The Life of π

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My plan

- This talk grew from a long series of talks I have given in Singapore to mainly arts students about the development of modern mathematical ideas and how they resonate today.
- These included development of the concept of computation, graph colouring,Alan Turing, and this one.
- Mathematics is this beautiful study in the origins of ideas and is more relevant today than ever, as we live in the most mathematical age ever.

Origins of π

- Humans became aware of the number we call π in ancient time.
- A natural thing is to put a peg in the ground, stretch some rope and draw a circle. How big is the circle? i.e. circumference, area etc.
- I wish to give you some idea as to how we clarified the calculation of π, and how this reflects parts of the development of modern mathematics, even to this day.
- I will in no way try to explain why some people are fascinated by π or memorize it to a gazillion places. That needs a psychology lecture.

π

- 3.14159265358979323846264338327950288419716939937510 . . .
- Not $\frac{22}{7}$, alas not.....
- We can define π as the area of a circle with radius of length 1.
- This turns out to be the circumference of a circle with diameter of length 1. (This needs proof)
- Indeed, π is the "ratio" of the circumference of a circle to its diameter, though this needs some proof.
- What does this mean? Does this number "exist" in the same way that we think "3" exists?

What is a Number?

- The standard philisophical view is that we are born with 1,2,3,....
- We want 4 things and mum gives us 2, so we "know" $\frac{1}{2}$.
- We have two things and sister takes them away, so we "know" 0.
- We buy a house and have a mortgage, so we "know" $-1,000,000$.
- We understand "fractions" $\frac{1}{7}$, $\frac{4}{1}\dots$, and we call these rational numbers.
- Surely we can put a stake in the ground and make circle so (whatever it is) we understand that a circle has a circumference.
- Is this length a number?

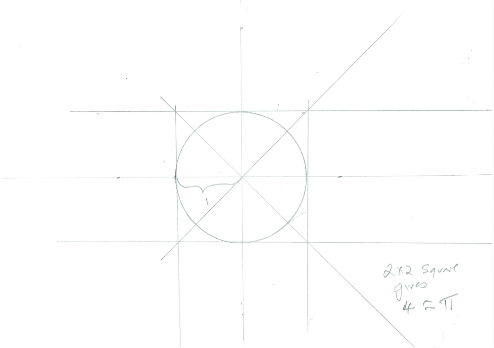
Really ancient π history

- Certainly, there were approximations to π known as early as 1900 BC (Babylon) as $\frac{25}{8}$ and $(\frac{16}{9})^2 = 3.1605$ (Egypt).
- In India, the text Shatapatha Brahmana used an approximation of $\frac{339}{108}$ correct to 4 decimal places. This work is dated (if you can believe it) *either* 600, 700 or 800 BC.
- That is, these geniuses from the past tried to calculate π as a rational number, and somehow got the above results.

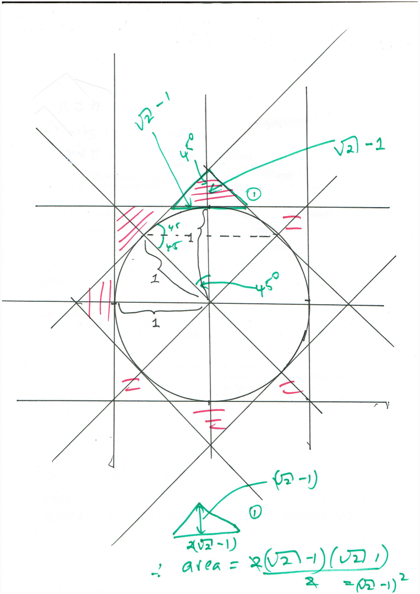
Archimedes

- The first iterative, and rigorous method of calculating π known to us is from Archimedes from around 250 BC.
- Uses approximations by polygons
- I'll illustrate using an octagon.
- This is using the area.

A square



An octagon



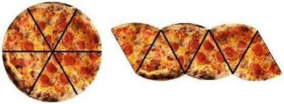
$\sqrt{2}-1$
45°
$\sqrt{2}-1$
①
1
1
1
45°
$(\sqrt{2}-1)$
①
$2(\sqrt{2}-1)$
∴ area = $\frac{2(\sqrt{2}-1)(\sqrt{2}-1)}{2}$
$=(\sqrt{2}-1)^2$

- The intersection of the two squares is on the diagonal which has length $\sqrt{2}$ what's that?.
- This is around 1.4142. The half way point generates a triangle with base length $2(\sqrt{2}-1)$ and height $\sqrt{2}-1$ and hence the area is $(\sqrt{2}-1)^2 = .1716$ approximately.
- But you've cheated there!
- There are 4 such sections to take away, so the total to be subtracted is around $4 \times .1716 = .6862$ giving an approximation of $4 - .6862 \approx 3.3137$ which is above the "real" area.

