Structural Properties of D.C.E. Degrees and Presentations of C.E. Reals

by

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To my wife Caixia

and my daughter Jiahui

Abstract

In this thesis, we are mainly concerned with the structural properties of the d.c.e. degrees and the distribution of the simple reals among the c.e. degrees.

In chapters 2 and 3, we study the relationship between the isolation phenomenon and the jump operator. We prove in chapter 2 that there is a high d.c.e. degree $\mathbf{d}$ isolated by a low$_2$ degree $\mathbf{a}$. We improve this result in chapter 3 by showing that the isolating degree $\mathbf{a}$ can be low.

Chapters 4 and 5 are devoted to the study of the pseudo-isolation in the d.c.e. degrees. We prove that pseudo-isolated d.c.e. degrees are dense in the c.e. degrees, and that there is a high d.c.e. degree pseudo-isolated by a low d.c.e. degree.

In chapter 6, we prove that there are two d.c.e. degrees between which there is exactly one c.e. degree and that there are two c.e. degree $\mathbf{a}_1 < \mathbf{a}_2$ and a d.c.e. degree $\mathbf{d} \in (\mathbf{a}_1, \mathbf{a}_2)$ such that the intervals $(\mathbf{a}_1, \mathbf{d})$ and $(\mathbf{d}, \mathbf{a}_2)$ contains no c.e. degrees.

In chapter 7, we construct an isolation pair $(\mathbf{a}, \mathbf{d})$ and a c.e. degree $\mathbf{c}$ such that $\mathbf{c} \cup \mathbf{d} = \mathbf{0}', \mathbf{c} \cap \mathbf{a} = \mathbf{0}$. This result gives an alternative proof of Downey's diamond embedding into the d.c.e. degrees.

Chapter 8 is a contribution to the study of the distribution of the simple reals in the c.e. degrees. We show that there is a noncomputable c.e. degree bounding no noncomputable the simple reals. Thus, simple reals are not dense in the structure of the computably enumerable degrees.

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Table of contents

CHAPTER 1

INTRODUCTION

Computability theory can be divided into three parts: the lattice of computably enumerable sets, degree theory and applied computability theory. The main part of this thesis (chapters 2-7) is concerned with degree theory. Generally speaking, the aim of degree theory is to understand the degree structures that arise naturally from various notions of relative computability. Our focuses here are $(\mathbf{R}, \leq)$ and $(\mathbf{D}_2, \leq)$, and the interactions between them, where $\mathbf{R}$ is the class of Turing degrees of computably enumerable (c.e.) sets, and $\mathbf{D}_2$ is the class of Turing degrees of 2-c.e. sets. The last chapter of this thesis is a contribution to computable analysis, where prefix-free presentations of c.e. reals are considered.

A set $A \subseteq \omega$ is computably enumerable (c.e. for short) if A can be listed effectively. Another way of defining c.e. sets is by computable approximations to their characteristic functions: A is c.e. iff there is a computable function f such that for all x:

(1) $f(x, 0) = 0$,

(2) $|\{s : f(x, s) \neq f(x, s+1)| \leq 1$,

(3) $A(x) = \lim_s f(x, s)$.

The intuition is that we can change our guesses about the membership of x in A at most once. An obvious variation of this definition is to allow more than one change: A is 2-c.e. iff it satisfies the same definition where (2) is replaced by

(2′) $|\{s : f(x, s) \neq f(x, s+1)| \leq 2$.

It is easy to see that A is 2-c.e. iff it is the difference of two c.e. sets $A_1 - A_2$, and therefore, 2-c.e. sets are also called d.c.e. sets. Similarly, we can define n-c.e.

sets by allowing at most n changes for any x. A is an ω-c.e. set if and only if there is a computable function bounding the number of such changes. Thus, the n-c.e. hierarchy of sets is a natural extension of the c.e. sets in that it classifies the set of all finite Boolean combinations of computably enumerable sets and their complements. As the c.e. sets are the sets many-one reducible to a given creative set K, n-c.e. sets are those $\leq_{\mathrm{btt}}$ (*bounded truth-table* reducible to) K, and ω-c.e. sets are just those $\leq_{\mathrm{tt}}$ (*truth-table* reducible to) K. Thus, there are Δ_2^0 sets which are not ω-c.e..

Interest in the d.c.e. degrees stems from:

(1) the light they shed on the c.e. degrees,

(2) understanding the bound of techniques used between the Δ_2^0 degrees and the c.e. degrees, and

(3) for their own sake as the first level of the Ershov hierarchy.

The d.c.e. degrees were first studied by Cooper [15] and Lachlan who showed that there is a *properly* d.c.e degree (a d.c.e. degree containing no c.e. sets), and that every noncomputable d.c.e. degree bounds a noncomputable c.e. degree, respectively. Thus, $\mathbf{R} \subset \mathbf{D}_2$. Lachlan's observation shows that downwards density holds in $\mathbf{D}_2$. Hence, as Jockusch noticed, the d.c.e. degrees are not complemented. This indicates some differences between $\mathbf{D}_2$ and the Δ_2^0 degrees, and at the same time, shows some similarities between $\mathbf{D}_2$ and $\mathbf{R}$. The elementary theories of $(\mathbf{R}, \leq, 0, 1)$ and $(\mathbf{D}_2, \leq, 0, 1)$ were shown to be undecidable by Harrington and Shelah [39] and Slaman (unpublished), respectively. $\mathbf{D}_2$ and $\mathbf{R}$ also share the splitting property:

Theorem 1.1 *(1) (Sacks [65]) Any noncomputable c.e. degree can be nontrivially split in* $\mathbf{R}$.

(2) (Cooper [17]) Any noncomputable d.c.e. degree can be nontrivially split in $\mathbf{D}_2$.

However, as proved by Cooper and Li in [20], there is no uniform proof in the d.c.e. degrees which can subsume (1) and (2) above simultaneously. Recently, in

[46], Ishmukhametov and Wu started the investigation of splittings of c.e. sets in the class of d.c.e. sets. This task is non-trivial because d.c.e. splittings do not have the same nice degree-theoretic properties as c.e. ones (we cannot have $D_0, D_1 \leq_{\mathrm{T}} A$ automatically in this case, where D_0, D_1 is a split of a c.e. set A). So far, most splitting theorems about d.c.e. splittings are degree theoretical rather than set theoretical. We prove in [46] that any noncomputable c.e. set has a properly d.c.e. splitting.

Theorem 1.2 *For any noncomputable c.e. set A, there are low, properly d.c.e. sets D_0, D_1 such that $A = D_0 \sqcup D_1$, and $A \equiv_{\mathrm{T}} D_0 \oplus D_1$.*

The first two structural differences between $(\mathbf{D}_2, \leq, 0, 1)$ and $(\mathbf{R}, \leq, 0, 1)$ were provided by Arslanov [1, 2] and Downey [24].

Theorem 1.3 *(Arslanov [1, 2]) For any noncomputable d.c.e. degree $\mathbf{d}$, there is some incomplete d.c.e. degree $\mathbf{c}$ such that $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$.*

Theorem 1.4 *(Downey [24]) There are two incomparable d.c.e. degrees $\mathbf{c}, \mathbf{d}$ such that $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$, $\mathbf{c} \cap \mathbf{d} = \mathbf{0}$.*

Theorem 1.3 shows that $\mathbf{0}'$ does not have the anticupping property in the d.c.e. degrees. In $\mathbf{R}$, as proved by Cooper and Yates (unpublished), $\mathbf{0}'$ does have this property. This is a Σ_3 deference between $(\mathbf{R}, \leq, 0, 1)$ and $(\mathbf{D}_2, \leq, 0, 1)$. Cooper, Lempp and Watson [19] extended Theorem 1.3 to show that each noncomputable d.c.e. degree cups to any high c.e. degree above it by a low d.c.e. degree.

Theorem 1.4 says that the diamond lattice can be embedded into the d.c.e. degrees with 0 and 1 preserved. This contrasts with Lachlan's nondiamond theorem for the c.e. degrees, and provides a difference between $(\mathbf{R}, \leq, 0, 1)$ and $(\mathbf{D}_2, \leq, 0, 1)$ at the Σ_2 level. Jiang [47] proved that Theorem 1.4 can be extended by replacing $\mathbf{0}'$ with any high c.e. degree.

Recently, Li and Yi [57] unified Theorem 1.3 and Theorem 1.4 as follows:

Theorem 1.5 *(Li and Yi [57]) There are two incomparable d.c.e. degrees $\mathbf{d}_1, \mathbf{d}_2$ such that any noncomputable c.e. degree cups one of them to $\mathbf{0}'$.*

Theorem 1.3 follows from Theorem 1.5 immediately. To prove Theorem 1.4, let $\mathbf{d}_1, \mathbf{d}_2$ as indicated in Theorem 1.5. Then $\mathbf{d}_1 \cup \mathbf{d}_2 = \mathbf{0}'$. Let $\mathbf{a}$ be any d.c.e. degree below $\mathbf{d}_1$ and $\mathbf{d}_2$. By Lachlan's observation mentioned above, we can assume that $\mathbf{a}$ is c.e.. Then, $\mathbf{a} = \mathbf{0}$ because otherwise, by Theorem 1.5, $\mathbf{a}$ cups one of $\mathbf{d}_1$ and $\mathbf{d}_2$ to $\mathbf{0}'$, which is impossible by our choice of $\mathbf{a}$.

The most striking difference between $(\mathbf{D}_2, \leq, 0, 1)$ and $(\mathbf{R}, \leq, 0, 1)$ is that $\mathbf{D}_2$ is not densely ordered:

Theorem 1.6 *(Cooper, Harrington, Lachlan, Lempp and Soare [18]) There is a maximal incomplete d.c.e. degree.*

This is another Σ_2-difference between $(\mathbf{R}, \leq, 0, 1)$ and $(\mathbf{D}_2, \leq, 0, 1)$. Recently, Arslanov, Cooper and Li [6] proved that for any d.c.e. degree $\mathbf{d}$, if $\mathbf{d}$ is low, then $\mathbf{0}'$ is splittable nontrivially above $\mathbf{d}$. As a consequence, there is no low maximal d.c.e. degree.

In contrast to the nondensity of the d.c.e. degrees, many weak density results have been shown to hold in $\mathbf{D}_2$. For example, Cooper showed in [16] that the low_2 d.c.e. degrees are dense, and Cooper, Lempp and Watson showed in [19] that the properly d.c.e. degrees are dense in the c.e. degrees. Cooper and Yi [21], and independently, Ishmukhametov [42], showed that for any c.e. degree $\mathbf{a}$, and d.c.e. degree $\mathbf{d}$, with $\mathbf{a} < \mathbf{d}$, there is a d.c.e. degree $\mathbf{c}$ such that $\mathbf{a} < \mathbf{c} < \mathbf{d}$. Also in [21], Cooper and Yi showed that $\mathbf{c}$ cannot be restricted to the c.e. degrees. The notion of an *isolated d.c.e. degree* was proposed by Cooper and Yi in this spirit.

Definition 1.1 *A d.c.e. degree* $\mathbf{d}$ *is isolated by a c.e. degree* $\mathbf{a}$*, if* $\mathbf{a} < \mathbf{d}$ *and any c.e. degree* $\mathbf{c}$ *below* $\mathbf{d}$ *is also below* $\mathbf{a}$. $(\mathbf{a}, \mathbf{d})$ *is called an isolation pair.* $\mathbf{d}$ *is isolated if there is a c.e. degree* $\mathbf{a}$ *such that* $\mathbf{a}$ *isolates* $\mathbf{d}$*, and nonisolated if there is no such c.e. degree.*

It is easy to see that $\mathbf{d}$ is isolated by $\mathbf{a}$ if and only if there is no c.e. degree between $\mathbf{a}$ and $\mathbf{d}$. Cooper and Yi [21] proved that both isoalted and nonisolated degrees exist. The existence of isolated degrees can also be obtained from Kaddah's infima theorem:

Theorem 1.7 *(Kaddah [49]) Each low c.e. degree is branching in the d.c.e. degrees.*

Let $\mathbf{a}$ be any low c.e. degree. By Theorem 1.7, there are two d.c.e. degrees $\mathbf{d}_1$ and $\mathbf{d}_2$ such that $\mathbf{a} = \mathbf{d}_1 \cap \mathbf{d}_2$. If $\mathbf{a}$ is nonbranching in the c.e. degrees, then there is no c.e. degree between $\mathbf{a}$ and $\mathbf{d}_1$, or between $\mathbf{a}$ and $\mathbf{d}_2$. That is, $\mathbf{a}$ isolates $\mathbf{d}_1$ or $\mathbf{d}_2$.

After Cooper and Yi's paper [21], the isolated degrees and the nonisolated degrees have been the objects of a significant amount of study, both for themselves and for their relationship of relative definability in general. Now it is well-known that both the isolated (and hence, the isolating degrees) and the nonisolated d.c.e. degrees are dense in the c.e. degrees.

Theorem 1.8 *(1) (Ding and Qian [23]; LaForte [54]) The isolated d.c.e. degrees are dense in the c.e. degrees.*

(2) (Arslanov, Lempp and Shore [9]) The nonisolated d.c.e. degrees are dense in the c.e. degrees.

LaForte's proof is based on the fact that d.c.e. degrees are 2-*c.e.a.* degrees, where the *c.e.a.*-hierarchy was proposed by Jockusch and Shore in [48]. A set A is 1-*c.e.a* if A is c.e., and $(n+1)$-*c.e.a.* if A is c.e. in and above some n-*c.e.a* set. Recently, Arslanov, LaForte and Slaman proved:

Theorem 1.9 *(Arslanov, LaForte and Slaman [7])* $\mathbf{D}_2 = (2\text{-}c.e.a.) \cap \mathbf{D}_\omega$*, where* $\mathbf{D}_\omega$ *is the class of Turing degrees of the* ω*-c.e. sets.*

Theorem 1.9 says that the class of the d.c.e. degrees is just the intersection of the class of the 2-*c.e.a.* degrees with that of the ω-c.e. degrees.

Fix D as a d.c.e. set. For any z, if z enters D, then let $s(z)$ denote this stage; otherwise, let $s(z)$ be undefined. Without loss of generality, we suppose that at each stage, only one number can enter or leave D. Thus, the function $z \mapsto s(z)$ is one to one. Define $A[D] = \{s(z) : s(z)\downarrow \ \& \ z \notin D\}$. Clearly, $A[D] \leq_T D$ and D is *c.e.a.* in $A[D]$, and as proved in Ishmukhametov [43], the degree of $A[D]$ is the least one in which D is computably enumerable. LaForte [54] constructed a d.c.e. set D which is isolated by $A[D]$. In this case, the degree of $A[D]$ is the unique c.e. degree below the degree of D in which D is computably enumerable. Ishmukhametov [43, 44] provide a comprehensive investigation of such topics.

In [21], Cooper and Yi also asked whether each c.e. degree isolates some d.c.e. degree. Arslanov, Lempp and Shore [9] provides a negative answer:

Theorem 1.10 *(Arslanov, Lempp and Shore [9]) There is a c.e. degree isolating no d.c.e. degree. Such c.e. degrees are called nonisolating degrees.*

Arslanov, Lempp and Shore [9] also proved that nonisolating degrees are downwards dense in the c.e. degrees, and realize each jump class.

Our thesis continues the study of isolation phenomena in the d.c.e. degrees. In the rest of this chapter, we outline the contents of this thesis, and highlight the main results.

In chapters 2 and 3, we investigate the relationship between isolation and the high/low hierarchy, where the high/low hierarchy is defined by

$$\mathrm{high}_n = \{\mathbf{a} \leq \mathbf{0}' : \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\}, \qquad \mathrm{low}_n = \{\mathbf{a} \leq \mathbf{0}' : \mathbf{a}^{(n)} = \mathbf{0}^{(n)}\}$$

for each $n \geq 1$. If $n = 1$, we usually omit the subscript.

We first prove in chapter 2 that there is a high d.c.e. degree isolated by a low_2 c.e. degree. This work is published in "Mathematical Logic Quarterly", vol. 47, 2001.

Theorem 1 *There is an isolation pair* $(\mathbf{a}, \mathbf{d})$ *such that* $\mathbf{a}$ *is* low_2 *and* $\mathbf{d}$ *is high.*

Thus, the isolated degrees and the corresponding isolating c.e. degrees can be quite far apart in the sense of high/low hierarchy. The proof of Theorem 1 provides a standard approach to the construction of isolated degrees, on which other chapters are based. In chapter 3, we improve Theorem 1 by showing that $\mathbf{a}$ can be low, not merely low_2. It is joint work with Ishmukhametov, and is published in "Archive for Mathematical Logic", Vol. 41 (2002), 259-266.

Theorem 2 (Ishmukhametov and Wu) *There is an isolation pair* $(\mathbf{a}, \mathbf{d})$ *such that* $\mathbf{a}$ *is low and* $\mathbf{d}$ *is high.*

In chapter 4, we define the *pseudo-isolated* degrees as nonisolated d.c.e. degrees with all c.e. predecessors bounded by a lesser d.c.e. degree. The existence of the pseudo-isolated degrees refutes the claim that nonisolated degrees are minimal upper bounds for the c.e. degrees below them. Using standard techniques, we prove in chapter 4 that the pseudo-isolated degrees are dense in the c.e. degrees.

Theorem 3 *Given two c.e. degrees,* $\mathbf{u} < \mathbf{v}$*, there is a pseudo-isolated degree* $\mathbf{d}$ *such that* $\mathbf{u} < \mathbf{d} < \mathbf{v}$.

By applying the technique developed in chapter 3, we show in chapter 5 that there is a high nonisolated degree with all c.e. predecessors low. This work will appear in "Annals of Pure and Applied Logic".

Theorem 4 *There is a high nonisolated d.c.e. degree such that all c.e. degrees below it are bounded by a low d.c.e. degree.*

In chapter 6, we consider bi-isolation phenomena in the d.c.e. degrees.

Say that a d.c.e. degree is *isolated from above* by a c.e. degree $\mathbf{a}$ if any c.e. degree larger than $\mathbf{d}$ is also larger than $\mathbf{a}$. Thus, we have the natural definition of bi-isolating degrees: a c.e. degree $\mathbf{a}$ is a *bi-isolating degree* if there are d.c.e. degrees $\mathbf{d}_1$ and $\mathbf{d}_2$ such that $\mathbf{d}_1 < \mathbf{a} < \mathbf{d}_2$, $\mathbf{a}$ isolates $\mathbf{d}_2$ from below and isolates $\mathbf{d}_1$ from above. Note that in this case, $\mathbf{a}$ is the unique c.e. degree between $\mathbf{d}_1$ and $\mathbf{d}_2$. We have:

Theorem 5 *Each high isolating degree (from below) is a bi-isolating degree, and hence, the bi-isolating degrees are dense in the high c.e. degrees.*

A d.c.e. degree $\mathbf{d}$ is said to be *bi-isolated* if there are c.e. degrees $\mathbf{a}_1$ and $\mathbf{a}_2$ such that $\mathbf{a}_1 < \mathbf{d} < \mathbf{a}_2$, $\mathbf{a}_1$ isolates $\mathbf{d}$ from below and $\mathbf{a}_2$ isolates $\mathbf{d}$ from above. We will show that bi-isolated degrees exist, and the proof is an improvement of Arslanov's cupping theorem (Theorem 1.3).

Theorem 6 *For any noncomputable c.e. degree* $\mathbf{c}$*, there is an isolated degree* $\mathbf{d}$ *such that* $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$.

Indeed, bi-isolated degrees occur in each jump class because of the following improvement of Theorem 6:

Theorem 7 *For any noncomputable c.e. degree* $\mathbf{c}$*, and for any* $\mathbf{e}$ *c.e.a. in* $\mathbf{0}'$*, there is an isolated degree* $\mathbf{d}$ *such that* $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$*, and* $\mathbf{d}' = \mathbf{e}$.

Chapter 7 is devoted to the construction of an isolation pair $(\mathbf{a}, \mathbf{d})$ and a c.e. degree $\mathbf{c}$ such that $\mathbf{c}$ cups $\mathbf{d}$ to $\mathbf{0}'$ and caps $\mathbf{a}$ to $\mathbf{0}$.

Theorem 8 *There is an isolation pair* $(\mathbf{a}, \mathbf{d})$ *and a c.e. degree* $\mathbf{c}$ *such that* $\mathbf{c}$ *cups* $\mathbf{d}$ *to* $\mathbf{0}'$ *and caps* $\mathbf{a}$ *to* $\mathbf{0}$.

Thus, $\{\mathbf{0}, \mathbf{c}, \mathbf{d}, \mathbf{0}'\}$ is a diamond lattice embedding into the d.c.e. degrees (the proof can be found in Chapter 7), which was first proved to exist by Downey in [24]. Furthermore, by Harrington-Soare's continuity of capping degrees, Theorem 8 gives a new proof of an N_5 embedding into the d.c.e. degrees with 0 and 1 preserved. This work will be published in "Journal of Symbolic Logic".

Chapter 8 is devoted to the study of prefix-free presentations of computably enumerable (c.e., for short) reals. In [11], Calude et al. proved that a real α is c.e. if and only if there is an infinite, computably enumerable prefix-free set V such that $\alpha = \sum_{\sigma \in V} 2^{-|\sigma|}$. Following Downey and LaForte [31], we call V a prefix-free presentation of α. Each computably enumerable real has a computable presentation. Say that a c.e. real α is simple if each presentation of α is computable. One might conjecture that the more complicated a real is, the more complicated the presentations can be. Surprisingly, Downey and LaForte [31] provided a counterexample to this conjecture by showing that there is a simple real with its Turing degree high.

Theorem 1.11 *(Downey and LaForte [31]) There is a simple c.e. real* α *such that* $\alpha' \equiv_{\mathrm{T}} \emptyset''$.

We will prove that the conjecture can be locally true:

Theorem 9 There is a noncomputable c.e. set A such that A bounds no noncomputable simple reals. That is, for any noncomputable c.e. real $\alpha \in [0,1]$, if $\alpha \leq_{\mathrm{T}} A$, then α has a noncomputable presentation.

Thus, the simple reals are not dense in the structure of the computably enumerable degrees. This work was presented in *COCOON'01* and was published in "Lecture Notes in Computer Science", vol. 2108.

Our notation and terminology are standard and generally follow Calude [10], Li-Vitányi [58], Odifreddi [61] and Soare [71]. Capital letters are used to denote partial computable functionals, with associated uses denoted by the corresponding lowercase Greek letters. In addition, the notation $\Phi^X(x)[s]$ represents the value of the functional Φ on argument x at stage s. A parameter p defined to be big at stage s means that p is bigger than any number being used so far. Particularly, $p > s$.

CHAPTER 2

ISOLATING A HIGH D.C.E. DEGREE BY A LOW$_2$ C.E. DEGREE

2.1 Introduction

One would have the impression that if **d** is isolated by **a**, then **d** would be "close" to **a**, because there is no c.e. degree between **d** and **a**. In this chapter, we show that in terms of the jump operator, this is not true.

We prove that there is a high d.c.e. degree **d** with all c.e. predecessors bounded by a low$_2$ degree **a**. The isolation technique presented here is standard and is formulated in detail.

Theorem 1 *There is an isolation pair* $(\mathbf{a}, \mathbf{d})$ *such that* $\mathbf{a}$ *is low*$_2$ *and* $\mathbf{d}$ *is high.*

To prove Theorem 1, we construct a c.e. set A, a d.c.e. set D, and partial functionals Λ, Σ in stages, satisfying the following requirements:

$$\mathcal{H}_e: \ \lim_x \Lambda^{A\oplus D}(e,x) = Tot(e), \tag{2.1}$$

$$\mathcal{R}_e: \ \Phi_e^{A\oplus D} = W_e \Rightarrow \exists\Gamma_e(\Gamma_e^A = W_e), \tag{2.2}$$

$$\mathcal{N}_e: \ Tot^A(e) = \Sigma^{\emptyset''}(e), \tag{2.3}$$

where Tot is the Π^0_2-complete index set $\{e : \varphi_e \text{ total}\}$, $Tot^A = \{e : \Phi_e^A \text{ total}\}$, and $\{\langle \Phi_e, W_e\rangle : e \in \omega\}$ is a standard enumeration of all pairs $\langle \Phi, W\rangle$, Φ a partial computable functional, W a c.e. set.

The priority tree will have 4 types of nodes:

- β nodes for the sake of $\mathcal{H}_e$ with outcome $0 <_{\mathrm{L}} 1$.

- η nodes for the sake of $\mathcal{R}_e$ with outcome $0 <_{\mathrm{L}} d <_{\mathrm{L}} 1$.
- τ nodes for the sake of $\mathcal{N}_e$ with outcome $0 <_{\mathrm{L}} 1$.
- α nodes living below $\tau^\frown\langle 0\rangle$ are also devoted to $\mathcal{N}_e$ via subrequirements $\mathcal{S}_{e,i}$. Such an α will try to preserve the computation $\Phi_e^A(i)$. α has two outcomes $0 <_{\mathrm{L}} 1$.

2.2 Basic strategies

In this section, we describe strategies meeting the requirements listed above.

$\mathcal{H}$-strategies

The basic idea of the $\mathcal{H}$-strategies is to approximate Tot via $\Lambda^{A\oplus D}$ at the limit. We ensure that $\Lambda^{A\oplus D}$ is total, and for almost all e,

$$\varphi_e \text{ total} \iff \lim_x \Lambda^{A\oplus D}(e,x) = 1.$$

By Shoenfield's Limit Lemma, $\emptyset'' \leq_{\mathrm{T}} Tot \leq_{\mathrm{T}} (A \oplus D)'$, and so $A \oplus D$ is high.

Our construction will proceed on a finitely branching tree T. Let $[T]$ be the set of infinite paths through T. Fix e. The construction of T ensures that on each $f \in [T]$, there is a unique $\mathcal{H}_e$-strategy, and the length of all $\mathcal{H}_e$-strategies is uniformly bounded. Thus we can number all $\mathcal{H}_e$-strategies on T in an effective way in the lexicographical order, and use $k(\beta)$ to denote that β is the $k(\beta)$-th one in this order. All $\mathcal{H}_e$-strategies work jointly to define $\Lambda^{A\oplus D}(e,-)$.

Fix β as an $\mathcal{H}_e$-strategy. Define s_β as the last stage at which β is initialized. β is only concerned with the definition of $\Lambda^{A\oplus D}(e,x)$ for those $x > s_\beta$. If β is on the true path, then we will ensure that for all $x > s_\beta$, $\Lambda^{A\oplus D}(e,x)$ is well-defined and equals $Tot(e)$. At stage s, define

$$\ell(\beta,s) = \max\{x < s : \varphi_e(y)[s]\downarrow \text{ for all } y \leq x\},$$

$$m(\beta, s) = \max\{0, \ell(\beta, t) : t < s \text{ and } t \text{ is a } \beta\text{-stage}\}.$$

Say s is β-expansionary if s is a β-stage and $\ell(\beta, s) > m(\beta, s)$.

Fix x with $s_\beta < x < \ell(\beta, s)$. If $\Lambda^{A\oplus D}(e, x)$ is currently undefined, then define $\Lambda^{A\oplus D}(e, x) = 1$ with use -1. If $\Lambda^{A\oplus D}(e, x)$ is defined and $\lambda(e, x) \neq -1$, then let s_0 be the stage at which $\Lambda^{A\oplus D}(e, x)$ was first defined. Enumerate $\langle e, k(\beta), x, s_0\rangle$ into D, and redefine $\Lambda^{A\oplus D}(e, x) = 1$ with new use $\langle e, k(\beta), x, s_0\rangle$.

For x with $\ell(\beta, s) \leq x < s$, if $\Lambda^{A\oplus D}(e, x)\downarrow$, then $\Lambda^{A\oplus D}(e, x) = 0$. If $\Lambda^{A\oplus D}(e, x)$ is currently undefined, then define $\Lambda^{A\oplus D}(e, x) = 0$ with use $\lambda(e, x) = \langle e, k(\beta), x, s\rangle$. Here numbers $\langle e, j, x, s\rangle$ ($j \leq k(\beta)$) are reserved for redefining $\Lambda^{A\oplus D}(e, x)$ (as 1) later. Since all $\mathcal{H}_e$-strategies $>_{\mathrm{L}} \beta$ are initialized at stage s, they don't (re)define $\Lambda^{A\oplus D}(e, x)$ afterwards. This makes $\Lambda^{A\oplus D}(e, x)$ well-defined.

If there is some $y < x$ such that $\varphi_e(y)\uparrow$, then $\Lambda^{A\oplus D}(e, x)$ will be defined as 0 first, and no $\mathcal{H}_e$-strategy can redefine it as 1. Thus, for all $n \geq x$, $\Lambda^{A\oplus D}(e, n) = 0$, and so $\lim_n \Lambda^{A\oplus D}(e, n) = 0$.

If φ_e is total, let β be the $\mathcal{H}_e$-strategy on the true path. We will ensure that for all $x > s_\beta$, $\Lambda^{A\oplus D}(e, x) = 1$. If $\Lambda^{A\oplus D}(e, x)$ is defined as 1 with use -1, then we are done, because we don't need to redefine it later. Suppose that $\Lambda^{A\oplus D}(e, x)$ is defined as 0 first with use $\langle e, k(\beta'), x, s_0\rangle$, then $\beta <_{\mathrm{L}} \beta'$, $k(\beta) < k(\beta')$, and by enumerating the number $\langle e, k(\beta), x, s_0\rangle$ into D we can redefine $\Lambda^{A\oplus D}(e, x)$ as 1 later. By the choice of s_β, $\langle e, k(\beta), x, s_0\rangle$ remains in D, and hence, later removal actions executed by $\mathcal{R}$-strategies below β cannot change $\Lambda^{A\oplus D}(e, x)$ back to 0. So $\lim_x \Lambda^{A\oplus D}(e, x) = 1$.

β has two outcomes, $0 <_{\mathrm{L}} 1$, where $0,\ 1$ denote the cases of whether or not there are infinitely many β-expansionary stages. In the case that β has outcome 0, β may enumerate numbers into D infinitely often. Thus, for any $\mathcal{R}$-strategy η below β, computations considered are chosen to be η-correct.

$\mathcal{R}$-strategies

An $\mathcal{R}$-strategy, η, works in a standard way to isolate $A \oplus D$ from A. Define

$$\ell(\eta, s) = \max\{x < s : (\forall y < x)(\Phi^{A\oplus D}_{e(\eta)}(y)[s] \text{ is } \eta\text{-correct, and}$$

$$\Phi^{A\oplus D}_{e(\eta)}(y)[s] = W_{e(\eta),s}(y))\},$$

$$m(\eta, s) = \max\{0, \ell(\eta, t) : t < s \text{ and } t \text{ is an } \eta\text{-stage}\}.$$

Stage s is η-expansionary if $s = 0$ or s is an η-stage and $\ell(\eta, s) > m(\eta, s)$. At an η-expansionary stage s, for any $y < \ell(\eta, s)$, if $\Gamma^A_\eta(y)[s] \uparrow$, define $\Gamma^A_\eta(y) = W_{e(\eta),s}(y)$ with $\gamma_\eta(y)[s] = s$.

Suppose that $\gamma_\eta(y)$ is defined at stage s. During the construction, numbers enumerated into D, z say, may injure the current computation of $\Phi^{A\oplus D}_{e(\eta)}(x)$. It may lift the use $\varphi_{e(\eta)}(A \oplus D; x)$, but with $\Gamma^A_\eta(x)$ unchanged. This provides chances for $W_{e(\eta)}$ to change at x, and if so, Γ^A_η becomes incorrect at x. Suppose that x enters $W_{e(\eta)}$ at stage s'. Since $\Phi^{A\oplus D}_{e(\eta)}(x)$ was η-correct at stage s, numbers enumerated since stage s by higher priority $\mathcal{H}$-strategies are larger than the use $\varphi_{e(\eta)}(A\oplus D; x)[s]$, and so numbers destroying this computation are enumerated by $\mathcal{H}$-strategies with lower priority. By moving out all these numbers, we can recover $\Phi^{A\oplus D}_{e(\eta)}(x)$ to $\Phi^{A\oplus D}_{e(\eta)}(x)[s]$. This action creates an inequality between $\Phi^{A\oplus D}_{e(\eta)}$ and $W_{e(\eta)}$ at x. By preserving this inequality, due to the c.e.-ness of $W_{e(\eta)}$, x is kept in $W_{e(\eta)}$, and hence $\mathcal{R}_{e(\eta)}$ is satisfied. We refer to this method as the disagreement strategy. On the other hand, if $\Phi^{A\oplus D}_{e(\eta)}$ is total and $\Phi^{A\oplus D}_{e(\eta)} = W_{e(\eta)}$, we ensure that Γ^A_η is totally defined and computes $W_{e(\eta)}$ correctly.

Now consider those $\mathcal{R}$-strategies η' above η with $\eta'^\frown\langle 0\rangle \subseteq \eta$. Suppose that z enters D before $\Gamma^A_{\eta'}(y)$ gets its definition, then as indicated above, $\gamma_{\eta'}(y)$ is defined larger than s. Since η's disagreement strategy takes z out, $\Phi^{A\oplus D}_{e(\eta')}(y)$ may be destroyed, and it is possible that before the next η'-expansionary stage, y enters $W_{e(\eta')}$, and so $\Gamma^A_{\eta'}$ becomes incorrect at y. To avoid this, whenever η performs its disagreement strategy, η will put s into A to undefine $\Gamma^A_{\eta'}(y)$. Such an enumeration causes no problem to η, because $s > \varphi_{e(\eta)}(A \oplus D; x)[s]$. As to η', at the next η'-expansionary stage, s' say, $\Gamma^A_{\eta'}(y)$ gets redefined again. Without loss of generality, suppose that $A_{s'} \upharpoonright (\varphi_{e(\eta')}(A \oplus D; x)[s']) = A \upharpoonright (\varphi_{e(\eta')}(A \oplus D; x)[s'])$, then $\Gamma^A_{\eta'}(y)[s']$

can only be undefined by those $\mathcal{R}$-strategies between η and η', and therefore, this process can repeat only finitely often, so $\Gamma^A_{\eta'}(y)\downarrow$.

η has three outcomes $0, d, 1$ with $0 <_{\mathrm{L}} d <_{\mathrm{L}} 1$, where d denotes the fact that the disagreement strategy succeeds, and 0 and 1 for the cases of whether there are infinitely many η-expansionary stages or not, if the disagreement strategy has not succeeded.

$\mathcal{N}_e$-strategies

Let τ be an $\mathcal{N}_e$-strategy. Define

$$\ell(\tau, s) = \max\{x < s : (\forall y < x)[\Phi^A_e(y)[s]\downarrow]\},$$

$$m(\tau, s) = \max\{0, \ell(\tau, t) : t < s \text{ and } t \text{ is a } \tau\text{-stage}\}.$$

Say a stage s is τ-expansionary, if $s = 0$ or s is a τ-stage and $\ell(\tau, s) > m(\tau, s)$. τ has two outcomes $0, 1$ with $0 <_{\mathrm{L}} 1$, according to whether there are infinitely many τ-expansionary stages or not.

