Courcelle's Theorem for Triangulations

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EXPLANATION

- In July, 2014 Vaughan invited me to talk at his (retrospective) 60th in Maui, and I had some work relevant to the theme of that meeting.
- But I will begin with the slides I used at that meeting itself.

THIS LECTURE:

- In this lecture I have chosen some material which is relevant to the theme of this meeting.
- It involves a combination of logic, complexity theory and low dimensional topology, so I will explain each bit separately.
- Before I do so, I would like to take this opportunity to talk about Vaughan's contributions to New Zealand mathematics.

Vaughan :

- When Vaughan won the Fields Medal, this fact was celebrated in New Zealand, and as we see, Vaughan applied his influence to Mathematics in NZ.
- When he did so there were far more sheep than mathematicians in New Zealand.
- Now....
- This is still true, and indeed there are also many more cows, but...

VAUGHAN :

- In our last “PBRF” exercise, mathematics was ranked as the strongest area of expertise in New Zealand.
- (2020 update) Still true, in 2020.
- Vaughan gave many public lectures and received a (paper) medal from the Prime Minister. This is now the Rutherford Medal, the premier award in New Zealand.
- He set up the NZMRI (a lucky coincidence was that the Marsden Fund was being extanblished in New Zealand).
- Finally we had summer meetings to raise the standards of NZ mathematics. (A model copied in e.g. Singapore)
- This lead to the NZIMA, which supported a huge amount of mathematics in New Zealand.
- He donates his own hard earned income to the NZMRI to this day.
- There are many excellent mathematicians, but Vaughan has shown us how to be an excellent person also.

Complexity Theory: the good

- In this talk I am interested in the running times for possible computations deciding questions (here, in low dimensional topology).
- Objects are given in some standard way (to be described) For example a graph G on n vertices needs $O(n^2)$ many entries to describe it.
- Classical complexity has "the good" as P, (polynomial time) languages (="yes, no questions") that some machine can decide in time $|G|^c$ for some constant c.
- For example, Euler cycle in a graph. (through all edges)
- It is now commonly known that there are many "high level" techiques to feed in some kind of formal description and automatically generate and algorithm.
- Algorithmic meta-mathematics.

Revisioning complexity theory

- When is the only thing you know about a problem is the input size
- Answer : only cryptography, and this is by design.
- For practical problems, the world comes equipped with many many additional parameters.
- For practical problems, the world comes equipped with many many additional parameters.
- As we soon see, sensitizing the run times to parameters allows the development of a distinctive and often useful toolkit.
- We are far from undrstanding the complexity of real data, and whether things like P vs NP even matters.

PARAMETERS

- Without even going into details, think of all the graphs you have given names to and each has a relevant parameter: planar, bounded genus, bounded cutwidth, pathwidth, treewidth, degree, interval, etc, etc. In numerical analysis the degree of precision etc.
- Also nature is kind in that for many practical problems the input (often designed by us) is nicely ordered.

- (First order Logic)
  1. **Atomic formulas:** $x = y$ and $R(x_1, ..., x_k)$, where R is a k-ary relation symbol and $x, y, x_1, ..., x_k$ are individual variables, are FO-formulas.
  2. **Conjunction, Disjunction:** If ϕ and ψ are FO-formulas, then $\phi \wedge \psi$ is an FO-formula and $\phi \vee \psi$ is an FO-formula.
  3. **Negation:** If ϕ is an FO-formula, then $\neg\phi$ is an FO-formula.
  4. **Quantification:** If ϕ is an FO-formula and x is an individual variable, then $\exists x\ \phi$ is an FO-formula and $\forall x\ \phi$ is an FO-formula.
- Eg We can state that a graph has a clique of size k using an FO-formula,

$$\exists x_1 ... x_k \bigwedge_{1 \leq i \leq j \leq k} E(x_i, x_j)$$

Monadic Second Order Logic

- Two sorted structure with variables for sets of objects.
- 1. **Additional atomic formulas:** For all set variables X and individual variables y, Xy is an MSO-formula.
  2. **Set quantification:** If ϕ is an MSO-formula and X is a set variable, then $\exists X\ \phi$ is an MSO -formula, and $\forall X\ \phi$ is an MSO-formula.
- Eg k-colorability

$$\exists X_1 \ldots \exists X_k \left(\forall x \bigvee_{i=1}^{k} X_i x \wedge \forall x \forall y \Big(E(x, y) \to \bigwedge_{i=1}^{k} \neg (X_i x \wedge X_i y) \Big) \right)$$

MODEL CHECKING

- Instance: A structure $\mathcal{A} \in \mathcal{D}$, and a sentence (no free variables) $\phi \in \Phi$.
  Question: Does $\mathcal{A}$ satisfy ϕ?
- PSPACE-complete for FO and MSO.

