Notes on Parametric Complexity

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When and Where did I meet Mike?

- When : almost certainly December 4th, 1990.
- Where : Palmerston North.
- At What : ACCMCC (a combinatorics conference mainly filled with block designers.)
- Palmerston North: John Cleese (of Monty Python fame) “If you ever do want to kill yourself, but lack the courage, I think a visit to Palmerston North will do the trick.”


DESTINATION MANAWATU

Why was I there?

- Well, there was essentially no research money in NZ for mathematicians/computer scientists.
- You needed to work on sheep, dairy or kiwifruit.
- So if there was any conference anywhere near we went. Who knows, you might pick up an unexpected idea. (I still believe this)
- Here is an ideal scientist of 1990 in New Zealand.

Why was Mike there?

- The Endless Summer, a famous surf movie had featured the exotic location of New Zealand in it.
- Mike wanted to surf New Zealand.
- Notably no surf in Palmerston North. But he did go to Castle Point.

Mike's Talk

- Mike gave a contributed talk probably about "Nonconstructive Advances in Polynomial Time", and I met him in the foyer after the talk.
- I said I have read something like this recently, and it turned out it was his paper I had read!
- We talked and talked over dinner, and he handed me the Abrahamson, Ellis, Fellows and Mata paper and asked me to try to prove a Ladner type density theorem for it, and maybe we would find interesting things to do with the material.
- Ladner's Density: $A \leq_T^p B$ implies there is a C with $A <_T^p C <_T^p B$.
- Often stated if NP$\neq$P then intermediate degrees. For us now $W[1] \neq$FPT then...

Background

- Mike Langston has a lot of material on the ideas he and Mike had up to AEFM.
- Funded by the Office of Naval Research to use Robertson-Seymour Theorems to design VLSI!
- In retrospect, the Database community, Vardi and others were looking at complexities, but the AEFM was the first to suggest asymptotic behaviour of the slices.
- The AEFM paper is a difficult read, and is concerned with more or less $W[P]$ completeness under logspace by the slice reductions, again more or less.

I WAS EXCITED!

- At a certain point I recall simplifying the notion of reduction as the AEFM one was hopeless.
- Why not study $L \subseteq \Sigma^* \times \Sigma^*$ or $\Sigma^* \times \mathbb{N}$, and have reductions as what we now see as parametric connections $(x, k) \mapsto (x', k')$.
- I was excitedly calling Mike and posting him letters.
- He invited me over to Victoria to work with him.
- It became clear that there are three definitions of being FPT, uniform, strongly uniform and nonuniform.
- Open Does Ladner's Theorem hold for uniform and nonuniform?

18th December 1990

- I have an e-mail where Mike summarizes the definitions so far.
- He sets out his ideas about using logical depth as a basis for hardness classification, via weft.
- That is $W[t]$ is the collection of problems fp reducible to the weighted sat problem for cicuits of fixed depth d, large gate depth t, the small gates allowing for easier inclusion.
- He had been thinking about (and now I was) Independent Set, Vertex Cover, and Dominating Set.
- When I visited Mike, maybe January 1991, and we spent maybe two weeks working out the details of Mike's weft vision.

- For that visit, Mike remembered to meet me off the plane, something he did not always do....
- We had the familiar flip chart, pots of coffee, in his house of teetering piles of books, and somewhat dubious home handymanship.
- There are still a number of very interesting open questions from that e-mail.

More open questions

- If $W[t] = W[t+1]$ does this cause collapse?
- If $W[2]$ =FPT collapse?
- Oracle separations.
- The notion of an oracle is interesting here: want FPT^A =FPT if $A \in$ FPT. We chose parametrized A and allow parametric queries so the access mechanism is essentially $\leq_T^{FPT}$.
- Oracle separation of the hierarchy.
- It might be that the hierarchy collapses to at least $W[2]$ under randomized reductions using a variant of Hastad's switching lemma

The modern view

- Whilst these are interesting, you can easily ask Do we care?
- First the XP optimality program suggests that the W-hierarchy (much as I love it) could be viewed as an artifact and $M[1]$ takes a central role.
- Second, we might ask why the practical FPT algorithms work so well anyway.
- And even things W-hard work effectively, like SAT-solvers. Gaspers and Szeider have a nice article looking at recent progress on parameterized analysis of SAT-solvers.
- I wonder if there is a coherent amalgam of PC with smoothed analysis or generic case complexity.