To make A low$_2$, basically, we need to argue that if $\ell(\tau, s) \to \infty$, then Φ^A_e is total. Hence, if outcome 0 is true, we need to preserve all A-uses of the computations involved. Namely, we need to preserve $\Phi^A_e(i)$ for all i. If we can do so, then on the true path f, we can read out the totality of Φ^A_e. By the fact that f is computable in $\emptyset''$ (as in the standard infinite injury argument), we have that A is low$_2$.

However, this task is too complicated to be achieved at a single node. The solution to this problem is to split this strategy into infinitely many substrategies, $\mathcal{S}_{e,i}$, $i \in \omega$, each of which preserves only one computation $\Phi^A_e(i)$. $\mathcal{N}_e$ is satisfied if for each i, $\mathcal{S}_{e,i}$ is satisfied.

$$\mathcal{S}_{e,i}: \ \Phi^A_e(i)\downarrow. \tag{2.4}$$

Thus, during the construction, τ does no action and so Σ is defined implicitly.

$\mathcal{S}_{e,i}$-strategies

An $\mathcal{S}_{e,i}$-strategy, α say, attempts to preserve the computation $\Phi_e^A(i)$. Let $\tau(\alpha)$ be the $\mathcal{N}_e$-strategy to which α is devoted. Note that here, we assume that α is below the infinite outcome of $\tau(\alpha)$, and $\tau(\alpha)$ guesses that $\Phi_e^A(i)$ converges. What we need to do is to preserve this computation forever via α's action (initializing all strategies with lower priority). Suppose α finds that $\Phi_e^A(i)\downarrow$ at stage s and α cannot be initialized afterwards, then after stage s, no $\mathcal{R}$-strategy with higher priority can execute the disagreement strategy. Also since all lower priority $\mathcal{R}$-strategies are initialized, their actions cannot later put numbers less than s into A. Thus, $A_s \upharpoonright s = A \upharpoonright s$, and $\Phi_e^A(i)[s]$ is preserved. So $\mathcal{S}_{e,i}$ is satisfied.

2.3 Construction

We define the priority tree, T, on which we will work, before giving the full construction. First, we assign the priority to the requirements as follows:

$$\mathcal{R}_0 < \mathcal{H}_0 < \mathcal{N}_0 < \mathcal{S}_{0,0} < \mathcal{R}_1 < \mathcal{H}_1 < \mathcal{N}_1 < \mathcal{S}_{0,1} < \mathcal{S}_{1,0} < \mathcal{S}_{1,1} < \mathcal{R}_2$$

$$< \cdots < \mathcal{R}_n < \mathcal{H}_n < \mathcal{N}_n < \mathcal{S}_{0,n} < \cdots < \mathcal{S}_{n-1,n} < \mathcal{S}_{n,0} < \cdots < \mathcal{S}_{n,n}$$

$$< \mathcal{R}_{n+1} < \cdots,$$

where for any $\mathcal{X}, \mathcal{Y}$, if $\mathcal{X} < \mathcal{Y}$, then $\mathcal{X}$ has higher priority than $\mathcal{Y}$.

The priority tree is defined by induction.

Definition 2.1 *(Satisfaction of strategies) Given $\xi \in T$,*

(1) $\mathcal{H}_e$ is satisfied at ξ, if there is an $\mathcal{H}_e$-strategy β such that $\beta \subset \xi$;

(2) $\mathcal{R}_e$ is satisfied at ξ, if there is an $\mathcal{R}_e$-strategy η such that $\eta \subset \xi$;

(3) $\mathcal{N}_e$ is satisfied at ξ, if there is an $\mathcal{N}_e$-strategy τ such that $\tau^\frown\langle 1\rangle \subseteq \xi$; $\mathcal{N}_e$ is active at ξ, if there is an $\mathcal{N}_e$-strategy τ such that $\tau \subset \tau^\frown\langle 0\rangle \subseteq \xi$. In this case, $\mathcal{N}_e$ is said to be active at ξ via τ;

(4) $\mathcal{S}_{e,i}$ *is satisfied at* ξ*, if either* $\mathcal{N}_e$ *is satisfied at* ξ*, or* $\mathcal{N}_e$ *is active at* ξ *via* τ*, and there is an* $\mathcal{S}_{e,i}$*-strategy* α *such that* $\tau^\frown\langle 0\rangle \subseteq \alpha \subset \xi$*.*

Definition 2.2 *(Construction of* T*)*

(1) Let λ*, the empty node, be an* $\mathcal{R}_0$*-strategy;*

(2) The immediate successors of a node are the possible outcomes of the corresponding strategy;

(3) For $\xi \in T$*,* ξ *works for the highest priority requirement which has not been satisfied and is not active at* ξ*.*

In the following, for a node, β say, we refer to $e(\beta)$ as the index of the $\mathcal{H}$-strategy to which β is devoted. For other strategies and nodes, we use the same notation.

Construction

Stage 0: Initialize all nodes first and let $A = D = \emptyset$ and let $\Lambda^{A\oplus D}$ be totally undefined.

Stage $s > 0$:

Step 1. We define a partial function δ_s of length $\leq s$ by substages. s is a ξ-stage if $\xi \subseteq \delta_s$.

Substage 0: Define $\delta_s(0) = \lambda$, the root node.

Substage $t > 0$. Given $\xi = \delta_s \restriction t$. Initialize all nodes $>_{\mathrm{L}} \xi$. If $t = s$, go to step 2. Otherwise, let s_ξ be the largest stage $\leq s$ at which ξ is initialized, take action for ξ, and determine ξ's outcome $\delta_s(t)$. There are four cases.

Case 1. $\xi = \beta$ is an $\mathcal{H}$-strategy.

1. If s is β-expansionary and $\ell(\beta, s) > s_\beta$, then define $\delta_s(t) = 0$.

2. Otherwise, define $\delta_s(t) = 1$.

Case 2. $\xi = \eta$ is an $\mathcal{R}$-strategy.

1. If η executed the disagreement strategy after stage s_η, then define $\delta_s(t) = d$.

2. If s is an η-expansionary stage, and Γ^A_η is correct, then define $\delta_s(t) = 0$.

3. If s is an η-expansionary stage, and there is some y s.t. $\Gamma^A_\eta(y)[s] \downarrow \neq W_s(y)$, then let y be the least one, and s_y be the stage at which $\Gamma^A_\eta(y)[s]$ was defined. Execute the disagreement strategy as follows:

   - remove all numbers associated with lower priority strategies which have entered D since stage s_y;
   - enumerate s_y into A and declare those $\Gamma^A(-)$ defined after stage s_y undefined;
   - define $\delta_s = \eta$, initialize all $\zeta > \eta$, perform the action of step 2, and go to the next stage.

4. Otherwise, define $\delta_s(t) = 1$.

Case 3. $\xi = \tau$ is an $\mathcal{N}$-strategy.

1. If s is a τ-expansionary stage, then define $\delta_s(t) = 0$.

2. Otherwise, define $\delta_s(t) = 1$.

Case 4. $\xi = \alpha$ is an $\mathcal{S}$-strategy.

1. If α is satisfied, then define $\delta_s(t) = 0$.

2. If $\Phi_{e(\alpha)}(i(\alpha))[s] \uparrow$, then define $\delta_s = \alpha$, initialize all $\xi > \alpha$, and go to step 2.

3. If $\Phi_{e(\alpha)}(i(\alpha))[s] \downarrow$, and α is not satisfied, then declare that α is satisfied. Define $\delta_s = \alpha$, initialize all $\xi > \alpha$, and go to step 2.

Step 2. For those $\mathcal{H}$-strategies β with $\beta^\frown\langle 0\rangle \subseteq \delta_s$,

1. if $x < \ell(\beta, s)$, either $\Lambda^{A\oplus D}(e(\beta), x) \downarrow$ with use -1, or $x < m(\beta, s)$, do nothing.

2. if $x < \ell(\beta, s)$ and $x \geq m(\beta, s)$, $\Lambda^{A\oplus D}(e(\beta), x) \uparrow$, then define $\Lambda^{A\oplus D}(e(\beta), x)$ with use -1.

3. if $x < \ell(\beta, s)$, $x \geq m(\beta, s)$ and $\Lambda^{A\oplus D}(e(\beta), x) \downarrow$ with use $\neq -1$, then enumerate $\langle e(\beta), k(\beta), x, s_0\rangle$ into D, where s_0 is the stage at which $\Lambda^{A\oplus D}(e(\beta), x)$ is defined. Redefine $\Lambda^{A\oplus D}(e(\beta), x)$ as 1 with use $\langle e(\beta), k(\beta), x, s_0\rangle$.

4. if $x \geq \ell(\beta, s)$, $x < s$, and $\Lambda^{A\oplus D}(e(\beta), x) \downarrow$, then redefine $\Lambda^{A\oplus D}(e(\beta), x)$ as 0 with use unchanged.

5. if $x \geq \ell(\beta, s)$, $x < s$, and $\Lambda^{A\oplus D}(e(\beta), x) \uparrow$, define $\Lambda^{A\oplus D}(e(\beta), x)$ as 0 with use $\langle e(\beta), k(\beta), x, s\rangle$.

Step 3. For $\mathcal{R}$-nodes, η say, with $\eta^\frown\langle 0\rangle \subseteq \delta_s$, if $y < \ell(\eta, s)$, and $\Gamma^A_\eta(y) \uparrow$, define $\Gamma^A_\eta(y)[s] = W_{e(\eta),s}(y)$ with use $\gamma_\eta(y)[s] = s$. Go the the next stage.

2.4 Verification

Let $f = \liminf_s \delta_s$. We call f the true path of the construction, and we will show that f is infinite.

Lemma 2.4.1 *For any node ξ on f,*

(1) ξ is initialized only finitely often;

(2) ξ will be visited infinitely often;

(3) If ξ is an $\mathcal{R}$, or $\mathcal{S}$-strategy, then ξ acts only finitely often;

Proof. We prove the lemma by induction.

For the root node λ, it can only be initialized at stage 0, and is visited at each stage. (1), (2) hold. By the construction of T, λ is an $\mathcal{R}_0$-strategy. So for (3), without loss of generality, suppose that λ does execute the disagreement strategy in the construction, then λ will set up a restraint preserving this inequality and keep $\mathcal{R}_0$ satisfied. After this, λ never acts again and (3) holds.

Now suppose that the lemma holds for all $\xi' \subset \xi$. Let ξ^- be the longest $\xi' \subset \xi$, and s_0 be the least stage after which no $\xi' \subset \xi$ can be initialized. By the induction hypothesis, s_0 exists. Since ξ is on f, we can assume that after stage $s_1 \geq s_0$, no $\zeta <_{\mathrm{L}} \xi$ can be accessible anymore.

Case 1. $\xi^- = \beta$ is an $\mathcal{H}$-strategy.

By the choice of s_0 and s_1, and the fact that an $\mathcal{H}$-strategy never initializes other nodes, (1) holds. (2) also holds because ξ is on f, and β does no initialization during the construction. By the definition of the tree, ξ is an $\mathcal{N}$-strategy, so (3) holds.

Case 2 $\xi^- = \eta$ is an $\mathcal{R}$-strategy.

η has three possible outcomes, $0,\ d,\ 1$. Let $\mathcal{O}$ be the correct outcome of η. That is, $\xi = \eta^\frown\langle\mathcal{O}\rangle$.

The case for $\mathcal{O} = 1$ is trivial.

If $\mathcal{O} = d$, suppose that η executes the disagreement strategy at stage $s_2 > s_1$ say. By the choice of s_0, no higher priority strategy can initialize η, and so η wins via this inequality, η is satisfied and never acts later. So (1) holds. Note that after stage s_2, $\xi = \eta^\frown\langle d\rangle$ will be visited at any η-stage. So (2) holds.

If $\mathcal{O} = 0$, then clearly, η never acts after stage s_0, so (1) holds. By the fact that ξ is on f, infinitely often, when η is visited, ξ will be visited too, so (2) holds.

Now by the definition of T, ξ is an $\mathcal{H}$-strategy, and so (3) holds.

Case 3 $\xi^- = \tau$ is an $\mathcal{N}$-strategy.

Note that τ never acts during the construction. By the choice of s_1, ξ will never be initialized after stage s_1 and so (1) holds. If $\xi = \tau^\frown\langle 0\rangle$, then 0 is the correct outcome of τ and infinitely often when τ is visited, ξ will be also visited. If $\xi = \tau^\frown\langle 1\rangle$, then after stage s_1, each τ-stage is also a ξ-stage. In any case, (2) holds.

Here, ξ is either an $\mathcal{S}$-strategy, $\mathcal{S}_{e,i}$ say, or an $\mathcal{R}$-strategy, η say. In the latter case, $\xi = \tau^\frown\langle 1\rangle$.

If $\xi = \mathcal{S}_{e,i}$, let s_2 be the first stage at which ξ is visited after s_1, then ξ acts and initializes all lower priority strategies. Therefore, these strategies are prevented from enumerating numbers $< s$ into A. Thus, for any $s \geq s_2, \Phi_e^A(i)[s] = \Phi_e^A(i)[s_2]$, $\Phi_e^A(i)\downarrow$, ξ is satisfied. After s_2, ξ never acts again. So (3) holds.

If $\xi = \eta$, as in case 2, ξ can act at most once more after stage s_1. (3) holds.

Case 4 $\xi^- = \alpha$ is an $\mathcal{S}$-strategy.

Let $\tau(\alpha)$ be the node to which α is devoted. Then $\tau(\alpha)^\frown\langle 0\rangle$ is on f, and so $\xi = \alpha^\frown\langle 0\rangle$.

Following the same discussion given in case 3, we can see that α acts only finitely often. By the choice of s_1, (1) holds. Note that, after the last stage at which α acts, whenever α is visited, ξ is also visited, so (2) holds.

As to (3), ξ can be either an $\mathcal{S}$-strategy or an $\mathcal{R}$-strategy, according to the definition of T. By the same argument as in case 3, (3) holds. □

The next two lemmas follow directly from Lemma 2.4.1.

Lemma 2.4.2 *For any ξ on f, there is an outcome $\mathcal{O}$ of ξ such that $\xi^\frown\langle\mathcal{O}\rangle$ is on f. Therefore, $|f| = \infty$.* □

Lemma 2.4.3 *If an $\mathcal{S}_{e,i}$-strategy α is on f, then $\Phi_e^A(i)\downarrow$.* □

Remark: By Lemma 2.4.2, f is total. By the definition of f, $f \leq_{\mathrm{T}} \emptyset''$ because "ξ is on f" is a Δ_3^0-proposition. Now we can decide whether $e \in Tot^A$ by finding the $\mathcal{N}_e$-strategy τ on f. If $\tau^\frown\langle 0\rangle$ is on f, then $e \in Tot^A$, and if $\tau^\frown\langle 1\rangle$ is on f, then $e \notin Tot^A$. Thus, $Tot^A \leq_{\mathrm{T}} f \leq_{\mathrm{T}} \emptyset''$ and A is low$_2$.

Lemma 2.4.4 *For all e, $\lim_x \Lambda^{A\oplus D}(e,x) = Tot(e)$.*

Proof. Fix e. Let β be the $\mathcal{H}_e$-strategy on the true path, and also let s_β be the last stage at which β is initialized. s_β exists by Lemma 2.4.1.

First, suppose that $Tot(e) = 0$, so φ_e is not total. Let $s_1 \geq s_\beta$ be the largest β-expansionary stage. Then, after stage s_1, whenever β is visited, $\beta^\frown\langle 1\rangle$ is also visited, and hence $\beta^\frown\langle 1\rangle \subseteq f$. Note that for any stage $s > s_1$ and for any $\mathcal{H}_e$-strategies $\beta' \geq \beta$, it follows that $\ell(\beta', s) < s_1$ because otherwise, β would observe such expansions later, which contradicts our choice of s_1. Therefore, for any $x > \ell(\beta, s_1)$, once $\Lambda^{A\oplus D}(e, x)$ is defined as 0, it cannot be redefined as 1 later.

Now suppose that $Tot(e) = 1$, so φ_e is total, and consequently, there are infinitely many β-expansionary stages. By the fact that any β-expansionary stage is a $\beta^\frown\langle 0\rangle$-stage, $\beta^\frown\langle 0\rangle$ is on f.

Fix $x \geq s_\beta$. Then $\langle e, k(\beta), x, -\rangle$ enters D only after stage s_β. Let $s > s_\beta$ be the least β-expansionary stage such that $\ell(\beta, s) > x$. It is possible that some $\mathcal{H}_e$-strategy $\beta' >_{\mathrm{L}} \beta$ has already redefined $\Lambda^{A\oplus D}(e, x)$ as 1 by enumerating $\langle e, k(\beta'), x, -\rangle$ into D. However, there is no guarantee that this number remains in D forever. Suppose that $\langle e, k(\beta'), x, -\rangle$ is taken out later (β' is initialized in this case). This may recover $\Lambda^{A\oplus D}(e, x)$ back to 0, and the corresponding use $\lambda(e, x)$ changes back to the original one automatically. By the construction, at stage s, $\langle e, k(\beta), x, -\rangle$ is enumerated into D, and so we can redefine $\Lambda^{A\oplus D}(e, x)$ as 1. By the choice of s_β, $\langle e, k(\beta), x, -\rangle$ remains in D, and no other $\mathcal{H}$-strategies can change this use. So $\Lambda^{A\oplus D}(e, x) = 1$ is well-defined. In this sense, β is the unique $\mathcal{H}_e$-strategy ensuring that the $\mathcal{H}_e$-requirement is satisfied. $\square$

Lemma 2.4.5 *If η is an $\mathcal{R}$-strategy on f, then*

(1) If $\eta^\frown\langle 1\rangle \subseteq f$ or $\eta^\frown\langle d\rangle \subseteq f$, then $\Phi_{e(\eta)}^{A\oplus D} \neq W_{e(\eta)}$.

(2) If $\eta^\frown\langle 0\rangle \subseteq f$, and $\Phi_{e(\eta)}^{A\oplus D}$ is total, then Γ_η^A is total, and $\Gamma_\eta^A = W_{e(\eta)}$.

Proof. Recall that s_η is the last stage at which η is initialized.

(1) If $\eta^\frown\langle 1\rangle \subseteq f$, then there are only finitely many η-expansionary stages, so $\Phi_{e(\eta)}^{A\oplus D} \neq W_{e(\eta)}$. As we can see, after s_η, η never executes the disagreement strategy, because otherwise, d would be the correct outcome.

If $\eta^\frown\langle d\rangle \subseteq f$, then η is satisfied via a disagreement strategy, at stage $s > s_\eta$ at y say. Let $\Gamma_\eta^A(y)$ be defined at stage s_1. Then $s_\eta < s_1 < s$. We show that we return the computation $\Phi_{e(\eta)}^{A\oplus D}(y)$ to that of stage s_1. That is,

$$A_s\restriction(\varphi_{e(\eta)}(A\oplus D;y)[s_1]) = A_{s_1}\restriction(\varphi_{e(\eta)}(A\oplus D;y)[s_1]);$$

$$D_s\restriction(\varphi_{e(\eta)}(A\oplus D;y)[s_1]) = D_{s_1}\restriction(\varphi_{e(\eta)}(A\oplus D;y)[s_1]).$$

Since $\Phi_{e(\eta)}^{A\oplus D}(y)$ is η-correct at stage s_1, the removal action ensures that the D-part is recovered. As to the A-part, since $\Gamma_\eta^A(y)$ remains defined at the end of stage $s-1$, we have that $A_{s-1}\restriction(\varphi_{e(\eta)}(A\oplus D;y)[s_1]) = A_{s_1}\restriction(\varphi_{e(\eta)}(A\oplus D;y)[s_1])$. Also note that the only number entering A at stage s is s_1. By $s_1 > \varphi_{e(\eta)}(A\oplus D;y)[s_1]$, we have the associated A-part is the same as the one at stage s_1. Therefore,

$$\Phi_{e(\eta)}^{A\oplus D}(y)[s] = \Phi_{e(\eta)}^{A\oplus D}(y)[s_1] = W_{e(\eta),s_1}(y) = 0 \neq 1 = W_{e(\eta),s}(y).$$

Since all lower priority strategies are prevented from enumerating small numbers into A and D, we have

$$\Phi_{e(\eta)}^{A\oplus D}(y) = \Phi_{e(\eta)}^{A\oplus D}(y)[s] \neq W_{e(\eta)}(y).$$

(2) Assume that $\eta^\frown\langle 0\rangle \subseteq f$ and that $\Phi_{e(\eta)}^{A\oplus D}$ is total. Then there are infinitely many η-expansionary stages. We show by induction that Γ_η^A is total and $W_{e(\eta)} = \Gamma_\eta^A$ as follows.

Fix x, and suppose that $\Gamma_\eta^A\restriction x$ has already settled down by stage $s_0 > s_\eta$ and computes $W_{e(\eta)}\restriction x$ correctly. Find the least stage $s \geq s_0$ such that s is η-expansionary, $\Gamma_\eta^A(x)[s]\downarrow$, and

$$A_s\restriction(\varphi_{e(\eta)}(A\oplus D;x)[s]) = A\restriction(\varphi_{e(\eta)}(A\oplus D;x)[s]);$$

$$D_s\restriction(\varphi_{e(\eta)}(A\oplus D;x)[s]) = D\restriction(\varphi_{e(\eta)}(A\oplus D;x)[s]).$$

s exists by the totality of $\Phi_{e(\eta)}^{A\oplus D}$. Then $\Gamma_\eta^A(x)$ can only be undefined later by some η' below, at stage s' say. By the choice of s, at the next η-expansionary stage, $\Gamma_\eta^A(x)$

is redefined again with use big, and now only strategies between η and η' have the chance to make $\Gamma_\eta^A(x)$ undefined. Since there are only finitely many such strategies between η and η' and each of them can act only finitely often, $\Gamma_\eta^A(x)$ is defined. By induction, Γ_η^A is total.

Now we prove that $\Gamma_\eta^A(x)$ computes $W_{e(\eta)}(x)$ correctly. The discussion above shows that there is a least stage s_1 such that $\lim_s \gamma_\eta(x)[s] = \gamma_\eta(x)[s_1]$. Then, $A\restriction(\varphi_{e(\eta)}(A \oplus D; x)[s_1]) = A_{s_1}\restriction(\varphi_{e(\eta)}(A \oplus D; x)[s_1])$. Let $s_2 \geq s_1$ be the stage such that the computation $\Phi_{e(\eta)}^{A\oplus D}(x)[s_2]$ is the final version of $\Phi_{e(\eta)}^{A\oplus D}(x)$. s_2 exists due to the totality of $\Phi_{e(\eta)}^{A\oplus D}$. We argue that

$$\Phi_{e(\eta)}^{A\oplus D}(x)[s_2] = \Phi_{e(\eta)}^{A\oplus D}(x)[s_1], \tag{$*$}$$

even though the use of $\Phi_{e(\eta)}^{A\oplus D}(x)$ can be lifted over $\varphi_{e(\eta)}(x)[s_1]$ by D-changes. Suppose that $(*)$ fails. By the choice of s_2, we have x entering $W_{e(\eta)}$ between s_1 and s_2. Then, at stage s_2, η would execute the disagreement strategy, and create an inequality between $W_{e(\eta)}$ and $\Phi_{e(\eta)}^{A\oplus D}$ at x. As described above, such an inequality would be preserved forever, making d the true outcome, a contradiction. $\square$

Now by Lemma 2.4.3, for all $e, i \in \omega$, the $\mathcal{S}_{e,i}$-requirements are all met, (2.4) holds and so by the remarks following Lemma 2.4.3, A is low$_2$. By Lemma 2.4.4, the $\mathcal{H}$-requirements are met, so (2.1) holds and $A \oplus D$ is high. By Lemma 2.4.5, the $\mathcal{R}$-requirements are met, so (2.2) holds and A isolates $A \oplus D$. Since A is low$_2$ and $A \oplus D$ is high, we have $A <_{\mathrm{T}} A \oplus D$.

This completes the proof of Theorem 1. $\square$

CHAPTER 3

ISOLATING A HIGH D.C.E. DEGREE BY A LOW C.E. DEGREE

3.1 Introduction

In this chapter, we improve Theorem 1 by proving that the isolating degree $\mathbf{a}$ in Theorem 1 can be low. We need to combine the lowness requirements with the usual high coding strategy.

Theorem 2 *There is an isolation pair* $(\mathbf{a}, \mathbf{d})$ *such that* $\mathbf{a}$ *is low and* $\mathbf{d}$ *is high.*

To prove Theorem 2, we construct a c.e. set A, a d.c.e. set D, and a partial computable functional Λ in stages, satisfying the following requirements:

$$\mathcal{R}_e: \ \Phi_e^{A \oplus D} = W_e \Rightarrow \exists \Gamma_e(\Gamma_e^A = W_e), \tag{3.1}$$

$$\mathcal{H}_e: \ \lim_n \Lambda^{A,D}(e, n) = Tot(e), \tag{3.2}$$

$$\mathcal{L}_e: \ \exists^\infty s(\Phi_e^A(e)[s] \downarrow) \Rightarrow \Phi_e^A(e) \downarrow, \tag{3.3}$$

where $\{\langle \Phi_e, W_e \rangle : e \in \omega\}$ is a standard enumeration of all pairs $\langle \Phi, W \rangle$, Φ a partial computable functional and W a c.e. set.

3.2 Basic strategies

In this section, we describe the basic strategies meeting the requirements listed above.

$\mathcal{H}$-strategies

As in section 2.2, $\mathcal{H}$-strategies attempt to approximate Tot via $\Lambda^{A\oplus D}$ at the limit, and we will ensure that $\Lambda^{A\oplus D}$ is total, and for all e,

$$\varphi_e \text{ total } \iff \lim_n \Lambda^{A\oplus D}(e,n) = 1.$$

By Shoenfield's Limit Lemma, $\emptyset'' \leq_{\mathrm{T}} Tot \leq_{\mathrm{T}} (A \oplus D)'$ and $A \oplus D$ is high.

The construction takes place on a tree of strategies, where for each e, the $3e$-th level of this tree will be devoted to the definition of $\Lambda^{A\oplus D}(e,-)$. We will coordinate the strategies on this level to ensure that $\Lambda^{A\oplus D}(e)$ is well-defined.

Fix e. Since each node on the tree has at most 3 outcomes, there are at most 3^{3e} many $\mathcal{H}_e$-strategies on the tree, arranged on the $3e$-th level, each of which tries to define $\Lambda^{A\oplus D}(e,n)$ for (almost) all n. List these strategies as $\alpha_1, \cdots, \alpha_m$ with $m \leq 3^{3e}$ in the lexicographical order. For $k \in [1,m]$, define $M_{\alpha_k} = \{\langle e, n\cdot 3^{3e}+k\rangle : n \in \omega\}$. Then, $M_{\alpha_{k_1}} \cap M_{\alpha_{k_2}} = \emptyset$, if $k_1 \neq k_2$, and M_α is reserved for α for the definition of $\Lambda^D(e,-)$. Also define for $n \geq 0$, $F_{e,n} = \{\langle e,j\rangle : j \in [n\cdot 3^{3e}+1, n\cdot 3^{3e}+m]\}$. Here, $F_{e,n}$ is reserved for all $\mathcal{H}_e$-strategies for the definition of $\Lambda^{A\oplus D}(e,n)$. Clearly, M_{α_k} and $F_{e,n}$ are computable and $M_{\alpha_k} \cap F_{e,n}$ contains exactly one number, $\langle e, n\cdot 3^{3e}+k\rangle$, which is denoted as $\lambda(\alpha_k, n)$.

For $n \geq 0$, if τ is a string with $|\tau| > \max F_{e,n}$, define

$$\Lambda^\tau(e,n) = \begin{cases} 0 & \text{if for each } x \in F_{e,n}, \tau(x) = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Such a definition ensures that $\Lambda^{A\oplus D}(e,n) = 1$ iff $D \cap F_{e,n} \neq \emptyset$.

The $\mathcal{H}_e$-strategy α works as follows. Let s_α be the last stage at which α was initialized. Recall that α is devoted to the definition of $\Lambda^{A\oplus D}(e,n)$ for $n \in \omega$ to make $\lim_n \Lambda^{A\oplus D}(e,n) = Tot(e)$. Whenever α sees that φ_e converges up to n, α enumerates into D the number $\lambda(\alpha,n)$, the only element in $M_\alpha \cap F_{e,n}$, provided that n is larger than s_α. This action forces $\Lambda^{A\oplus D}(e,n)$ to take value 1. $\lambda(\alpha,n)$ can never be taken out afterwards, unless α is initialized.

Define the length of convergent computations as follows:

$$\ell_{\mathcal{H}}(\alpha, s) = \max\{x < s : (\forall y < x)(\varphi_{e(\alpha)}(y)[s]\downarrow)\},$$

$$m_{\mathcal{H}}(\alpha, s) = \max\{0, \ell_{\mathcal{H}}(\alpha, t) : t < s \ \&\ t \text{ is } \alpha\text{-stage}\}.$$

Say that s is α-expansionary, if s is an α-stage, and $\ell_{\mathcal{H}}(\alpha, s) > m_{\mathcal{H}}(\alpha, s)$.

α has two outcomes $0, 1$ with $0 <_{\mathrm{L}} 1$, where 0, 1 stand for the cases whether $\lim_s \ell_{\mathcal{H}}(\alpha, s)$ is infinite or not. Clearly, α has outcome 0 if and only if there are infinitely many α-expansionary stages.

Let α be the $\mathcal{H}_e$-strategy on the true path. α is responsible for the definition of $\Lambda^{A\oplus D}(e, n)$ for almost all n, namely, $n > s_\alpha$, where s_α is the last stage at which α is initialized. Note that though other $\mathcal{H}_e$-strategies work jointly with α to define $\Lambda^{A\oplus D}(e, -)$, α does not believe that other strategies can make $\Lambda^{A\oplus D}(e, -)$ compute $Tot(e)$ correctly at the limit, because the witnesses enumerated by these strategies can be taken out of D later. Because of this, α itself acts by enumerating $\lambda(\alpha, -)$ into D whenever α-expansionary stages appear.

Suppose that α has outcome 1. Let s_1 be the largest α-expansionary stage. That is, $n_1 = \ell_{\mathcal{H}}(\alpha, s_1)$ is the least point at which $\varphi_{e(\alpha)}$ diverges, and so, for any $\mathcal{H}_e$-strategy, α' say, $\ell_{\mathcal{H}}(\alpha', s)$ cannot exceed n_1 during the whole construction. Hence, for $n > n_1$, $F_{e,n} \cap D = \emptyset$, $\Lambda^{A\oplus D}(e, n) = 0$. This makes $\lim_n \Lambda^{A\oplus D}(e, n) = 0$.

If α has outcome 0, then there are infinitely many α-expansionary stages. Since α is on the true path, the limit of s_α exists. We simply denote this limit by s_α. Thus for any $n > s_\alpha$, $\lambda(\alpha, n)$ will be enumerated into D at some α-expansionary stages. That is, for all $n > s_\alpha$, $\Lambda^{A\oplus D}(e, n)$ is defined as 1, $\lim_n \Lambda^{A\oplus D}(e, n) = 1$.

By requiring that $\lambda(\alpha, n)$ be enumerated into D only when $s_\alpha < n < \ell_{\mathcal{H}}(\alpha, s)$, and the fact that $\lambda(\alpha, n)$ cannot be removed unless α is initialized later, $\lambda(\alpha, n)$ can be enumerated into D at most once. This makes D d.c.e..

$\mathcal{R}$-strategies

An $\mathcal{R}_e$-strategy, β, works in the same way as η presented in section 2.2.

$\mathcal{L}$-strategies

An $\mathcal{L}_e$-strategy, η say, attempts to make $\Phi_e^A(e)\downarrow$, if $\Phi_e^A(e)[s]$ converges for infinitely many s. To this aim, whenever η observes that $\Phi_e^A(e)$ becomes convergent, at stage s say, η will set a restraint to preserve this computation. That is, η will initialize all strategies with lower priority, particularly those $\mathcal{R}$-strategies, because only these strategies can take numbers out of D_s and make $A_s \upharpoonright s$ change, but keep η untouched. Due to the fact that the construction is finite injury (relative to A), η's action will succeed, provided that η can observe that $\Phi_e^A(e)$ converges infinitely often.

One possible case is that $\Phi_e^A(e)$ converges infinitely often between successive η-stages, but when η is accessible, $\Phi_e^A(e)$ becomes divergent. In this way, even though η is on the true path, we cannot guarantee that η can observe that $\Phi_e^A(e)$ converges infinitely often, and hence the argument mentioned above fails.

Our strategy is to combine the usual tree construction with the allowing of the (finite injury) lowness strategies to act prior to the calculation of the current state of the construction that determines which of the infinitary coding and properly-d.c.e.-diagonalization strategies act at a given stage. That is, at each stage s, first check whether there is some $e < s$ such that $\Phi_e^A(e)[s]\downarrow$ and $\mathcal{N}_e$ is not satisfied. If so, choose e as the least one, and the leftmost $\mathcal{N}_e$-strategy ever being visited, η^* say, and initialize all strategies $> \eta^*$. Notice that this action is off the true path.

Due to the finitary nature of the the construction (relative to A), for each ξ on the true path, only finitely many $\mathcal{N}$-strategies with higher priority can be visited during the whole construction, and hence ξ can be injured only finitely often.

η has one outcome, 1.

3.3 Construction

First, we construct the priority tree as follows:

Let $\mathcal{O} = \{0, d, 1\}$, with $0 <_{\mathrm{L}} d <_{\mathrm{L}} 1$. Recall that $\mathcal{N}$-strategies have only one outcome, 1, $\mathcal{H}$-strategies have two possible outcomes, $0 <_{\mathrm{L}} 1$, and $\mathcal{R}$-strategies have three possible outcomes, $0 <_{\mathrm{L}} d <_{\mathrm{L}} 1$.

The priority tree T is a subtree of $\mathcal{O}^{<\omega}$, which is defined inductively as follows. Given $\xi \in T$,

(1) $|\xi| = 3e$. Then define ξ as an $\mathcal{H}_e$-strategy.

(2) $|\xi| = 3e+1$. Then define ξ as an $\mathcal{R}_e$-strategy.

(3) $|\xi| = 3e+2$. Then define ξ as an $\mathcal{N}_e$-strategy.

(4) The immediate successors of a node are the possible outcomes of the corresponding strategy.

(5) All $\delta \in T$ are ordered lexicographically.

Construction

In the following, we give the whole construction. At each stage s, define δ_s as the current approximation of the true path. Say that s is a ξ-stage, if $\xi \subseteq \delta_s$.

Stage 0: Initialize all nodes on T, and let $D = A = \emptyset$. For any e, n, define $\Lambda^{A,D}(e,n)[0] = 0$ with use $\lambda(e,n) = \max\{\lambda(\alpha, n) : \alpha \text{ is an } \mathcal{H}_e\text{-strategy}\}$.

Stage $s > 0$:

Step 1. Find some (least) $e < s$ such that $\Phi_e^A(e)[s]\downarrow$, and no $\mathcal{N}_e$-strategy has succeeded in satisfying the $\mathcal{N}_e$-requirement. Let η be the leftmost $\mathcal{N}_e$-strategy being visited so far. Initialize all nodes $\xi > \eta$, and declare that $\mathcal{N}_e$ is satisfied via η at stage s. If there is no such e or such η, do nothing.