- Logicians such as Büchi, Rabin and others realized the conection between MSO and complexity, especially on formal languages. Below we will discuss this in the situation where objects of interest have parse languages, meaning that they are build up in an inductive manner.
- MSO is central to program verification, and makes logic the calculus of computer science. See articles by Morshe Vardi.
- That is, the connection is way beyond mere complexity.

- Graphs constructed inductively. Treewidth, Pathwidth, Branchwidth, Cliquewidth mixed width etc. k-Inductive graphs, plus old favourites such as planarity etc, which can be viewed as local width.
- Example:

DEFINITION

[Tree decomposition and Treewidth] Let $G = (V, E)$ be a graph. A tree decomposition, TD, of G is a pair $(T, \mathcal{X})$ where
1. $T = (I, F)$ is a tree, and
2. $\mathcal{X} = \{X_i \mid i \in I\}$ is a family of subsets of V, one for each node of T, such that

(i) $\bigcup_{i \in I} X_i = V$,
(ii) for every edge $\{v, w\} \in E$*, there is an* $i \in I$ *with* $v \in X_i$ *and* $w \in X_i$*, and*
(iii) for all $i, j, k \in I$*, if* j *is on the path from* i *to* k *in* T*, then* $X_i \cap X_k \subseteq X_j$.

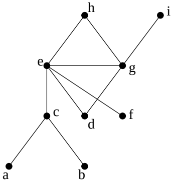
- This gives the following well-known definition.

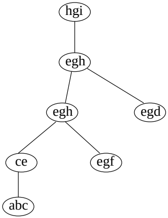
DEFINITION

The width of a tree decomposition $((I, F), \{X_i \mid i \in I\})$ is $\max_{i \in I} |X_i| - 1$. The treewidth of a graph G, denoted by $tw(G)$, is the minimum width over all possible tree decompositions of G.

- The following refers to any of these inductively defined graphs families. Notes that many commercial constructions of, for example chips are inductively defined.
    1. Find a bounded-width tree (path) decomposition of the input graph that exhibits the underlying tree (path) structure.
    2. Perform dynamic programming on this decomposition to solve the problem.

AN EXAMPLE FOR INDEPENDENT SET





$\emptyset$	a	b	c	ab	ac	bc	abc
0	1	1	1	2	-	-	-

BODLAENDER'S THEOREM

- The following theorem is shows that treewidth is FPT. Improves many earlier results showing this. The constant is about 2^{35k^3}.

THEOREM (BODLAENDER)

k-TREEWIDTH *is linear time FPT*

- Not practical because of large hidden O term.
- Unknown if there is a practical FPT treewidth algorithm
- Nevertheless approximation and algorithms specific to known decomps run well at least sometimes.

- Graphs of treewidth k can be parsed by operators
  1. Create a set of vertices labelled $\{1, \ldots, k+1\}$. $\emptyset$
  2. Join vertex labelled j to the one labelled j, join(i, j).
  3. Erase the label i but create a new vertex with label i disjoint from the current graph. Push(i).
  4. Glue two labelled graphs at the labels, making any multiple edges single. $\oplus$.
- Hence a graph of bounded treewidth is simply a sequence of parse operations in a formal language.
- These can be processed by tree automata.

Courcelle's and Seese's Theorems

Theorem (Courcelle 1990)

The model-checking problem for MSO restricted to graphs of bounded treewidth is linear-time fixed-parameter tractable. That is, model checking can be done in (linear) time.

Detleef Seese has proved a converse to Courcelle's theorem.

Theorem (Seese 1991)

Suppose that $\mathcal{F}$ is any family of graphs for which the model-checking problem for MSO is decidable, then there is a number n such that, for all $G \in \mathcal{F}$, the treewidth of G is less than n.