Other methods

- Before I return to $\sqrt{2}$ I mention one other method, known to be used by Satō Moshun and Leonardo da Vinci (both $\sim$ 15th Century)
- It uses subdividing the circle into approximate triangles.

A rectangle

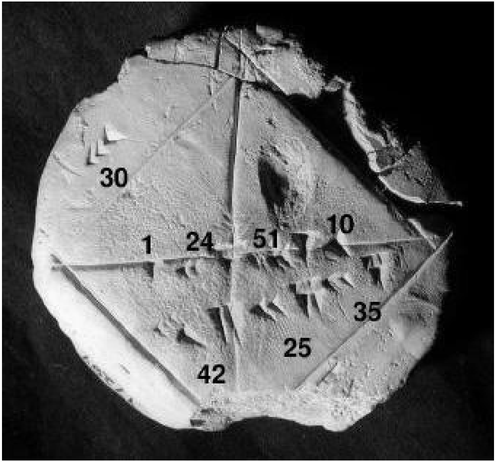


You cheated: $\sqrt{2}$

- Well this is just as old!
- Babylonian tablet of around 1700BC gives an approximation of $1.4142296296266\ldots$ (interpreting the base 60 approximation)
- What is a base?
- We work base 10, but if we only had two fingers we might work base 2, using only $0, 1$.
- To illustrate 634 in base 10 means 6×10^2 plus 3×10^1 plus $4 \times 10^0 (= 1)$. That is, $600 + 30 + 4$.
- 101 in base 2 would mean 1×2^2 plus 0×2^1 plus 1×2^0, that is $4 + 0 + 1 = 5$.
- Ancients (e.g. Sumerians in 3000 BC!) used all kinds of bases. (So is base 10 "natural"?) Also the Chinese calender.
- Other civilizations used e.g. Base 24, 32, 12, 4, 5, etc!
- Computers are based around base 2.

Cuniform tablet

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \approx 1.414213.$$



$\sqrt{2}$, again.

- The Babylonians used the following method: Called Hero's Method (Sorry this is a bit technical.)
- Guess some $x \approx \sqrt{2}$ (with $x < \sqrt{2}$) so that the error E is small $\sqrt{2} = x + E$.
- $2 = (x+E)^2 = x^2 + 2xE + E^2 = x^2 + E(2x+E)$.
- Therefore $E = \frac{2-x^2}{2x+E} \approx \frac{2-x^2}{2x}$.
- Therefore $\sqrt{2} = x + E \approx x + \frac{2-x^2}{2x} = \frac{x+\frac{2}{x}}{2}$.
- Guess x and use $\frac{x+\frac{2}{x}}{2}$ as the next guess.
- Start at 1 certainly $1 < \sqrt{2}$. Call this a_0.
- Let $a_1 = \frac{a_0 + \frac{a_0}{2}}{2} = \frac{3}{2}$.
- Let $a_{n+1} = \frac{a_n + \frac{a_n}{2}}{2}$...
- $1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \ldots$

Archimedes

- Archimedes used a 96-gon. That is, regular, and has 96 sides.
- He proves $\frac{223}{71} < \pi < \frac{22}{7}$ by approximating from the inside and outside.
- Archimedes understood that you could calculate areas by successive approximations, and invented the "method of exhaustion" using an analog of infinitesimals which resembles the early forms of integration, developed in the 1700's.
- Invented the famous "Archimedes screw", a ray gun!, and "Archimedes principle" ("eureka"-likely apocryphal).
- Famously supposedly killed by a Roman whilst working on a problem. Again likely apocryphal.

The polygonal people

- People continued to try for better approximations using more sides.
- 265 AD Liu Hui, used a 3072 sided polygon to get 3.1416, and then in 480 AD Zu Chongzhi, used this algorithm applied to a 12,288 sided (!) polygon to get $\pi \approx 3.141592920 \approx \frac{355}{113}$. This is the best approximation for 800 years thereafter!
- Then others with the zenith being Christoph Grienberger (Austria) in 1630 using a polygon of 10^{40} (!!) many sides, giving a value known to be correct to 39 decimal places.

A question

- What is π? Is it a fraction?
- A number is called rational if it can be expressed as $\frac{p}{q}$ with p, q integers ($q \neq 0$) like $0, 1, 2, 3, \ldots$ or their negatives.
- Ancient Greek question: Is π rational.
- Historical context: Up to the Greek age the only numbers thought to exist were the rationals.