Step 2. Define a partial function δ_s of length $\leq s$ by substages.

Substage 0. Define $\delta_s(0) = \lambda$, the root node.

Substage $t > 0$. Given $\xi = \delta_s \upharpoonright t$. Initialize all nodes $>_{\mathrm{L}} \xi$. If $t = s$, define $\delta_s = \xi$, initialize all nodes $> \xi$ and go to the next stage. Otherwise, take action for ξ and determine $\delta_s(t)$ as follows. There are three cases:

Case 1. $\xi = \alpha$ is an $\mathcal{H}$-strategy.

If s is α-expansionary, choose n as the least number such that $n < \ell_{\mathcal{H}}(\alpha, s)$, $\lambda(\alpha, n) \notin D_s$, and $n > s_\alpha$, enumerate $\lambda(\alpha, k)$ with $n \leq k < \ell_{\mathcal{H}}(\alpha, s)$ into D. Define

$\delta_s(t) = 0$. Otherwise, do nothing and define $\delta_s(t) = 1$.

Case 2. $\xi = \beta$ is an $\mathcal{R}$-strategy.

If β has executed the disagreement strategy after stage s_β, where s_β is the last stage at which β gets initialized, then define $\delta_s(t) = d$.

If s is β-expansionary, and Γ^A_β is correct on its domain, then for each $y < \ell_{\mathcal{R}}(\beta, s)$ with $\Gamma^A_\beta(y)[s] \uparrow$, define $\Gamma^A_\beta(y) = W_{\beta,s}(y)$ with use $\gamma_\beta(y)[s] = s$. Define $\delta_s(t) = 0$.

If s is β-expansionary, and there is some disagreement between $\Gamma^A_\beta[s]$ and $W_{\beta,s}$, then let y be the least one, and s_y be the stage at which $\Gamma^A_\beta(y)[s]$ was defined. Execute the disagreement strategy as follows:

- remove all numbers associated with lower priority strategies and being enumerated into D since stage s_y out of D;
- enumerate s_y into A;
- declare that all $\Gamma^A_{\beta'}(-)$ defined after stage s_y are undefined.

Define $\delta_s = \beta$, initialize all $\xi > \beta$, and go to the next stage.

Otherwise, define $\delta_s(t) = 1$.

Case 3. $\xi = \eta$ is an $\mathcal{N}$-strategy.

If $\Phi_{e(\eta)}(e(\eta))[s] \uparrow$ or $\mathcal{N}_{e(\eta)}$ is satisfied via some $\mathcal{N}_e$-strategy $\leq \eta$, do nothing and define $\delta_s(t) = 1$.

Otherwise, define $\delta_s = \eta$, declare that $\mathcal{N}_{e(\eta)}$ is satisfied via η at stage s. Initialize all nodes $> \eta$, and go to the next stage.

This completes the construction.

3.4 Verification

We define the *true path* $f \in [T]$ of the construction inductively by

$$f(n) = \mu k \in \mathcal{O}((f \upharpoonright n)^\frown\langle k\rangle \text{ is visited infinitely often}).$$

By the construction, f is well-defined. Clearly, for each n, $f \upharpoonright n$ can be initialized only finitely often, and can act only finitely often. For $\xi \subset f$, let s_ξ be the last stage (during the whole construction) at which ξ is initialized.

Lemma 3.4.1 *(Highness Lemma) For all e, $\lim_s \Lambda^{A\oplus D}(e,x) = Tot(e)$ and thus $A \oplus D$ is high.*

Proof. Fix α as the $\mathcal{H}_e$-strategy on the true path.

First, assume that there are only finitely many α-expansionary stages, then there is some (least) n_0 such that $\varphi_e(n_0)\uparrow$. Thus, φ_e is not total, and $Tot(e) = 0$.

Find $s_0 > s_\alpha$ such that $\alpha \subseteq \delta_{s_0}$ with $n_0 = \ell_{\mathcal{H}}(\alpha, s_0)$. Then no $\mathcal{H}_e$-strategy α' can have $\ell_{\mathcal{H}}(\alpha', s)$ larger than n_0 during the whole construction, and hence for all $n \geq n_0$, no element from $F_{e,n}$ can be enumerated into D. This makes $\lim_n \Lambda^{A\oplus D}(e,n) = 0$.

Now suppose that there are infinitely many α-expansionary stages. Then for any n, α can observe that the convergence length of φ_e exceeds n at some α-expansionary stage, and so α enumerates all $\lambda(\alpha, n)$ $(n > s_\alpha)$ into D at the expansionary stages. This defines $\Lambda^{A\oplus D}(e,n) = 1$ for all $n > s_\alpha$. Therefore, $\lim \Lambda^{A\oplus D}(e,n) = 1$. In this case, it's easy to see that φ_e is total, $e \in Tot$. □

Lemma 3.4.2 *(D.c.e. Lemma) D is a d.c.e. set.*

Proof. During the construction, only $\mathcal{H}$-strategies can enumerate numbers into D, and hence we only need to verify that for all α, n, $\lambda(\alpha, n)$ can be enumerated into D at most once.

Fix α and n. Recall that $\lambda(\alpha, n)$ is reserved by α for the definition of $\Lambda^{A\oplus D}(e(\alpha), n)$. Without loss of generality, suppose that $\lambda(\alpha, n)$ is enumerated into D at stage s_0. Then s_0 is α-expansionary, and $s_\alpha[s_0] \leq n < \ell_{\mathcal{H}}(\alpha, s_0)$. Obviously, $\lambda(\alpha, n)$ cannot be put into D again unless α is initialized after stage s_0. So, suppose that α is initialized at stage $s_1 > s_0$. $\lambda(\alpha, n)$ can be taken out of D after s_1. But since s_α is defined to be larger than s_1, which is larger than $s_0 > \ell_{\mathcal{H}}(\alpha, s_0) > n$, we also have that $\lambda(\alpha, n)$ cannot be put into D afterwards.

Hence, D is d.c.e. □

Lemma 3.4.3 *($\mathcal{R}_e$-satisfaction Lemma) If $\Phi_e^{A\oplus D} = W_e$, then $W_e \leq_{\mathrm{T}} A$.*

Proof. Fix the $\mathcal{R}_e$-strategy $\beta \subset f$. By $\Phi_e^{A\oplus D} = W_e$, there are infinitely many β-expansionary stages, so $\beta^\frown\langle 0\rangle \subset f$. We prove that Γ_β^A is total and computes W_e correctly.

(1) Γ_β^A is total.

Suppose not. Then there is some (least) n_0 s.t. $\Gamma_\beta^A(n_0)\uparrow$. Recall that $\Gamma_\beta^A(n_0)$ can become undefined in two ways.

(a) $A \upharpoonright \varphi_e(n_0)$ changes,

(b) $\mathcal{R}$-strategies below β excute the disagreement strategy.

Since $\Phi_e^{A\oplus D}(n_0)\downarrow$, (a) happens only finitely often. Choose s_0 (least) such that $\Phi_e^{A\oplus D}(n_0)[s_0]$ is the correct computation of $\Phi_e^{A\oplus D}(n_0)$. Then $\Gamma_\beta^A(n_0)$ can only be undefined via (b). Without loss of generality, let $s_1 > s_0$ be the first stage at which some $\mathcal{R}$-strategy β' performs the disagreement strategy, and the least associated stage (enumerated into A automatically by this action) is less than $\gamma_\beta(n_0)$, $\Gamma_\beta^A(n_0)$ is declared to be undefined in this way, and will get redefined at the next β-expansionary stage, s_2 say. Then, only $\mathcal{R}$-strategies between β and β' can undefine $\Gamma_\beta^A(n_0)$ again, and such an action can happen only finitely often. Therefore, after a sufficiently large stage, s_3 say, $\Gamma_\beta^A(n_0)$ has settled down with its use fixed forever. Therefore $\Gamma_\beta^A(n_0)\downarrow$, contradicting the choice of n_0.

(2) Γ_β^A is correct.

Otherwise, β would perform the disagreement strategy at some (least) m at stage s_0 say. This disagreement (between $W_e(m)$ and $\Phi_e^{A\oplus D}(m)$) is preserved forever because $\Phi_e^{A\oplus D}(m)$ is chosen as a β-correct computation when the agreement function $\ell_{\mathcal{R}}(\beta, s)$ is defined, and also because all strategies with lower priority are initialized at stage s_0. Thus, $\Phi_e^{A\oplus D}(m) = 0 \neq 1 = W_e(m)$, a contradiction. $\square$

Lemma 3.4.4 *($\mathcal{N}_e$-satisfaction Lemma) If $\Phi_e^A(e)$ converges infinitely often, then $\Phi_e^A(e)\downarrow$.*

Proof. Fix the $\mathcal{N}_e$-strategy $\eta \subset f$, and let $\eta = \eta^{-\frown}\langle \mathcal{O} \rangle$. Then η^- is the $\mathcal{R}_e$-strategy by the construction of T. Let $s_0 > s_\eta$ be the least stage such that $\eta \subseteq \delta_{s_0}$ and let $s' \geq s_0$ be the least stage at which $\Phi_e^A(e)[s'] \downarrow$. There are two cases:

(1) $\mathcal{N}_e$ is satisfied currently.

Suppose that $\mathcal{N}_e$ is satisfied via η^* at stage s^*. Then $s^* < s_0$, and $\eta^* \leq \eta$ has not been initialized between stages s^* and s_0. Recall that at stage s^*, all strategies $\xi > \eta^*$ are initialized. By the assumption that $\mathcal{N}_e$ is satisfied at stage s^*, and the choice of s_η and s_1, no number entering D before s^* can be taken out, and so no number less than s^* can be put into A. Therefore, $A_{s^*} \upharpoonright s^* = A \upharpoonright s^*$ and $\Phi_e^A(e) \downarrow$.

(2) Otherwise.

By the construction, η initializes all strategies with lower priority. This ensures that no strategies with lower priority can enumerate numbers less than s' into A. By the choice of s_α, $A_{s'} \upharpoonright s' = A \upharpoonright s'$, and hence $\Phi_e^A(e) \downarrow = \Phi_e^A(e)[s']$. □

This completes the proof of Theorem 2. □

CHAPTER 4

PSEUDO-ISOLATED DEGREES ARE DENSE IN THE C.E. DEGREES

4.1 Introduction

Let $\mathbf{d}$ be a nonisolated d.c.e. degree. Then for any c.e. degree $\mathbf{a} < \mathbf{d}$, there is a c.e. degree $\mathbf{c}$ such that $\mathbf{a} < \mathbf{c} < \mathbf{d}$. Thus, it is natural to conjecture that nonisolated d.c.e. degrees are minimal upper bounds of c.e. degrees below them. In this chapter, we provide a negative answer to this conjecture. We will prove that there are d.c.e. degrees $\mathbf{b} < \mathbf{d}$ such that $\mathbf{d}$ is nonisolated and $\mathbf{b}$ bounds all c.e. degrees below $\mathbf{d}$. We call $\mathbf{d}$ a *pseudo-isolated* degree, and we say that $\mathbf{d}$ is *pseudo-isolated* by $\mathbf{b}$.

In this chapter, we show that pseudo-isolated degrees are densely distributed in the c.e. degrees.

Theorem 3 *Given c.e. degrees,* $\mathbf{u} < \mathbf{v}$*, there is a pseudo-isolated degree* $\mathbf{d}$ *such that* $\mathbf{u} \leq \mathbf{d} \leq \mathbf{v}$.

Note that by Sacks density theorem [66], we may obtain $\mathbf{u} < \mathbf{d} < \mathbf{v}$ in Theorem 3. Given $\mathbf{v}$ strictly above $\mathbf{u}$, we use Sacks density theorem to obtain $\mathbf{u}^*, \mathbf{v}^*$ with $\mathbf{u} < \mathbf{u}^* < \mathbf{v}^* < \mathbf{v}$. By applying Theorem 3 to $\mathbf{u}^*$ and $\mathbf{v}^*$, there is a pseudo-isolated degree $\mathbf{d}$ between $\mathbf{u}^*$ and $\mathbf{v}^*$, and hence, strictly between $\mathbf{u}$ and $\mathbf{v}$.

To prove Theorem 3, we will construct d.c.e. sets B, D satisfying the following requirements:

$$\mathcal{G}\colon\ B \oplus D \leq_{\mathrm{T}} U \oplus V, \tag{4.0}$$

$$\mathcal{P}_e\colon\ B \neq \Phi_e^{W_e}\ \vee\ W_e \neq \Psi_e^{U,B}, \tag{4.1}$$

$$\mathcal{Q}_e\colon\ \Theta_e^{U,B} = W_e \Rightarrow (\exists \text{ c.e. } C_e \leq_{\mathrm{T}} U \oplus B)(\forall i)(C_e \neq \Omega_i^{W_e}), \tag{4.2}$$

$$\mathcal{R}_e\colon\ \Phi_e^{U,B,D} = W_e \Rightarrow \exists \Gamma_e(\Gamma_e^{U,B} = W_e), \tag{4.3}$$

$$\mathcal{T}_e\colon\ D \neq \Phi_e^{U,B} \tag{4.4}$$

where $\{\langle \Phi_e, \Psi_e, \Theta_e, W_e\rangle : e \in \omega\}$ is a standard enumeration of all $\langle \Phi, \Psi, \Theta, W\rangle$, Φ, Ψ, Θ partial computable functionals and W a c.e. set.

Let $\mathbf{b}, \mathbf{d}$ be the Turing degrees of $U \oplus B, U \oplus B \oplus D$ respectively. By the $\mathcal{P}$-strategies, $\mathbf{b}$ is a properly d.c.e. degree. By the $\mathcal{T}$-strategies, $\mathbf{b} < \mathbf{d}$. The $\mathcal{Q}$-requirements guarantee that $\mathbf{b}$ is nonisolated, and the $\mathcal{R}$-strategies ensure that $\mathbf{b}$ bounds all c.e. degrees below $\mathbf{d}$. By the global requirement $\mathcal{G}$, $\mathbf{u} \leq \mathbf{b}, \mathbf{d} \leq \mathbf{v}$.

We now show that $\mathbf{d}$ is nonisolated. Let $\mathbf{c}$ be any c.e. degree below $\mathbf{d}$. By the fact that $\mathbf{b}$ bounds all c.e. degrees below $\mathbf{d}$, $\mathbf{c} < \mathbf{b}$. Since $\mathbf{b}$ is nonisolated, there is a c.e. degree $\mathbf{e}$ between $\mathbf{c}$ and $\mathbf{b}$. Therefore, $\mathbf{c} < \mathbf{e} < \mathbf{d}$ and $\mathbf{d}$ is nonisolated.

4.2 Basic strategies

In this section, we describe strategies designed to satisfy a single requirement.

$\mathcal{P}$-strategies

A $\mathcal{P}$-strategy, α say, working for a $\mathcal{P}$-requirement, $\mathcal{P}_e$, attempts to find a number witnessing the failure of $B \neq \Phi_e^{W_e}$ or $W_e \neq \Psi_e^{U,B}$. α works in the way given in Cooper, Lempp and Watson [19]. That is, α runs cycles (j,k) for $j,k \in \omega$, with cycle $(0,0)$ starting first, and where each cycle (j,k) may start cycles $(j,k+1)$ or $(j+1,0)$, and stop (or cancel) cycles (j',k') if $(j',k') > (j,k)$ (in the lexicographical ordering). Cycle (j,k) attempts to define $\Gamma_{\alpha,j}^U(k)$ and $\Delta_\alpha^U(j)$, and runs as follows:

(1) Choose x as a fresh number,

(2) Wait for a stage s_1 such that for some least v_1,

$$0 = B_{s_1}(x) = \Phi_e^{W_e}(x)[s_1], \text{and}$$

$$W_{e,s_1} \upharpoonright \varphi_{e,s_1}(x) = \Psi_{e,s_1}^{U_{s_1}, B_{s_1} \upharpoonright v_1} \upharpoonright \varphi_{e,s_1}(x).$$

(If this never happens, then x is a witness to the success of $\mathcal{P}_e$.)

(3) Restrain $B \upharpoonright v_1$ from other strategies from now on.

(4) Set $\Gamma_{\alpha,j}^U(k) = V_{s_1}(k)$ with use $\gamma_{\alpha,j}(k) = v_1$ and start cycle $(j, k+1)$ to run simultaneously.

(5) Wait for $U \upharpoonright v_1$ or $V(k)$ to change.

- If $U \upharpoonright v_1$ changes first then cancel cycles $(j', k') > (j, k)$, drop the B-restraint of cycle (j, k) to 0, and go back to (2).
- If $V(k)$ changes first then stop cycles (j', k') if $(j', k') > (j, k)$. Go to (6).

(6) Put x into B.

(7) Wait for a stage $s_2 > s_1$ such that for some v_2,

$$1 = B_{s_2}(x) = \Phi_e^{W_e}(x)[s_2], \text{and}$$

$$W_{e,s_2} \upharpoonright \varphi_{e,s_2}(x) = \Psi_{e,s_2}^{U_{s_2}, B_{s_2} \upharpoonright v_2} \upharpoonright \varphi_{e,s_2}(x).$$

(8) Restrain $B \upharpoonright v_2$ from other strategies from now on.

(9) Set $\Delta_\alpha^U(j) = V_{s_2}(j)$ with use $\delta_\alpha(j) = v_2$ and start cycle $(j+1, 0)$ to run simultaneously.

(10) Wait for $U \upharpoonright v_2$ or $V(j)$ to change.

- If $U \upharpoonright v_2$ changes first then cancel cycles $(j', k') \geq (j+1, 0)$, drop the B-restraint of cycle (j, k) to v_1, and go back to (7).
- If $V(j)$ changes first then stop cycles (j', k') if $(j', k') \geq (j+1, 0)$. Go to (11).

(11) Remove x from B.

(12) Wait for $U \upharpoonright v_1 \neq U_{s_1} \upharpoonright v_1$ or $U \upharpoonright v_2 \neq U_{s_2} \upharpoonright v_2$.

- If $U \upharpoonright v_1 \neq U_{s_1} \upharpoonright v_1$, then redefine $\Gamma^U_{\alpha,j}(k) = V(k)$ with use 0, cancel cycles $(j', k') > (j, k)$, start cycle $(j, k+1)$, and halt.
- If $U \upharpoonright v_2 \neq U_{s_2} \upharpoonright v_2$, then redefine $\Delta^U_\alpha(j) = V(j)$ with use 0, cancel cycles $(j', k') \geq (j+1, 0)$, start cycle $(j+1, 0)$, and halt.

Here, whenever a cycle (j, k) is started, any previous version of it has been canceled and the corresponding functionals are undefined because of the U-changes. Therefore, $\Gamma^U_{\alpha,j}$ and Δ^U_α are well-defined. We say that cycle (j, k) acts if it chooses a fresh number x as its attacker *(step (1))*, or it changes the value of $B(x)$ *enumerate x into B (step (6)) or remove x from B (step (11))*. The actions of other strategies are defined in the same way. Thus, infinite cycling does not count as action.

As explained in Cooper, Lempp and Watson [19], α has two outcomes:

A. There is a stage s after which no cycle runs.

(That is, some cycle (j_0, k_0) eventually waits at (2), (7), or (12) forever, which is a finitary outcome.)

B. Some cycle, (j_0, k_0) say, runs infinitely often but no cycle $< (j_0, k_0)$ does so.

(Then, cycle (j_0, k_0) goes from (5) to (2), or from (10) to (7), infinitely often, and hence, $\Phi^{W_e}_e$ or $\Psi^{U,B}_e$ is partial.)

α also has two pseudo-outcomes:

C. There is a j_0 and there are stages $s_j (j < j_0)$ and $t_k (k \in \omega)$ such that no cycle (j, k) runs after stage s_j and such that cycle (j_0, k) does not run after stage t_k; but there is no stage s such that no cycle (j_0, k) runs after stage s. ("Row j_0 runs infinitely often but rows $j < j_0$ run finitely.")

(Then every cycle (j_0, k) eventually waits at (5) or (12) forever, and together these show that $\Gamma^U_{\alpha,j_0} = V$, contrary to hypothesis.)

D. There are stages $s_j (j \in \omega)$ such that no cycle (j,k) runs after stage s_j; but there are infinitely many stages at which some cycle runs. ("Every row runs finitely often.")

(Then for every j there is a cycle (j, k_j) that eventually waits at (10) or (12) forever, and together these show that $\Delta_\alpha^U = V$, contrary to hypothesis.)

$\mathcal{Q}$-strategies

Let ζ be a $\mathcal{Q}$-strategy. To satisfy $\mathcal{Q}$, ζ will construct a c.e. set C_e such that if $\Theta_e^{U,B} = W_e$, then $C_e \leq_{\mathrm{T}} U \oplus B$ and for all $i \in \omega, C_e \neq \Omega_i^{W_e}$. We break the $\mathcal{Q}$-requirement into infinitely many subrequirements:

$$\mathcal{S}_{e,i} : \Theta_e^{U,B} = W_e \;\Rightarrow C_e \neq \Omega_i^{W_e}, \tag{4.5}$$

The construction of C_e is carried out by the $\mathcal{S}$-strategies, which are arranged below ζ.

Let β be an $\mathcal{S}_{e,i}$-strategy. β proceeds in the same way as in Arslanov, Lempp and Shore [9], which also runs $\omega \times \omega$ many cycles, (j,k), $j,k \in \omega$ with cycle $(0,0)$ starting first. Again, cycle (j,k) can start cycles $(j,k+1)$ or $(j+1,0)$ and stop, or cancel, cycles $(j',k') > (j,k)$ (in the lexicographical ordering). Cycle (j,k) defines $\Gamma_{\beta,j}^U(k)$ and $\Delta_\beta^U(j)$, and runs as follows:

(1) Choose x and $x+1$ as fresh numbers.

(2) Wait for a stage s_1 such that

$$\Omega_i^{W_e}(x)[s_1]\downarrow = 0,$$

and for some least v_1 such that

$$W_{e,s_1} \restriction \omega_{i,s_1}(x) = \Psi_{e,s_1}^{(U\oplus B)_{s_1} \restriction v_1} \restriction \omega_{i,s_1}(x).$$

(3) Protect $B \restriction v_1$ from other strategies from now on.

(4) Set $\Gamma^U_{\beta,j}(k) = V_{s_1}(k)$ with use $\gamma_{\beta,j}(k) = v_1$, and start cycle $(j, k+1)$ to run simultaneously.

(5) Wait for $U \upharpoonright v_1$ or $V(k)$ to change.

- If $U \upharpoonright v_1$ changes first, then cancel cycles $(j', k') > (j, k)$, drop the B-restraint of cycle (j, k) to 0, and go back to (2).
- If $V(k)$ changes first, then stop all $(j', k') > (j, k)$ and go to (6).

(6) Put x into B and C_e.

(7) Wait for a stage s_2 such that

$$\Omega_i^{W_e}(x)[s_2]\downarrow = B_{s_2}(x) = 1,$$

and for some least v_2 such that

$$W_{e,s_2} \upharpoonright \omega_{i,s_2}(x) = \Psi_{e,s_2}^{(U \oplus B)_{s_2} \upharpoonright v_2} \upharpoonright \omega_{i,s_2}(x).$$

(8) Protect $B \upharpoonright v_2$ from other strategies from now on.

(9) Set $\Delta^U_\beta(j) = V_{s_2}(j)$ with use $\delta_\beta(j) = v_2$, and start cycle $(j+1, 0)$ simultaneously.

(10) Wait for $U \upharpoonright v_2$ or $V(j)$ to change.

- If $U \upharpoonright v_2$ changes first, then cancel cycles $(j', k') \geq (j+1, 0)$, drop the B-restraint of cycle (j, k) to v_1, and go back to (7).
- If $V(j)$ changes first, then stop all cycles $(j', k') \geq (j+1, 0)$, and go to step (11).

(11) Remove x from B.

(12) Wait for $U \upharpoonright v_1 \neq U_{s_1} \upharpoonright v_1$.

(13) Reset $\Gamma^U_{\beta,j}(k) = V(k)$, put $x+1$ into B, cancel cycles $(j', k') > (j, k)$, drop the B-restraint of cycle (j, k) to 0, stop cycle (j, k) and start cycle $(j, k+1)$.

Again, whenever cycle (j,k) is started, any previous version of it has been canceled and the corresponding functionals have become undefined through the U-changes and therefore, $\Gamma_{\beta,j}$ and Δ_β are well-defined. Note that here, when x is moved out of B, the permission for the enumeration of x into C_e goes away. However, such a diagonalization is a global win for $\mathcal{Q}_e$, and C_e is not needed, unless $U \upharpoonright v_1$ changes later. In the latter case, $x+1$ is enumerated into B, and so $U \oplus B$ can recognize this. In the construction, the enumeration of $x+1$ is done by ζ (not by β, because it is possible that after (11), β cannot be eligible to be accessible anymore).

β has two outcomes:

A. There is a stage s after which no cycle runs.

(Then, some cycle, (j_0,k_0) say, eventually waits at step at (2), (7) or (12) forever, and β succeeds through cycle (j_0,k_0).)

B. Some cycle, (j_0,k_0) say, runs infinitely often, but no cycle $< (j_0,k_0)$ does so.

(Then, cycle (j_0,k_0) goes from (5) to (2), or from (10) to (7) infinitely often, $\Theta_e^{U,B}$ or $\Omega_i^{W_e}$ is partial.)

β also has two pseudo-outcomes:

C. There is a j_0 and there are stages $s_j (j < j_0)$ and $t_k (k \in \omega)$ such that no cycle (j,k) runs after stage s_j and such that cycle (j_0,k) does not run after stage t_k; but there is no stage s such that no cycle (j_0,k) runs after stage s. ("Row j_0 runs infinitely often but rows $j < j_0$ run finitely.")

(Then every cycle (j_0,k) eventually waits at (5) or (13) forever, and together these show that $\Gamma^U_{\beta,j_0} = V$, contrary to hypothesis.)

D. There are stages $s_j (j \in \omega)$ such that no cycle (j,k) runs after stage s_j; but there are infinitely many stages at which some cycle runs. ("Every row runs finitely often.")

(Then for every j there is a cycle (j, k_j) that eventually waits at (10) forever, and together these show that $\Delta_\beta^U = V$, contrary to hypothesis.)

$\mathcal{R}$-strategies

An $\mathcal{R}_e$-strategy, η say, is devoted to the construction of a partial functional, Γ_η, such that if $\Phi_e^{U,B,D} = W_e$, then $\Gamma_\eta^{U,B}$ is well-defined and computes W_e correctly. Define

$$\ell_{\mathcal{R}}(\eta, s) = \max\{x < s : (\forall y < x)(\Phi_e^{U,B,D}(y)[s] \downarrow= W_{e,s}(y))\},$$
$$m_{\mathcal{R}}(\eta, s) = \max\{0, \ell_{\mathcal{R}}(\eta, t) : t < s \ \&\ t \text{ is an } \eta\text{-stage}\}.$$

Here, all parameters involved in a computation at stage s are bounded by s. Say that stage s is η-expansionary if $s = 0$ or s is an η-stage and $\ell_{\mathcal{R}}(\eta, s) > m_{\mathcal{R}}(\eta, s)$.

η runs ω many cycles, (j), $j \in \omega$ with cycle (0) starting first. Cycle (j) can start cycle $(j+1)$ and can stop or cancel cycle $(l) > (j)$.

Cycle (j) attempts to define $\Delta_\eta^U(j)$, and runs as follows:

(1) Wait for an η-expansionary stage, s say.

(2) If $\Gamma_\eta^{U,B}(x)[s] = W_{e,s}(x)$ for each x with $\Gamma_\eta^{U,B}(x)[s]\downarrow$, then go to (3). Otherwise, let y be the least one such that $\Gamma_\eta^{U,B}(y)[s] \neq W_{e,s}(y)$. Say that the error at y is recognized. Go to (4).

(3) For all $x < \ell_{\mathcal{R}}(\eta, s)$, if $\Gamma_\eta^{U,B}(x)[s] \uparrow$, then define $\Gamma_\eta^{U,B}(x)[s] = W_{e,s}(x)$ with use $\gamma(x) = s$. Go back to (1).

(4) Protect $B \upharpoonright s, D \upharpoonright s$ from other strategies from now on.

(5) Set $\Delta_{\eta,s}^U(j) = V_s(j)$ with use $\delta_{\eta,s}(j) = \gamma(y)[s]$, and start cycle $(j+1)$ to run simultaneously.

(6) Wait for $U \upharpoonright \delta_{\eta,s}(j)$ or $V(j)$ to change.

- If $U \upharpoonright \delta_{\eta,s}(j)$ changes first, then cancel all cycles $(l) > (j)$, drop the the B, D-restraint of cycle (j) to 0, declare that y is not the error point anymore. Go back to (1).

- If $V(j)$ changes first, then stop all cycles $(l) > (j)$, remove from D all numbers enumerated into D after stage $\gamma_\eta(y)[s]$, and put $\gamma_\eta(y)[s]$ into B. Go to (7).

(7) Wait for $U \upharpoonright \delta_{\eta,s}(j)$ to change.

(8) Reset $\Delta_\eta^U(j) = V(j)$ with use 0, cancel all cycles $(l) \geq (j)$, drop the B, D-restraint of cycle (j) to 0, start cycle $(j+1)$.

η has two possible outcomes:

A. There is a stage s after which no cycle runs.

(Then, some cycle (j_0) eventually waits at (1) or (7) forever, and η succeeds through cycle (j_0).)

B. Some cycle (j_0) runs infinitely often but no cycle $< (j_0)$ does so.

(Then, cycle (j_0) goes from (3) or (6) to (1) infinitely often. In this case, if $\Phi_e^{B,D}$ is total, then Γ_η is totally defined and computes W_e correctly. $\mathcal{R}_e$ is satisfied.)

η also has one pseudo-outcome:

C. There are stages $s_j (j \in \omega)$ such that cycle (j) does not run after stage s_j; but there are infinitely many stages at which some cycle runs. ("Every cycle runs finitely often.")

(Then for every j, cycle (j) eventually waits at (6) or (8) forever, and together these show that $\Delta_\eta^U = V$, contrary to hypothesis.)

$\mathcal{T}$-strategies

Let τ be a $\mathcal{T}$-strategy. τ is a modification of Sacks' coding strategy. Define

$$\ell_{\mathcal{T}}(\tau, s) = \max\{x < s : (\forall y < x)(\Phi_e^{U,B}(y)[s] \downarrow = D_s(y))\},$$

$$m_{\mathcal{T}}(\tau, s) = \max\{0, \ell_{\mathcal{T}}(\tau, t) : t < s \;\&\; t \text{ is a } \tau\text{-stage}\}.$$

Stage s is τ-expansionary if $s = 0$ or s is a τ-stage and $\ell_{\mathcal{T}}(\tau, s) > m_{\mathcal{T}}(\tau, s)$.

τ runs ω many cycles to define a functional Σ_τ such that if $D = \Phi_e^{U,B}$, then $V = \Sigma_\tau^U$. That is, for each j, cycle (j) attempts to define $\Sigma_\tau^U(j)$. Cycle (0) starts first, and each cycle (j) can start cycle $(j+1)$ or stop (cancel) cycles $(l) > (j)$.

Cycle (j) runs as follows:

(1) Choose $x(j)$ as a fresh number.

(2) Wait for a stage s_1 such that $\ell_{\mathcal{T}}(\tau, s_1) > x(j)$.

(3) Define $\Sigma_\tau^U(j) = 0$ with use $\sigma(j) = \varphi_e(x(j))[s_1]$.

(4) Protect $B \upharpoonright s_1$ from other strategies from now on. Start cycle $(j+1)$ simultaneously.

(5) Wait for $U \upharpoonright \varphi_e(x(j))[s_1]$ or $V(j)$ to change.

- If $U \upharpoonright \varphi_e(x(j))[s_1]$ changes first, then cancel all cycles $(l) > (j)$, drop the B-restraint of cycle (j) to 0, and go back to (2).
- If $V(j)$ changes first, then stop all cycles $(l) > (j)$, and go to (6).

(6) Put $x(j)$ into D.

(7) Wait for $U \upharpoonright \varphi_e(x(j))[s_1]$ to change.

(8) Redefine $\Sigma_\tau^U(j)$, cancel all cycles $(l) \geq (j)$, stop cycle (j) and start cycle $(j+1)$.

Again, τ has two outcomes:

A. There is a stage s after which no cycle runs.

(Then, some cycle, (j_0) say, eventually waits at (2), (5) or (7) forever. τ succeeds through cycle (j_0).)

B. Some cycle (j_0) runs infinitely often but no cycle $< (j_0)$ does so.

(Then, cycle (j_0) goes from (5) to (2) infinitely often, and hence, $\Phi_e^{U,B}$ is partial, $\mathcal{T}_e$ is satisfied.)

τ also has one pseudo-outcome:

C. There are stages $s_j (j \in \omega)$ such that cycle (j) does not run after stage s_j; but there are infinitely many stages at which some cycle runs. ("Every cycle runs finitely often.")

(Then for every j, cycle (j) eventually waits at (5) or (7) forever, and together these show that $\Sigma_\tau^U = V$, contrary to hypothesis.)

4.3 Construction

In this section, we give the full construction of B and D. First we define the priority tree, T, on which the construction will be carried out. The priority of requirements are listed as follows:

$$\mathcal{G} < \mathcal{P}_0 < \mathcal{R}_0 < \mathcal{T}_0 < \mathcal{Q}_0 < \mathcal{S}_{0,0} < \mathcal{P}_1 < \mathcal{R}_1 < \mathcal{T}_1 < \mathcal{Q}_1 < \mathcal{S}_{0,1} < \mathcal{S}_{1,0}$$
$$< \mathcal{S}_{1,1} < \mathcal{P}_2 < \cdots < \mathcal{P}_n < \mathcal{R}_n < \mathcal{T}_n < \mathcal{Q}_n < \mathcal{S}_{0,n} < \cdots < \mathcal{S}_{n-1,n} < \mathcal{S}_{n,0}$$
$$< \mathcal{S}_{n,1} < \cdots < \mathcal{S}_{n,n} < \mathcal{P}_{n+1} < \cdots,$$

where for any $\mathcal{X} < \mathcal{Y}$, if $\mathcal{X} < \mathcal{Y}$, then $\mathcal{X}$ has higher priority than $\mathcal{Y}$. Requirement $\mathcal{G}$ is a global requirement and we don't put it on the tree of strategies.

Definition 4.1 *(Outcomes of strategies) Let ξ be any node on T, ξ's outcomes are defined as follows:*

(1) If ξ is a $\mathcal{P}$-strategy or an $\mathcal{S}$-strategy, then ξ's outcomes are (j, k), $j, k \in \omega$, with lexicographical ordering.

(2) If ξ is an $\mathcal{R}$-strategy or a $\mathcal{T}$-strategy, then ξ's outcomes are (j), $j \in \omega$ with natural ordering.

(3) If ξ is a $\mathcal{Q}$-strategy, then ξ's outcomes are $0, 1$ with $0 <_{\mathrm{L}} 1$, where 1 denotes that some substrategy of α reaches (11), and is waiting in (12). ($\mathcal{Q}$ is satisfied temporarily.)

