Effectivity in Abelian Group Theory

Rod Downey
Victoria University
Wellington
New Zealand
mostly joint with Alexander Melnikov
Also currently at VUW

Kobe, April 2013

Our Concern

- Mal'cev 1962 A computable abelian group is computably presented if we have $G = (G, +, 0)$ has $+$ and $=$ computable functions/relations on $G = \mathbb{N}$.
- When can an abelian group be computably presented? (Relative to an oracle) Is there any reasonable answer?
- Do different computable presentations have different computable properties?
- Mal'cev produced examples presentations of $\mathbb{Q}^\infty$ that were not computably isomorphic, as we see later.
- Along with Rabin and Frölich and Shepherdson, began the theory of presentations of computable structures, though arguably back to Emmy Noether as recycled in van der Waerden (first edition).
- See Metakies and Nerode "Effective Content of Field Theory".

Why should we care?

- We are logicians after all, and hence its our calling,....but:
- If we are interested in actual processes on algebraic structures then surely we need to understand the extent to which they are algorithmic.
- Effective algorithmics requires more detailed understanding of the model theory. Witness the resurrection of the study of invariants despite Hilbert's celebrated "destruction" of the programme. The Hilbert basis (or nulstellensatz) theorem(s) are fine, but suppose we need to calculate the relevant basis.
- Examples of this include the whole edifice of combinatorial group theory. The theory of Gröbner bases etc. Ashenbrunner's Thesis.
- As we will see a backdoor into estabishing classical results about the existence/nonexistence of invariants in mathematics. Computability is used to establish classical result.
- Establishing calibrations of complexity of algebraic constructions.... reverse mathematics.

While we are on the subject of logic

- Thanks to Moshe Vardi for this and the next quote (my highlighting).
- Cosma R. Shalizi, Santa Fe Institute (A famous US think-tank).

> *If, in 1901, a talented and sympathetic outsider had been called upon (say by a granting agency) to survey the sciences an name a branch that would the the least fruitful in the century ahead, his choice might well have settled upon mathematical logic, and exceedingly recondite field whose practicioners could all have fit into a small auditorium. It had no practical applications, and not even that much mathematics to show for itself: its crown was an exceedingly obscure definition of cardinal numbers.*

More recently

- Martin Davis (1988) Influences of mathematical Logic on Computer Science.

  > *When I was a student, even the topologists regarded mathematical logicians as living in outer space. Today the connections between logic and computers are a matter of engineering practice at every level of computer organization.*

- Yuri Gurevich (Microsoft) quoted as saying engineers need logic not calculus!
- Read a somewhat dated but wonderful collection in the Bulletin of Symbolic Logic: On the Unusual Effectiveness of Logic in Computer Science (Halpern, Harper, Immerman, Kolaitis, and Vardi).
- Echoes Wigner's 1960 article "The unreasonable effectiveness of mathematics in the natural sciences," and Galileo's "The book of nature is writ in the language of mathematics."

Computable abelian groups

- Describe computably presentable Abelian groups.

Theorem (Khisamiev 1970's, Ash-Knight-Oates 1980's)

A certain characterization of computable reduced abelian p-groups of finite Ulm type in terms of limitwise monotonic approximations of functions.

- Recall that a set S is limitwise monotonic iff $S = \mathrm{ra}(f)$ for some computable $f = f(\cdot, \cdot)$, where for $\lim_s f(n, s)$ exists for all n, and $f(n, s+1) \geq f(n, s)$ for all s.
- Sometimes the function f has only elements of ω in its range and sometimes for convenience we have ∞ there.
- Fact: the finite members of the range of one of these functions is a Σ^0_2 set.

Equivalence relations/structures

- Will be of relevance and interest later.
- E is a structure with cells c_i for $i \in \omega$. As above, note that they only get bigger.

Theorem (Calvert, Cenzer, Harizanov, and Morozov 2006)

An equivalence structure $\mathcal{E}$ with infinitely many classes is computable if and only if there is a limitwise monotonic function F (with range $\omega \cup \{\infty\}$) for which there are exactly $|\{x : F(x) = \kappa\}|$ many classes of size κ (for each $\kappa \in \omega \cup \{\infty\}$) in $\mathcal{E}$.