Generic Case

- Look at algorithms which don't always halt but if they do they most be correct. (+coarse variations)
- They are correct a lot:

$$\lim_{n\to\infty} \frac{|\{\sigma \mid |\sigma| \leq n \wedge \Phi(\sigma)\downarrow\}|}{|\Sigma^n|} \to 1.$$

- This should happen exponentially fast and and the running time of Φ should be fast. (eg group theory)
- It is more easily applied than things using distributions and the like. Borel density is not a measure.

Back to the early years I

- Mike took me to a place called *Sombrio Beach*.
- In those days, only ex-hippies in shacks lived in sombrio, now they have been “moved on” and it is a crazy busy part of the west coast trail.
- We always ate at Shakies; Oyster Burgers.
- In a later trip, I remember Niel Koblitz all in black looking very out of place (and that’s where FPT=kernelizable came from).
- Shakies is where Mike tried to kill one of his visitors with an oyster.


wannasurf.com

Back to the early years II

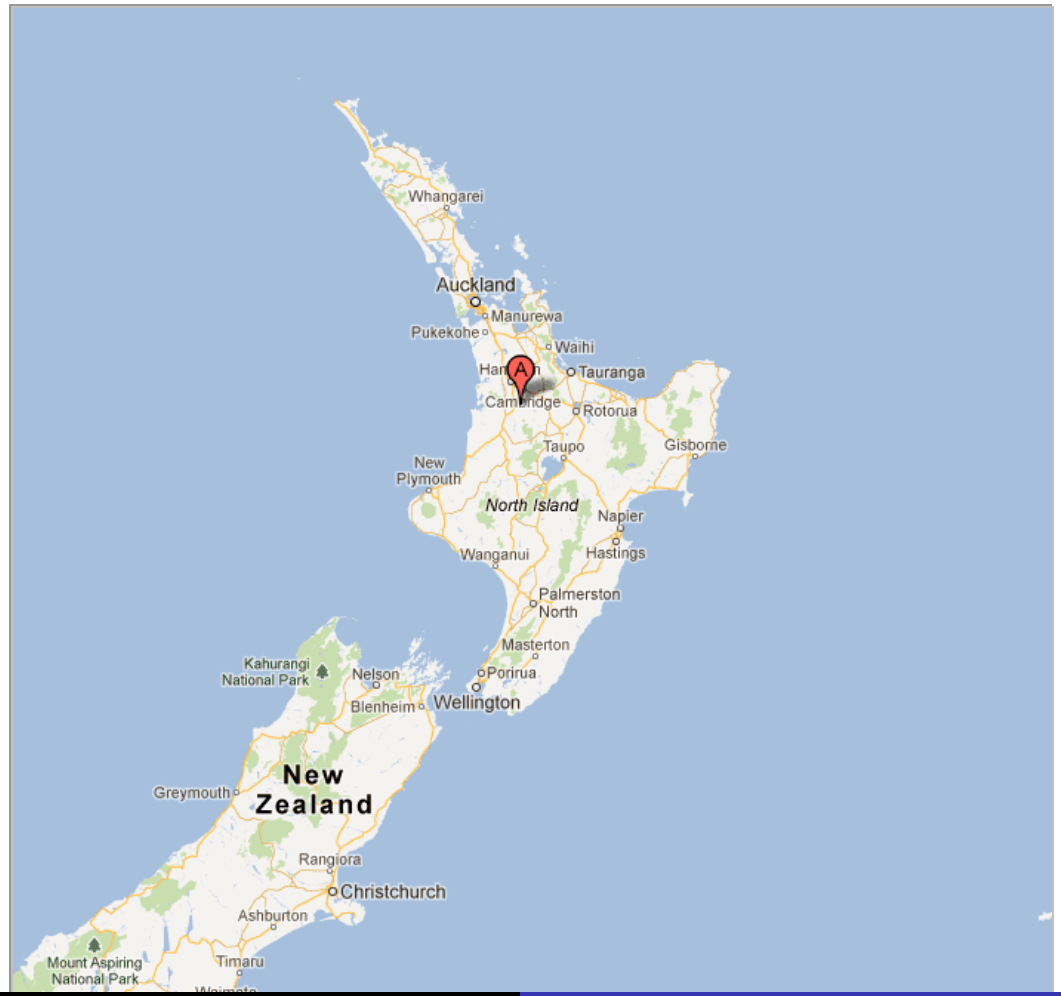
- We had a first draft "A Completeness Theory for Fixed Parameter Problems"
- New definitions, weft ideas, some hardness proofs,
- k-Perfect Code, k-Not All Equal Sat, k-CNF Sat, k-Dominating Set, k-Independent Dominating Set, mostly correct.
- FTP examples like Feedback Vertex Set, Planar Face Cover Number, Min Cut Linear Arrangement, Graph Genus
- Mike gave the first talk on this around this time in Manitoba. Plus we submitted probably the best paper ever in Congressus Numerantium.
- Submitted to FOCS.