Definition 4.2 *(Satisfaction of strategies) Given $\xi \in T$,* *(1) $\mathcal{P}_e$ is satisfied at ξ, if there is an $\mathcal{P}_e$-strategy α such that $\alpha \subset \xi$;*

(2) $\mathcal{R}_e$ is satisfied at ξ, if there is an $\mathcal{R}_e$-strategy η such that $\eta \subset \xi$;

(3) $\mathcal{T}_e$ is satisfied at ξ, if there is a $\mathcal{T}_e$-strategy τ such that $\tau \subseteq \xi$;

(4) $\mathcal{Q}_e$ is satisfied at ξ, if there is a $\mathcal{Q}_e$-strategy $\zeta ^\frown \langle 1 \rangle \subseteq \xi$; $\mathcal{Q}_e$ is active at ξ, if there is an $\mathcal{Q}_e$-strategy ζ such that $\zeta ^\frown \langle 0 \rangle \subseteq \xi$. In this case, $\mathcal{Q}_e$ is said to be active at ξ via ζ;

(5) $\mathcal{S}_{e,i}$ is satisfied at ξ, if either $\mathcal{Q}_e$ is satisfied at ξ, or $\mathcal{Q}_e$ is active at ξ via ζ, and there is an $\mathcal{S}_{e,i}$-strategy β such that $\zeta ^\frown \langle 0 \rangle \subseteq \beta \subset \xi$.

T is defined by induction as follows.

Definition 4.3 *(Construction of T)*

(1) Let λ, the root node, be a $\mathcal{P}_0$-strategy;

(2) The immediate successors of a node are the possible outcomes of the corresponding strategy;

(3) For $\xi \in T$, ξ works for the highest priority requirement which has not been satisfied and is not active at ξ.

Construction

The construction will proceed by stages, and at the end of each stage s, define δ_s as the current approximation of the true path. Say that s is a ξ-stage if $\xi \subseteq \delta_s$.

Stage 0: Initialize all nodes on T, and let $B = D = \emptyset$.

Stage $s > 0$:

Substage 0: Let $\delta_s(0) = \lambda$, the root node.

Substage t. Given $\xi = \delta_s \upharpoonright t$. Initialize all nodes $>_{\mathrm{L}} \xi$. If $t = s$, then define $\delta_s = \xi$, initialize all nodes $> \xi$ and go to stage $s + 1$.

Otherwise, for $t < s$, define $\delta_s(t)$ and take action for $\xi = \delta_s(t)$. If ξ is a $\mathcal{Q}$-strategy, and some substrategy β of ξ has some cycle (j,k) reaching (11), and U has no change below v_1, then let $\xi^\frown 1$ be eligible to be accessible at the next substage. If (j,k) reaches (11), and U changes below v_1, then put $x+1$ (defined by β's cycle (j,k)) into B, redefine $\Gamma^U_{\beta,j}(k) = V(k)$ with use 0, cancel β's cycles $> (j,k)$, drop the restraint of cycle (j,k) to 0, and initialize all $\xi > \beta^\frown(j,k)$. Let $\xi^\frown 0$ be eligible to be accessible at the next substage. Otherwise, do nothing, and let $\xi^\frown 0$ be eligible to be accessible at the next substage.

If ξ is a strategy of other types, then run program α, β, η or τ according to which strategy ξ works for.

(1) If ξ has no cycles started, then start ξ's least cycle. That is, if ξ is an α or β-strategy, then start cycle $(0,0)$, and if ξ is an η or τ-strategy, then start cycle (0). Define $\delta^t_s = \xi^\frown(0,0)$ or $\xi^\frown(0)$. Otherwise, let (j,k) or (j) be the largest cycle active.

(2) If ξ has no cycle running at stage s, then define $\delta^t_s = \xi^\frown(j,k)$ or $\xi^\frown(j)$.

(3) If ξ has some cycle $(j',k') \leq (j,k)$ or $(j') \leq (j)$ making progress at stage s, and cycle (j',k') or (j') goes to the final point, then define $\delta^t_s = \xi^\frown(j'+1,0)$ or $\xi^\frown(j'+1)$, according to the type of strategy of ξ.

(4) If ξ has some cycle $(j',k') \leq (j,k)$ or $(j') \leq (j)$ making progress at stage s, and cycle (j',k') does not go to the final point, then define $\delta^t_s = \xi^\frown(j',k')$ or $\xi^\frown(j')$, according to the type of strategy of ξ.

Furthermore, in case (4), if ξ enumerates numbers into B or D, or moves numbers out of B or D, then define $\delta_s = \delta^t_s$, initialize all strategies $\geq \delta_s$, and go to stage $s+1$ directly. Otherwise, at the end of stage s, define $\delta_s = \delta^t_s$. In this case, if $\xi \subseteq \delta_s$ is an $\mathcal{R}$-strategy, and s is a ξ-expansionary stage, then for all $x < \ell_{\mathcal{R}}(\xi, s)$,

- If $\Gamma^{U,B}_\xi(x)[s] \uparrow$, and the computation $\Phi^{U,B,D}_\xi(x)$ has changed from the last ξ-expansionary stage (if it exists), then define $\Gamma^{U,B}_\xi(x)[s] = W_{\xi,s}(x)$ with use $\gamma_\xi(x)[s] = s$.

- If $\Gamma_\xi^{U,B}(x)[s]\uparrow$, and the computation $\Phi_\xi^{U,B,D}(x)$ is the same as that at the last ξ-expansionary stage, then redefine $\Gamma_\xi^{U,B}(x)[s] = W_{\xi,s}(x)$ with use $\gamma_\xi(x)[s] = \gamma_\xi(x)[s']$, where $s' < s$ is the last ξ-expansionary stage.

- Otherwise, do nothing.

Go the stage $s+1$.

This completes the construction.

4.4 Verification

Define the true path δ of the construction by induction as follows:

$$\delta = \liminf_s \delta_s.$$

Lemma 4.4.1 *δ is well-defined.*

Proof. We prove this by induction. Assume that the lemma holds for all $m < n$. Let $\xi = \delta \upharpoonright n = \liminf_s(\delta_s \upharpoonright n)$, and let s_0 be the least stage such that for all $s \geq s_0$, $\delta_s \geq \delta \upharpoonright n$. We prove that $\delta(n)$ is defined.

Suppose that $\delta(n)$ does not exist, and suppose that ξ is a $\mathcal{P}$-strategy. For other strategies, the argument is almost the same or simpler. By our assumption, $\delta(n)$ does not exist, ξ runs infinitely many cycles in the construction, and each of them runs only finitely often. There are two cases.

Case 1: There is some (the least) j_0 such that every cycle $(j_0, k), k \in \omega$, eventually waits at (5), or (12) forever. *(Row j_0 runs infinitely often.)*

For $j < j_0$, let s_j be the last stage at which cycle $(j, -)$ runs. By our assumption, s_j exists. For each $k \in \omega$, let t_k be the last stage at which some cycle (j_0, k) runs. Again, t_k exists. Then $\Gamma^U_{\xi,j_0}(k)$ gets defined at stage t_k, and computes $V(k)$ correctly. $\Gamma^U_{\xi,j_0} = V$, contradicting the hypothesis $U <_{\mathrm{T}} V$.

Case 2: For every j there is a cycle (j, k_j) that eventually waits at step (10) or (14) forever. *(Every row runs finitely.)*

Fix j. Let s_j be the last stage at which some cycle, (j, k_j) say, runs. By our assumption, s_j exists. Then cycle (j, k_j) defines $\Delta_\xi^U(j)$ at stage s_j, and computes $V(j)$ correctly. Therefore, $\Delta_\xi^U = V$, $V \leq_{\mathrm{T}} U$, another contradiction. □

Lemma 4.4.2 *For all $n \in \omega$,*

(1) $\delta \upharpoonright n$ can be initialized at most finitely often.

(2) $\delta \upharpoonright n$ acts at most finitely often.

(3) If $\delta \upharpoonright n$ is a $\mathcal{P}, \mathcal{S}$, $\mathcal{R}$, or $\mathcal{T}$-strategy, then $\delta \upharpoonright n$ is satisfied.

Proof. We prove Lemma 4.4.2 by induction. Assume that the lemma holds for all $m < n$. By (2), we can assume that there is stage s_0 after which no $\delta \upharpoonright m$ with $m < n$ acts. Let $\xi = \delta \upharpoonright n$, and ξ^- be the immediate predecessor of ξ. Without loss of generality, we assume that $\xi = \xi^- {}^\frown (j, k)$. Then by the fact that $U <_{\mathrm{T}} V$, only finitely many cycles $(j', k') < (j, k)$ can be accessible during the whole construction. Since ξ is on the true path, we know that each such cycle (j', k') can be visited at most finitely many times, and hence each such cycle can initialize cycle (j, k) at most finitely many times. Therefore, we have a stage $s_1 > s_0$ after which ξ cannot be initialized anymore. (1) holds.

For (2) and (3), there are 4 cases:

Case 1: ξ is a $\mathcal{P}$-strategy.

Let cycle (j_0, k_0) be the least one being visited infinitely often. By Lemma 4.4.1, (j_0, k_0) exists. Let $s_2 \geq s_1$ be the last stage at which cycle (j_0, k_0) is canceled, and let $s_3 \geq s_2$ be the stage at which $x = x_{j_0,k_0}$ is defined. Then x cannot be redefined anymore. Let $\Gamma^U_{\xi,j_0}(k_0)$ be defined equal to $V(k_0)$ at stage $s_4 > s_3$. Then at stage s_4,

$$B_{s_4}(x) = 0 = \Phi_\xi^{W_\xi}(x)[s_4],$$

and there is a (least) v_1 such that

$$\Psi_{\xi}^{(U\oplus B)_{s_4}\upharpoonright v_1} \upharpoonright \varphi_{\xi}(x) = W_{\xi,s_4} \upharpoonright \varphi_{\xi}(x).$$

Without loss of generality, suppose that cycle (j_0, k_0) acts at stage $s_5 > s_4$. Then at stage s_5, x is put into B (action at (6)) making

$$B_{s_5}(x) = 1 \neq 0 = \Phi_{\xi}^{W_{\xi}}(x)[s_5].$$

Note that $k_0 \in V_{s_5}$. If later, U has a change below v_1, then we can redefine $\Gamma_{\xi,j_0}^{U}(k_0) = 1 = V(k_0)$ with use 0, and therefore, (j_0, k_0) will start a new cycle, (j_0, k_0) cannot be visited infinitely often, contradicting our choice of (j_0, k_0).

Let $s_6 > s_5$ be the stage at which cycle (j_0, k_0) goes to (7). Then

$$B_{s_6}(x) = 1 = \Phi_{\xi}^{W_{\xi}}(x)[s_6],$$

and there is a (least) v_2 such that

$$\Psi_{\xi}^{(U\oplus B)_{s_6}\upharpoonright v_2} \upharpoonright \varphi_{\xi,s_6}(x) = W_{\xi,s_6} \upharpoonright \varphi_{\xi,s_6}(x),$$

$B \upharpoonright v_2$ is restrained, and $\Delta_{\xi}^{U}(j_0)$ is defined as $V(j_0)$ with use v_2. Now if (j_0, k_0) acts at stage $s_7 > s_6$, then $j_0 \in V_{s_7}$, and x is taken out of B, making $B_{s_7} \upharpoonright v_1 = B_{s_4} \upharpoonright v_1$. Note that between stages s_4 and s_7, W_{ξ} must have changed below $\varphi_{\xi,s_4}(x)$, at z say. The removal of x recovers the computation $\Psi_{\xi}^{(U\oplus B)\upharpoonright v_1}(z)$ to $\Psi_{\xi}^{(U\oplus B)\upharpoonright v_1}(z)[s_4]$, and hence $\Psi_{\xi}^{(U\oplus B)\upharpoonright v_1}(z)[s_7] \neq W_{\xi,s_7}(z)(x)$. ξ is satisfied, and (3) holds. Again, if $U \upharpoonright v_1$ changes later, then cycle (j_0, k_0) would be stopped, and at the same time, (j_0, k_0) will start a larger cycle, contradicting the choice of (j_0, k_0) again.

Case 2: ξ is an $\mathcal{S}$-strategy.

The same as discussed in case 1.

Case 3: ξ is an $\mathcal{R}$-strategy.

Then ξ acts only when a confirmed error gets a V-permission. By Lemma 4.4.1, let (j_0) be the least cycle being visited infinitely often during the construction, and let stage s_0 be the last stage at which cycle (j_0) gets canceled. Then cycle (j_0) starts to construct functional $\Gamma_\xi^{U,B}$ such that if $\Phi_\xi^{U,B,D} = W_\xi$, then $\Gamma_\xi^{U,B}$ is well-defined and computes W_ξ correctly. If cycle (j_0) does no action after stage s_0 (hence (2) holds), then either there is some y such that $\Phi_\xi^{U,B,D}(y) \uparrow$ or $\Gamma_\xi^{U,B}$ is well-defined. In both cases, ξ is satisfied and (3) holds. Without loss of generality, suppose that cycle (j_0) acts after stage s_0, and the associated error is y. Let $\Gamma_\xi^{U,B}(y)$ be defined at stage s_1 with use $\gamma_\xi(y)[s_1] = s_1$. Suppose that at stage $s_2 > s_1$, ξ confirms that y is an error. Then y enters W_ξ between stages s_1 and s_2, and $\Phi_\xi^{U,B,D}(y)[s_1] = 0$. By the η-strategy, at stage s_2, cycle (j_0) defines $\Delta_\xi^U(j_0) = V(j_0)$ with use $\delta_\xi(j_0) = \varphi_\xi(y)[s_1]$, and waits for $U \upharpoonright \varphi_\xi(y)[s_1]$ or $V(j_0)$ to change. If $U \upharpoonright \gamma_\xi(y)[s_1]$ changes first, then $\Gamma_\xi^{U,B}(y)$ is undefined and will be redefined equal to $W_\xi(y)[s_2]$, which is 1, and cycle (j_0) cannot act at y, contradicting our choice of y. Thus, $V(j_0)$ changes first, and ξ executes the diagonalization procedure (i.e., remove from D numbers entering after stage s_1, and also puts $s_1 = \gamma_\xi(y)[s_1]$ into B, for the sake of other $\mathcal{R}$-strategies). Now we claim that $\Phi_\xi^{U,B,D}(y) \neq W_\xi(y)$, where $W_\xi(y) = 1$, since the computation $\Phi_\xi^{U,B,D}(y)$ is recovered to $\Phi_\xi^{U,B,D}(y)[s_1]$, which is 0, and $\Phi_\xi^{U,B,D}(y)$ is protected from the injury of lower priority strategies. On the other hand, by the choice of s_0, no higher priority strategies and cycles can act after s_0, and hence, can enumerate or remove numbers from B or D. The only way to change the computation $\Phi_\xi^{U,B,D}(y)$ is the U-change on $\varphi_\xi(y)[s_1]$. However, if it happens, then cycle (j_0) will redefine $\Delta_\xi^U(j_0)$ such that $\Delta_\xi^U(j_0) = V(j_0) = 1$ with use 0. $\xi^\frown(j_0)$ cannot be visited anymore, contradicting our choice of j_0. Therefore, $\Phi_\xi^{U,B,D}(y) \neq W_\xi(y)$, ξ is satisfied, and ξ has no cycles acting after stage s_2. (2) and (3) hold.

Case 4: ξ is a $\mathcal{T}$-strategy.

Similar to Case 3. □

By Theorem 4.4.2, $|\delta| = \infty$ and each strategy on δ is satisfied, except the $\mathcal{Q}$-strategies.

Lemma 4.4.3 *For all e, $\mathcal{Q}_e$ is satisfied.*

Proof. Fix e. Without loss of generality, suppose that $\Theta_e^{U,B} = W_e$. Then there are infinitely many $\mathcal{Q}_e$-expansionary stages and for each $i \in \omega$, there is an $\mathcal{S}_{e,i}$-strategy on the true path. By Lemma 4.4.2, all $\mathcal{S}_{e,i}$ are satisfied, and hence, $C_e \neq \Omega_i^{W_e}$.

Fix x. We claim that $x \in C_e$ iff

(1) x and $x+1$ are chosen to be attackers by some $\mathcal{S}_{e,i}$-strategy.

(2) exactly one of x and $x+1$ is in B.

Therefore, $C_e \leq_{\mathrm{T}} U \oplus B$. The "if" direction is obviously, because x enters B at the same stage when x enters C_e, and $x+1$ enters B only after x is taken out of B. To show that "only if" holds, suppose that x and $x+1$ are chosen by cycle (j,k) of $\mathcal{S}_{e,i}$-strategy β, and let x be enumerated into C_e and B at stage s_1. That is, (6) of cycle (j,k) happens at stage s_1. If cycle (j,k) stops at (6), then, x cannot be taken out of B, and $x+1$ has no chance to be put into B. So (2) holds. Otherwise, suppose that x is removed from B at stage $s_2 > s_1$. Then the diagonalization executed at (6) fails, and cycle (j,k) sees an inequality between $\Theta_e^{U,B}$ and W_e, which is a possible global win for $\mathcal{Q}_e$. However, by our assumption that $W_e = \Theta_e^{U,B}$, eventually, U will change below the associated use, and so cycle (j,k) goes to (13), redefining $\Gamma_{\beta,j}^U(k) = V(k) = 1$, putting $x+1$ into B. Since (j,k) is stopped, no other cycles or strategies can take number $x+1$ out of B. So (2) holds. $\square$

Now we prove that the requirement $\mathcal{G}$ is satisfied. We need the following lemma about our construction. Fix $\xi \in T$. Define the $permission - bound(\xi)$ as

$$\max(\{j : \exists \xi'[\xi' {}^\frown (j) \subseteq \xi]\} \cup \{j+k : \exists \xi'[\xi' {}^\frown (j,k) \subseteq \xi]\}),$$

which is an upper bound for all numbers used in outcomes of nodes $\subset \xi$. Let $s(\xi)$ be the least stage such that

$$V_{s(\xi)} \restriction permission - bound(\xi) = V \restriction permission - bound(\xi).$$

Then $permission - bound(\xi)$ is computable and $s(\xi)$ is V-computable. The next lemma is from LaForte [54], which says that the delayed permission cannot be delayed too long. This is the essential point of the density argument.

Lemma 4.4.4 *Let $\xi \subseteq \delta_{s_0}$ and $s_1 \geq s_0, s(\xi)$ be any U-true stage. If ξ is not visited at stage s_1, then after stage s_1, ξ is initialized before being accessible again.*

Proof. We prove the lemma by induction on the length of ξ. Obviously, the lemma is true for the root node, λ, since each stage is a λ-stage.

Assume that the lemma holds for ξ. Without loss of generality, suppose that ξ is a $\mathcal{P}$-strategy and suppose that $\xi^\frown(j,k) \subseteq \delta_{s_0}$. Let $s_1 > s_0$ be a U-true stage and let $s_2 > s_1$ be the first stage at which $\xi^\frown(j,k)$ is accessible. We now show that $\xi^\frown(j,k)$ is initialized between stages s_0 and s_2. Note that we can assume that ξ is not initialized between stages s_0 and s_2 (otherwise, $\xi^\frown(j,k)$ would be initialized automatically) and hence $\xi \subseteq \delta_{s_1}$.

By the fact that $\xi^\frown(j,k)$ is visited at stage s_0, we know that cycle (j,k) was started before stage s_0+1. There are two cases:

Case 1: Cycle (j,k) is canceled between stages s_0 and s_2.

Let s_3 be the stage at which (j,k) is canceled by some cycle $(j',k') < (j,k)$. We show that the initialization of $\xi^\frown(j,k)$ occurs before stage s_3.

Obviously, if cycle (j',k') progresses but does not go to (12), then $\xi^\frown(j',k')$ is visited at stage s_3 and hence, $\xi^\frown(j,k)$ is initialized. Thus, we can assume that at stage s_3, cycle (j',k') goes to (12).

Without loss of generality, suppose that cycle $(j',k'+1)$ is canceled and restarted at stage s_3 (as to the case that cycle $(j'+1,0)$ is started, the argument is almost the same). Thus, the nontrivial case is when $j=j', k=k'+1$. By our assumption that $\xi^\frown(j,k)$ is visited at stage s_0, cycle (j',k') must be waiting at step (5) or (10) for a V-permission at stage s_0. However, to reach (12) at stage s_3, cycle (j',k') must have already acted at some stage between s_0 and s_3, s_4 say. Therefore, at stage s_4, $\xi^\frown(j',k')$ is visited, and $\xi^\frown(j,k)$ is initialized.

Case 2: Otherwise.

Remember that we are assuming that $\xi \subseteq \delta_{s_1}$ and $\xi^\frown(j,k) \not\subseteq \delta_{s_1}$. Since $\xi^\frown(j,k)$ is not canceled at stage s_1, there must be some $(j'',k'') > (j,k)$ such that $\xi^\frown(j'',k'') \subseteq \delta_{s_1}$. Thus, cycle (j,k) must be waiting at (5), (10), or (12). If (j,k) is waiting at (12), then ξ can never have (j,k) as its outcome unless it is canceled, which is impossible by our assumption. If (j,k) is waiting at (5), then it is waiting for a V-permission at k. By the choice of s_1, $k \in V_{s_1}$ iff $k \in V$, and the computations $\Psi_{e,s_1}^{U_{s_1},B_{s_1}} \upharpoonright \varphi_{e,s_1}(x)$ cannot be changed by the U-changes. Thus, if cycle (j,k) enters (6) before being canceled, then such an action must happen at stage s_1, and hence (j,k) itself is initialized. Otherwise, (j,k) cannot reach (6) anymore without being canceled first. Again, by the choice of s_1, as long as cycle (j,k) remains in (5), ξ will have an outcome larger than (j,k). Thus, by the assumption that $\xi^\frown(j,k)$ is visited at stage s_2, (j,k) must have already been canceled between stages s_1 and s_2. A contradiction. $\square$

Lemma 4.4.5 *$B, D \leq_{\mathrm{T}} V$.*

Proof. First, we prove that $D \leq_{\mathrm{T}} V$. Note that during the whole construction, only $\mathcal{T}$-strategies can enumerate numbers into D and only $\mathcal{R}$-strategies can remove numbers out of D.

Fix x. Then we can computably decide whether x is chosen as an attacker of some cycle of a $\mathcal{T}$-strategy ξ. If not, then $x \notin D$. Otherwise, suppose that x is chosen by cycle (j) at stage s_0 say. If $j \notin V$, then x can never get V-permission to enter D, so $x \notin D$. If $j \in V$, and $j \in V_{s_0}$, then cycle (j) goes to the final point directly, and again, $x \notin D$.

Suppose that j enters V at stage $s_1 > s_0$. By Lemma 4.4.4, x can only enter D before the next U-true stage s_2, which is larger than s_0, s_1 and $s(\xi)$. Because s_0 is computable, both s_1 and $s(\xi)$ are V-computable, and $U \leq_{\mathrm{T}} V$, s_2 is V-computable. Thus, if $x \notin D_{s_2}$, then $x \notin D$, and so, we only need to consider the case that x enters D before stage s_2. Let s_3 be the stage at which x enters D. Then $s_3 \leq s_2$.

Now, x can only be removed from D by $\mathcal{R}$-strategies β with $\beta \subset \xi$. Note that at stage s_3, only a finite number of $\mathcal{R}$-strategies are active, each of which runs cycle (j_β). Let $s'(\xi)$ be the maximum of all $s(\beta^\frown(k))$. Then by Lemma 4.4.4, there is a U-true stage s_5 after which x cannot be moved out by these βs, where s_5 is V-computable as before. Therefore, $x \in D$ iff $x \in D_{s_5}$. Therefore, $D \leq_{\mathrm{T}} V$.

Now we prove that $B \leq_{\mathrm{T}} V$. The main idea is almost the same as the proof of $D \leq_{\mathrm{T}} V$, and hence, in the following, we just consider two different ways of enumerating numbers into B.

The first is the enumeration of the $\mathcal{R}$-strategies. In this case, only when some number $y < x$ is moved out of D can x be enumerated into B. By the discussion above, V can tell whether a number $y < x$ can be enumerated into D at stage s and whether $y \in D_s$ can be moved out after stage s, and hence V can determine whether x can be enumerated into B by $\mathcal{R}$-strategies.

The second is the enumeration of the $\mathcal{Q}$-strategies, ζ say. Suppose that x and $x+1$ are selected by cycle (j,k) of $\mathcal{S}$-strategy β, a substrategy of ζ. Then $x+1$ must enter B when ζ finds that U has a change below v_1 *(at (13) of cycle* (j,k)*)*. That is, after j and k enter V and U has a change below the use v_1 in (3). By the fact that $U \leq_{\mathrm{T}} V$, V knows whether U has a change below v_1, and hence, V knows whether $x+1$ enters B or not.

Now we conclude that $B \leq_{\mathrm{T}} V$. $\square$

This completes the proof of Theorem 3. $\square$

CHAPTER 5

PSEUDO-ISOLATED DEGREES AND THE JUMP OPERATOR

5.1 Introduction

In this chapter, we continue the study of pseudo-isolation phenomena in the d.c.e. degrees. We utilize the technique developed in chapter 3 to show that there are d.c.e. degrees $\mathbf{b} < \mathbf{d}$ such that $\mathbf{b}$ pseudo-isolates $\mathbf{d}$, $\mathbf{d}$ is high and $\mathbf{b}$ is low.

Theorem 4 *There are a high degree* $\mathbf{d}$ *and a low nonisolated (properly) d.c.e. degree* $\mathbf{b} < \mathbf{d}$ *such that* $\mathbf{b}$ *bounds all c.e. degrees below* $\mathbf{d}$.

This theorem shows a surprising gap between the jump properties of the c.e. information below a d.c.e. set even when such a set cannot be isolated by a single c.e. degree below it.

To prove Theorem 4, we build d.c.e. sets B, D, and a partial computable functional Λ in stages, satisfying the following requirements:

$$\mathcal{P}_e: \ B \neq \Phi_e^{W_e} \ \vee \ W_e \neq \Psi_e^B, \tag{5.1}$$

$$\mathcal{Q}_e: \ \Theta_e^B = W_e \Rightarrow (\exists \text{ c.e. } U_e \leq_{\mathrm{T}} B)(\forall i)(U_e \neq \Omega_i^{W_e}), \tag{5.2}$$

$$\mathcal{R}_e: \ \Phi_e^{B\oplus D} = W_e \Rightarrow \exists \Gamma_e(\Gamma_e^B = W_e), \tag{5.3}$$

$$\mathcal{H}_e: \ \lim_n \Lambda^{B\oplus D}(e,n) = Tot(e), \tag{5.4}$$

$$\mathcal{L}_e: \ \exists^\infty s(\Phi_e^B(e)[s]\downarrow) \ \Rightarrow \ \Phi_e^B(e)\downarrow, \tag{5.5}$$

where Tot is the Π^0_2-complete index set $\{e : \varphi_e \text{ total}\}$, and $\{\langle \Phi_e, \Psi_e, \Theta_e, W_e \rangle : e \in \omega\}$ is a standard enumeration of all $\langle \Phi, \Psi, \Theta, W \rangle$ for which Φ, Ψ, Θ are partial computable functionals and W is a c.e. set.

Let $\mathbf{b}, \mathbf{d}$ be the Turing degrees of B and $B \oplus D$ respectively. By the $\mathcal{P}$-strategies, $\mathbf{b}$ is a properly d.c.e. degree. By the $\mathcal{Q}$-strategies, $\mathbf{b}$ is nonisolated. The $\mathcal{H}$ and $\mathcal{L}$-strategies guarantee that $\mathbf{d}$ is high and $\mathbf{b}$ is low and so $\mathbf{b} < \mathbf{d}$. The $\mathcal{R}$-strategies ensure that $\mathbf{b}$ bounds all c.e. degrees below $\mathbf{d}$. As explained in chapter 4, $\mathbf{d}$ in Theorem 4 is nonisolated.

5.2 Basic strategies

In this section, we describe the basic strategies designed to satisfy the requirements listed above. All strategies will be arranged on a priority tree, and a node on the tree is identical to a strategy.

$\mathcal{P}$-strategies

A $\mathcal{P}$-strategy, α say, working for a $\mathcal{P}$-requirement, $\mathcal{P}_e$, attempts to find a number witnessing $B \neq \Phi_e^{W_e}$ or $W_e \neq \Psi_e^B$.

α works as follows:

(1) Choose x as a fresh number,

(2) Wait for a stage s such that

$$B_s(x) = 0 = \Phi_e^{W_e}(x)[s] \text{ and}$$

$$W_{e,s} \upharpoonright \varphi_{e,s}(x) = \Psi_{e,s}^{B_s} \upharpoonright \varphi_{e,s}(x).$$

(If this never happens, then x is a witness to the success of $\mathcal{P}_e$.)

(3) Put x into B. Protect $B_s \upharpoonright s$ from other strategies.

(4) Wait for a stage s' such that

$$B_{s'}(x) = 1 = \Phi_e^{W_e}(x)[s'], \text{and}$$

$$W_{e,s'} \upharpoonright \varphi_{e,s}(x) = \Psi_{e,s'}^{B_{s'}} \upharpoonright \varphi_{e,s}(x).$$

(If this never happens, then again, we will have found a witness to the success of $\mathcal{P}_e$. If it happens, then the change in $\Phi_e^{W_e}(x)$ between stages s and s' can only be brought about by a change in $W_e \upharpoonright \varphi_{e,s}(x)$, which is irreversible since W_e is a c.e. set.)

(5) Remove x from B and restrain $B \upharpoonright s$ from other strategies.

(Now x is a permanent witness to the success of $\mathcal{P}_e$ because $W_e \upharpoonright \varphi_{e,s}(x) \neq \Psi_e^B \upharpoonright \varphi_{e,s}(x)$.)

α has two outcomes, $0 <_{\mathrm{L}} 1$, where 0 stands for the case that α stops at step (5), and 1 for the cases that α waits at step (2) or step (4).

$\mathcal{Q}$-strategies and substrategies

A $\mathcal{Q}_e$-strategy, ζ say, attempts to construct a c.e. set U_e such that if $W_e = \Theta_e^B$, then $U_e \leq_T B$ and for all $i \in \omega$, $U_e \neq \Omega_i^{W_e}$. Define

$$\begin{aligned}
\ell_{\mathcal{Q}}(\zeta, s) &= \max\{x < s : (\forall y < x)(\Theta_e^B(y)[s] \downarrow= W_{e,s}(y))\},\\
m_{\mathcal{Q}}(\zeta, s) &= \max\{0, \ell_{\mathcal{Q}}(\zeta, t) : t < s \ \&\ t \text{ is a } \zeta\text{-stage}\}.
\end{aligned}$$

Stage s is ζ-expansionary if $s = 0$ or s is a ζ-stage and $\ell_{\mathcal{Q}}(\zeta, s) > m_{\mathcal{Q}}(\zeta, s)$. ζ has two outcomes: $0 <_{\mathrm{L}} 1$, where 0 stands for the case that there are infinitely many ζ-expansionary stages, and 1 for the other case.

Suppose that ζ has outcome 0. Then the construction of U_e will be carried out by $\mathcal{Q}_e$'s substrategies, $\mathcal{S}_{e,i}$, $i \in \omega$, which are arranged in the cone below $\zeta ^\frown \langle 0 \rangle$.

$$\mathcal{S}_{e,i} : U_e \neq \Omega_i^{W_e}, \tag{5.6}$$

Let β be an $\mathcal{S}_{e,i}$-strategy. Then β tries to figure out a disagreement between U_e and $\Omega_i^{W_e}$ or between W_e and Θ_e^B. β proceeds as follows:

(1) Choose x as a fresh number.

(2) Wait for a stage s such that

$$\Omega_{i,s}^{W_{e,s}}(x) \downarrow = 0, \quad \text{and}$$

$$W_{e,s} \upharpoonright \omega_{i,s}(x) = \Theta_{e,s}^{B_s} \upharpoonright \omega_{i,s}(x).$$

(If this never happens, then x is a witness to the success of $\mathcal{S}_{e,i}$.)

(3) Put x into U_e and B. Protect $B \upharpoonright s$ from other strategies.

(4) Wait for a stage s' such that

$$\Omega_{i,s'}^{W_{e,s'}}(x) \downarrow = 1, \quad \text{and}$$

$$W_{e,s'} \upharpoonright \omega_{i,s}(x) = \Theta_{e,s'}^{B_{s'}} \upharpoonright \omega_{i,s}(x).$$

(If this never happens, then again x is a witness to the success of $\mathcal{S}_{e,i}$. If it happens, then the change in $\Omega_i^{W_e}(x)$ between stages s and s' can only be brought about by a change in $W_e \upharpoonright \omega_{i,s}(x)$, which is irreversible since W_e is a c.e. set.)

(5) Remove x from B and protect $B \upharpoonright s$ from other strategies.

(Now x is a permanent witness to the success of $\mathcal{S}_{e,i}$ because

$$\Theta_e^B \upharpoonright \omega_{i,s}(x) = \Theta_{e,s}^B \upharpoonright \omega_{i,s}(x) = W_{e,s} \upharpoonright \omega_{i,s}(x) \neq W_e \upharpoonright \omega_{i,s}(x).$$

Note that taking x from B leads to a global win on $\mathcal{Q}_e$. U_e is no longer needed, and so we don't need to care about the loss of B-permission for x (which is left in U_e).)

β has one outcome 1. Note that if β reaches step (5), then an inequality between W_e and $\Theta_e(B)$ is created, and so β achieves a global win.

$\mathcal{R}$-strategies

An $\mathcal{R}_e$-strategy, η say, constructs a partial functional Γ_η such that if $\Phi_e^{B\oplus D} = W_e$, then $\Gamma_\eta^B = W_e$. First, define

$$\ell_{\mathcal{R}}(\eta, s) = \max\{x < s : (\forall y < x)(\Phi_e^{B\oplus D}(y)[s] \downarrow = W_{e,s}(y))\},$$
$$m_{\mathcal{R}}(\eta, s) = \max\{0, \ell_{\mathcal{R}}(\eta, t) : t < s \,\&\, t \text{ is an } \eta\text{-stage}\}.$$

Here, all parameters involved in a computation at stage s are bounded by s. Say that stage s is η-expansionary if $s = 0$ or s is an η-stage and $\ell_{\mathcal{R}}(\eta, s) > m_{\mathcal{R}}(\eta, s)$.

At an η-expansionary stage s, for any $y < \ell_{\mathcal{R}}(\eta, s)$, if $\Gamma_\eta^B(y)[s] \uparrow$, then define $\Gamma_\eta^B(y)[s] = W_{e,s}(y)$ with $\gamma_\eta(y)[s] = s$. Obviously, any change on $B \upharpoonright \varphi_{e,s}(y)$ allows us to redefine $\gamma_\eta(y)$ because $\varphi_{e,s}(y) < s$. During the construction, $\mathcal{H}$-strategies may enumerate numbers (z say) into D, and injure the computation $\Phi_e^{B\oplus D}(x)[s]$. This provides chances for W_e to change at x, making $\Gamma_\eta^B(x)$ incorrect. Suppose that s' is the first η-expansionary stage after x enters W_e. By taking out these numbers (z, for example), we can recover the computation $\Phi_e(B \oplus D; x)$ to $\Phi_e(B \oplus D; x)[s]$. Such an action creates a (permanent) inequality between $\Phi_e^{B\oplus D}$ and W_e (at x). Because W_e is c.e., $\mathcal{R}_e$ is satisfied.