- Limitwise monotonic approximations found applications:
- in computable linear orders (Downey-Khoussainov, Harris, Kach-Turetsky),
- in computable models of $\aleph_1$-categorical theories (Khoussainov, Nies, Shore),
- in computable equivalence structures (Harizanova et al.),
- in a characterization of high c.e. degrees (Downey, Kach, Turetsky).
- Groups as we soon see:

Ulm's Theorem

- G is a p-group if each element has order p^n for some n. G is reduced if no element of infinite height. The height of g is the largest n with $p^n x = g$ having a solution (or ∞).
- Ulm Sequence $G_0 = G$, $G_{\alpha+1} = pG_\alpha$, and for limit $G_\alpha = \cap_{\beta<\alpha} G_\beta$. There is some $\alpha = \lambda(G)$ with $G_\alpha = G_{\alpha+1}$.
- This $\alpha = \lambda(G)$ is called the length. If G is computable then $\alpha < \omega_1^{CK}$ by general results.
- Let $P = \{a \in G \mid pa = 0\}$ and considering $\frac{G_\beta \cap P}{G_{\beta+1} \cap P}$ as a vector space over $\mathbb{Z}_p$, we get a sequence $(u_\beta(G))_{\beta<\alpha}$, called the Ulm sequence.

Theorem (Ulm, 1933)

If $(u_\beta(G))_{\beta<\alpha}$ is a countable sequence of elements of $\omega \cup \{\infty\}$, then there is a countable group with this sequence iff (i) if $\alpha = \beta + 1$, $u(\beta) \neq 0$ and (ii) for any limit $\beta < \alpha$, there is an increasing $\beta_n \neq 0$ and $\beta_n \to \beta$.

Theorem (Khisamiev; Ash, Knight, Oates)

Let G be a countable reduced abelian p-group with length $\lambda(G) < \omega^2$, the G has a computable copy iff

1. *the relation $R_i = \{n, k) \mid u_{\omega.i+n}(G) \geq k\}$ is Σ^0_{2i+2}, and*
2. *There is a Δ^0_{2i+1} function such that for each i, $f_i(n, s)$ is a limitwise monotomic with finite limit m and $u_{\omega.i+m}(G) \neq \emptyset$.*

We remark that if we are given any length $\nu < \omega_1^{CK}$ and the Δ^0_{2i+1} functions uniformly, then we have a group G corresponding to the functions.

- Question (Khisameiv, Ash et al.) Does this hold for ordinals $\geq \omega^2$? If not what is a possible characterization?
- The problem is that the proof is nonuniform, and works by induction on ordinals below ω^2. It appears to lack uniformity.

Theorem (Downey, Menikov, Ng)

There is a computable abelian p-group of Ulm length ω^2 which does not satisfy the uniform version of Khisamiev-Ash-Knight-Oates theorem. Therefore, their proof can not be pushed up to ω^2.

- Strangely, the proof filters through computable trees.
- Laurel Rogers gave an analysis of Ulm's Theorem in TAMS in the 1960's demonstrating that you can obtain it via trees.
- Question: Is there a computable reduced p-group with no corresponding computable tree? Conj Yes (Downey), No (Melnikov), No Clue (current state of affairs).

Rogers' analysis

- $T = (\omega^{<\omega}, p, \emptyset)$, p predecessor.
- $G(T)$ via $\emptyset = 0$, $pa = b$ iff $p(a) = b$ $b \in G(T)$ represented by $\sum_{i=1}^{n} k_i a_i$ with $a_i \in T$ and $k_i \in \omega$.
- (Rogers) If T has no infinite branches then $G(T)$ is a reduced abelian p-group. The converse is also true.
- Trees are not unique, but there is an equivalence relation which is $T_1 \equiv T_1$, then $G(T_1) \cong G(T_2)$, and conversely. Equivalence relation= sequences of "strippings"
- Example: T is the tree with one node p at level 1, and infinitely many successors c_k such that each is a chain and for each n there are infinitely many c_k of length $\geq n$. $\hat{T}$ is the same as T except that for each n there is a node a_n of length 1 with a chain of length n below it. $\hat{T}$ stripped them off p. $T \equiv \hat{T}$. Same Ulm invariants.
- If T is computable, so is $G(T)$. Open : Converse?
- The Ash, Knight, Oates proof shows how to construct a computable tree from the given information.

A minor victory

- Problem [Khisamiev 1990's] Describe computable groups of the from $\bigoplus_{p \in P} Q^{(p)}$, where P is a set of primes, and $Q^{(p)} = \{\frac{n}{p^k} : n \in Z \text{ and } k \in N\}$.