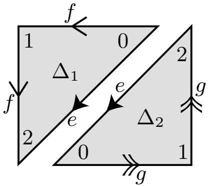
GRAPHS?

- The treewidth methodology has been applied in lots of areas. Finite model theory, braids, knots (Makowski, Rotics etc), matroids,
- Can be used for hard problems like counting.
- No general metheorem.
- Generalized to other parse notions like CLIQUEWIDTH d-DEGERACY, the idea being that formal languages and automata are basic.
- Here I will look at a new application.

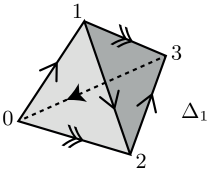
Triangulations

- 1. A *d-dimensional triangulation* consists of a collection of d-dimensional simplices $\Delta_1, \ldots, \Delta_n$ some or all of whose facets (ie. $d-1$-dimensional faces) are affinely identified.
  2. Each facet F may only be identified with at most one other facet F' of a d-simplex. (This could be the same d=simplex but not the same facet.)
- There are $\binom{d+1}{i+1}$ many i-faces of the d-simplices (where a 0-face is a vertex, a 1-face an edge etc).
- A *d-manifold triangulation* is simply a d-dimensional triangulation whose underlying topological space is a d-manifold when using the quotient topology.
- There are $\binom{d+1}{i+1}$ many i-faces of the d-simplices (where a 0-face is a vertex, a 1-face an edge etc).
- The idea is to have a metatheorem for such triangulations.

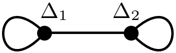
An example-Klein Bottle



(a) A Klein bottle $\mathcal{K}$



(b) A one-tetrahedron solid torus



(c) The dual graph $\mathcal{D}(\mathcal{K})$

FIGURE: Examples of triangulations

$$\Delta_1 : 02 \longleftrightarrow \Delta_2 : 20, \quad \Delta_1 : 01 \longleftrightarrow \Delta_1 : 12, \quad \Delta_2 : 01 \longleftrightarrow \Delta_2 : 12.$$

- The resulting triangulation has one vertex (since all three vertices of Δ_1 and all three vertices of Δ_2 become identified together), and three edges (labelled e, f, g in the diagram).
- The dual graph $\mathcal{D}(T)$ of a triangulation T is the multigraph whose nodes correspond to simplices and whose edges correspond to identified pairs of facets.
  Above we have the dual graph of the Klein bottle.

The language

- Standard boolean operations.
- for each $i \in [0, d]$ variables for i-faces of a triangulation, and ones for sets of them.
- for each $i \in [0, d]$, and each ordered sequence $\pi_0, \ldots, \pi_i$ of distinct integers from $\{0, \ldots, d\}$, a subface relation $\leq_{\pi_0 \ldots \pi_i}$.
- The interpretation $f \leq_{\pi_0 \ldots \pi_i} s$ indicates that f is a subface of the triangulation, s a simplex of the triangulation, and f is identified with th subface of s formed by the simplicial vertices $\pi_0, \ldots, \pi_i$ in the way that vertices of the face $0, \ldots, i$ of the face f correspond to vertices $\pi_0, \ldots, \pi_i$ of the simplex s.

ORIENTABILITY

Recall: 2-dimensional triangulation is orientable if and only if each triangle can be assigned an orientation (clockwise or anticlockwise) so that adjacent triangles have compatible orientations, as illustrated in Figure 2 below.

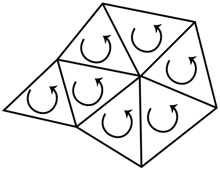


FIGURE: Adjacent triangles have compatible orientations

Theorem (Burton and D)

For any fixed dimension d, let K denote the class of d-dimensional triangulations whose dual graphs have universally bounded treewidth. Then for a fixed MSO ϕ, and triangulation $T \in K$, we can decide if T satisfies ϕ in linear time.

- We also show that the optimization problems can be solved.
- The method is one of reduction to the graph version.
- We apply this to various problems on these objects. including things called Taut Angle Structure, Discrete Morse Matching, Turaev-Viro Invariants, of which I know nothing.
- Ben has implemented and these methods give the currently best performing algorithms!

- Many Thanks
- I feel proud to have known Vaughan, and the world is a lesser place with his passing.