In our construction, since we are making B properly d.c.e., the isolation strategy is a little more complicated than the standard ones. Suppose that $\Gamma_\eta^B(x)$ gets defined at stage s_0, and a $\mathcal{P}$ (or $\mathcal{S}$)-strategy ξ with lower priority enumerates a number $z < \varphi_\eta(x)$ into B at stage $s_1 > s_0$. As mentioned above, such an enumeration forces (and allows) us to lift $\gamma_\eta(x)$ to a larger number, $\gamma_\eta(x)[s_1]$. Later, at stage s_2, to get a disagreement, ξ takes z out, and thus, $\gamma_\eta(x)$ returns to its definition at stage s_0. Such a variation does no harm to the disagreement strategy of η. Suppose that η observes at some stage between s_1 and s_2 that $\Gamma_\eta^B(x)$ is incorrect and performs the disagreement strategy, then ξ will be initialized, and so ξ has no chance to take z out. In this case, s_2 does not exist. In case that s_2 does exist, and η finds an incorrectness of $\Gamma_\eta^B(x)$ at stage $s_3 > s_2$, since at stage s_2, $\gamma_\eta(x)$ returns to that of stage s_0, we have that

$$B_{s_2} \upharpoonright \varphi_{e,s_0} = B_{s_0} \upharpoonright \varphi_{e,s_0}.$$

Now by the fact that $\gamma_\eta(x)[s_3] = s_0$, we have

$$B_{s_3} \upharpoonright \varphi_{e,s_0} = B_{s_0} \upharpoonright \varphi_{e,s_0}.$$

This guarantees the success of η's disagreement strategy.

η has three outcomes, $0 <_{\mathrm{L}} d <_{\mathrm{L}} 1$, where d stands for the fact that the disagreement strategy succeeds, and 0 and 1 for the cases of whether or not there are infinitely many η-expansionary stages.

Since $\mathcal{R}_e$'s disagreement strategy initializes all strategies of lower priority, we only need to consider the consistency between η and the higher priority $\mathcal{R}$-strategies. Fix $\eta' < \eta$ and let $s_0 < s$ be the stage at which $\Gamma_\eta(y)$ was defined. For $\Gamma^B_{\eta'}(y)$ defined before stage s_0, the removal action just recovers the computation $\Phi^{B\oplus D}_{\eta'}(y)$ back to that of stage s_0, and so does not injure the correctness of $\Gamma^B_{\eta'}(y)$. If $\Gamma^B_{\eta'}(y)$ is defined after s_0, taking z out does affect the computation $\Phi^{B\oplus D}_{\eta'}(y)$, but the enumeration of s_0 into B allows us to redefine $\Gamma^B_{\eta'}(y)$. Thus, η's action does not injure η'.

$\mathcal{H}$-strategies

The basic idea of $\mathcal{H}$-strategies is to construct a functional Λ such $\Lambda^{B\oplus D}$ approximates Tot at the limit. We ensure that $\Lambda^{B\oplus D}$ is total, and for all e,

$$\varphi_e \text{ is total } \iff \lim_{n\to\infty} \Lambda^{B\oplus D}(e,n) = 1.$$

By Shoenfield's Limit Lemma, $\emptyset'' \leq_{\mathrm{T}} Tot \leq_{\mathrm{T}} (B \oplus D)'$ and $B \oplus D$ is high.

Our construction will be carried out on a finitely branching tree T. Let $[T]$ be the set of infinite paths through T. Fix e. The construction of T ensures that on each $f \in [T]$, there is a unique $\mathcal{H}_e$-strategy, and so all $\mathcal{H}_e$-strategies on T are above certain level, $8e^2$ say, working jointly to define $\Lambda^{B\oplus D}(e,-)$. Thus we can number all $\mathcal{H}_e$-strategies on T in an effective way in the lexicographical order, and use $k(\tau)$ to denote that τ is the $k(\tau)$-th one in this order. Define $k(e) = \max\{k(\tau) : \tau \text{ is an } \mathcal{H}_e\text{-strategy on } T\}$. For each n, we first define $\Lambda^{B\oplus D}(e,n)[0] = 0$ with use $\langle e, k(e), n\rangle$.

The number $\langle e, k(\tau), n\rangle$ is reserved for τ to redefine $\Lambda^{B\oplus D}(e,n)$ (as 1) later.

Fix τ as an $\mathcal{H}_e$-strategy. Define s_τ as the last stage at which τ is initialized so far. τ is only concerned with the definition of $\Lambda^{B\oplus D}(e,x)$ for those $x > s_\tau$. If τ is on the true path, then we will ensure that for all $x > s_\tau$, $\Lambda^{B\oplus D}(e,x)$ is well-defined and the limit is equal to $Tot(e)$. Define

$$\ell_{\mathcal{H}}(\tau, s) = \max\{x < s : \varphi_e(y)[s]\downarrow \text{ for all } y \leq x\},$$

$$m_{\mathcal{H}}(\tau, s) = \max\{0, \ell_{\mathcal{H}}(\tau, t) : t < s \text{ and } t \text{ is a } \tau\text{-stage}\}.$$

Say that s is τ-expansionary if $s = 0$ or s is a τ-stage and $\ell_{\mathcal{H}}(\tau, s) > m_{\mathcal{H}}(\tau, s)$.

τ has two outcomes, $0 <_{\mathrm{L}} 1$, where 0, 1 stand for the cases of whether or not there are infinitely many τ-expansionary stages. Clearly, τ has outcome 0 if and only if φ_e is total.

Let τ be the $\mathcal{H}_e$-strategy on the true path. Then s_τ is defined as the last stage at which τ is initialized.

Suppose that τ has outcome 0. Then there are infinitely many τ-expansionary stages, and τ will define $\Lambda^{B\oplus D}(e,n)$ for all $n > s_\tau$. At any τ-expansionary stage, τ acts by enumerating numbers $\langle e, k(\tau), n\rangle$, where $m_{\mathcal{H}}(\tau, s) \leq n < \ell_{\mathcal{H}}(\tau, s)$, into D to redefine $\Lambda^{B\oplus D}(e,n) = 1$. Thus, for almost all n, $\Lambda^{B\oplus D}(e,n)$ is defined as 1, and so $\lim_n \Lambda^{B\oplus D}(e,n) = 1 = Tot(e)$.

If τ has outcome 1, then let s_1 be the last τ-expansionary stage. That is, $n_1 = \ell_{\mathcal{H}}(\tau, s_1)$ is the least point at which φ_e diverges. Then, for any $\mathcal{H}_e$-strategy, τ' say, $\ell_{\mathcal{H}}(\tau', s)$ cannot exceed n_1 during the whole construction. Hence, for any $n > n_1$, $\Lambda^{B\oplus D}(e,n)$ cannot be redefined later, $\Lambda^{B\oplus D}(e,n)\downarrow= 0$. This makes $\lim_n \Lambda^{B\oplus D}(e,n) = 0$.

Interactions between $\mathcal{R}$ and $\mathcal{H}$-strategies

Consider the case that η works below some higher priority $\mathcal{H}$-strategy τ with $\tau^\frown\langle 0\rangle \subseteq \eta$. Then, η guesses that τ will enumerate infinitely many numbers into D. Thus, some computations involved in the η-strategy may alternately converge and diverge infinitely often and cause us to falsely believe in a divergent computation. To avoid this, we take η-correct computations whenever we define $\ell_{\mathcal{R}}(\eta)$ to make sure that τ's enumeration cannot destroy η-correct computations.

$\mathcal{L}$-strategies

An $\mathcal{L}_e$-strategy, σ say, attempts to make $\Phi_e^B(e)\downarrow$, if $\Phi_e^B(e)[s]$ converges for infinitely many s. Whenever σ observes that $\Phi_e^B(e)$ converges, at stage s say, σ will settle down a restraint to protect this computation from other strategies. Due to the fact that the construction is finite injury (relative to B), σ's action will succeed, provided that σ can observe that $\Phi_e^B(e)$ converges infinitely often.

One possible case is that $\Phi_e^B(e)$ converges infinitely often between successive σ-stages, but when σ is accessible, $\Phi_e^B(e)$ appears divergent. In this way, even though σ is on the true path, we cannot ensure that σ can observe any convergence and hence the argument mentioned above fails.

Our strategy is to assign the highest priority to the lowness strategies. That is, at each stage s, first check whether there is some $e < s$ such that $\Phi_e^B(e)[s]\downarrow$ and $\mathcal{L}_e$ is not satisfied. If so, choose e as the least one, and the leftmost $\mathcal{L}_e$-strategy ever being accessible, σ^* say, initialize all strategies $> \sigma^*$. Notice that this action is off the true path.

Due to the finitary nature of the construction (relative to B), for each ξ on the true path, only finitely many $\mathcal{L}$-strategies with higher priority can be visited during the whole construction, and hence ξ can be injured in this way only finitely often.

σ has one outcome 1.

5.3 Construction

Before giving the full construction, we define the priority tree, T, on which the construction will be carried out. First, we assign the priority to the requirements as follows:

$$\mathcal{H}_0 < \mathcal{L}_0 < \mathcal{P}_0 < \mathcal{Q}_0 < \mathcal{R}_0 < \mathcal{S}_{0,0} < \mathcal{H}_1 < \mathcal{L}_1 < \mathcal{P}_1 < \mathcal{Q}_1 < \mathcal{R}_1 < \mathcal{S}_{0,1}$$
$$< \mathcal{S}_{1,0} < \mathcal{S}_{1,1} < \mathcal{H}_2 < \cdots < \mathcal{H}_n < \mathcal{L}_n < \mathcal{P}_n < \mathcal{Q}_n < \mathcal{R}_n < \mathcal{S}_{0,n} < \cdots$$
$$< \mathcal{S}_{n-1,n} < \mathcal{S}_{n,0} < \cdots < \mathcal{S}_{n,n} < \mathcal{H}_{n+1} < \cdots,$$

where for any $\mathcal{X}, \mathcal{Y}$, if $\mathcal{X} < \mathcal{Y}$, then $\mathcal{X}$ has higher priority than $\mathcal{Y}$.

The priority tree is defined by induction.

Definition 5.1 *(Satisfaction of strategies) Given $\xi \in T$,*

(1) $\mathcal{H}_e$ is satisfied at ξ, if there is an $\mathcal{H}_e$-strategy τ such that $\tau \subset \xi$;

(2) $\mathcal{L}_e$ is satisfied at ξ, if there is an $\mathcal{L}_e$-strategy σ such that $\sigma \subset \xi$;

(3) $\mathcal{P}_e$ is satisfied at ξ, if there is a $\mathcal{P}_e$-strategy α such that $\alpha \subset \xi$;

(4) $\mathcal{Q}_e$ is satisfied at ξ, if there is a $\mathcal{Q}_e$-strategy ζ such that $\zeta^\frown\langle 1\rangle \subseteq \xi$; $\mathcal{Q}_e$ is active at ξ, if there is a $\mathcal{Q}_e$-strategy $\zeta^\frown\langle 0\rangle \subseteq \xi$. In this case, $\mathcal{Q}_e$ is said to be active at ξ via ζ.

(5) $\mathcal{R}_e$ is satisfied at ξ, if there is a $\mathcal{R}_e$-strategy η such that $\eta \subset \xi$;

(6) $\mathcal{S}_{e,i}$ is satisfied at ξ, if either $\mathcal{Q}_e$ is satisfied at ξ, or $\mathcal{Q}_e$ is active at ξ via ζ, and there is an $\mathcal{S}_{e,i}$-strategy β such that $\zeta \subset \zeta^\frown\langle 0\rangle \subseteq \beta \subset \xi$.

Definition 5.2 *(Construction of T)*

(1) Let λ, the root node, be an $\mathcal{H}_0$-strategy;

(2) The immediate successors of a node are the possible outcomes of the corresponding strategy;

(3) For $\xi \in T$, ξ works for the highest priority requirement which has not been satisfied nor is active at ξ.

For a node, β say, we refer to $e(\beta)$ as the index of the $\mathcal{H}$-strategy to which β is devoted. For other strategies and nodes, we use the same notation.

Construction

In the following, we give the whole construction. At the end of each stage s, define δ_s as the current approximation of the true path. Say that s is a ξ-stage, if $\xi \subseteq \delta_s$.

Stage 0: Initialize all nodes on T, let $B = D = \emptyset$. Define $\Lambda^{B\oplus D}(e,n)[0] = 0$ for all $e, n \in \omega$ with use $\lambda(e,n)[0] = \langle e, k(e), n\rangle$.

Stage $s > 0$:

Step 1. Find some (least) $e < s$ such that $\Phi_e^B(e)[s] \downarrow$, and no $\mathcal{L}_e$-strategy succeeds in satisfying the $\mathcal{L}_e$-requirement. Let σ be the leftmost $\mathcal{L}_e$-strategy being accessible so far. Initialize all nodes $\xi > \sigma$, and declare that $\mathcal{L}_e$ is satisfied via σ at stage s. If there is no such e or σ, do nothing.

Step 2. Define a partial function δ_s of length $\leq s$ by substages.

Substage $t \geq 0$. Given $\xi = \delta_s \upharpoonright t$. Initialize all nodes $>_{\mathrm{L}} \xi$. If $t = s$, define $\delta_s = \xi$, initialize all nodes $> \xi$ and go to the next stage. Otherwise, take action for ξ and determine $\delta_s(t)$ as follows. There are six cases:

Case 1. $\xi = \tau$ is an $\mathcal{H}$-strategy.

If s is τ-expansionary, then for each n with $m_{\mathcal{H}}(\tau, s) \leq n < \ell_{\mathcal{H}}(\tau, s)$, enumerate $\langle e(\tau), k(\tau), n\rangle$ into D, and redefine $\Lambda^{B\oplus D}(e(\tau), n) = 1$ with use unchanged. Define $\delta_s(t) = 0$. Otherwise, do nothing and define $\delta_s(t) = 1$.

Case 2. $\xi = \sigma$ is an $\mathcal{L}$-strategy.

If $\Phi^B_{e(\sigma)}(e(\sigma))[s] \uparrow$ or $\mathcal{L}_{e(\sigma)}$ is satisfied via some $\mathcal{L}_e$-strategy $\leq \sigma$, do nothing and define $\delta_s(t) = 1$.

Otherwise, define $\delta_s = \sigma$, declare that $\mathcal{L}_{e(\sigma)}$ is satisfied via σ at stage s, initialize all strategies with lower priority and go to the next stage.

Case 3. $\xi = \alpha$ is a $\mathcal{P}$-strategy.

1. If $x(\alpha) \uparrow$, then define $x(\alpha)$ as fresh. Define $\delta_s = \alpha$, initialize all strategies with lower priority and go to the next stage.

2. If $x(\alpha)\downarrow$, $\Phi_{e(\alpha),s}^{W_{e(\alpha),s}}(x(\alpha))\downarrow= 0 = B_s(x(\alpha))$, and

$$W_{e(\alpha),s} \upharpoonright \varphi(B_s; e(\alpha), x(\alpha), s) = \Psi_{e(\alpha),s}^{B_s} \upharpoonright \varphi(B_s; e(\alpha), x(\alpha), s),$$

   then enumerate $x(\alpha)$ into B, define $\delta_s = \alpha$, initialize all strategies with lower priority and go to the next stage.

3. If $x(\alpha)\downarrow$, $\Phi_{e(\alpha),s}^{W_{e(\alpha),s}}(x(\alpha))\downarrow= 1 = B_s(x(\alpha))$, and

$$W_{e(\alpha),s} \upharpoonright \varphi(B_{s'}; e(\alpha), x(\alpha), s') = \Psi_{e(\alpha),s}^{B_s} \upharpoonright \varphi(B_{s'}; e(\alpha), x(\alpha), s'),$$

   where s' is the stage at which $x(\alpha)$ is enumerated into B, then take $x(\alpha)$ out of B, define $\delta_s = \alpha$. Declare that α is satisfied, and initialize all strategies with lower priority. Go to the next stage.

4. If α is satisfied, then let $\delta_s(t) = 0$.

5. Otherwise, define $\delta_s(t) = 1$.

Case 4. $\xi = \zeta$ is a $\mathcal{Q}$-strategy.

If s is ζ-expansionary, then define $\delta_s(t) = 0$. Otherwise, define $\delta_s(t) = 1$.

Case 5. $\xi = \eta$ is an $\mathcal{R}$-strategy.

1. If η has already executed the disagreement strategy after stage s_η, define $\delta_s(t) = d$.

2. If s is an η-expansionary stage, and Γ_η^B is correct, then define $\delta_s(t) = 0$, and for those n with $m_{\mathcal{H}}(\eta, s) \leq n < \ell_{\mathcal{H}}(\eta, s)$, define $\Gamma_\eta^B(n)[s] = W_{\eta,s}(n)$ with use $\gamma_\eta(n)[s] = s$.

3. If s is an η-expansionary stage, and there is some y s.t. $\Gamma_\eta^B(y)[s]\downarrow\neq W_{\eta,s}(y)$, then let y be the least one, and s_y be the stage at which $\Gamma_\eta^B(y)[s]$ was defined. Execute the disagreement strategy as follows:

- remove all numbers associated with lower priority strategies that entered D since stage s_y;
- enumerate s_y into B and declare those $\Gamma^B_{\eta'}(-)$ defined after stage s_y undefined;
- define $\delta_s = \eta$, initialize all strategies with lower priority. Go to the next stage.

4. Otherwise, define $\delta_s(t) = 1$.

Case 6. $\xi = \beta$ is an $\mathcal{S}$-strategy.

1. If $x(\beta) \uparrow$, then define $x(\beta)$ as a fresh number, define $\delta_s = \beta$, initialize all strategies with lower priority, go to the next stage.
2. If $x(\beta) \downarrow$, $\Omega^{W_{e(\beta)}}_{i(\beta)}(x(\beta))[s] \downarrow= 0 = U_{e(\beta),s}(x(\beta))$ and
$$W_{e(\beta),s} \upharpoonright \omega_{i(\beta),s}(x(\beta)) = \Theta^{B_s}_{e(\beta),s} \upharpoonright \omega_{i(\beta),s}(x(\beta)),$$
then enumerate $x(\beta)$ into B and U_β. Define $\delta_s = \beta$, initialize all strategies with lower priority and go to the next stage.
3. If $x(\beta) \downarrow$, $\Omega^{W_{e(\beta)}}_{i(\beta)}(x(\beta))[s] \downarrow= 1 = U_{e(\beta),s}(x(\beta))$, and
$$W_{e(\beta),s} \upharpoonright \omega_{i(\beta),s}(x(\beta)) = \Theta^{B_s}_{e(\beta),s} \upharpoonright \omega_{i(\beta),s}(x(\beta)),$$
then take $x(\beta)$ out of B, define $\delta_s = \beta$, initialize all strategies with lower priority, go to the next stage.
4. Otherwise, define $\delta_s(t) = 1$.

This completes the construction.

5.4 Verification

Define the true path f of the construction by induction as follows:

$$f(n) = \mu k(f \upharpoonright n^\frown \langle k \rangle \text{ is visited infinitely often}).$$

Obviously, f is well-defined. For each n, $f \upharpoonright n$ can be initialized only finitely often, and acts finitely often.

For $\xi \subset f$, let s_ξ be the last stage at which ξ is initialized.

Lemma 5.4.1 *($\mathcal{H}_e$-satisfaction Lemma) For all e, $\lim_n \Lambda^{B\oplus D}(e,n) = Tot(e)$.*

Proof. Fix e and let τ be the $\mathcal{H}_e$-strategy on f. The nontrivial case is when $e \in Tot$. That is when φ_e is total. Then there are infinitely many τ-expansionary stages.

Fix $n > s_\tau$. We shall show that $\Lambda^{B\oplus D}(e,n) = 1$ and so $\lim_n \Lambda^{B\oplus D}(e,n) = 1$. Let $s_1 > s_\tau$ be the first stage such that $\ell_{\mathcal{H}}(\tau, s_1) > n$. Then τ enumerates the number $\langle e, k(\tau), n\rangle$ into D and redefine $\Lambda^{B\oplus D}(e,n)[s_1]$ as 1. By the choice of s_τ, $\langle e, k(\tau), n\rangle$ remains in D and so for all $s > s_1$, $\Lambda^{B\oplus D}(e,n)[s] = 1$. Therefore, $\Lambda^{B\oplus D}(e,n) = \lim_s \Lambda^{B\oplus D}(e,n)[s] = 1$.

Thus, $\lim_n \Lambda^{B\oplus D}(e,n) = 1 = Tot(e)$. □

Lemma 5.4.2 *$B \oplus D$ is high.*

Proof. For any e, by Lemma 5.4.1,

$$\varphi_e \text{ is total} \iff \lim_n \Lambda^{B\oplus D}(e,n) = 1.$$

By Shoenfield's Limit Lemma, $Tot \leq_{\mathrm{T}} (B \oplus D)'$. Since Tot is Π^0_2-complete, we have $\emptyset'' \leq_{\mathrm{T}} Tot \leq_{\mathrm{T}} (B\oplus D)'$ and so $B \oplus D$ is high. □

Lemma 5.4.3 *($\mathcal{L}_e$-satisfaction Lemma) Suppose that $\Phi^B_e(e)$ converges infinitely often. Then $\Phi^B_e(e)\downarrow$.*

Proof. Fix σ as the unique $\mathcal{L}_e$-strategy on f. Let $s_0 > s_\sigma$ be the least σ-stage, and $s_1 \geq s_0$ be the least stage at which $\Phi^B_e(e)[s_1]\downarrow$. If $\mathcal{L}_e$ is not satisfied via any $\mathcal{L}_e$-

strategy $\sigma^* \leq \sigma$ so far, then σ initializes all strategies with lower priority. Therefore, no strategy changes $B \upharpoonright s_1$ afterwards, $\Phi_e^B(e) \downarrow = \Phi_e^B(e)[s_1]$.

Otherwise, let s^* be the stage at which $\mathcal{L}_e$ is satisfied via σ^*. Then σ^* can not have been initialized till s_1, because otherwise, σ^*'s actions would be destroyed, and $\mathcal{L}_e$ becomes unsatisfied again, contradicting our assumption. By the choice of $s_1 > s_\sigma$, σ (and hence σ^*) cannot be initialized later, and so the computation $\Phi_e^B(e)[s^*]$ will be preserved forever, and $\Phi_e^B(e) \downarrow$. $\square$

Lemma 5.4.4 *($\mathcal{P}_e$-satisfaction Lemma) $B \neq \Phi_e^{W_e}$ or $W_e \neq \Phi_e^B$.*

Proof. Let α be the $\mathcal{P}_e$-strategy on f and $s_0 > s_\alpha$ be the stage at which $x(\alpha)$ gets its definition. Without loss of generality, suppose that at stage $s_1 > s_0$, α observes the following equations:

$$\Phi_e^{W_e}(x(\alpha))[s_1] \downarrow = 0 = B_{s_1}(x(\alpha)) \text{ and}$$

$$\Psi_{e,s_1}^{B_{s_1}} \upharpoonright \varphi_{e,s_1}(x(\alpha)) = W_{e,s_1} \upharpoonright \varphi_{e,s_1}(x(\alpha)).$$

Then α enumerates $x(\alpha)$ into B and initializes all lower priority strategies. Thus, $B - \{x(\alpha)\} \upharpoonright \varphi_{e,s_1}(x(\alpha)) = B_{s_1} \upharpoonright \varphi_{e,s_1}(x(\alpha))$. Suppose that at α-stage $s_2 > s_1$, $\Phi_e^{W_e}(x(\alpha))[s_2] \downarrow = 1$. (If there is no such a stage, then $\mathcal{P}_e$ is satisfied by the first attempt.) Then, $W_e \upharpoonright \varphi_{e,s_1}(x(\alpha))$ changes between stages s_1 and s_2. For the equality to be restored at a later stage, $\Psi_e^B \upharpoonright \varphi_{e,s_1}(x(\alpha))$ must also change. The only reason for such changes is the enumeration of $x(\alpha)$ into B, and $x(\alpha)$ is less than $\psi_{e,s_1}(\varphi_{e,s_1}(x(\alpha))$. Taking $x(\alpha)$ out at stage s_2 recovers the computations $\Psi_e^B \upharpoonright \varphi_{e,s_1}(x(\alpha))$ to $\Psi_{e,s_1}^{B_{s_1}} \upharpoonright \varphi_{e,s_1}(x(\alpha))$, and so creates a permanent inequality between W_e and Ψ_e^B. So $\mathcal{P}_e$ is satisfied. $\square$

Lemma 5.4.5 *($\mathcal{Q}_e$-satisfaction Lemma) If $\Theta_e^B = W_e$, then there is a c.e. set $U_e \leq_T B$ such that for all $i, U_e \neq \Phi_i^{W_e}$.*

Proof. Let ζ be the $\mathcal{Q}_e$-strategy on f. Suppose that $\Theta_e^B = W_e$, then there are infinitely many ζ-expansionary stages and so $\zeta ^\frown \langle 0 \rangle \subset f$. U_e is constructed by $\mathcal{Q}_e$'s substrategies $\mathcal{S}_{e,i}$, $i \in \omega$, which are arranged below $\zeta^\frown\langle 0\rangle$ by the construction of T.

First, we show that $U_e \leq_T B$. In the construction, when $x(\beta)$ is enumerated into U_e, by β at stage s_0 say, $x(\beta)$ is also enumerated into B. Note that $x(\beta)$ can only be taken out by β, at stage s_2 say. Then such a removal action creates an inequality $\Theta_e^B(x(\beta))[s_2] \neq W_e(x(\beta))$. Remember that β's action initialized all strategies with lower priority strategies, particularly, those $\xi \supseteq \zeta^\frown\langle 1\rangle$. Let $s_3 > s_2$ be any ζ-stage, then by $\Theta_e^B(x(\beta))[s_3] = \Theta_e^B(x(\beta))[s_2] = 0 \neq 1 = W_{e,s_3}(x(\beta))$, s_3 is not ζ-expansionary, and so $\zeta^\frown\langle 1\rangle$ is visited at stage s_3. This is why β cannot be initialized later, and hence the computation $\Theta_e^B(x(\beta))[s_2] = 0$ is preserved forever. Thus, $\zeta^\frown\langle 1\rangle \subset f$, contradicting our assumption.

Now we prove that for any i, $U_e \neq \Phi_i^{W_e}$. Fix i. Let β be the $\mathcal{S}_{e,i}$-strategy on f, and let $x(\beta)$ be defined at stage $s_1 > s_\beta$. Then $U_e(x(\beta)) \neq \Phi_i^{W_e}(x(\beta))$. Without loss of generality, suppose that at stage $s_2 > s_1$, β enumerates $x(\beta)$ into B and U_e, and initializes all lower priority strategies. By the discussion above, $\Phi_i^{W_e}(x(\beta))$ cannot converge to 1 at any β-stage, because otherwise, β will remove $x(\beta)$ from B, make $\zeta^\frown\langle 1\rangle \subset f$, a contradiction. Therefore, $U_e(x(\beta)) = 1 \neq \Phi_i^{W_e}(x(\beta))$. $\square$

Lemma 5.4.6 *($\mathcal{R}_e$-satisfaction strategy) If $W_e = \Phi_e^{B\oplus D}$, then there is a total functional Γ such that $W_e = \Gamma^B$.*

Proof. Let η be the unique $\mathcal{R}_e$-strategy on f and suppose that $\Phi_e^{B\oplus D} = W_e$. Then there are infinitely many η-expansionary stages.

For the sake of a contradiction, we assume that at stage $s_1 > s_\eta$, η finds that $\Gamma_\eta^B(x)$ is not correct. Let $s_0 > s_\eta$ be the stage at which $\Gamma_\eta^B(x)[s_1]$ is defined. Then $\gamma_\eta(x)[s_1] = s_0$, and $W_e(x)$ changes from 0 to 1 between s_0 and s_1. By $\gamma_\eta(x)[s_1] = s_0$, we know that $B_{s_0} \upharpoonright \varphi_{e,s_0}(x) = B_{s_1} \upharpoonright \varphi_{e,s_0}(x)$, and hence the only reason leading to the change of the computation $\Phi_e^{B\oplus D}(x)$ is the numbers enumerated into D by $\mathcal{H}$-strategies. Since at stage s_0, $\Phi_e^{B\oplus D}(x)$ is chosen as η-correct, any τ with higher priority cannot injure this computation. This means that the injury comes from the lower priority $\mathcal{H}$-strategies. η's action at stage s_1 is to take all these numbers out of D, in order to recover the computation $\Phi_e^{B\oplus D}(x)$ to $\Phi_e^{B\oplus D}(x)[s_0]$. Thus, an inequality between $\Phi_e^{B\oplus D}(x)$ and $W_{e,s}(x)$ is created because

$$\Phi_e^{B\oplus D}(x)[s_0] = W_{e,s_0}(x) = 0 \neq 1 = W_{e,s_1}(x).$$

By protecting η from other strategies, we have for any $s > s_1$,

$$\Phi_e^{B\oplus D}(x)[s] = \Phi_e^{B\oplus D}(x)[s_0] \neq W_{e,s_1}(x) = W_{e,s}(x).$$

By the construction, we have that $\eta^\frown\langle d\rangle$ is on f, a contradiction. Note that at stage s_1, we also put s_0 into B to undefine all $\Gamma_{\eta'}^B(x')$ defined after stage s_0. Such an enumeration does no harm to the disagreement strategy of η, because $s_0 > \varphi_{e,s_0}(x)$, but just keeps η consistent with other $\mathcal{R}$-strategies with higher priority.

Thus, we only need to show that Γ_η^B is total.

Fix x. Let $\Gamma_\eta^B(x)$ be defined at stage s_0. Then $\gamma_\eta(x)$ is defined as s_0. Without loss of generality, suppose that at stage $s_1 > s_0$, some $\eta' > \eta$ executes the disagreement strategy, making $\Gamma_\eta^B(x)$ undefined. Then at the next η-expansionary stage s_2, $\Gamma_\eta^B(x)$ gets redefined again with use $\gamma_\eta(x)[s_2] = s_2$. By the construction, all nodes $> \eta'$ are initialized at stage s_1 and cannot undefine $\Gamma_\eta^B(x)[s_2]$ again. Since only finitely many nodes below $\eta^\frown\langle 0\rangle$ with priority higher than η' are visited before stage s_2, and only these strategies' actions can undefine the newly defined $\Gamma_\eta^B(x)$, there is a large enough stage s_x at which $\Gamma_\eta^B(x)$ gets defined and this definition cannot be destroyed afterwards. Thus $\Gamma_\eta^B(x)\downarrow = \Gamma_\eta^B(x)[s_x]$. □

This completes the proof of Theorem 4. □

CHAPTER 6

BI-ISOLATION IN THE D.C.E. DEGREES

6.1 Introduction

Say that a d.c.e. degree $\mathbf{d}$ is *isolated from above* if there is a c.e. degree $\mathbf{e} > \mathbf{d}$ such that any c.e. degree above $\mathbf{d}$ is also above $\mathbf{e}$. Thus, strictly speaking, isolated degrees defined by Cooper and Yi are degrees isolated from below. Let $\mathbf{d}$ be the maximal incomplete d.c.e. degree constructed by Cooper et al. in [18], then $\mathbf{0}'$ isolates $\mathbf{d}$ from above. We have learned from Prof. Marat Arslanov recently that the isolation from above phenomena was also studied by A. Efremov in his papers [34, 35], where Efremov proved the following results:

Theorem 6.1 *(Efremov, [34], also see [3]) For any c.e. degree $\mathbf{b} < \mathbf{0}'$, there are a c.e. degree $\mathbf{a}$ and a d.c.e. degree $\mathbf{d}$ such that $\mathbf{b} < \mathbf{d} < \mathbf{a}$, $\mathbf{b}' = \mathbf{a}'$, and $\mathbf{a}$ isolates $\mathbf{d}$ from above.*

Theorem 6.2 *(Efremov, [35] also see [3]) There is a c.e. degree $\mathbf{b} > \mathbf{0}$ such that any d.c.e. degree below $\mathbf{b}$ cannot be isolated from above. As a consequence, $\mathbf{b}$ isolates no d.c.e. degree below it, and the d.c.e. degrees isolated from above are not dense in the c.e. degrees.*

In this chapter, we investigate the *bi-isolation* phenomenon in the d.c.e. degrees. Say that a c.e. degree $\mathbf{a}$ is a *bi-isolating* degree if there are d.c.e. degrees $\mathbf{d}_1, \mathbf{d}_2$ such that $\mathbf{d}_1 < \mathbf{a} < \mathbf{d}_2$ and $\mathbf{a}$ isolates $\mathbf{d}_2$ from below and isolates $\mathbf{d}_1$ from above respectively. In this case, $\mathbf{a}$ is the unique c.e. degree between $\mathbf{d}_1$ and $\mathbf{d}_2$.

Theorem 5 *Each high isolating degree (from below) is a bi-isolating degree. Therefore, the bi-isolating degrees are dense in the high c.e. degrees.*

Proof. Let $\mathbf{h}$ be any high c.e. degree. By Harrington's result that each high c.e. degree has the anticupping property, there is a c.e. degree $\mathbf{a} < \mathbf{h}$ such that for any c.e. degree $\mathbf{c} < \mathbf{h}$, $\mathbf{a} \cup \mathbf{c} < \mathbf{h}$. Fix $\mathbf{a}$. Now, by Cooper, Lempp and Watson [19], there is a d.c.e. degree $\mathbf{d} < \mathbf{h}$ such that $\mathbf{a} \cup \mathbf{d} = \mathbf{h}$. Obviously, there is no c.e. degree between $\mathbf{d}$ and $\mathbf{h}$, so $\mathbf{d}$ is isolated from above by $\mathbf{h}$.

Furthermore, if $\mathbf{h}$ is an isolating degree (from below), then there is a d.c.e. degree $\mathbf{e}$ isolated by $\mathbf{h}$ from below. Therefore, $\mathbf{h}$ is a bi-isolating degree. $\square$

Corollary 6.1 *There is an interval* $(\mathbf{d}_1, \mathbf{d}_2)$ *of d.c.e. degrees containing exactly one c.e. degree.*

Proof. Suppose that $\mathbf{a}$ isolates $\mathbf{d}_1$ from above, and $\mathbf{d}_2$ from below. Then $\mathbf{d}_1 < \mathbf{d}_2$. By Theorem 5, such $\mathbf{a}$, $\mathbf{d}_1$ and $\mathbf{d}_2$ exist. Let $\mathbf{e}$ be any c.e. degree between $\mathbf{d}_1$ and $\mathbf{d}_2$. By the fact that $\mathbf{a}$ isolates $\mathbf{d}_1$ from above, $\mathbf{e} \geq \mathbf{a}$, and by the fact that $\mathbf{a}$ isolates $\mathbf{d}_2$ from below, $\mathbf{e} \leq \mathbf{a}$. Thus, $\mathbf{e} = \mathbf{a}$. $\square$

The existence of intervals of d.c.e. degrees containing no c.e. degrees is well-known. By Theorem 3, such intervals are dense in the c.e. degrees, and by Theorem 4, such intervals can be large from the high/low hierarchy point of view.