A Bombshell

Theorem (in Euclid-300BC, but likely Hippasus of Metapontum or Theodorus-500BC)

$\sqrt{2}$ *is irrational.*

[Courant and Robbins] " This revelation was a scientific event of the highest importance. Quite possibly it marked the origin of what we consider the specifically Greek contribution to rigorous procedure in mathematics. Certainly it has profoundly affected mathematics and philosophy from the time of the Greeks to the present day. "

Pythagoreans

- This was actually a quasi-religious cult.
- It is certainly thought that Hippasus's proof was suppressed.
- There are stories about him being executed for the heresy. (Tossed out of a boat.)
- The cat came out of the bag with the proof in Euclid's Elements, one of the greatest achievements of intellectual thinking in history. (This took at least 100 years.)

The Elements

“The findings of over two hundred years of Greek geometry and number theory, and fifteen hundred years of Babylonian mathematics, were rigorously established from first principles. Starting from a handful of simple statements, and proceeding exorably one small step at a time, Euclid obtains result after result of genuine depth. Achievements wrung over the ages with great difficults were made to seem inevitable.”
....Two thousand years of geometry in a few short compelling pages”‘
Donald O’Shea “The Poincaré Conjecture”

Notably the third most read book in history after the Bible and the Q’ran. The most editions of any book except the Bible.

- Einstein "The holy little geometry book."
- Lincoln kept a copy in his saddle bag.

  "You never can make a lawyer if you do not understand what demonstrate means; and I left my situation in Springfield, went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight"

- Ian Wright:

  " For over 2,000 years, his work was considered the definitive textbook not only for geometry, but also for the entirety of mathematics."

A proof from Euclid

- Proof by contradiction. Suppose that $\sqrt{2} = \frac{a}{b}$ such that a, b have no common factors. ($\frac{3}{4}$ not e.g. $\frac{6}{8}$. as 2 is a factor of 6 and also of 8.)
- Then $2 = (\frac{a}{b})^2 = \frac{a^2}{b^2}$, so $a^2 = 2b^2$.
- Thus a is even.
- So $a = 2c$. Thus $2b^2 = (2c)^2 = 4c^2$.
- So $b^2 = 2c^2$. Thus b is also even.
- This is a contradiction.

My wife's question

- What do you mean by these numbers “exist?”
- Do such numbers “exist” in “reality?”, rather than as “mathematical constructs”.
- Well is the universe “smooth”? Can you keep halving lengths and having something of that size “existing” in the universe, or is it “granular”?
- Zeno's paradox....
- We have no idea about the physical universe in lengths below 10^{-20} mm.
- Would you fly in a plane whose flying depended on the “true value of $\sqrt{2}$”?

Well, what about π?

Theorem (Lambert, 1761)

π is irrational.

This result was much harder to prove that $\sqrt{2}$.

Greek thinking, continued

- What is a "number?"
- The Greeks recognized that π existed in the sense that we could give an approximation to arbitrary accuracy.
- We now define the real number system as the collection of numbers with this property. (Cauchy, 18th C)
- The real numbers are the "completion" of the rational numbers.
- The "same" as as number with a decimal expansion. (or any base for that matter)

Analysis

- We now think about procedures only defined by infinite processes.
- Think about Archimedes polygons but continuing 96-gon, 97-gon, 98-gon,. . . , n-gon, . . .
- This leads to calculus, then analysis, differential equations, a whole new world of mathematics, Newton, Cauchy, Leibnitz, etc.
- This models almost any process we regard as smooth, or continuous.
- A circle is “smooth” but is a limit of Archimedes polygons, none of which are smooth.

What about π vs $\sqrt{2}$

- We somehow "like" $\sqrt{2}$ more as it obeys a simple equation $x^2 - 2 = 0$. Although it is irrational it seems closer to being rational.
- Numbers that can be obtained using a finite number rationals and $+$, -, rationals n-th roots we call algebraic.
- For example $\frac{\frac{\sqrt{\sqrt[5]{36}-\sqrt[8]{\frac{77}{9009}}}^{1200}}{324}}{\sqrt[11]{\frac{33}{6}}}$.
- If a number is not algebraic it is called transcendental.

Theorem (Lindemann, 1882)

π is transcendental. So there is no finite process which gives π using the symbols above and rationals.

- Cantor showed that if you throw a dart at the real line it will land on a transcendental with probability 1. "Most numbers are transcendental".

Solutions to equations

- You are taught at school that your can solve the quadratic $ax^2 + bx + c = 0$.
- Ingredients: numbers, $+, - \times$, division, $\sqrt{\ }$, maybe cube roots, powers etc.
- Operations : Combine in sensible ways.
- Can we do the same for degree 3, the "cubic"

$$ax^3 + bx^2 + cx + d = 0?,$$

  what about degree 4, etc.
- This was one of the many questions handed to us by the Greeks.
- The answer is yes for degree 3 and degree 4; there are such formuale, such "algorithms" using these ingredients for solving them.

