join irreducible.*

The following is being worked up but seesm correct. Perhaps “Theorem” is a bit strong.

THEOREM (D, KO KAI JUN)

S_8 *does not embed into the c.e. Q-degees.*

Back to the Z-degrees.

- There have been some recent investigations of the Ziegler degrees.

Theorem ((Jacobsen-Grocott)-Lempp-Scott)

There is a minimal Ziegler degree. Furthermore, every finite distributive lattice is an initial segment of $\mathcal{D}_Z$, and hence the three quantifier theory is undecidable.

- Note that the undecidability of $\mathcal{D}_Z$ follows from the earlier work of D Laforte and Nies since $X \leq_Z \emptyset'$ implies X is a c.e., and hence you can take the undecidability result of $\leq_Q$ on the c.e. degrees and interpret models of arithmetic using parameters.
- This is much harder than it looks, as $\leq_Z$ is quite tricky to adapt to Spector-like forcing. In the end certain kinds of uniform trees a la Lachlan were needed.

- Following the story, we could re-interpret into $\leq_Z$ old results of Lerman, Lachlan, Nerode and Shore, etc:

THEOREM (D, LEMPP AND SCOTT)

Every countable ideal I in $\mathcal{D}_Z$ has an exact pair; i.e., a pair of degrees $\{\mathbf{a}, \mathbf{b}\}$ such that $\mathbf{c} \in I$ iff $\mathbf{c} \leq \mathbf{a}, \mathbf{b}$.

THEOREM (D, LEMPP AND SCOTT)

Every countable distributive lattice embeds as an initial segment in $\mathcal{D}_Z$.

THEOREM (D, LEMPP AND SCOTT)

The first order theory of $\mathcal{D}_Z$ is bi-interpretable with true second-order arithmetic.

Local Theory

- We know the first order theory of the c.e. Q-degrees is undecidable but have not proof we can interpret arithmetic.
- Do this.
- With Scott and Lempp we can show e.g. the Π^0_3 Theory is undecidable. The proof is horrible.
- Known techniques don't seem to generalize.

- Thank You.