In the rest of this chapter, we show that the degree isolated by $\mathbf{0}'$ from above can also be a degree isolated from below, and hence, *bi-isolated* degrees exist. Our observation is based on the following theorem, which is an improvement of Arslanov's Cupping Theorem (Theorem 1.3):

Theorem 6 *Each noncomputable c.e. degree can be cupped to* $\mathbf{0}'$ *via an isolated degree.*

Let $\mathbf{c}$ be the an anticupping witness of $\mathbf{0}'$. By Theorem 6, there is an isolated d.c.e. degree $\mathbf{d}$ such that $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$, and hence, $\mathbf{d}$ is isolated from above by $\mathbf{0}'$. Let $\mathbf{a}$ be the c.e. degree isolating $\mathbf{d}$ (from below). Then $\mathbf{d}$ is isolated by $\mathbf{0}'$ from above, and by $\mathbf{a}$ from below.

Corollary 6.2 *There is a bi-isolated d.c.e. degree. Furthermore, there is a c.e. degree* $\mathbf{a}$ *and a d.c.e. degree* $\mathbf{d} > \mathbf{a}$ *such that* $\mathbf{d}$ *is incomparable with any c.e. degree between* $\mathbf{a}$ *and* $\mathbf{0}'$.

Proof. The existence of bi-isolated d.c.e. degrees follows from the discussion immediately above. Now, let $\mathbf{e}$ be any c.e. degree between $\mathbf{a}$ and $\mathbf{0}'$. If $\mathbf{e} > \mathbf{d}$, then $\mathbf{e} = \mathbf{0}'$, and if $\mathbf{e} < \mathbf{d}$, then $\mathbf{e} = \mathbf{a}$. Therefore, $\mathbf{d}$ is incomparable with $\mathbf{e}$. □

We mention here that recently, Lempp et al. [55] proved that there are d.c.e. degrees $\mathbf{a} > \mathbf{b} > \mathbf{0}$ such that any d.c.e. degree below $\mathbf{a}$ is comparable with $\mathbf{b}$.

The proof of Theorem 6 can be modified to prove that in every jump class, there is a bi-isolated degree.

Theorem 7 *For any noncomputable c.e. degree* $\mathbf{c}$*, and for any* $\mathbf{e}$ *c.e.a. in* $\mathbf{0}'$*, there is an isolated degree* $\mathbf{d}$ *such that* $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$*, and* $\mathbf{d}' = \mathbf{e}$.

In [5], Arslanov asked whether the bi-isolating degrees also occur in each jump class. By Theorem 5, bi-isolating degrees are dense in the high c.e. degrees. We conjecture that Arslanov's question has a positive answer.

Conjecture: *The bi-isolating degrees occur in any jump class.*

Now fix $C \in \mathbf{c}$. We will construct a d.c.e. set D and a c.e. set A satisfying the following requirements:

$$\mathcal{G} : K = \Gamma(C, D);$$

$$\mathcal{P}_e : D \neq \Phi_e^A;$$

$$\mathcal{R}_e : \Phi_e^{A,D} = W_e \Rightarrow \exists \Delta_e(\Delta_e^A = W_e).$$

Let $\mathbf{a}, \mathbf{d}$ be the Turing degrees of $A, A \oplus D$ respectively. By $\mathcal{G}$, we have $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$. By the $\mathcal{P}$-requirements, $\mathbf{a} < \mathbf{d}$. By the $\mathcal{R}$-requirements, $(\mathbf{a}, \mathbf{d})$ is an isolation pair, and hence, $\mathbf{d}$ is incomplete.

We now prove Theorem 6.

6.2 Basic strategies

In this section, we describe strategies satisfying requirements listed above.

$\mathcal{G}$-strategy

To satisfy the $\mathcal{G}$-requirement, we will construct a partial computable functional Γ such that $K = \Gamma^{C,D}$, where K is a fixed creative set. The construction of Γ proceeds as follows:

(a) If there is an x such that $\Gamma(C, D; x) \downarrow \neq K(x)$, then let k be the least such x, enumerate $\gamma(k)$ into D, and for any $x \geq k$, let $\Gamma(C, D; x)$ be undefined.

(b) If k is the least number x such that $\Gamma(C, D; x) \uparrow$, then define $\Gamma(C, D; k) = K(k)$ with $\gamma(k)$ fresh.

However, as C is a given c.e. set, we cannot put numbers into C to lift up the γ-uses to protect a computation from being injured. We will use the method developed by Arslanov in paper [1]. For instance, suppose that we want to protect a computation $\Phi^{A,D}(y)[s]$ to fulfill a diagonalization procedure. We first put a small number, $\gamma(b)[s]$ say, into D to lift up all $\gamma(x)$, $x \geq b$, above $\varphi(y)[s]$, and at the same time, wait for C to change below $\gamma(b)[s]$. Here, we assume that $\gamma(b)[s]$ is less than $\varphi(y)[s]$, and hence, the enumeration of $\gamma(b)[s]$ may injure the computation $\Phi^{A,D}(y)[s]$. Such an injury is called *a capricious injury.* In the construction, we will combine such a method with isolation strategies. While waiting for C to change below $\gamma(b)[s]$, we start a new attack. If $C \upharpoonright \gamma(b)[s]$ changes, then we take $\gamma(b)[s]$ out of D, recovering the computation $\Phi^{A,D}(y)$ to $\Phi^{A,D}(y)[s]$. C's change ensures that $\Phi^{A,D}(y)[s]$ cannot be injured by the $\mathcal{G}$-strategy anymore, and we succeed in protecting the computation $\Phi^{A,D}(y)[s]$. On the other hand, if $C \upharpoonright \gamma(b)[s]$ has no change, then by defining a computable funtion g, which computes $C \upharpoonright \gamma(b)[s]$ correctly, we threaten the noncomputability of C.

$\mathcal{P}$-strategies

Let α be a $\mathcal{P}$-strategy. α is a Friedberg-Muchnik procedure and works as follows:

(a) Appoint a witness, x say, as a big number. Wait for a stage s such that $\Phi^A(x)[s]\downarrow= 0 = D_s(x)$.

(b) Enumerate x into D, and stop.

α satisfies the $\mathcal{P}$-requirement by one of the following cases:

Case 1. $\Phi^A(x)\downarrow= 0 \neq 1 = D(x)$ for some x.

Case 2. $\Phi^A(x)\downarrow\neq 0 = D(x)$ or $\Phi^A(x)\uparrow$ for some x.

α has two possible outcomes d, w with priority ordering $d <_{\mathrm{L}} w$, where d denotes for the fact that $\mathcal{P}$ is satisfied via case 1, and w for the other case.

$\mathcal{R}$-strategies

An $\mathcal{R}$-strategy, β say, works to satisfy an $\mathcal{R}$-requirement, which is a variation of the isolation strategy described in chapter 2.

Define

$$\ell(\beta, s) = \max\{x < s : (\forall y < x)(\Phi_\beta^{A,D}(y)[s]\downarrow= W_{\beta,s}(y))\},$$

$$m(\beta, s) = \max\{0, \ell(\beta, t) : t < s \text{ and } t \text{ is a } \beta\text{-stage}\}.$$

Say that stage s is β-expansionary, if $s = 0$ or s is a β-stage and $\ell(\beta, s) > m(\beta, s)$.

First, we introduce a parameter $k(\beta)$ as a threshold for the enumeration of γ-markers, and whenever K changes below $k(\beta)$, reset β. Since $k(\beta)$ is fixed, such a reset procedure can happen at most $k(\beta)$ many times.

β runs cycles (n), $n \in \omega$, each of which defines a partial computable functional, $\Delta_{\beta,n}$.

Cycle (n) runs as follows:

(1) Wait for a β-expansionary stage.

(2) Let s_1 be a β-expansionary stage. For any $y < \ell(\eta, s_1)$ with $\Delta^A_{\beta,n}(y)[s_1]\uparrow$, define $\Delta^A_{\beta,n}(y)[s_1] = W_{\beta,s_1}(y)$ with $\delta_\beta(y)[s_1] = s_1$. Go back to (1) and simultaneously, wait for $W_\beta(y)$ to change *(from 0 to 1)*.

(3) Let $s_2 > s_1$ be the stage at which $W_\beta(y)$ changes from 0 to 1. Put $\gamma(k(\beta))[s_2]$ into D. For those $z < \gamma(k(\beta))[s_2]$, if $g_\beta(z)\uparrow$, define $g_\beta(z) = C_{s_2}(z)$. Start cycle $(n+1)$, and simultaneously, wait for C to change below $\gamma(k(\beta))[s_2]$.

(4) Take $\gamma(k(\beta))[s_2]$ out of D, perform the disagreement strategy for β and declare that β is satisfied via the disagreement at y.

β satisfies the $\mathcal{R}$-requirement by one of the following cases:

Case 1. There is a cycle waiting forever at (1).

(There is some y such that $\Phi^{A,D}_\beta(y)\uparrow$ or $\Phi^{A,D}_\beta(y)\downarrow\neq W_\beta(y)$.)

Case 2. There is a cycle, (n) say, going from (2) to (1) infinitely often.

(If $\Phi^{A,D}_\beta = W_\beta$, then $\Delta^A_{\beta,n}$ is totally defined, and computes W_β correctly.)

Case 3. There is a cycle, (n) say, reaches (4) via number y.

(Then, $\Phi^{A,D}_\beta(y)\downarrow = 0 \neq 1 = W_\beta(y)$.)

Note that the noncomputability of C ensures that it cannot be the case that for each $n \in \omega$, cycle (n) can be started, because otherwise, each cycle would wait at (3) forever, making g_β totally defined, and hence C is computable.

Thus, β has the following outcomes:

$$(0,\infty) <_{\mathrm{L}} (0,w) <_{\mathrm{L}} (1,\infty) <_{\mathrm{L}} (1,w) <_{\mathrm{L}} \cdots <_{\mathrm{L}} (n,\infty) <_{\mathrm{L}} (n,w) <_{\mathrm{L}} \cdots <_{\mathrm{L}} d,$$

where (n,w) means that cycle (n) waits at (1) forever (Case 1), (n,∞) means that cycle n goes from (2) to (1) infinitely often (Case 2), and d means that some cycle reaches (4) (Case 3).

6.3 Construction

The construction will proceed on a priority tree, T. First, assign the priority to the requirements as follows:

$$\mathcal{G} < \mathcal{P}_0 < \mathcal{R}_0 < \mathcal{P}_1 < \mathcal{R}_1 < \cdots < \mathcal{P}_n < \mathcal{R}_n < \cdots$$

where for any $\mathcal{X}, \mathcal{Y}$, if $\mathcal{X} < \mathcal{Y}$, then $\mathcal{X}$ has higher priority than $\mathcal{Y}$.

For $\xi \in T$, say that requirement $\mathcal{X}$ is *satisfied at* ξ, if there is an $\mathcal{X}$-strategy ζ such that $\zeta \subset \xi$. $\mathcal{G}$ is a global requirement, and we do not list $\mathcal{G}$ on T.

Definition 6.3 *(1) Define the root node λ as an $\mathcal{P}_0$-strategy;*

(2) The immediate successors of a node are the possible outcomes of the corresponding strategy;

(3) For $\xi \in T$, ξ works for the highest priority requirement which is not satisfied at ξ.

In the construction, if ξ is initialized, then any parameters will be canceled. If ξ is an $\mathcal{R}$-strategy, say that ξ is reset, when any parameters, except $k(\xi)$, will be canceled. If $\xi < \xi'$ and ξ is reset or initialized, then ξ' will be initialized simultaneously and automatically.

Construction

Without loss of generality, suppose that K is enumerated at odd stages, and that exactly one element is enumerated into K at such stages.

Stage 0: Set $A = D = \emptyset$, and initialize all strategies.

Stage $s+1$:

(I) $s+1$ odd. Let k be the number in $K_{s+1} - K_s$.

1. If $\Gamma(C, D; k) \downarrow$, then

- Enumerate $\gamma(k)$ into D.
- For any strategy ξ, if $k(\xi) > k$, then reset ξ.
- Go to stage $s+2$.

2. Otherwise, let x be the least one such that $\Gamma(C, D; x) \uparrow$, define $\Gamma(C, D; x) = K(x)$ with $\gamma(x)$ fresh, and go to stage $s+2$.

(II) $s+1$ even.

We define a partial function σ_{s+1} of length $\leq s+1$ by substages. Say that a strategy ξ is *visited at* stage $s+1$, if ξ is eligible to act at a substage t of stage $s+1$.

First, let λ, the root node, be eligible to act at substage 0.

Substage $t > 0$: Given $\xi = \sigma_{s+1} \upharpoonright t$, initialize all nodes $>_{\mathrm{L}} \xi$. If $t = s+1$, then define $\sigma_{s+1} = \xi$, initialize all $\zeta > \sigma_{s+1}$, and go to the next stage.

Otherwise, there are two cases:

Case 1. $\xi = \alpha$ is an $\mathcal{P}$-strategy.

α1. If $x(\alpha)$ has no definition, then define $x(\alpha)$ as a fresh number, define $\sigma_{s+1} = \alpha$, initialize all $\xi > \sigma_{s+1}$, and go to the next stage.

α2. If $x(\alpha)$ is defined, and $x(\alpha) \in D$, then let $\alpha^\frown d$ be eligible to act at the next substage.

α3. If $x(\alpha)$ is defined, $\Phi^A_\alpha(x(\alpha)) \downarrow = D(x(\alpha)) = 0$, then enumerate $x(\alpha)$ into D, define $\sigma_{s+1} = \alpha$, initialize all $\xi > \sigma_{s+1}$ and go to the next stage.

α4. Otherwise, let $\alpha^\frown w$ be eligible to act at the next substage.

Case 2. $\xi = \beta$ is an $\mathcal{R}$-strategy.

If $k(\beta)$ is not defined, then define $k(\beta)$ as a fresh number, define $\sigma_{s+1} = \beta$, initialize all $\xi > \sigma_{s+1}$, and go to the next stage.

If β is satisfied via some number z, then let $\beta^\frown d$ be eligible to act at the next substage.

If β is not satisfied, and there is some z such that $g_\beta(z) \neq C_{s+1}(z)$, let z be the least such disagreement point, and let $s' \leq s$ be the stage at which $g_\beta(z)$ is defined. Remove from D all numbers entering after stage s' (including $\gamma(k(\beta))[s']$), put s' into A, and for all $y \geq k(\beta)$, let $\Gamma^{C,D}(y)$ be undefined. Declare that β is satisfied via z. Define $\sigma_{s+1} = \beta$ and initialize all $\xi > \sigma_{s+1}$. Go to the next stage.

Otherwise, suppose that β is in cycle (n). There are three subcases:

Subcase 1: $s+1$ is not β-expansionary.

Action: Let $\beta^\frown(n, w)$ be eligible to act at the next substage.

Subcase 2: $s+1$ is β-expansionary and $\Delta^A_{\beta,n}$ is correct.

Action: For any $y < \ell(\beta, s)$ with $\Delta^A_{\beta,n}(y)[s] \uparrow$, define $\Delta^A_{\beta,n}(y)[s+1] = W_{\beta,s+1}(y)$ with use $\delta_{\beta,n}(y) = s+1$. Let $\beta^\frown(n, \infty)$ be eligible to act at the next substage.

Subcase 3: $s+1$ is β-expansionary and $\Delta^A_{\beta,n}$ is incorrect at some (least) y.

Action: Check whether $\gamma(k(\beta))[s]$ is larger than $\varphi_\beta(y)[\delta_{\beta,n}(y)[s]]$.

If yes, then remove from D all numbers entering after stage $\delta_{\beta,n}(y)[s]$, put number $\delta_{\beta,n}(y)[s]$ into A, and declare that β is satisfied via z.

Otherwise, put $\gamma(k(\beta))[s]$ into D. For $y \geq k(\beta)$, let $\Gamma^{C,D}(y)$ be undefined and for $z < \gamma(k(\beta))[s]$, if $g_\beta(z)$ has no definition, define $g_\beta(z) = C_{s+1}(z)$. Stop cycle (n) and start cycle $n+1$.

In both cases, define $\sigma_{s+1} = \beta$ and initialize all $\xi > \sigma_{s+1}$. Go to the next stage.

This completes the construction.

6.4 Verification

We now verify that the construction above satisfies all requirements. First we have:

Lemma 6.4.1 *For any x, s,*

(1) If $\Gamma(C, D; x)[s] \downarrow$, then $\gamma(x)[s] \notin C_s \cup D_s$;

(2) If $\Gamma(C, D; x+1) \downarrow$, then $\Gamma(C, D; x) \downarrow$ and $\gamma(x)[s] < \gamma(x+1)[s]$;

(3) If $s < v$, $\Gamma(C, D; x)[s] \downarrow$, $C_v \upharpoonright (\gamma(x)[s] + 1) = C_s \upharpoonright (\gamma(x)[s] + 1)$ *and* $D_v \upharpoonright (\gamma(x)[s]+1) = D_s \upharpoonright (\gamma(x)[s]+1)$, *then* $\Gamma(C, D; x)[v] \downarrow$, *and* $\gamma(x)[v] = \gamma(x)[s]$;

(4) If $\Gamma(C, D; x)$ *is undefined at stage* $s + 1$ *and* $\Gamma(C, D; x)$ *is defined at stage* s, *then* $C_{s+1} \upharpoonright (\gamma(x)[s]+1) \neq C_s \upharpoonright (\gamma(x)[s]+1)$ *or* $D_{s+1} \upharpoonright (\gamma(x)[s]+1) \neq D_s \upharpoonright (\gamma(x)[s] + 1)$. □

Define $f = \liminf_s \sigma_s$. f is the true path of the construction.

Lemma 6.4.2 *For* $\xi \subset f$,

(1) ξ *is initialized or reset only finitely often;*

(2) ξ *acts only finitely often.*

Proof. We prove the lemma by induction. Let ξ^- be the immediate predecessor of ξ. By the induction hypothesis, there is some (least) stage s_0 such that

(a) ξ^- cannot be initialized or reset afterwards;

(b) ξ^- does not act afterwards.

Then, since ξ is on f, there is some stage $s_1 \geq s_0$ such that $\forall s \geq s_1(\sigma_s \geq \xi)$. By the choice of s_0, ξ cannot be initialized after s_1. Furthermore, if ξ is an $\mathcal{R}$-strategy, then let $k(\xi)$ be defined at stage $s_2 \geq s_1$. From now on, $k(\xi)$ cannot be cancelled and ξ cannot be reset. So (1) holds.

For (2), there are two cases:

(i) ξ is a $\mathcal{P}$-strategy.

Suppose that at stage $s_3 > s_1$, ξ chooses a big number $x(\xi)$ as an attacker for ξ. If there is a ξ-stage $s_4 > s_3$ such that $\Phi_\xi^A(x(\xi)) \downarrow = 0$, then $x(\xi)$ is enumerated into D, and the computation $\Phi_\xi^A(x(\xi))[s_4]$ is protected from lower priority strategies. By the choice of stage s_1, no strategy with higher priority can act after s_4. Hence, $\Phi_\xi^A(x(\xi))[s_4]$ is protected forever, $\Phi_\xi^A(x(\xi)) \downarrow = \Phi_\xi^A(x(\xi))[s_4] = 0 \neq 1 = D(x(\xi))$. ξ is satisfied and does not act anymore. So (2) holds.

(ii) ξ is an $\mathcal{R}$-strategy.

Suppose that ξ acts infinitely often. Then since each cycle of ξ acts at most once, we know that ξ starts infinitely many cycles, and each cycle extends the definition of g_ξ. g_ξ is totally defined. We now show that g_ξ computes C correctly. Suppose not. Let y be the least disagreement between g_ξ and C. Assume that $g_\xi(y)$ is defined at stage $s_4 > s_2$. That is, there is a (least) z confirmed as an error at stage s_4. Let s_z be the stage at which $\Delta_\xi^A(z)$ is defined. Then by moving out all numbers enumerated into D after stage s_z, we get a disagreement between $\Phi_\xi^{A,D}(z)$ and $W_\xi(z)$ successfully. By the construction, $\Phi_\xi^{A,D}(z)[s_z]$ is preserved forever, and hence, no more cycles will be started later, a contradiction. However, since g_ξ computes C correctly, C is computable, which is another contradiction. So (2) holds for ξ. $\square$

Lemma 6.4.3 *For any $\xi \subset f$, there is some $\mathcal{O}$ such that $\xi^\frown\mathcal{O} \subset f$. Thus, $|f| = \infty$.*

Proof. The lemma is obviously true when ξ is a $\mathcal{P}$-strategy.

Let $\xi \subset f$ is an $\mathcal{R}$-strategy. The argument in the proof of lemma 6.4.2 shows that ξ can start at most finitely many cycles. Let cycle (n) be the largest one being started. Suppose that cycle (n) is started at stage s_1. Then, after stage s_1, for $m < n$, no $\xi^\frown(m,-)$ can be visited again. There are three cases:

Case 1: There is some (least) y such that $g_\xi(y)\downarrow$ and $g_\xi(y) \neq C(y)$, where $g_\xi(y)$ is defined by cycle $(m) \leq (n)$ at stage s_2. By the construction, we will take $\gamma(k(\beta))[s_2]$, together with other numbers, out of D to perform the disagreement strategy ξ is satisfied via this disagreement, which makes d the true outcome of ξ.

Case 2: There are only finitely many ξ-expansionary stages. Then there is no ξ-expansionary stages, after stage $s_3 > s_1$ say. Then for any stage $s > s_3$, if s is a ξ-stage, then s is also a $\xi^\frown(n,w)$ stage. Therefore, (n,w) is the true outcome of ξ.

Case 3: There are infinitely many ξ-expansionary stages. Then $\Delta_{\xi,n}^A$ equals W_ξ on each argument, because otherwise, cycle (n) would start cycle $(n+1)$, contradicts our choice of n. Thus, each ξ-expansionary stage larger than s_1 is a $\xi^\frown(n,\infty)$-stage, and (n,∞) is the true outcome of ξ. $\square$

Lemma 6.4.4 *For any $\xi \subset f$,*

(1) If ξ is a $\mathcal{P}$-strategy, then $\mathcal{P}$ is satisfied.

(2) If ξ is an $\mathcal{R}$-strategy, then $\mathcal{R}$ is satisfied.

Proof. (1) Follows from the proof of (i) in Lemma 6.4.2.

(2) Suppose that $\Phi_\xi^{A,D} = W_\xi$. By the discussion in Lemma 6.4.2, there is a largest cycle n being started, and there is no disagreement between $\Delta_{\xi,n}^A$ and W_ξ. We now prove that Δ_ξ^A is total, and therefore, $W_\xi \leq_{\mathrm{T}} A$ via $\Delta_{\xi,n}$.

Fix x. Let $\Delta_{\xi,n}^A(x)$ be defined at stage s_0. Then $\delta_{\xi,n}^A(x)[s_0]$ is defined as s_0. Without loss of generality, suppose that at stage $s_1 > s_0$, some $\xi' > \xi$ executes the disagreement strategy, and puts a number less than s_0 into A. Thus, $\Delta_{\xi,n}^A(x)$ becomes undefined. At the next ξ-expansionary stage s_2, $\Delta_{\xi,n}^A(x)$ is redefined with use $\delta_{\xi,n}^A(x)[s_2] = s_2$. By the construction, all nodes $> \xi'$ are initialized at stage s_1 and cannot undefine $\Delta_{\xi,n}^A(x)[s_2]$ again. Thus, only strategies below $\xi^\frown(n, \infty)$ with priority higher than ξ' can act and undefine $\Delta_{\xi,n}^A(x)[s_2]$. Since there are only finitely many such strategies being visited before stage s_2, and only these strategies' actions can undefine the newly defined $\Delta_{\xi,n}^A(x)$, there is a sufficiently large stage, s_x say, at which $\Delta_{\xi,n}^A(x)$ gets defined and this definition can never be destroyed afterwards. Then $\Delta_{\xi,n}^A(x)\downarrow$ with use $\delta_{\xi,n}(x)[s_x]$. $\square$

Lemma 6.4.5 $\Gamma(C, D) = K$.

Proof. By Lemma 6.4.1, $\Gamma(C, D)$ is a partial computable functional. By actions during odd stages, if $\Gamma(C, D)$ is total, then $\Gamma(C, D) = K$.

Fix k. Let $\xi \subset f$ be the first $\mathcal{R}$-strategy extending $f \restriction k$. Then ξ can only be visited after stage k. Since ξ is on the true path, by Lemma 6.4.2, there is some (least) stage s_ξ such that ξ cannot be initialized or reset afterwards. Let $s' \geq s_\xi$ be the stage at which $k(\xi)$ gets defined. Then $k(\xi) > k$ and $k(\xi)$ cannot be canceled later. Note that all lower priority strategies are initialized at stage s', by the argument given in Lemma 6.4.2, ξ can put $\gamma(k(\xi))$ (different values) into D at most finitely often, and so $\gamma(k(\xi))$ is lifted up in this way only finitely often.

Let s be the last stage at which $\gamma(k(\xi))$ is lifted up via D-enumerations. If C has no change below $\gamma(k(\xi))[s]$, then $\gamma(k(\xi))\downarrow = \gamma(k(\xi))[s+1]$. (*Here, we keep* $\gamma(k(\xi))$ *unchanged even if* D *changes between* $\gamma(k(\xi))[s]$ *and* $\gamma(k(\xi))[s+1]$*, which can happen only finitely often.)* If C has a change below $\gamma(k(\xi))[s]$, then by the construction, $\gamma(k(\xi))[s]$ will be taken out of D, and $\gamma(k(\xi))$ can be lifted up to a big number, $\gamma(k(\xi))[s']$ say. Again, $\gamma(k(\xi))\downarrow = \gamma(k(\xi))[s']$. $\square$

This completes the proof of Theorem 6. $\square$

6.5 Remarks on the proof of Theorem 7

In this section, we describe how the proof of Theorem 6 can be modified to give a proof of Theorem 7. We follow the usual proof of the Sacks Jump Theorem by starting with a c.e. set F such that

(1) $n \in E \Rightarrow F^{[n]}$ is finite;

(2) $n \notin E \Rightarrow F^{[n]} = \omega^{[n]}$.

For technical convenience, we assume that a number m is enumerated into $F^{[n]}$ at some stage s, then m is less than s and every $k < s$ which is in $\omega^{[n]}$ and not already in $F^{[n]}$ is enumerated into $F^{[n]}$ at stage s. We will make D a thick subset of F. That is, D should meet the following thickness requirements:

$\mathcal{T}_e$: $D^{[n]} \subseteq F^{[n]}$ and $F^{[n]} - D^{[n]}$ is finite.

These requirements ensure that $E \leq_{\mathrm{T}} D'$. Indeed, we will make D satisfy $\mathcal{T}_e$ for all $e \geq 3$, and leave $\omega^{[0]}, \omega^{[1]}, \omega^{[2]}$ for us from which we can choose the γ-uses for the requirement $\mathcal{G}$, the attackers for the requirements $\mathcal{P}$, and the δ-uses for the requirements $\mathcal{R}$, respectively. Fix $e \geq 3$ and suppose that $\mathcal{T}_e$ is assigned to α, then as usual, α has outcomes: i (infinite) and t, $t \in \omega$, where t is the last stage at which α notices that there is some number enumerated into $F^{[e]}$. Let s be any α-stage. Check whether there has been a number enumerated into $F^{[e]}$ since the last stage at which α is accessible (since stage 0 if s is the first stage at which α is accessible). If so, then α guesses that $F^{[e]}$ is infinite, and α has outcome i. Declare that $\alpha^\frown i$ is accessible, and simultaneously, enumerate every $z > R(\alpha, s)$ which is in $F_s^{[e]}$ into D. Otherwise, α guesses that $F^{[e]}$ is finite, and α has outcome t, where t is the last stage at which there is some number enumerated into $F^{[e]}$.

The priority of requirements can be assigned as follows:

$$\mathcal{G} < \mathcal{P}_0 < \mathcal{R}_0 < \mathcal{P}_1 < \mathcal{R}_1 < \mathcal{P}_2 < \mathcal{R}_2 < \mathcal{P}_3 < \mathcal{R}_3 < \mathcal{T}_3 < \cdots < \mathcal{P}_n < \mathcal{R}_n < \mathcal{T}_n < \cdots$$

where for any $\mathcal{X}, \mathcal{Y}$, if $\mathcal{X} < \mathcal{Y}$, then $\mathcal{X}$ has higher priority than $\mathcal{Y}$. Now the construction of D follows from the one given in section 6.3 easily except that computations $\Phi^{B,D}(x)$ considered at α are always α-believable. Here, we say that a computation $\Phi^{B,D}(x)[s]$ is α-believable if for all $k \geq 3$, if $\alpha(3k-3) = i$, then for all $z \in \omega^{[k]}$, if $z < \varphi(x)[s]$ and $z > R(\alpha \restriction (3k-3))$, then $z \in D_s$.

$\square$

CHAPTER 7

ISOLATION AND LATTICE EMBEDDINGS

7.1 Introduction

In [24], Downey proved that the diamond lattice can be embedded into the d.c.e. degrees with 0 and 1 preserved. Ding and Qian [23] proved that one of the atoms in Downey's diamond can be c.e. By Lachlan's nondiamond theorem, $\mathbf{d}$ is properly d.c.e. In [57], Li and Yi proved that there are a c.e. degree $\mathbf{a}$ and a d.c.e. degree $\mathbf{d}$ such that any d.c.e degree $\mathbf{c} \in (\mathbf{0}, \mathbf{a})$ cups $\mathbf{d}$ to $\mathbf{0}'$ and caps $\mathbf{d}$ to $\mathbf{0}$.

In this chapter, we prove that there is an isolation pair $(\mathbf{a}, \mathbf{d})$ and an incomparable c.e. degree $\mathbf{c}$ such $\mathbf{c}$ cups the isolated degree $\mathbf{d}$ to the degree of the halting problem and caps the isolating degree $\mathbf{a}$ to the trivial degree of computable sets. This appears to be as close as one can get to overcoming the obstacle to constructing an embedding of the four element Boolean algebra preserving 0 and 1 into the c.e. degrees which first became apparent from Lachlan's nondiamond theorem [51].

Theorem 8 *There are an isolation pair* $(\mathbf{a}, \mathbf{d})$ *and a c.e. degree* $\mathbf{c}$ *such that* $\mathbf{c}$ *cups* $\mathbf{d}$ *to* $\mathbf{0}'$*, and caps* $\mathbf{a}$ *to* $\mathbf{0}$*.*

Theorem 8 provides a diamond embedding into the d.c.e. degrees with 0 and 1 preserved. We only need to show that $\mathbf{c} \cap \mathbf{d} = \mathbf{0}$. Suppose that there is some nonzero d.c.e. degree $\mathbf{x}$ below both $\mathbf{c}$ and $\mathbf{d}$. By Lachlan's observation that any nonzero d.c.e. degree bounds a nonzero c.e. degree, we can assume that $\mathbf{x}$ is c.e. By the fact that $\mathbf{a}$ bounds all c.e. degree below $\mathbf{d}$, $\mathbf{x} < \mathbf{a}$ and hence $\mathbf{x} \leq \mathbf{a} \cap \mathbf{c} = \mathbf{0}$. A contradiction.

On the other hand, by using Harrington-Soare's continuity of capping degrees, we have that N_5 can be embedded into d.c.e. degrees with 0 and 1 preserved. First note that $\{\mathbf{0}, \mathbf{a}, \mathbf{c}, \mathbf{d}, \mathbf{0}'\}$ in Theorem 8 is not an N_5 embedding because by Lachlan's nondiamond theorem, $\mathbf{a} \cup \mathbf{c} < \mathbf{0}'$. However, since the c.e. degrees $\mathbf{a}$ and $\mathbf{c}$ form a minimal pair, by Harrington-Soare's continuity of capping degrees (see [40]), there is a c.e. degree $\mathbf{f} > \mathbf{c}$ such that $\mathbf{f}$ and $\mathbf{a}$ form a minimal pair. As discussed above, $\mathbf{f} \cap \mathbf{d} = \mathbf{0}$. Thus, $\{\mathbf{0}, \mathbf{c}, \mathbf{f}, \mathbf{d}, \mathbf{0}'\}$ is an N_5 embedding.

We remark here that the proof presented below can be easily modified to prove that M_5 can be embedded into d.c.e. degrees with 0 and 1 preserved. The main idea is to construct two isolation pairs $(\mathbf{a}_1, \mathbf{d}_1)$, $(\mathbf{a}_2, \mathbf{d}_2)$ and a c.e. degree $\mathbf{c}$ such that

(1) $\mathbf{d}_1 \cup \mathbf{c} = \mathbf{d}_2 \cup \mathbf{c} = \mathbf{d}_1 \cup \mathbf{d}_2 = \mathbf{0}'$,

(2) $\mathbf{a}_1 \cap \mathbf{c} = \mathbf{a}_2 \cap \mathbf{c} = \mathbf{a}_1 \cap \mathbf{a}_2 = \mathbf{0}$.

Then $\{\mathbf{0}, \mathbf{c}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{0}'\}$ is an M_5 embedding. Such a method can be improved further to prove that M_5 can be embedded below any high degree with 0 and 1 preserved, and hence, S_8 can be embedded into d.c.e. degrees with 0 and 1 preserved. $\square$

7.2 Requirements and basic strategies

To prove the theorem, we construct c.e. sets A, C, a d.c.e. set D and a partial computable functional Γ meeting the following requirements:

$$\mathcal{G} : K = \Gamma(C, D);$$

$$\mathcal{P}_e^C : C \neq \Phi_e^{A \oplus D};$$

$$\mathcal{P}_e^D : D \neq \Phi_e^C;$$

$$\mathcal{N}_e : \Phi_e(A) = \Phi_e(C) = g \text{ total} \Rightarrow g \text{ computable};$$

$$\mathcal{R}_e : \Phi_e(A \oplus D) = W_e \Rightarrow \exists \Delta_e(\Delta_e^A = W_e);$$

where $e \in \omega$, $\{(\Phi_e, W_e) : e \in \omega\}$ is an effective enumeration of all pairs (Φ, W) such that Φ is a partial computable functional, and W is a computably enumerable set. K is a fixed creative set.

Let $\mathbf{a}, \mathbf{c}, \mathbf{d}$ be the Turing degrees of $A, C, A \oplus D$ respectively. By $\mathcal{G}$, we have $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$. By the $\mathcal{P}$-requirements, $\mathbf{c}$ and $\mathbf{d}$ are incomparable. By the $\mathcal{N}$-requirements, $\mathbf{a}$ and $\mathbf{c}$ form a minimal pair. By Lachlan's nondiamond theorem, and the $\mathcal{G}$-requirement, the $\mathcal{P}$- and $\mathcal{N}$-requirements, $A \not\equiv_T A \oplus D$. Thus, $\mathbf{a} < \mathbf{d}$. The $\mathcal{R}$-requirements guarantee that $(\mathbf{a}, \mathbf{d})$ is an isolation pair.

The $\mathcal{G}$-strategy

The $\mathcal{G}$-strategy proceeds as follows:

1. If there is an x such that $\Gamma(C, D; x) \downarrow \neq K(x)$, then let k be the least such x, enumerate $\gamma(k)$ into D, and for any $x \geq k$, let $\Gamma(C, D; x)$ be undefined.

2. If k is the least number x such that $\Gamma(C, D; x) \uparrow$, then define $\Gamma(C, D; k) = K(k)$ with $\gamma(k)$ fresh.