Theorem (Khisamiev 2002)

The group G_P is computable with some extra condition if and only if P is n ot in a certain proper subclass of hh-immune sets.

Theorem (Downey, Goncharov, Knight et al. 2010)

The group G_P is computable if and only if P is Σ_3^0.

Computable Categoricity

- The effective classification tool.
- A computable structure $\mathcal{A}$ is computably categorical iff for all $\mathcal{B} \cong \mathcal{A}$, $\mathcal{A} \cong_{\text{computable}} \mathcal{B}$.
- relatively if it works for all oracles.
- There is a longstanding program to understand the relationship between $\cong, \cong_{\text{comp}}$, classical structure of $\mathcal{A}$ and logical structure of $\mathcal{A}$ in terms of definability.
- These all also have "higher up" versions, like Δ^0_α categocity, definability etc.

Theorem (Goncharov, 1975)

If $\mathcal{A}$ is $2-$decidable, then $\mathcal{A}$ is computable cat iff it is relatively computably cat iff it has an effective naming, that is a c.e. Scott family of existential formulae with parameters $\overline{c}$, such that for all $\overline{a}, \overline{b}$ if they satisfy the same ϕ, then they are automorphic.

More Recent Metatheorems

Theorem (Downey, Kach, Lempp, Turetsky-Fund. Math)

If $\mathcal{A}$ is 1-decidable and it is computably cat, then it is relatively Δ_2^0 cat, as it has a Σ_2 Scott family.

Theorem (Downey, Kach, Lempp, Lewis, Montalbán, Turetsky-submitted Annals of Math)

For each $\alpha < \omega_1^{CK}$ there is a computably cat $\mathcal{A}$ which is not relatively Δ_α^0 cat.

Example-Equivalence relations

- Computably cat equivalence structures are rare. Basically finitary.

Theorem (Calvert, Cenzer, Harizanov, Morozov)

A computable equivalence relation is comput. cat iff

1. *it has only finitely many finite cells, or*
2. *has finitely many infinite classes, bounded character, and at most one finite $k > 0$ with infinitely many equivalnce classes of size k.*

- character $\chi(E) = \{\langle n, k\rangle \mid E$ has at least n classes of size $\geq k$. bounded if k is bounded.
- More interesting we look at Δ^0_2 categoricity. General classification seems hard.

Case study: coding a set

- The singleton case is interesting.

Definition

For a set $X \subset \omega$, let $E(X)$ be an equivalence structure with ω-many infinite classes and exactly one class of size n for each $n \in X$.
Say that an infinite Σ^0_2 set X is categorical if the computable $E(X)$ is Δ^0_2-categorical.

- There are infinite X which are categorical.
- If an infinite Σ^0_2 set X is limitwise monotonic then X is not categorical.
- There exists an infinite set which is not categorical and not limitwise monotonic.
- The general intuition is that being not categorical is a "non-uniform version" of being limitwise monotonic.
- Question How much do these notions differ?

Categoricity bounding vs. (non-)l.m. bounding

- Being limitwise monotonic is not a degree-invariant property. The same is true about being categorical.
- Which c.e. degrees bound a categorical set?

Theorem (Downey, Melnikov, Ng)

For a c.e. degree $\mathbf{a}$, the following are equivalent:

1. $\mathbf{a}$ *is high (i.e.* $\mathbf{a}' = \mathbf{0}''$*).*
2. *There exists an infinite categorical set* $X \leq_T \mathbf{a}$*.*
3. *(Downey, Kach, Turetsky) There exists an infinite* $X \leq_T \mathbf{a}$ *such that* X *is not limitwise monotonic.*

- Thus, c.e. degrees do not see the difference. The proof of $1 \Leftrightarrow 2$ has nothing to do with limitwise monotonicity.

The general case of multi-sets

- Question Can we at least reduce the general problem to the set case (remove repetitions)?
- Given an equivalence structure E, remove repetitions of finite classes from E. Call the resulting E_0 the condensation of E.

Theorem (DMN)

If E is Δ_2^0-categorical, then its condensation is Δ_2^0-categorical as well.

Theorem (Downey, Melnikov, Ng)

There is a computable E which is not Δ_2^0-categorical but whose condensation is Δ_2^0-categorical.

Proof uses a $\mathbf{0}'''$ priority argument.'