The great Kiwi roadtrip

map new zealand north island - Google Maps https://maps.google.co.uk/maps?hl=en&client=firefox-a&q=m...

Google

Address **Waikato**
New Zealand




WANNASURF.COM

- After Blacks Reef, I drove the tricky bit of road from Mahia to Napier.
- Of course Mike slept, having run out of eggs to eat, only waking to complain about my choice of music (Lou Reed).
- We arrived late in Napier and stopped for a drink at the first pub on the road, not noticing that there were only trucks outside.
- The characters in the bar:

Maths on the trip

- We figured out the planar gadget.
- We thought that the $W[1,t]$ classes would stratify.
- Recall $W[1,t]$ is weighted t-CNF
- Got the reduction for Red/Blue Nonblocker and hence the completeness for Clique and Independent Set.
- Recently recycled in the Turing lower bound completeness paper of Hermelin, Kratsch, Soltys, Wahlström, and Wu, which is a great program of miniature miniatures.
- Natural basic hardness class: $W[1]$.

- Notice that circuits were the original basis.
- Only later with Liming and Jianer did we get the completeness of SHORT NTM ACCEPTANCE.
- Open What about $k \log n$-NTM satisfiability and $M[1]$?
- Then lots of concrete reductions, rejection from FOCS and (later) acceptance by CCC (then called Structures), 1992.
- The basic papers I and II.
- I spoke on this at Schloß Dagstuhl 9.00 am on Monday the 3rd of February 1992 having come in on a plane from NZ the night before.

What does the material from then show?

- I think we wrongly focused too much on Robertson-Seymour and hardness. The big selling point is tractability, and especially industrial strength.
- There were some amazing successes: notably k-Processor Scheduling is a prominent problem in the back of Garey and Johnson. Hans and Mike showed it is $W[2]$-hard. This means that, assuming $FPT \neq W[2]$, there should be no feasible algorithm for large k. Later Alenknovich and Razborov.
- 25th February, 1991, Mike said "As for practical, I don't know. It's a bad news theory. Apart from completeness there are some fun positive results..."
- The big change was Mr Feasible, *Parameterized computational feasibility*, and then its heirs particularly those with Ulrike.
- "the extent to which FPT is really useful us unclear."

Maybe on Ulrike's visit

The modern incarnation

- This is the exciting thing of the last decade.
- The development of tools to match (up to O) upper and lower bounds, beginning with Cai and Juedes.
- $M[1] \neq$FPT or ETH implying tight membership of XP.
- Perhaps not known by everyone:
  $s_d = \inf\{\epsilon \mid \exists O^*(2^{\epsilon n})$ algorithm for n variable $d-$CNFSAT$\}$.
- Clearly $s_d \leq s_{d+1}$. We can define $s_\infty = \lim_{d\to\infty} s_d$.
- Impagliazzo and Paturi noted that ETH means that infinitely many $s_d < s_{d+1}$.
- SETH is that $s_\infty = 1$.
- Can be used for strong lower bounds, see Cygan, Dell, Lokshtanov, Marx, Nederlof, Okamoto, Paturi, Saurabh, Wahlström applied to Set Splitting and Hitting Set.