Note that the $\mathcal{G}$-strategy only enumerates numbers into D, so the $\mathcal{G}$-strategy itself is not sufficient to satisfy the requirement $\mathcal{G}$ (otherwise K will be coded into D, making D complete).

The use function γ of Γ will have the following properties:

1. Whenever we define $\gamma(k)$, we define it as fresh;

2. For any k, s, if $\Gamma(C, D; k)[s] \downarrow$, then $\gamma(k)[s] \notin C_s \cup D_s$;

3. For any x, y, if $x < y$, and $\gamma(y) \downarrow$, then $\gamma(x) \downarrow$, and $\gamma(x) < \gamma(y)$;

4. If $\Gamma(C, D; k)[s] \downarrow = 0$, and $k \in K_{s+1} - K_s$, then there is an $n \leq k$ such that $\gamma(n)[s] \in C - C_s$ or $\gamma(n)[s] \in D - D_s$;

5. $\Gamma(C, D; x)$, $x < s$, is undefined at stage s iff at stage s, there is $y \leq x$ such that $\gamma(y)$ is enumerated into C or D.

If $(1) - (5)$ are met and $\Gamma(C, D)$ is total, then $\Gamma(C, D) = K$. So $\mathcal{G}$ is satisfied.

$\mathcal{N}$-strategies

An $\mathcal{N}$-strategy, α say, attempting to satisfy an $\mathcal{N}$-requirement, $\mathcal{N}_e$ say, is a standard minimal pair strategy, which involves enumerating just one element into the union of A and C at each stage. That is, α tries to preserve computations up to the largest agreement between $\Phi_e(A)$ and $\Phi_e(C)$ until a longer agreement appears.

Define the length of agreement functions as follows:

$$\ell_{\mathcal{N}}(\alpha, s) = \max\{x : \forall y < x(\Phi_e(A; y)[s] \downarrow = \Phi_e(C; y)[s] \downarrow)\},$$
$$m_{\mathcal{N}}(\alpha, s) = \max\{0, \ell_{\mathcal{N}}(\alpha, t) : t < s \ \& \ t \text{ is an } \alpha - \text{stage}\}.$$

Say that stage s is α-expansionary if $s = 0$ or s is an α-stage and $\ell_{\mathcal{N}}(\alpha, s) > m_{\mathcal{N}}(\alpha, s)$.

α has two outcomes, $0, 1$, with $0 <_{\mathrm{L}} 1$, where 0 indicates that there are infinitely many α-expansionary stages and 1 otherwise.

Let α be the $\mathcal{N}_e$-strategy on the true path, and suppose that $\mathcal{P}$ and $\mathcal{R}$-strategies with higher priority act only finitely often (hence, α is initialized). We show that α satisfies $\mathcal{N}_e$. Assume that after stage s_0 (where s_0 is large enough) α cannot be initialized anymore. If α has outcome 1, then there are only finitely many α-expansionary stages, $\Phi_e(A) \neq \Phi_e(C)$, $\mathcal{N}_e$ is satisfied. Now suppose that $\Phi_e(A) = \Phi_e(C) = g$ is total. Then there are infinitely many α-expansionary stages, and so $\alpha^\frown\langle 0\rangle$ is on the true path. We compute g as follows.

Given x, find the least α-expansionary stage, s_1 say, such that $s_1 > s_0$, $\ell_{\mathcal{N}}(\alpha, s)[s_1] > x$. We claim that

$$g(x) = \Phi_e(A; x)[s_1] = \Phi_e(C; x)[s_1].$$

Let $\{s_i : i \geq 1, s_i < s_{i+1}\}$ be the set of all α-expansionary stages. Since at stage $s_1 + 1$, numbers can only be enumerated into at most one of A and C, we have:

$$A_{s_1} \upharpoonright s_1 = A_{s_1+1} \upharpoonright s_1 \ \text{ or } \ C_{s_1} \upharpoonright s_1 = C_{s_1+1} \upharpoonright s_1.$$

Note that any number enumerated into $A \cup C$ between stages $s_1 + 1$ and s_2 are larger than s_1 (because they are associated with strategies to the right of $\alpha ^\frown \langle 0\rangle$, and these strategies were initialized at stage $s_1 + 1$), we have

$$A_{s_2} \upharpoonright s_1 = A_{s_1} \upharpoonright s_1 \text{ or } C_{s_2} \upharpoonright s_1 = C_{s_1} \upharpoonright s_1.$$

Therefore, $g(x)[s_2] = g(x)[s_1]$.

By induction, we can prove that for any $n \geq 1$,

$$A_{s_n} \upharpoonright s_n = A_{s_{n+1}} \upharpoonright s_n \text{ or } C_{s_n} \upharpoonright s_n = C_{s_{n+1}} \upharpoonright s_n,$$

and $g(x)[s_{n+1}] = g(x)[s_n]$. Thus, for any $n \geq 1$, $g(x)[s_n] = g(x)[s_1]$.

$\mathcal{R}$-strategies

An $\mathcal{R}$-strategy, β say, is designed to satisfy an $\mathcal{R}$-requirement, $\mathcal{R}_e$ say. It is a variation of the $\mathcal{R}$-strategy given in chapter 2.

Define

$$\ell_{\mathcal{R}}(\beta, s) = \max\{x < s : (\forall y < x)(\Phi_e^{A\oplus D}(y)[s] \downarrow= W_{e,s}(y))\},$$

$$m_{\mathcal{R}}(\beta, s) = \max\{0, \ell_{\mathcal{R}}(\beta, t) : t < s \text{ and } t \text{ is a } \beta\text{-stage}\}.$$

Stage s is β-expansionary, if $s = 0$ or s is a β-stage and $\ell_{\mathcal{R}}(\beta, s) > m_{\mathcal{R}}(\beta, s)$. At a β-expansionary stage s, for any $y < \ell_{\mathcal{R}}(\beta, s)$, if $\Delta_\beta^A(y)[s] \uparrow$, define $\Delta_\beta^A(y) = W_{e,s}(y)$ with $\delta_\beta(y)[s] = s$.

Suppose that $\delta_\beta(y)$ is defined at stage s. During the construction, numbers enumerated into D, z say, may injure the current computation of $\Phi_e^{A\oplus D}(x)[s]$, and lift the use $\varphi_e(A \oplus D; x)$. This provides a chance for $W_e(x)$ to change, making $\Delta_\beta^A(x)$ incorrect. Suppose that $W_e(x)$ changes at stage s'. Then extracting all these numbers, z, from D will restore the computation $\Phi_e^{A\oplus D}(x)$ to $\Phi_e^{A\oplus D}(x)[s]$. Such an action creates an inequality between $\Phi_e^{A\oplus D}$ and W_e at x. Due to the c.e.-ness of W_e,

$$\Phi_e^{A\oplus D}(x) = \Phi_e^{A\oplus D}(x)[s] = W_{e,s}(x) = 0 \neq 1 = W_e(x),$$

so $\mathcal{R}_e$ is satisfied.

The nontrivial case is when some of the removed numbers, z say, are γ-markers. Taking z out might lead to the failure of the $\mathcal{G}$-strategy. On the other hand, since $\mathcal{G}$ has the highest priority, the enumeration of γ-markers into D may destroy the computation $\Phi_e^{A\oplus D}(x)$ and hence injure the disagreement strategy (finite action) described above. Our strategy is to introduce a parameter $k(\beta)$ as a threshold for the enumeration of γ-markers. This is necessary because of the essentially uncontrollable amount of information introduced by coding K into the supremum of the degrees being constructed.

$k(\beta)$ works as follows:

(1) Define $k(\beta)$ as a fresh number.

(2) If some $n < k(\beta)$ enters K, then reset β by undefining all parameters of β, except $k(\beta)$.

(β can be reset in this way only finitely often because $k(\beta)$ is fixed.)

(3) If β requests to perform the disagreement strategy, then enumerate $\gamma(k(\beta))$ into C first, and say that β's request is confirmed.

(4) If β's request is confirmed, then perform the disagreement strategy as described above.

Thus, if $z = \gamma(y)[s_0]$ is put into D at stage s_0+1 to record $y \in K_{s_0+1} - K_s$ where $y \geq k(\beta)$, then the enumeration of $\gamma(k(\beta))[s]$ into C allows us to redefine $\Gamma(C, D; y)$ again, and we are eligible to take z out without injuring $\mathcal{G}$. Also since all $\gamma(n)$ with $n \geq k(\beta)$ are lifted above the use $\varphi_e(x)[s]$, the disagreement is preserved forever and $\mathcal{R}_e$ is satisfied.

On the other hand, if $\Phi_e^{A\oplus D}$ is total and $\Phi_e^{A\oplus D} = W_e$, then β has no chance to initiate the disagreement strategy. Thus, if Δ_β^A is total, then Δ_β^A can compute W_e correctly.

Now consider the consistency between two $\mathcal{R}$-strategies, $\mathcal{R}_1 < \mathcal{R}_2$ say. When a number, z say, is extracted from D by $\mathcal{R}_2$-strategy, some computations involved in the $\mathcal{R}_1$-strategy, β_1 say, are destroyed, and lead to the mistakes of $\Delta^A_{\beta_1}$. Such incorrectness cannot be corrected by enumerating z into D again, because we are making D d.c.e. To avoid this, whenever the $\mathcal{R}_2$ strategy β_2 executes the disagreement strategy at y (extracting z out of D), we put s_y, the stage at which $\Delta^A_{\beta_2}(y)$ was defined, into A to undefine $\delta_{\beta_1}(y')$ which was defined after stage s_y. Clearly, such an action does no harm to those $\delta_{\beta_1}(y'')$ defined before stage s_y. β_2 is consistent with β_1.

β has three outcomes, $0 <_{\mathrm{L}} d <_{\mathrm{L}} 1$, where d stands for the fact that the disagreement strategy succeeds, and 0 and 1 for the cases of whether or not there are infinitely many β-expansionary stages unless we force a disagreement.

A $\mathcal{P}^C$-strategy

A $\mathcal{P}^C$-strategy, η say, will satisfy a $\mathcal{P}^C$-requirement, $C \neq \Phi_e^{A\oplus D}$ say. η is the Friedberg-Muchnik strategy, with some modifications to cooperate with the $\mathcal{G}$-strategy.

η works as follows:

(1) Define $k(\eta)$ as a fresh number.

(2) Appoint a witness, $x(\eta)$, as fresh.

(3) Wait for a stage s at which $\Phi_e^{A\oplus D}(x)[s]\downarrow= 0$.

(4) Enumerate $x(\eta)$ and $\gamma(k(\eta))[s]$ into C, and stop.

Since $x(\eta) > k(\eta)$, enumerating of $\gamma(k(\eta))[s]$ into C lifts all $\gamma(n)(n \geq k(\eta))$ and so $\Phi_e^{A\oplus D}(x(\eta))$ is protected from the injury of the $\mathcal{G}$-strategy. Therefore,

$$\Phi_e^{A\oplus D}(x(\eta)) = \Phi_e^{A\oplus D}(x(\eta))[s] = 0 \neq 1 = C_{s+1}(x(\eta)) = C(x(\eta)).$$

So $\mathcal{P}^C$ is satisfied.

η has two possible outcomes, $0, 1$ with $0 <_L 1$, where 0 indicates that η arrives at step 4 eventually, and 1 indicates that η keeps waiting at step 3.

A $\mathcal{P}^D$-strategy

A $\mathcal{P}^D$-strategy behaves like the $\mathcal{P}^C$-strategy described above, except that at stage s, we only put $x(\eta)$ into D. We don't put $\gamma(k(\eta))$ into D because $\mathcal{G}$ does not enumerate γ-markers into C directly. Thus, $\Phi_e^C(x(\eta))[s]$ is injured only when η is initialized.

7.3 Construction

Before describing the construction, we define the priority tree, T say, effectively. First define the *priority* of the requirements as follows:

$$\mathcal{G} < \mathcal{N}_0 < \mathcal{R}_0 < \mathcal{P}_0^C < \mathcal{P}_0^D < \mathcal{N}_1 < \mathcal{R}_1 < \mathcal{P}_1^C < \mathcal{P}_1^D < \cdots$$
$$< \mathcal{N}_n < \mathcal{R}_n < \mathcal{P}_n^C < \mathcal{P}_n^D < \cdots,$$

where for any requirements, $\mathcal{X}, \mathcal{Y}$ say, if $\mathcal{X} < \mathcal{Y}$, then $\mathcal{X}$ has priority higher than $\mathcal{Y}$.

For $\xi \in T$, say that requirement $\mathcal{X}$ is *satisfied at* ξ, if there is an $\mathcal{X}$-strategy τ such that $\tau \subset \xi$. $\mathcal{G}$ is a global requirement, and we do not list $\mathcal{G}$ on T.

Definition 7.1 *(1) Define the root node λ as an $\mathcal{N}_0$-strategy;*

(2) The immediate successors of a node are the possible outcomes of the corresponding strategy;

(3) For $\xi \in T$, ξ works for the highest priority requirement which is not satisfied at ξ.

Each $\xi \in T$ has two parameters attached, $k(\xi)$ and $n(\xi)$. If ξ is a $\mathcal{P}$-strategy, then ξ has another parameter, $x(\xi)$.

In the construction, if ξ is initialized, then any parameters will be canceled. If ξ is reset, then any parameters, except $k(\xi)$, will be canceled. If $\xi < \xi'$ and ξ is reset or initialized, then ξ' will be initialized simultaneously and automatically.

Construction

Without loss of generality, suppose that K is enumerated at odd stages, and that exactly one element is enumerated into K at such stage.

Stage 0: Set $C = D = A = \emptyset$, and initialize all nodes.

Stage $s+1$:

(I) $s+1$ odd. Let k be the number in $K_{s+1} - K_s$.

1. If $\Gamma(C, D; k)\downarrow$, then
   - Enumerate $\gamma(k)$ into D.
   - For any strategy ξ, if $k(\xi) > k$, then reset ξ.
   - Go to stage $s+2$.
2. Otherwise, let x be the least y such that $\Gamma(C, D; y)\uparrow$, define $\Gamma(C, D; x) = K(x)$ with $\gamma(x)$ fresh, and go to stage $s+2$.

(II) $s+1$ even.

We define a partial function σ_{s+1} of length $\leq s+1$ by substages. Say that a strategy ξ is *visited at* stage $s+1$, if ξ is eligible to act at a substage t of stage $s+1$.

First, let λ, the root node, be eligible to act at substage 0.

Substage $t \geq 0$: Given $\xi = \sigma_{s+1} \upharpoonright t$, initialize all nodes $>_{\mathrm{L}} \xi$. If $t = s+1$, then define $\sigma_{s+1} = \xi$, initialize all $\zeta > \sigma_{s+1}$, and go to the next stage.

If $k(\xi)[s]\uparrow$, then define $k(\xi)[s+1]$ as a fresh number, define $\sigma_{s+1} = \xi$, initialize all $\zeta > \sigma_{s+1}$, and go to the next stage. Otherwise, there are four cases:

Case 1. $\xi = \alpha$ is an $\mathcal{N}$-strategy.

α1. If $s+1$ is a α-expansionary stage, then let $\alpha^\frown\langle 0\rangle$ be eligible to act at the next substage;

α2. Otherwise, let $\alpha^\frown\langle 1\rangle$ be eligible to act at the next substage.

Case 2. $\xi = \beta$ is an $\mathcal{R}$-strategy.

β1. If $s+1$ is β-expansionary and Δ^A_β is correct, then for any $y < \ell_{\mathcal{R}}(\beta, s)$ with $\Delta^A_\beta(y)[s] \uparrow$, define $\Delta^A_\beta(y)[s+1] = W_{\beta,s+1}(y)$ with use $\delta_\beta(y) = s+1$. Let $\beta \frown \langle 0 \rangle$ be eligible to act at the next substage;

β2. If $s+1$ is β-expansionary and Δ^A_β is incorrect at some (least) y, $n(\beta) \uparrow$, then put $\gamma(k(\beta))[s]$ into C, define $n(\beta) = y$, $s(\beta) = s+1$, and also define $\sigma_{s+1} = \beta$, initialize all $\zeta > \sigma_{s+1}$, and go to the next stage.

β3. If $s(\beta) \downarrow, n(\beta) \downarrow = y$, and $d(\beta) \uparrow$, then remove the numbers enumerated into D between $s(\beta)$ and the stage at which $\Delta^A_\beta(y)[s]$ was defined, put $\delta_\beta(y)[s]$ into A, and undefine those $\Delta^A_{\beta'}(y)$ defined after the stage at which $\Delta^A_\beta(y)[s]$ gets defined. Define $d(\beta) \downarrow$, and define $\sigma_{s+1} = \beta$. Initialize all $\zeta > \sigma_{s+1}$, and go to the next stage.

β4. If $s+1$ is not β-expansionary, $d(\beta) \downarrow$, then let $\beta \frown \langle d \rangle$ be eligible to act at the next substage;

β5. Otherwise, let $\beta \frown \langle 1 \rangle$ be eligible to act at the next substage.

Case 3. $\xi = \eta$ is a $\mathcal{P}^C$-strategy.

η1. If $x(\eta) \uparrow$, then define $x(\eta)$ as a fresh number, define $\sigma_{s+1} = \eta$, initialize all $\zeta > \sigma_{s+1}$, and go to the next stage

η2. If $x(\eta) \downarrow$ and $x(\eta) \in C$, then let $\eta \frown \langle 0 \rangle$ be eligible to act at the next substage;

η3. If $x(\eta) \downarrow, \Phi^{A\oplus D}_\eta(x(\eta)) \downarrow = C(x(\eta)) = 0$, then enumerate $x(\eta)$ and $\gamma(k(\eta))$ into C, define $\sigma_{s+1} = \eta$, initialize all $\zeta > \sigma_{s+1}$, and go to the next stage.

η4. Otherwise, let $\eta \frown \langle 1 \rangle$ be eligible to act at the next substage.

Case 4. $\xi = \tau$ is a $\mathcal{P}^D$-strategy.

τ1. If $x(\tau) \uparrow$, then define $x(\tau)$ as a fresh number, define $\sigma_{s+1} = \tau$, initialize all $\zeta > \sigma_{s+1}$, and go to the next stage.

τ2. If $x(\tau) \downarrow$ and $x(\tau) \in D$, then let $\tau \frown \langle 0 \rangle$ be eligible to act at the next substage;

$\tau 3$. If $x(\tau)\downarrow, \Phi_\tau^C(x(\tau))\downarrow = D(x(\tau)) = 0$, then enumerate $x(\tau)$ into D, define $\sigma_{s+1} = \tau$, initialize all $\zeta > \sigma_{s+1}$, and go to the next stage.

$\tau 4$. Otherwise, let $\tau^\frown\langle 1\rangle$ be eligible to act at the next substage.

This completes the construction.

7.4 Verification

In this section, we verify that our construction satisfies all requirements. First we have:

Lemma 7.4.1 *For any x, s,*

(1) If $\Gamma(C, D; x)[s]\downarrow$, then $\gamma(x)[s] \notin C_s \cup D_s$;

(2) If $\Gamma(C, D; x+1)\downarrow$, then $\Gamma(C, D; x)\downarrow$ and $\gamma(x)[s] < \gamma(x+1)[s]$;

(3) If $s < v$, $\Gamma(C, D; x)[s]\downarrow$, $C_v \upharpoonright (\gamma(x)[s]+1) = C_s \upharpoonright (\gamma(x)[s]+1)$ and $D_v \upharpoonright (\gamma(x)[s]+1) = D_s \upharpoonright (\gamma(x)[s]+1)$, then $\Gamma(C, D; x)[v]\downarrow$, and $\gamma(x)[v] = \gamma(x)[s]$;

(4) If $\Gamma(C, D; x)$ is undefined at stage $s+1$ and $\Gamma(C, D; x)$ is defined at stage s, then $C_{s+1} \upharpoonright (\gamma(x)[s]+1) \neq C_s \upharpoonright (\gamma(x)[s]+1)$ or $D_{s+1} \upharpoonright (\gamma(x)[s]+1) \neq D_s \upharpoonright (\gamma(x)[s]+1)$. $\square$

Define $f = \liminf_s \sigma_s$. f is the true path of the construction.

Lemma 7.4.2 *For $\xi \subset f$,*

(1) ξ is initialized or reset only finitely often;

(2) ξ acts only finitely often.

Proof. We prove the lemma by induction. Let ξ^- be the immediate predecessor of ξ. By the induction hypotheses, there is some (least) stage s_0 such that

(a) ξ^- cannot be initialized or reset afterwards;

(b) ξ^- does not act afterwards.

Since ξ is on f, there is some stage $s_1 \geq s_0$ such that $\forall s \geq s_1(\sigma_s \geq \xi)$. By the choice of s_0, ξ cannot be initialized after s_1. Let $k(\xi)$ be defined at stage $s_2 \geq s_1$. Then $k(\xi)$ cannot be canceled later and ξ can be reset only finitely often. So (1) holds.

For (2), there are four cases:

(i) ξ is an $\mathcal{N}$-strategy.

ξ does no action after stage s_2. So (2) follows immediately.

(ii) ξ is an $\mathcal{R}$-strategy.

Without loss of generality, suppose that ξ acts after stage s_2. Then some disagreement between Δ_ξ^A and W_ξ at y say, appears at stage s_y say. Let $\Delta_\xi^A(y)[s_2]$ be defined at stage $s' < s_2$. By the construction, ξ requests to perform the disagreement strategy. By enumerating $\gamma(k(\xi))[s_y]$ into C, ξ's request is confirmed and all $\gamma(n)$ with $n \geq k(\xi)$ are lifted above s_y. (Here the choice of s_2 guarantees that no $y < k(\xi)$ enters K later.) Let $s_4 > s_2$ be the next ξ-stage. Then ξ executes the disagreement strategy by taking out all numbers enumerated into D between stages s' and s_y, and so the computation $\Phi_\xi^{A\oplus D}(y)$ is recovered to $\Phi_\xi^{A\oplus D}(y)[s']$. By preserving this inequality, we have:

$$\Phi_\xi^{A\oplus D}(y) = \Phi_\xi^{A\oplus D}(y)[s'] = W_{\xi,s'}(y) = 0 \neq 1 = W_{\xi,s_y}(y) = W_\xi(y).$$

ξ is satisfied and does not act anymore. So (2) holds.

(iii) ξ is a $\mathcal{P}^C$-strategy.

Suppose that at stage $s_5 > s_2$, ξ chooses a fresh number $x(\xi)$ as an attacker for C, and suppose at stage $s_6 > s_5$ at which ξ is accessible again, $\Phi_\xi^{A\oplus D}(x(\xi))[s_0] \downarrow= 0$. That is, ξ is ready to perform the diagonalization action. By the construction, $x(\xi)$ and $\gamma(k(\xi))$ are put into C. The enumeration of $\gamma(k(\xi))$ lifts all $\gamma(n)$ for $n \geq k(\xi)$, and the computation $\Phi_\xi^{A\oplus D}(x(\xi))[s_6]$ is preserved forever. So, ξ is satisfied and does not act anymore.

(iv) ξ is a $\mathcal{P}^D$-strategy.

Similar to (iii), except that at stage s_6, we just put $x(\xi)$ into D. Since the $\mathcal{G}$-strategy always puts the markers into D, $C \upharpoonright s_6$ can be changed only when ξ is initialized. By the choice of s_2, no such initialization can happen, and so $\Phi_\xi^C(x(\xi))[s_6]$ is preserved forever and ξ's requirement is satisfied. □

By Lemma 7.4.2, we have the following lemma immediately.

Lemma 7.4.3 *For any $\xi \subset f$, there is some $\mathcal{O}$ such that $\xi \frown \langle \mathcal{O} \rangle \subset f$. Thus, $|f| = \infty$.* □

Lemma 7.4.4 *For any $\xi \subset f$,*

(1) If ξ is a $\mathcal{P}$-strategy, then $\mathcal{P}$ is satisfied.

(2) If ξ is an $\mathcal{N}$-strategy, then $\mathcal{N}$ is satisfied.

(3) If ξ is an $\mathcal{R}$-strategy, then $\mathcal{R}$ is satisfied.

Proof. (1) By the proof of (iii), (iv) in Lemma 7.4.2.

(2) Assume that $\Phi_\xi(A) = \Phi_\xi(C) = g$ is total. Since ξ is on the true path, we know from Lemma 7.4.2 that ξ can be initialized or reset only finitely often. Let s_0 be the least stage such that ξ cannot be initialized or reset afterwards. By our assumption, there are infinitely many ξ-expansionary stages bigger than s_0. We compute g as follows.

Given x, find the least ξ-expansionary stages s_1 say, $s_1 > s_0$ and $\ell_{\mathcal{N}}(\xi, s_1) > x$. Then $g(x) = \Phi_\xi(A; x)[s_1] = \Phi_\xi(C; x)[s_1]$ by the argument in the $\mathcal{N}$-strategy. Since s_1 can be found effectively, g is computable.

(3) Suppose that $\Phi_\xi(A \oplus D) = W_\xi$. By the discussion in Lemma 7.4.2, ξ has no chance to initiate the disagreement strategy, because otherwise, a disagreement between W_ξ and $\Phi_\xi(A \oplus D)$ would be created and preserved forever, making $\Phi_\xi(A \oplus D) \neq W_\xi$, a contradiction. Thus, we only need to prove that Δ_ξ^A is total.

Fix x. Let $\Delta_\xi^A(x)$ be defined at stage s_0. Then $\delta_\xi^A(x)[s_0] = s_0$. W.l.o.g., suppose that at stage $s_1 > s_0$, some $\xi' > \xi$ executes the disagreement strategy, and $\Delta_\xi^A(x)$ is declared to be undefined. Then at the next ξ-expansionary stage s_2, $\Delta_\xi^A(x)$ gets redefined with use $\delta_\xi^A(x)[s_2] = s_2$. By the construction, all nodes $> \xi'$ is initialized at stage s_1 and hence, cannot undefine $\Delta_\xi^A(x)[s_2]$ again. Since only finitely many nodes below $\xi^\frown\langle 0\rangle$ with priority higher than ξ' have been visited before stage s_2, and only these strategies' actions can undefine the newly defined $\Delta_\xi^A(x)$, there is a large enough stage s_x at which $\Delta_\xi^A(x)$ gets defined and this definition cannot be destroyed afterwards. Thus, $\Delta_\xi^A(x)\downarrow = \Delta_\xi^A(x)[s_x]$. □

Lemma 7.4.5 $\Gamma(C, D) = K$.

Proof. By Lemma 7.4.1, $\Gamma(C, D)$ is a p.c. functional. By actions during odd stages, if $\Gamma(C, D)$ is total, then $\Gamma(C, D) = K$.

Fix k. Let $\zeta = f \upharpoonright k$. Then ζ can only be visited after stage k. Since ζ is on the true path, by Lemma 7.4.2, there is some (least) stage s_ζ such that ζ cannot be initialized or reset afterwards. Let $s' \geq s_\zeta$ be the stage at which $k(\zeta)$ gets defined. Then $k(\zeta) > k$ and $k(\zeta)$ cannot be canceled later. Since all lower priority strategies are initialized at stage s', $C \upharpoonright \gamma(k(\zeta))[s']$ (and so $C \upharpoonright \gamma(k)[s']$) remains unchanged. Also by the choice of s_ζ, ζ cannot be reset again, so no γ-markers less than $\gamma(k(\zeta))[s']$ can enter D. Thus,

$$C \upharpoonright \gamma(k)[s'] = C_{s'} \upharpoonright \gamma(k)[s'] \text{ and } D \upharpoonright \gamma(k)[s'] = D_{s'} \upharpoonright \gamma(k)[s'],$$

$\Gamma(C, D; k)\downarrow = \Gamma(C, D; k)[s']$. □

This completes the proof of Theorem 8. □

CHAPTER 8

PRESENTATIONS OF C.E. REALS

8.1 Introduction

Prefix-free languages play an important role in the study of randomness, see Martin-Löf [60] or Chaitin [12]. Let $\Sigma = \{0, 1\}$ be the binary alphabet, and Σ^* be the set of finite binary strings. For $\sigma \in \Sigma^*$, let $|\sigma|$ be the length of σ. $A \subseteq \Sigma^*$ is prefix-free if for any $\sigma_1, \sigma_2 \in A$, if σ_1 is a prefix of σ_2, then $\sigma_1 = \sigma_2$. Following Solovay [73], we say that C is a *Chaitin machine* (self-delimiting Turing machine) if C is a Turing machine processing binary strings such that its program set (domain) $PROG_C = \{\eta \in \Sigma^* : C(\eta) \text{ halts}\}$ is prefix-free. Clearly, $PROG_C$ is computably enumerable (c.e. for short). Conversely, each prefix-free c.e. set of strings is the domain of some Chaitin machine. The *program-size complexity* of the string $\sigma \in \Sigma^*$ (relatively to C) is defined as $H_C(\sigma) = \min\{|\eta| : \eta \in \Sigma^*, C(\eta) = \sigma\}$, where $\min \emptyset = \infty$.

Theorem 8.1 *(Invariance Theorem) There is a Chaitin machine U (called universal) such that for every Chaitin machine C and all σ, $H_U(\sigma) \leq H_C(\sigma) + O(1)$.*

Note that $PROG_U$ is c.e., but not computable.

Another property of a prefix-free language, A say, is that $\sum_{\sigma \in A} 2^{-|\sigma|}$ converges to a real in $[0, 1]$ (by Kraft's Inequality), which is denoted as 2^{-A}. This exhibits some relationship between prefix-free languages and reals. Say that a real α is computably enumerable if there is a computable increasing sequence $(r_n)_{n\in\omega}$ of rational numbers such that $(r_n)_{n\in\omega}$ converges to α. Clearly, 2^{-PROG_U} is a c.e. real. We use Ω_U to denote this real. Ω_U is the so-called *Chaitin Ω number.* If the rate of convergence of $(r_n)_{n\in\omega}$ can be controlled by a computable function, then we say that α is computable. All algebraic numbers, π, the Euler number e, and all numbers commonly used in numerical analysis are computable. Since $PROG_U$ is not computable, Ω_U is a noncomputable real. This provides a natural example of a noncomputable c.e. real. Calude, Hertling, Khoussainov and Wang [11], Downey, Hirschfeldt and Nies [30], Slaman [50], and Solovay [73] provide extensive investigations of the randomness of Ω.

Calude et al. [11] also gives several equivalent definitions of the c.e. reals. Indeed, they prove the following theorem:

Theorem 8.2 *(Calude et al., [11]) The following are equivalent for a real α:*

(1) α is computably enumerable.

(2) The left Dedekind cut of α is computably enumerable.

(3) There is an infinite computably enumerable prefix-free set $V \subset \Sigma^*$ such that $\alpha = \Sigma_{\sigma \in V} 2^{-|\sigma|}$.

(4) There is a computable prefix-free set $V \subset \Sigma^*$ such that $\alpha = \Sigma_{\sigma \in V} 2^{-|\sigma|}$.

(5) There is a computable function $f(x, y)$ of two variables such that

(5a) for all k, s, if $f(k, s) = 1$ and $f(k, s+1) = 0$, then there exists $k' < k$ such that $f(k', s) = 0$ and $f(k', s+1) = 1$.

(5b) $\alpha = 0.a_1 a_2 \cdots$, where $a_i = \lim_s f(i, s)$.

(6) There is a computable increasing sequence of rationals with limit α.

Based on characterization (3), Downey and LaForte [31] proposed the notion of presentations as follows:

Definition 8.1 *(Downey and LaForte [31]) Say that $V \subseteq \Sigma^*$ is a prefix-free presentation of a c.e. real α if*

(1) V is a prefix-free c.e. set;

(2) $\alpha = 2^{-V}$.

By (4) of Theorem 8.2, each c.e. real has a computable presentation. Say that a c.e. real α is *simple* if all presentations of α are computable. One might conjecture that the more complicated a real is, the more complicated the presentations can be. Surprisingly, Downey and LaForte [31] provided a counterexample to this conjecture by showing that there is a simple real of high Turing degree.

Theorem 8.3 *(Downey and LaForte [31]) There is a simple real α with $\alpha' \equiv_{\mathrm{T}} \emptyset''$.*

In this paper, we prove that the conjecture can be true locally:

Theorem 9 *There is a noncomputable c.e. set A such that A bounds no noncomputable simple reals. That is, for any noncomputable c.e. real $\alpha \in [0,1]$, if $\alpha \leq_{\mathrm{T}} A$, then α has a noncomputable presentation.*

Thus, the simple reals are not dense in the structure of the computably enumerable degrees. Here, $\alpha \leq_{\mathrm{T}} A$ means that there is a (partial) computable functional Φ such that $\alpha = \Phi(A)$, namely, the real $0.\Phi(A;0)\Phi(A;1)\cdots$ is a binary expansioanry of α. Theorem 9 is proved by induction as follows:

Theorem 8.4 *There is a noncomputable c.e. set A s.t. for any $n \geq 0$, for any noncomputable c.e. real $\alpha \in [1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}]$, if $\alpha \leq_{\mathrm{T}} A$, then α has a noncomputable presentation.*

The main part of this chapter is devoted to the proof of the base step $n = 0$:

Theorem 8.5 *There is a noncomputable c.e. set A s.t. for any noncomputable c.e. real $\alpha \in [0, \frac{1}{2}]$, if $\alpha \leq_{\mathrm{T}} A$, then α has a noncomputable presentation.*

The successor step of the proof is as follows: fix A as in Theorem 8.4, and assume that we have Theorem 8.4 for n. We now consider the case $n+1$. Fix $\alpha \in [1-\frac{1}{2^{n+1}}, 1-\frac{1}{2^{n+2}}]$, $\alpha \leq_{\mathrm{T}} A$ as a noncomputable c.e. real. Then $\beta = 2\alpha - 1 \in [1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}}]$ and $\beta \leq_{\mathrm{T}} A$ is also a noncomputable c.e. real. By the induction hypothesis, β has a noncomputable presentation V_β. Define $V_\alpha = \{0\sigma : \sigma \in V_\beta\} \cup \{1\}$. Then V_α is c.e. and noncomputable. Prefix-freeness of V_α comes from that of V_β directly. Also we have

$$2^{-V_\alpha} = \sum_{\eta \in V_\alpha} 2^{-|\eta|} = \sum_{\sigma \in V_\beta} 2^{-|0\sigma|} + \frac{1}{2} = \frac{1}{2} \sum_{\sigma \in V_\beta} 2^{-|\sigma|} + \frac{1}{2} = \frac{1}{2}\beta + \frac{1}{2} = \alpha.$$

Therefore, V_α is a presentation of α. $\square$

The proof of Theorem 8.5 involves the use of the "gap-cogap" argument and the whole proof is a $0'''$-argument. By combining the construction in this chapter with the high$_2$ strategy as in Downey, Lempp and Shore [32], we can make A high$_2$.

8.2 Requirements

In this section we list the requirements necessary to prove Theorem 8.5. We will construct a c.e. set A satisfying for all i the requirements:

$\mathcal{P}_i$: $A \neq \overline{W}_i$;

$\mathcal{R}_i$: $[\alpha_i = \Phi_i(A) \in [0, \frac{1}{2}]$ & α_i is c.e.$] \Rightarrow [\alpha_i$ is computable $\vee$ $(\exists$ noncomputable c.e. $U_i)(U_i$ presents $\alpha_i)]$.