Back to groups

Definition (Multi-cyclic groups)

A multi-cyclic group is a direct sum of cyclic ($\mathbb{Z}_{p^n}$) and quasi-cyclic ($\mathbb{Z}_{p^\infty}$) abelian p-groups.

Theorem

A multi-cyclic group with infinitely many infinite quasi-cyclic summands is effectively Δ^0_2-categorical if, and only if, the naturally associated equivalence structure is effectively Δ^0_2-categorical.

Corollary

There exists a Δ^0_2-categorical multi-cyclic group having infinitely many quasi-cyclic summands. (Answers a question left open by CCHM)

Multi-cyclic groups

- Comments on the proof:
  1. (Uniform) Δ^0_2-categoricity in such groups is regulated by the complexity of height-function. (The proof uses a refinement of the first half of Kaplansky's book.)
  2. We don't know if the theorem holds for plain Δ^0_2-categoricity (conjecture: no).
  3. A direct proof of the Corollary, without using the Theorem, would be problematic.
- (Remark) In the context of c.e. degrees, effective Δ^0_2-categoricity bounding is equivalent to being complete (a pretty proof).

Categoricity questions for abelian groups

- When we specialize to specific structures within which it is hard to code graphs questions become more complex. You actually have to do some algebra!
- This is not too hard if you have torsion, and in particular p-groups.
- These have proven useful in lots of areas, $\aleph_1$ categorical theories, equivalence relations, linear orderings, etc.

Theorem (Goncharov, Smith)

A computable p-group is computably categorical iff it can be written in one of the following forms.

1. $(\mathbb{Z}(p^\infty))^\ell \oplus G$ *for* $\ell \in \omega \cup \{\infty\}$ *and* G *finite;*
2. $(\mathbb{Z}(p^\infty))^n \oplus (\mathbb{Z}_{p^k})^\infty \oplus G$ *where* G *is finite, and* $n, k \in \omega$.

- (Calvert-Cenzer-Harizanov-Morozov) A p-group is computably categorical iff it is uniformly computably categorical (and hence iff relatively computably categorical) (and hence has a simple algebraic structure by results of Goncharov-Smith)
- We remark that this phenomenom (uniform=nonuniform) is kind of rare.
- The reason is that the uniform cases can be dealt with by forcing whereas the non-uniform ones use priority arguments.
- Example: The algorithmic dimension if a structure $\mathcal{A}$ is the number of computable isomorphism types it has. Goncharov showed that finite dimensions are possible. Ash-Knight-Manasse-Slaman, and Chisholm showed that only 1 and ∞ are possible in the relative case.

Torsion-Free Abelian Groups

- Here we will study torsion-free abelian groups. That is, they have no elements z with z^n trivial.
- Some kind of good behaviour.

Theorem (Khisamiev)

Every Π^0_{n+1} presentable torsion-free abelian group is isomorphic to one which is Δ^0_n-presentable.

- In general the isomorphism problem is very complex:

Theorem (Downey and Montalbán)

The isomorphism problem for torsion-abelian groups is Σ^1_1 complete.

Consequences of Σ_1^1-completeness

- The idea of an invariant is that is ought to make the problem simpler.
- Classical isomorphism is always Σ_1^1. "There is a function such that ...."
- Invariants make this easier, you would expect. Dimension in a vector space makes the problem Δ_3^0.
- The point is that a Σ_1^1-completeness result result means that the cannot be reasonable invariants for the isomorphism problem.
- This methodology understands invariant theory computationally.
- There are other programmes like this as we now will see.

The Borel game

- This is related to work by the descriptive set theorists who seek to have a notion of Borel cardinality for isomorphism types.
- One class $\mathcal{C}$ is reducible to another $\mathcal{D}$ if there is a Borel mapping injectively taking the isomorphism types of $\mathcal{C}$ into $\mathcal{D}$.
- For example, rank 3 torsion free groups are above rank 2 groups here.
- H. Friedman, Kechris, Thomas, Hjorth etc.

Better algebraic classes

- The idea is to look at algebraically more tractable classes; this is what is done classically anyway.
- Recall that if G is a torsion-free then G embeds into $\oplus_{i \in F}(\mathbb{Q}, +)$. The cardinality of the least such F is called the (Prüfer) rank of G.
- Khisamiev proved that there is an effective embedding.