Omar Khayyam's Solution

- Islam saved vast amounts of knowledge through the middle ages, from the Greeks and others.
- This is where algebraic notation comes from.
- Omar Khayyam (1044-1123) gave a *geometric* solution to the cubic question, which can be interpreted as a solution with these ingredients; though lacked all appropiate notation. See `https://www.youtube.com/watch?v=2t-Gv8RoXSM`
- His methods are extendable to degree 4, as shown by M. Hachtroudi as reported in Amir-Moéz, in 1962. (Mathematics Magazine Vol 35).
- It was 400 hundred years till the next solution.

The Sorry Tale

- For degree 3 this was first proven by Ferro (1500), modulo the work of Khayyam.
- Ferro left it to his son-in-law Nave and pupil Fiore.
- (In those days, maths was like a trade, and information was secret; a bit like much modern research supported by drug companies.)
- Fiore challenged Tartaglia (in 1535) who then re-discovered the solution with a few days to spare, leaving Foire in ignomy.
- Tartaglia also kept it secret, but told Cardano, who promised by his Christian faith to keep it secret, but....
- in 1545 Cardano published it in his great text Ars Magna
- Additionally Cardano published how to extend to degree 4, being discovered by a student Ferrari.

Abel

- Finally, in 1823, a young Norwegian mathematician, Abel proved that there is no recipe using the given ingredients for the degree 5 case, the quintic.
- (The paper was called "Memoir on algebraic purifications..." rather than "Memoir on algebraic equations..." due to a typsetting error.)
- (My favourite error in one of my own papers referred to a journal "Annals of Mathematical Logic" as "Animals of Mathematical Logic."It made me think of some of my colleagues!)
- Nobody believed him, for a long time. (There had been an earlier announcement by Ruffini, which contained "gaps".)

Galois

- Evariste Galois (1811-32) eventually gave a general methodology for deciding if a given degree *n* equation admits a solution with the ingredients described.
- This work laid the basis for group theory. which has seen applications ranging from electronics, to cosmic physics and to scheduling teams in a tournament. Group theory is the mathematics of symmetries.
- Galois method is to associate a "group", a mathematical structure, with each equation, so that the equation is solvable in terms of the given ingredients (arithmetic operations and radicals) iff the group has a certain structure on its subgroups. This is one of the gems of mathematics.

Calculating π

- Many many series have been found that compute π.
- For example, the triangle approximation methods can be used.
- (Gregory-Leibnitz, 17th C) $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots)$.
- (Ramanujan, 1916) $\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}(\sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{k!^4(396^{4k})})$.
- Most recent new algorithm due to Borwein and others circa 2005, of great interest because of the feature that it allows the computation of the the n-th bit without the others in a certain sense.
- That is, we have developed new methods of constructing algorithms for problems having nothing to do with π, quite as a by-product.

Is that it?

- Of course not!
- For example, what can be said about the "distribution" of numbers in π.
- Suppose that you tossed a 10-sided coin, and recorded the (infinite number of) outcomes. If you assumed the coin not biased, you would expect the same proportion of 0's, 1's etc.
- (Borel) An expansion is called normal to base b if we have the expected distribution of bits base b.
- Thus $.012345678901234567890123456789\ldots$ is normal base 10.
- This is a kind of randomness. Borel showed that almost every number is normal to every base, called absolutely normal.
- We think π, e, $\sqrt{2}$, are all absolutely normal but have no proof that they are normal to any base, in spite of the work of number theorists for a century.

- Turing was the first person to give an “explicit” absolutely normal number.
- Recent work has constructed absolutely normal numbers of “low complexity”

Explicit numbers

- Also we can "construct" π. We know an algorithm which calculates π to any desired level of accuracy.
- If you throw a dart at the real line the number you hit will almost certainly not have this property.
- What can be said about such numbers. Quite intense development since the initial analysis by Turing.
- Are all numbers in "reality" like this? Or are there really noncomputable numbers in the universe?
- Church-Turing-Kleene-Post showed that there are undecidable statements in formal systems of mathematics: there is no algorithm to decide if they are true.
- T. Cubitt, D. Perez-Garcia and M. M. Wolf proved recently that, for a physically reasonable class of systems, no algorithm can decide whether a given system has a "spectral gap."

Coda: Why should we care?

- I don't subscribe to the view that the only reason we learn maths is to do your tax form; is the only reason we do english so that we can read road signs?
- What is maths anyway? Mathematics builds models of the the universe.
- Models of motion: physics, scanning, etc; models of thought: logic computers, etc, models of counting: combinatorics, DNA analysis, algorithms, models of randomness: probability theory, statistics,
- Modern society would be impossible without the mathematics underpinning it.
- Today's story here is that ideas from the Greeks and earlier led to the development of modern analysis, and these ideas underpin almost all of modern matematics and hence modern society; because they are fundamental.

Thank You