Here we assume that $\{W_i\}_{i\in\omega}$ is a listing of all c.e. sets of natural numbers, and $\{(\alpha_i, \Phi_i)\}_{i\in\omega}$ is a listing of all pairs (α, Φ) s.t. α is a c.e. real, and Φ is a partial computable functional.

Since $\mathcal{R}_i$ is a complicated requirement, we break it into simpler subrequirements $\mathcal{S}_{i,j}$ as follows:

$\mathcal{S}_{i,j}$: $U_i \neq \overline{V}_j$.

Here $\{V_i\}_{i\in\omega}$ is a listing of all c.e. sets of strings in Σ^*.

8.3 Basic modules for each requirement

We now describe the atomic strategies (*modules*) for each type of requirement $\mathcal{P}_i, \mathcal{R}_i, \mathcal{S}_{i,j}$.

The ρ-module for $\mathcal{P}_i$

A $\mathcal{P}$-strategy, ρ say, which tries to find a number x such that $A(x) \neq \overline{W}_i(x)$, is the standard Friedberg-Muchnik procedure. ρ runs as follows:

(1) Choose a big number, x say.

(2) Wait for $\Phi_i(x)\downarrow$.

(3) Put x into A, and stops.

ρ has two finitary outcomes: w and s, where w denotes the fact that ρ waits at (2) forever, and s denotes the fact that ρ reaches (3) and stop. In both cases, $A(x) \neq \overline{W}_i(x)$, and $\mathcal{P}_i$ is satisfied.

The τ-module for $\mathcal{R}_i$

Let $\alpha_i[s]$ be the approximation of α_i at stage s via F_i, a c.e. set of strings presenting α_i. By the way we choose to present the α_i's, F_i exists. Then, $\alpha_i[s] = \sum_{\sigma \in F_{i,s}} 2^{-|\sigma|}$. Define the distance functions between α_i and Φ_i^A as follows:

$$\ell(i,s) = \max\{x < s : (\forall y < x)[\Phi_i^A(y)[s]\downarrow]\},$$
$$m\ell(i,s) = \max\{0, \ell(i,t) : t < s \text{ is an } i\text{-expansionary stage}\},$$
$$d(i,s) = |\textstyle\sum_{y<\ell(i,s)} \Phi_i^A(y)[s] \cdot 2^{-y} - \alpha_i[s]|,$$
$$md(i,s) = \min\{0, d(i,t) : t < s \text{ is an } i\text{-expansionary stage}\}.$$

Say that stage s is i-expansionary if $s = 0$ or

(1) $\ell(i,s) > \max\{m\ell(i,s), |\sigma| : \sigma$ is assigned to an $\mathcal{S}_{i,j}$-strategy$\}$, and also

(2) $d(i,s) < \min\{md(i,s), 2^{-m\ell(i,s)}, 2^{-|\sigma|} : \sigma$ is assigned to an $\mathcal{S}_{i,j}$-strategy$\}$.

The τ-module has a (*finitary*) outcome 1, in the case when there are only finitely many i-expansionary stages, and an (*infinitary*) outcome 0, in the case when there are infinitely many i-expansionary stages.

If the τ-module has outcome 0, then τ is responsible for constructing an auxiliary c.e. set of strings U_τ such that U_τ is noncomputable, and such that U_τ presents

α_τ. U_τ is constructed by τ's substrategies. Links from τ to these substrategies will be created and removed alternatingly in the whole construction. We will also ensure that $\sum_{\sigma\in U_{\tau,s}} 2^{-|\sigma|} \leq \alpha_{\tau,s}$, and that infinitely often we can update U_τ such that $\sum_{\sigma\in U_{\tau,s}} 2^{-|\sigma|} = \alpha_{\tau,s}$. By the Chaitin-Kraft inequality, we can always do this. Thus, we can effectively expand the prefix-free set $U_{\tau,s}$ to a larger one with the prefix-freeness preserved. Since $\alpha_i \in [0, \frac{1}{2}]$, we can always choose the strings for the updating process as 1σ, and leave the strings of the form $00\cdots 01$ for the $\mathcal{S}_{i,j}$-strategies, $j \in \omega$. This arrangement avoids the interaction between the diagonalization and the updating process when prefix-freeness is considered.

The η-module for $\mathcal{S}_{i,j}$

The η-module is the "gap-cogap" argument, which is the key part of the proof. The η-module requires various parameters such as $\sigma(\eta), r(\eta)$, whose values at the end of stage s will be denoted by $\sigma(\eta)[s], r(\eta)[s]$. The η-module has two objectives. One is to find some string σ witnessing $U_\eta \neq \overline{V}_\eta$, and if this fails, η will ensure that the real α_η (namely, $\alpha_{\tau(\eta)}$) is computable. In the first case, η itself is satisfied. In the second case, $\mathcal{R}_i$ is satisfied and no further subrequirements $\mathcal{S}_{i,j'}, j' \neq j$, need to be examined.

The η-module consists of the following steps:

(1) Choose a string $\sigma = 00\cdots 01$, with $|\sigma|$ big.

(2) Wait for a stage s such that $\sigma \in V_j[s]$. At stage $s+1$ *open a gap* by setting $r[s+1] = 0$, and go to step (3). **Stop updating U_i until this gap is closed.**

(3) Wait for the least i-expansionary stage $t \geq s+1$ such that $\ell(i,t) > |\sigma|$. At stage $t+1$ *close the gap* by performing step (3a) or (3b) according to which case applies:

(3a) (**Successful close**) Suppose that $\alpha_i[t] - \alpha_i[s] > 2^{-|\sigma|}$. Enumerate σ into U_i, update U_i and stop.

(3b) (**Unsuccessful close**) Otherwise, define $r[t+1] = t$ (to preserve $\Phi_i^A[t] \upharpoonright |\sigma|$). Cancel σ, update U_i and go to step (1).

Note that if $\alpha_i = \Phi_i^A$, then every gap must eventually be closed, and so U_i will be updated correctly. The η-module has three possible outcomes:

w. In this outcome, the η-module opens at most finitely many gaps and never closes one successfully.

We may suppose that each gap once opened is eventually closed, else there are at most finitely many i-expansionary stages, contradicting the assumption that τ has outcome 0.

Let s be the last stage at which η closes its gap, and let σ be chosen at stage $t > s$. Then σ never enters V_j, because otherwise, η will create another link, which is contrary to the choice of s. Thus,

$$\overline{V}_j(\sigma) = 1 - V_j(\sigma) = 1 \neq 0 = U_i(\sigma).$$

$\mathcal{S}_{i,j}$ is satisfied.

s. There is some stage $s+1$ at which η closes a gap successfully.

In this case, $\sigma = \sigma[s]$ is enumerated into U_i at stage $s+1$. Suppose that at stage $v_1 < s$, σ was defined as $\sigma[s]$, and at stage $v_2 \in [v_1, s)$, η opened a gap trying to enumerate σ into U_i. By the module, $\alpha_i[s] - \alpha_i[v_2] > 2^{-|\sigma|}$, and the updating process does not act during the gap, hence, we are eligible to put σ into U_i. Thus,

$$U_i(\sigma) = 1 \neq 0 = 1 - \overline{V}_j(\sigma),$$

$\mathcal{S}_{i,j}$ is satisfied.

g. In this outcome η opens infinitely many gaps and closes each one unsuccessfully.

Obviously, the limit of the length of the candidates (strings) is ∞. Suppose that the infinitely many gaps are begun at stages $\{t_n\}_{n\in\omega}$ and closed at stages $\{v_n\}_{n\in\omega}$, where

$$t_0 < v_0 < t_1 < v_1 < \cdots.$$

The intervals of stages $\{s : t_n \leq t < v_n\}$ are called gaps, since $r = 0$ in these intervals and we are free to enumerate any number into A to make A noncomputable, while the intervals $\{s : v_n \leq s < t_{n+1}\}$ are called cogaps and we have $r = v_n$.

α_i is computable by the following argument:

Given n, choose a stage t large enough such that a gap is begun at stage $t+1$, and the corresponding string σ has length $|\sigma| > n+3$. Since the gap is closed unsuccessfully at stage $v+1 > t+1$, we have $\alpha_i[v] - \alpha_i[t] \leq 2^{-|\sigma|}$. If the next gap is begun at stage $t'+1 > v+1$, then the A-restraint r ensures that $\Phi^A \upharpoonright |\sigma|[v] = \Phi^A \upharpoonright |\sigma|[t']$. Thus,

$$\begin{aligned}
& \alpha_i[t'] - \alpha_i[v] \\
\leq\ & |\alpha_i[t'] - \textstyle\sum_{y<\ell(i,v)} \Phi_i^A(y)[v] \cdot 2^{-y}| + |\textstyle\sum_{y<\ell(i,v)} \Phi_i^A(y)[v] \cdot 2^{-y} - \alpha_i[v]| \\
\leq\ & |\alpha_i[t'] - \textstyle\sum_{y<\ell(i,t')} \Phi_i^A(y)[t'] \cdot 2^{-y}| + |\textstyle\sum_{y<\ell(i,t')} \Phi_i^A(y)[t'] \cdot 2^{-y} - \\
& \textstyle\sum_{y<\ell(i,v)} \Phi_i^A(y)[v] \cdot 2^{-y}| + |\textstyle\sum_{y<\ell(i,v)} \Phi_i^A(y)[v] \cdot 2^{-y} - \alpha_i[v]| \\
\leq\ & 2^{-|\sigma|} + 2^{-|\sigma|} + 2^{-|\sigma|} \\
=\ & 3 \cdot 2^{-|\sigma|}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\alpha_i[t'] - \alpha_i[t] &= (\alpha_i[t'] - \alpha_i[v]) + (\alpha_i[v] - \alpha_i[t]) \\
&\leq 3 \cdot 2^{-|\sigma|} + 2^{-|\sigma|} \\
&= 2^{-(|\sigma|-2)}.
\end{aligned}$$

Now the argument continues with σ replaced by a new one, σ' say. This establishes the following:

$$\begin{aligned}
\alpha_i - \alpha_i[t] &\leq 2^{-(|\sigma|-2)} + 2^{-(|\sigma'|-2)} + \cdots \\
&\leq 2^{-(|\sigma|-3)}.
\end{aligned}$$

Since $|\sigma| > n+3$, we have $\alpha_i - \alpha_i[t] < 2^{-n}$. Thus, α_i is computable.

In this case, $\liminf_s r[s] = 0$, because r is defined as 0 during gaps.

g is an infinitary outcome, while w, s are both finitary. In the construction, we will use $F(\eta)[s]$ to approximate the outcome of η, $F(\eta)$. Thus, if $\lim_s F(\eta)[s] = a$ exists, then the outcome is a, which is finitary and if $\lim_s F(\eta)[s]$ does not exist, then the outcome is g, in which case there are infinitely many gaps.

8.4 The priority tree

In this section, we construct the priority tree T on which our construction will proceed.

Let $\Lambda = \{0, 1, s, g, w\}$ with the order $<_\Lambda$ in the order (from left to right) as listed. T is defined as follows as a subset of the finite sequences of Λ.

Definition 8.2 *(The priority tree)*

(1) Define $T = \{\xi \in \Lambda^{<\omega} : (\forall n)[\xi(3n) \in \{0,1\}, \xi(3n+1) \in \{s,w\} \ \& \ \ \xi(3n+2) \in \{s,g,w\}]\}$.

(2) Define $[T] = \{h \in \Lambda^{\omega} : (\forall n)[h \upharpoonright n \in T]\}$, *the set of all infinite paths through* T.

Every node $\tau \in T$ with $|\tau| \equiv 0 \pmod 3$, will be assigned to an $\mathcal{R}$-requirement. The outcome 1 for τ denotes that there are at most finitely many τ-expansionary stages, so τ believes that $\mathcal{R}$ is satisfied. The outcome 0 denotes that there are infinitely many τ-expansionary stages, and that further actions to satisfy $\mathcal{R}$ must be taken by nodes $\eta \supset \tau$.

Every node $\rho \in T$ with $|\rho| \equiv 1 \pmod 3$, will be assigned to a $\mathcal{P}$-requirement.

Every node $\eta \in T$ with $|\eta| \equiv 2 \pmod 3$, will be assigned to some subrequirement $\mathcal{S}_{i,j}$. The η-strategy will be a variant of the basic module described in the section above, having outcomes from $\{s, g, w\}$.

Definition 8.3 *(Satisfication of strategies) Define (partial) functions $i, j : T \to \omega$ and lists L_0, L_1, L_2, by induction on $n = |\xi|$ as follows.*

(1) $n = 0$. Define as above, $i(\lambda) = 0$, and set $L_0(\lambda) = L_1(\lambda) = L_2(\lambda) = \omega$.

(2) $n > 0$. Let $\xi = \xi^- {}^\frown \langle a \rangle$ for $a \in \Lambda$, and assume that $i(\xi^-)$, $L_0(\xi^-)$, $L_1(\xi^-)$, $L_2(\xi^-)$ are defined, and also that $j(\xi^-)$ is defined if ξ^- is an $\mathcal{S}$-strategy.

First define $L_0(\xi), L_1(\xi)$ and $L_2(\xi)$ as in the next paragraph. If ξ is an $\mathcal{R}$-strategy, define $i(\xi) = \mu i[i \in L_0(\xi)]$, and let $j(\xi)$ be undefined. If ξ is a $\mathcal{P}$-strategy, define $i(\xi) = \mu i[i \in L_1(\xi)]$, and also let $j(\xi)$ be undefined. If ξ is an $\mathcal{S}$-strategy, define $i(\xi) = i_0, j(\xi) = j_0$ where $\langle i_0, j_0 \rangle = \mu n[n \in L_2(\xi)]$.

Case 1: $|\xi^-| \equiv 0 \pmod 3$. Adopt the subcase below.

(i): $a = 0$. Define $L_0(\xi) = L_0(\xi^-) - \{i(\xi^-)\}$, $L_1(\xi) = L_1(\xi^-)$, and $L_2(\xi) = L_2(\xi^-)$.

(ii): $a = 1$. Define $L_0(\xi) = (L_0(\xi^-) - \{i(\xi^-)\}) \cup \{i' : i' > i(\xi^-)\}$, $L_1(\xi) = L_1(\xi^-)$, and $L_2(\xi) = (L_2(\xi^-) - \{\langle i(\xi^-), k \rangle : k \in \omega\}) \cup \{\langle i', k \rangle : i' > i(\xi^-) \;\&\; k \in \omega\}$.

Case 2: $|\xi| \equiv 1 \pmod 3$. Define $L_0(\xi) = L_0(\xi^-)$, $L_1(\xi) = L_1(\xi^-) - \{i(\xi^-)\}$, and $L_2(\xi) = L_2(\xi^-)$.

Case 3: $|\xi| \equiv 2 \pmod 3$. Adopt the subcase below.

(i): $a = s$ or $a = w$. Define $L_0(\xi) = L_0(\xi^-)$, $L_1(\xi) = L_1(\xi^-)$, and $L_2(\xi) = L_2(\xi^-) - \{\langle i(\xi^-), j(\xi^-) \rangle\}$.

(ii): $a = g$. Define $L_0(\xi), L_1(\xi)$ and $L_2(\xi)$ as in case 1(ii).

This completes the priority assignment.

8.5 Construction

We shall denote the reals $\alpha_{i(\xi)}$ by α_ξ, and the funtionals $\Phi_{i(\xi)}$ by Φ_ξ. During the construction we regard α, Φ, A, W, V and so on as being in a state of formation.

At the end of the construction, we define the *true path* $f \in [T]$ of the construction. f is approximated via $\delta[s] \in T$ at each stage s, and we will have $f = \liminf_s \delta[s]$. During stage s, we define $\delta[s]$ via a series of substages $t < s$ at which we define a string δ_t such that $\delta_t \subseteq \delta_{t+1}$ and $\delta[s+1]$ is the last δ_t defined. Only string δ_t is allowed to act at substage t.

Definition 8.4 *(Genuine stages) For $\xi \in T$,*

(1) Say that $s+1$ is a ξ-stage if $\xi \subseteq \delta[s+1]$. In addition, 0 is a ξ-stage. Let S^ξ be the set of all ξ-stages.

(2) Say that $s+1$ is a genuine ξ-stage if $\xi = \delta_t$ for some substage t of stage $s+1$, where δ_t will be defined during the construction. Let G^ξ be the set of genuine ξ-stages.

Definition 8.5 *(Expansionary stages) Fix τ. At stage $s+1$, define the length functions as:*

$$\ell(\tau, s) = \max\{x < s : (\forall y < x)[\Phi_\tau^A(y)[s] \downarrow]\},$$

$$m\ell(\tau, s) = \max\{0, \ell(\tau, t) : t \leq s \text{ is a } \tau\text{-expansionary stage } \& \ t+1 \in S^\tau\}.$$

Also define the distance functions as:

$$d(\tau, s) = |\textstyle\sum_{y<\ell(\tau,s)} \Phi_\tau^A(y)[s] \cdot 2^{-y} - \alpha_\tau[s]|,$$

$$md(\tau, s) = \min\{0, d(\tau, t) : t \leq s \text{ is a } \tau\text{-expansionary stage } \& \ t+1 \in S^\tau\}.$$

Say that stage v is a τ-expansionary stage, if $v = 0$, or if $v = s+1$, $s+1 \in S^\tau$, and

(1) $\ell(\tau, s) > \max\{m\ell(\tau, s), |\sigma| : \sigma$ is assigned to an $\mathcal{S}_{i(\tau),j}$-strategy$\}$,

(2) $d(\tau, s) < \min\{md(\tau, s), 2^{-m\ell(\tau,s)}, 2^{-|\sigma|} : \sigma$ is assigned to an $\mathcal{S}_{i(\tau),j}$-strategy$\}$.

Suppose that at stage s, an $\mathcal{S}$-strategy η opens a gap. Then we shall construct a link (τ, η) with bottom η and top $\tau = \tau(\eta)$. By the construction of T, we have $\tau^\frown\langle 0\rangle \subseteq \eta$ and $F(\eta)[s] = g$. The link (τ, η) remains in force until either η closes the gap (either a successful or an unsuccessful close), or η is initialized. If at stage $s+1$, there is a link (τ, η) and there is a substage t of stage $s+1$ such that $\delta_t = \tau$ and $\delta_{t+1} = \eta$, then we say that the link (τ, η) is travelled at stage $s+1$. The purpose of the link is to allow the flow of control of the construction to pass directly from the top node τ to the bottom node η at τ-expansionary stages, to close the gap. The hypothesis $\alpha_\tau = \Phi_\tau(A)$ guarantees that each link will be travelled (at most once) and the corresponding gap will be closed.

Note that traveling the link (τ, η) may lead the construction to pass over some nodes $\xi, \tau \subset \xi \subset \eta$ of lower global priority, i.e., $i(\xi) > i(\eta) = i(\tau)$, which would otherwise receive attention before η if we simply descended the tree without using links. For such ξ, $s+1$ is a ξ-stage, but not a genuine ξ-stage. This can result in the failure of ξ. Thus, to consider whether a given ξ-strategy succeeds or not, we must examine the genuine ξ-stages rather than merely the ξ-stages.

Construction

The construction is as follows:

Stage $s = 0$: Initialize all nodes $\xi \in T$. Define $\delta[0] = \lambda$.

Stage $s+1$: The construction will proceed by substages $t \leq s+1$. We refer to substage t of stage $s+1$ as stage $(s+1, t)$. The value of a parameter p at the end of substage t will be denoted by p_t, rather than the complete specification $p[s+1, t]$. At the end of substage t we shall have $\delta_t \in T$ such that $\delta_t \subseteq \delta_{t+1} \subseteq \delta[s+1]$ for $t \leq s$, and only δ_t may act during substage t. At the end of stage $s+1$, if $s+1$ is a τ-expansionary stage, and there is no link with top τ created at stage $s+1$, where τ is an $\mathcal{R}$-strategy, then perform the updating process for τ by enumerating into U_τ appropriate strings to make $\sum_{\sigma \in U_{\tau,s+1}} 2^{-|\sigma|} = \alpha_{\tau,s+1}$. This process keeps the resulting c.e. set $U_{\tau,s+1}$ prefix-free.

Substage $t = 0$. Define $\delta_0 = \lambda$. Go to substage 1.

Substage $t+1 \leq s+1$. Given δ_t and for all $\mathcal{S}$-strategies $\eta \in T$, $F_t(\eta) \in \{s, g, w\}$, the current state of the η-module, define δ_{t+1} as follows:

Case 1. δ_t is an $\mathcal{R}$-strategy. If $s+1$ is not δ_t-expansionary, then define $\delta_{t+1} = \delta_t^\frown\langle 1\rangle$. If $s+1$ is δ_t-expansionary, and there is no link with top δ_t, then define $\delta_{t+1} = \delta_t^\frown\langle 0\rangle$. Otherwise, $s+1$ is δ_t-expansionary, and there is a link with top δ_t and bottom $\eta \supseteq \delta_t^\frown\langle 0\rangle$, define $\delta_{t+1} = \eta$.

Case 2. δ_t is a $\mathcal{P}$-strategy. If δ_t is satisfied, define $\delta_{t+1} = \delta_t^\frown\langle s\rangle$. Otherwise, define $\delta_{t+1} = \delta_t^\frown\langle w\rangle$.

Case 3. δ_t is an $\mathcal{S}$-strategy. Define $\delta_{t+1} = \delta_t^\frown\langle F_t(\delta_t)\rangle$.

Let $\xi = \delta_{t+1}$. Say that ξ requires attention at substage $(s+1, t+1)$ if ξ has not been satisfied, and:

1. ξ is a $\mathcal{P}$-strategy, and one of the following holds:

   (1a) $x(\xi)$ is not defined, or

   (1b) $\Phi_\xi(x(\xi))[s]\downarrow$.

2. ξ is an $\mathcal{S}$-strategy, and one of the following holds:

   (2a) $\sigma(\xi)$ is not defined, or

   (2b) ξ is ready to open a gap, or

   (2c) ξ is ready to close a gap.

If ξ does not require attention, and $t < s$, then go to substage $t+2$. Otherwise, choose the first case which applies and perform the action as follows. If $t < s$, and ξ opens an ξ-gap, then go to substage $t+2$. Otherwise, define $\delta[s+1] = \delta_{t+1}$, go to stage $s+2$.

Case 1: $\xi = \rho$ is a $\mathcal{P}$-strategy. If (1a) applies, then define $x(\rho)$ as a big number. If (1b) applies, then enumerate $x(\rho)$ into A and say that $\mathcal{P}_\rho$ is satisfied. In any case, initialize all $\zeta > \rho$.

Case 2: $\xi = \eta$ is an $\mathcal{S}$-strategy.

If (2a) applies, define $\sigma(\eta) = 00\cdots01$ with $|\sigma(\eta)|$ big. *(Clearly,* $\{\sigma(\eta)\}\cup U_{\tau(\eta),s}$ *is prefix-free.)* Initialize all $\zeta > \eta^\frown\langle w\rangle$.

If (2b) applies (*This action is called* **step 1**), open an η-gap by defining $F(\eta) = g$, and $r(\eta) = 0$. Initialize all $\zeta \geq \eta^\frown\langle w\rangle$. Create a link (τ, η) with top $\tau = \tau(\eta)$, and bottom η.

If (2c) applies (*This action is called* **step 2**), Let $v + 1 < s + 1$ be the stage at which the current η-gap was opened, and let $\sigma = \sigma(\eta)[s]$. Close the η-gap as follows:

Step 2a (*Successful Close*) If $\alpha_\eta[s] - \alpha_\eta[v] \geq 2^{-|\sigma(\eta)|}$, then enumerate $\sigma(\eta)$ into $U_{\tau(\eta)}$, and define $F(\eta) = s$. Initialize all $\zeta \geq \eta^\frown\langle g\rangle$. Set $r(\eta) = 0$.

Step 2b (*Unsuccessful Close*) If $\alpha_\eta[s] - \alpha_\eta[v] < 2^{-|\sigma(\eta)|}$, then define $F(\eta) = w$ and $r(\eta) = s$ (*to preserve* $\Phi^A_\eta \upharpoonright |\sigma(\eta)|$), cancel $\sigma(\eta)$, and initialize all $\zeta \geq \eta^\frown\langle w\rangle$.

This completes the construction.

8.6 Verification

In this section, we verify that the construction above satisfies the requirements.

First, we have the following properties of the links:

Lemma 8.6.1 *(1) If (τ, η) is a link, then $\eta \supset \tau^\frown\langle 0\rangle$, $i(\eta) = i(\tau)$, $\tau = \tau(\eta)$, $|\eta| \equiv 2$ (mod 3) and $|\tau| \equiv 0$ (mod 3).*

(2) There is at most one link (τ, η) with bottom η or top τ existing at the end of any substage.

(3) Any link (τ, η) may be travelled at most once before it is canceled.

(4) Suppose that (τ_1, η_1) and (τ_2, η_2) are links both existing at the end of stage s. Suppose that (τ_2, η_2) is created at stage s, but (τ_1, η_1) already exists at stage s. Then s is not τ_1-expansionary, $\tau_1^\frown\langle 0\rangle \not\subset \tau_2$ but either $\tau_1^\frown 0 <_{\mathrm{L}} \tau_2$ or $\tau_2^\frown 0 \subset \tau_1$. In either case if (τ_1, η_1) is travelled at some stage $t > s$, then either (τ_2, η_2) is

canceled at stage t or (τ_2, η_2) has been removed at stage t. Finally, if τ_2 is not initialized at stage t, then $\tau_2^\frown 0 \subset \tau_1$.

(5) Suppose that (τ_1, η_1) and (τ_2, η_2) are links existing at the end of stage s with $\tau_1^\frown 0 \subset \tau_2$. Then if (τ_2, η_2) is created at stage s_2 and (τ_1, η_1) at stage s_1, we have $s_1 \geq s_2$. Furthermore (τ_1, η_1) must be removed before (τ_2, η_2).

Proof. We only give the proof of (2). For others, see Soare [71].

Since each η-node works for a unique $\mathcal{R}$-strategy, τ, we only need to prove that there are no two links (τ, η_1) and (τ, η_2) existing at the end of a substage, (s, t) say. Without loss of generality, assume that (τ, η_1) is created earlier than (τ, η_2). Let (s_1, t_1) and (s_2, t_2) be the substages at which (τ, η_1) and (τ, η_2) are created respectively. Then $s_1 < s_2 \leq s$ or $s_1 = s_2 \leq s$ and $t_1 < t_2$, and if $s_1 = s_2 = s$, then $t_1 < t_2 \leq t$. First assume that $s_1 < s_2$. Note that s_2 is a τ-expansionary stage, by our assumption, the link (τ, η_1) has not been traveled until stage s, and hence, at stage s_2, (τ, η_1) will be traveled, and hence, will be canceled. This contradicts the assumption that (τ, η_1) keeps active until the end of stage s. Now suppose that $s_1 = s_2$. Then $\delta_{t_1} = \eta_1$, and η_1 opens a gap at stage (s_1, t_1). By $t_1 > t_1$, we know $\eta_1 = \delta_{t_1} \subset \delta_{t_2} = \eta_2$, and hence, $\eta_1^\frown g \subseteq \eta_2$. By the construction of the priority tree, we know that η_2 does not work for τ. Thus, it is impossible to create a link between τ and η_2. A contradiction. $\square$

Let f be the leftmost path. That is, the leftmost $f \in [T]$ such that for all strings $\xi \in T$, if $\xi \leq_{\mathrm{L}} f$ and $\xi \not\subseteq f$, then there are only finitely many stages where $\xi \subseteq \delta[s]$. The following lemma establishes that f is 'genuine'.

Lemma 8.6.2 *Fix n and $\xi \subset f$ with $|\xi| = n$. Then,*

(1) $(\exists^\infty s)(\xi \subsetneq \delta_s)$,

(2) $|G^\xi| = \infty$.

(3) If $|\xi| \equiv 1$ (mod 3), then ξ receives attention only finitely often.

(4) For any $\tau \subset \xi$, *if* $|\tau| \equiv 0$ *(mod 3), and* $\tau^\frown\langle 0\rangle \subset \xi$, *then every link with top* τ *is removed and there are infinitely many genuine* ξ*-stages at which there are no links with top* τ. □

Lemma 8.6.3 *Let* $\rho \subset \xi$ *with* $|\rho| \equiv 1$ *(mod 3), and let* x *be the follower of* ρ *with* $x = \lim_s x(\rho)[s]$, *the last follower of* ρ *appointed at stage* s_1 *say, where* $W_{e,s_1} \cap A_{s_1} = \emptyset$. *Then*

(1) $(\forall s \in G^\rho)[s > s_1 \to x > R(\rho, s)]$, *where* $R(\rho, s) = \max\{r(\zeta, s) : \zeta^\frown 0 <_{\mathrm{L}} \rho\}$.

(2) $\mathcal{P}_{i(\rho)}$ *is satisfied.*

Proof. At stage s_1, x is chosen as a big bumber, namely, x exceeds all $R(\rho, s_1)$ by convention. It suffices to prove that $R(\rho, s) = R(\rho, s_1)$ for all $s > s_1$ with $s \in G^\rho$. Indeed we claim that for all $\zeta^\frown 0 <_{\mathrm{L}} \rho$, $r(\zeta, s_1) = r(\zeta, s)$ for all such s. Suppose that $r(\zeta, s_1) \neq r(\zeta, s)$, then it can only be that $\zeta^\frown\langle w\rangle$ or $\zeta^\frown\langle s\rangle \subseteq \rho$. In the first case, $r(\zeta, s)$ would only be reset after a $\zeta^\frown\langle g\rangle$-stage s' and this would cancel x since $s' > s_1$, and $\zeta^\frown\langle g\rangle <_{\mathrm{L}} \rho$ with $\zeta^\frown\langle g\rangle \not\subseteq \rho$. In the latter case, if $r(\zeta, s_1) \neq r(\zeta, s)$, it must be that ζ receives attention at stage s. Thus in this case also we cancel x. Thus these cases cannot occur and so $r(\zeta, s_1) = r(\zeta, s)$. This gives (1) and (2) by the standard argument. □

Lemma 8.6.4 *Suppose that* $\tau \subset f$ *with* $|\tau| \equiv 0$ *(mod 3),* $\tau^\frown\langle 1\rangle \subset f$, *and that after stage* s_τ, τ *cannot be injured along the true path anymore. Then* $\alpha_{i(\tau)} \neq \Phi^A_{i(\tau)}$. □

Lemma 8.6.5 *Suppose that* $\eta \subset f$ *with* $|\eta| \equiv 2$ *(mod 3).*

(1) If $\eta^\frown\langle s\rangle \subset f$, *then* $U_{\tau(\eta)} \cap V_{j(\eta)} \neq \emptyset$. *Indeed,* $\sigma(\eta) \in U_{\tau(\eta)} \cap V_{j(\eta)}$, *where* $\sigma(\eta)$ *is the (unique) string enumerated into* $U_{\tau(\eta)}$ *when the (last)* η*-gap is closed (successfully).*

(2) If $\eta^\frown\langle g\rangle \subset f$, *then* α_η *is computable.*

(3) If $\eta^\frown\langle w\rangle \subset f$, *then* $U_{\tau(\eta)} \cup V_{j(\eta)} \neq \Sigma^*$.

Proof. (1) If $\eta^\frown\langle s\rangle \subset f$, then at some stage s, η closed the η-gap successfully, and enumerated $\sigma = \sigma(\eta)[s]$ into $U_{\tau(\eta)}$. But necessarily, $\sigma \in V_{j(\eta),s}$ since at the stage s' say, at which η opens this gap, σ has already been enumerated into $V_{j(\eta),s'}$.

(2) Suppose that $\eta^\frown\langle g\rangle \subset f$. Let $\eta^+ = \eta^\frown\langle g\rangle$. Choose s_0 such that $\eta^+ \leq \delta[s]$ for all $s \geq s_0$, and that η is never initialized at any stage $s \geq s_0$. Now as in the basic module, we have that α_η is computable.

(3) If $\eta^\frown\langle w\rangle \subset f$, then $\sigma = \lim_s \sigma(\eta)[s]$ exists. Choose s_0 as the stage at which the last candidate of σ was chosen. By the construction, σ can never be enumerated into $U_{\tau(\eta)}$, and hence, $U_{\tau(\eta)}(\sigma) = 0$.

Now suppose that $\sigma \in V_{j(\eta)}$. Then for almost every stage, s, η is ready to open a gap. By Lemma 8.6.3, there are infinitely many genuine η-stages, and so η opens a gap at some stage $s > s_0$, contrary to the hypothesis. $\square$

Lemma 8.6.6 *Suppose that $\tau \subset f$ with $|\tau| \equiv 0$ (mod 3) and $\tau^\frown\langle 0\rangle \subset f$, and that after stage s_τ, τ cannot be injured along the true path anymore. Then U_τ is a presentation of α_τ. Namely, U_τ is prefix-free, and $\sum_{\sigma\in U_\tau} 2^{-|\sigma|} = \alpha_\tau$. Moreover, if α_τ is not computable, then U_τ is also not computable.*

Proof. By the hypothesis and the definition of f, we have that there are infinitely many $\tau^\frown\langle 0\rangle$ genuine stages, and that there is no $\xi \supset \tau^\frown\langle 0\rangle$ such that $i(\xi) \leq i(\tau), \xi \subset f$, and either ξ is an $\mathcal{R}$-strategy with outcome 1, or ξ is an $\mathcal{S}$-strategy, with outcome g.

U_τ is the collection of $\sigma(\eta)[s] = \sigma(\eta)$ enumerated at the success of η-gaps where $\tau = \tau(\eta)$. Clearly, U_τ is c.e..

Since $\tau^\frown\langle 0\rangle \subset f$, by Lemma 8.6.1, there are infinitely many τ-expansionary stages at which no links with top τ are created, and so we can update U_τ infinitely often. Thus

$$\sum_{\sigma\in U_\tau} 2^{-|\sigma|} = \alpha_\tau.$$

By the fact that each updating process ensures that each resulting set, $U_{\tau,s}$, is prefix-free, we have that U_τ is prefix-free. Therefore, U_τ is a presentation of α_τ.

Suppose that α_τ is noncomputable. Assume that U_τ is computable. Find some η, $\tau \subset \eta \subset f$ such that $i(\eta) = i$ and $V_{j(\eta)} = \overline{U}_\tau$. This is possible since by Lemma 8.6.4, there is no $\xi \subset f$ with $i(\xi) < i$ and $f(|\xi|) \in \{g, 1\}$. Hence for almost every j there exists an $\mathcal{S}$-strategy η, $\tau \subset \eta \subset f$ with $i(\eta) = i$ and $j(\eta) = j$. Let $\eta = f \upharpoonright j$. By the fact that α_τ is noncomputable, we have that $f(n) \in \{s, w\}$. By Lemma 8.6.4, if $f(n) = s$, then $\sigma(\eta) \in U_\tau \cap V_{j(\eta)}$ and if $f(n) = w$, then $\sigma(\eta) \notin U_\tau \cup V_{j(\eta)}$. Hence $\overline{U}_\tau \neq V_{j(\eta)}$. A contradiction. $\square$

This completes the proof of Theorem 8.5. $\square$

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