Rank One Groups

- The only groups we understand well are the rank one groups (and certain mild generalizations) If $g \in G$, define $t(g) = (a_1, a_1, \dots)$ where $a_i \in \{\infty\} \cup \omega$ and represents the maximum number of times p_i divides g. Say that $t(g) = t(h)$ if they are $=^*$, meaning that they must be ∞ in the same places, but otherwise are finitely often different. Thus we can write $t(G)$.
- For example, a divisible group would have $(\infty, \infty, \dots)$ as its type.

Theorem (Baer, Levi)

For rank 1 torsion-free abelian groups, $G \cong H$ iff they have the same type.

- One corollary is that if we consider $T(G) = \{\langle x, y\rangle \mid x \leq t(G)_y\}$, then G is computably presentable iff $T(G)$ is c.e.. (Mal'tsev)

Two Corollaries

- G is a computably categorical torsion-free abelian group iff it has finite rank.

Definition

A structure $\mathcal{A}$ has a degree iff $\min\{\deg(\mathcal{B}) \mid \mathcal{B} \cong \mathcal{A}\}$ exists.

- Strictly speaking, we would mean the isomorphism type here.
- (Jockusch) Can define jump degree by replacing $\deg(\mathcal{B})$ by $\deg(\mathcal{B})'$. The same for α-th jump degree. Proper if no β-th jump degree for $\beta < \alpha$.
- (Coles, Downey and Slaman) Every torsion free abelian group of finite rank has first jump degree.
- (Anderson, Kach, Melnikov, Solomon) For each computable α and $\mathbf{a} > \mathbf{0}^{\alpha}$ there is a torsion-free abelian group with proper α-th jump degree $\mathbf{a}$.

The infinite rank case

- It could be hoped that if G has infinite rank, then $G \cong \bigoplus_{i \in \omega} H_i$ with H_i of rank one.
- Alas, this is not true, however, there is a class of groups for which this is true, called completely decomposable for which this does happen.
- What about categoricity for such groups?
- We cannot hope for computable categoricity, but can hope for things "higher up".

The homogeneous case

- If $G \cong \bigoplus H$ for a fixed H then G is called homogeneous

Theorem (Downey and Melnikov)

Homogeneous computable torsion free abelian groups are Δ_3^0 categorical.

- The proof relies on a new notion of independence called S-independence generalizing a notion of Fuchs to sets S of primes.
- B, a set of elements, is S-independent (in G) iff for all $p \in S$ and $b_1, \ldots, b_k \in G$,

$$p \Big| \sum_{i=1}^{k} m_i b_i \text{ implies } p | m_i \text{ for all } i.$$

- This bound is tight.

But when can it be Δ^0_2 categorical?

- Recall that a set S is called semilow if $\{e \mid W_e \cap S \neq \emptyset\} \leq \emptyset'$.
- Semilow sets allow for a certain kind of local guessing, and aroze in (i) automorphisms of the lattice of computably enumerable sets (Soare) and in (ii) computational complexity as non-speedable ones. (Soare, Blum-Marques, etc.)

Theorem (Downey and Melnikov)

G is Δ^0_2 categorical iff the type of H consists of only 0's and ∞'s and the position of the 0's is semilow.

- The proof is tricky and splits into 5 cases depending on "settling times".
- We remark that this is one of the very few known examples of when Δ^0_2 categoricity of structures has been classified.

The general completely decomposable case

Theorem (Downey and Melnikov)

A completely decomposable G is Δ^0_5 categorical. The bound is tight.

The proof uses methods from the homogeneous case, plus some new ideas. The sharpness is a coding argument. For sharpness we use copies of $\oplus_{i\in\omega}\mathbb{Z}\oplus\oplus_{i\in\omega}\mathbb{Q}^{(p)}\oplus\oplus_{i\in\omega}\mathbb{Q}^{(q)}$., where $p\neq q$ primes and $\mathbb{Q}^{(r)}$ denotes the additive group of the localization of $\mathbb{Z}$ by r. Then a relation θ on this group which is decidable in one copy and very bad in another.
With some extra work we can also prove the following. We don't know if the bound is sharp here.

Corollary (Downey and Melnikov)

The index set of completely decomposable groups is Σ^0_7.

References

- Computable completely decomposable groups, (DM) to appear TAMS
- Effectively Categorical Abelian Groups, (DM) to appear J. Algebra.
- Δ^0_2 Categorical Equivalence Relations and *p*-groups (DMN), in preparation
- On presentations of *p*-groups (DM), in preparation

Thank You