For example

Theorem (Lokshtanov, Marx and Saurabh)

If Independent Set *can be solved in time* $O^*((2-\epsilon)^{tw(G)})$ *for some* $\epsilon > 0$*, then for some* $\delta > 0$ *we can solve* Sat *in time* $O^*((2-\delta)^n)$.

AW AND PSPACE

- Mike in an e-mail of February 27, 1991. Mike says he noticed this "weird thing"; which was that a certain problem whose unparameterized version was in Σ_2^P did not seem to fit the model we had.
- "Maybe the whole hierarchy is some kind of analog of the polynomial time hierarchy..." "Or maybe there is some kind of weird combinatorial reduction placing this above the current hierarchy."
- Developed into the AW-hierarchy.
- In case you forgot... $\exists^{\text{weight}_{k_1}} x_1 \forall^{\text{weight}_{k_2}} x_2 \ldots$.
- Home of k move games.

- COMPACT NTM (COMPUTATION)
  *Instance:*A nondeterministic Turing machine M and a word x.
  Parameter: A positive integer k.
  *Question:*Is there an accepting computation of M on input x that visits at most k work tape squares?
- Cai, Chen Downey and Fellows COMPACT NTM (COMPUTATION) is $AW[SAT]$ -hard.
- Open; Is it $AW[P]$ complete/hard? Falsely claimed in the DF book.
- Open What is the correct treatment of parameterized space?
- Open Is there any analog of QBFSAT aligning to space?
- Open What is the parameterized version of interaction?

Other 90's things

- Downey-Fellows-Regan development of $\oplus P$, parameterized BW[1] etc.
- Proof that $W[t]$ reduces to unique $W[t]$ under randomized parameterized reductions.
- Later (not 90's) Müller proved the same for e.g. unique independent set etc.
- Open Parameterized Toda's Theorem?
- Possibility $AW[P]_k$ reduces to $\#W[P]$ under randomized FPT reductions. Possibly the A-hierarchy of Flum and Grohe.
- Open paramerized PCP. Perhaps this can be used for parameterized approximation.

- Also Cesati easier membership, his Turing way.
- The W^*-hierarchy. In case you forgot: the depth of $W^*[t]$ is a function of the parameter k instead of a constant as per $W[t]$.
- Mike proved with Taylor that $W^*[1] = W[1]$ and $W^*[2] = W[2]$, to solve a question of Yannakakis and Papadimitriou.
- Open Is $W^*[t] = W[t]$ for $t \geq 3$?

Complexity post-DF

- Entry of Martin Grohe, Jorg Flum, Venkatesh Raman, Rolf Nidermeier and others.
- Now the next generation.
- Flum-Grohe approach basing the whole thing on model checking. Make the logical depth more apparent. The A-hierarchy and E-hierarchies (with Weyer)
- Parameterized approximation Three groups of authors, but an old question of Mike: is there an FPT algo for $(k, 2k)$-Dominating Set (Open).
- Complete inapproximability.
- Flum-Grohe-Grüber reductions. Marx the best results.
- Open no general theory.
- Complexity and completeness for kernels.
- Open what about iterative compression? What about incremental computation? Open What about bounded search trees, though Daniel has a completeness progarm here

Reflections

- Those early years were incredibly fun and productive. Mike and I have 1 book, 2 edited volumes and 34 papers together now, kind of like a marriage.
- The decade after the 90's has seen the vision of the extended discourse with a problem being realized with some amazing positive techniques and negative toolkit becoming very polished.
- Mike has been at the heart of much of this, as have many of the "old troupers" (who I won't name, as maybe some don't think they are so old).
- So I finish with
  "Congratulations Mike!!