A Hierarchy Degrees

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MOTIVATION

- Understanding the dynamic nature of constructions, and definability in the natural structures of computability theory such as the computably enumerable sets and degree classes.
- Beautiful examples: (i) definable solution to Post's problem of Harrington and Soare
  (ii) definability of the double jump classes for c.e. sets of Cholak and Harrington

- (iii) (Nies, Shore, Slaman) Any relation on the c.e. degrees invariant under the double jump is definable in the c.e. degrees iff it is definable in first order arithmetic.
- The proof of (i) and (ii) come from analysing the way the automorphism machinery fails. (ii) only gives $L_{\omega_1,\omega}$ definitions.

Natural definability

- This work is devoted to trying to find “natural” definitions.
- For instance, the NSS Theorem involves coding a standard model of arithmetic into the c.e. degrees, using parameters, and then dividing out by a suitable equivalence relation to get the (absolute) definability result.
- As articulated by Shore, we seek natural (e.g something that a lattice theorist might come up with) definable classes as per the following.

- (Ambos-Spies, Jockusch, Shore, and Soare) A c.e. degree **a** is promptly simple iff it is not cappable. (Ambos-Spies, Jockusch, Shore, and Soare)
- (Downey and Lempp) A c.e. degree **a** is contiguous iff it is locally distributive, meaning that

$$
\begin{aligned}
\forall \mathbf{a}_1, \mathbf{a}_2, \mathbf{b}(\mathbf{a}_1 \cup \mathbf{a}_2 = \mathbf{a} \wedge \mathbf{b} \leq \mathbf{a} \rightarrow \\
\exists \mathbf{b}_1, \mathbf{b}_2(\mathbf{b}_1 \cup \mathbf{b}_2 = \mathbf{b} \\
\wedge \mathbf{b}_1 \leq \mathbf{a}_1 \wedge \mathbf{b}_2 \leq \mathbf{a}_2)),
\end{aligned}
$$

  holds in the c.e. degrees.
- (Ambos-Spies and Fejer) A c.e. degree **a** is contiguous iff it is not the top of the non-modular 5 element lattice in the c.e. degrees.

- (Downey and Shore) A c.e. truth table degree is low_2 iff it has no minimal cover in the c.e. truth table degrees.
- (Ismukhametov) A c.e. degree is array computable iff it has a strong minimal cover in the degrees.

Second (and main) motivation: Unification

- It is quite rare in computability theory to find a single class of degrees which capture precisely the underlying dynamics of a wide class of apparently similar constructions.
- Example: promptly simple degrees again.
- Martin identified the high c.e. degrees as the ones arizing from dense simple, maximal, hh-simple and other similar kinds of c.e. sets constructions.
- low_2 degrees and lattice properties.
- K-trivials; lots of people, especially Nies and Hirschfeldt.

- Our inspiration was the the array computable degrees.
- These degrees were introduced by Downey, Jockusch and Stob
- This class was introduced by those authors to explain a number of natural “multiple permitting” arguments in computability theory.

- Definition: A degree **a** is called array noncomputable iff for all functions $f \leq_{wtt} \emptyset'$ there is a a function g computable from **a** such that
$$\exists^\infty x(g(x) > f(x)).$$
- Looks like "non-low$_2$."
- Indeed many nonlow$_2$ constructions can be run with only the above. For example, every anc degree bounds a 1-generic.

- c.e. anc degree are those that:
- (Kummer) Contain c.e. sets of infinitely often maximal Kolmogorov complexity
- (Downey, Jockusch, and Stob) bound disjoint c.e. sets A and B such that every separating set for A and B computes the halting problem
- Exactly those that have integer valued randoms (D-Barmpalias) and have packing dimension 1 (D-Greenberg).
- (Cholak, Coles, Downey, Herrmann) The array noncomputable c.e. degrees form an invariant class for the lattice of Π^0_1 classes via the thin perfect classes

The first class

- (Downey, Greenberg, Weber, also J. Miller) We say that a c.e. degree **a** is totally ω-c.a. iff for all functions $g \leq_T \mathbf{a}$, g is ω-c.a.. That is, there is a computable approximation $g(x) = \lim_s g(x, s)$, and a computable function h, such that for all x,

$$|\{s : g(x, s) \neq g(x, s+1)\}| < h(x).$$

- array computability is a uniform version of this notion where h can be chosen independent of g. Since **a** is not totally ω-c.e. means that there is a function $g \leq \mathbf{a}$, such that for all $f \leq_{wtt} \emptyset'$, $\exists^{\infty} n(g(n) > f(n))$. Note the quantifier swap from anc.
- So every array computable degree (and hence every contiguous degree) is totally ω-c.a..

AND LATTICE EMBEDDINGS

- Lattice embedding into the c.e. degrees. (Lerman, Lachlan, Lempp, Solomon etc.)
- One central notion:
- (Downey, Weinstein) Three incomparable c.e. degrees $\mathbf{a}_0, \mathbf{b}, \mathbf{a}_1$ form a weak critical triple iff $\mathbf{a}_0 \cup \mathbf{b} = \mathbf{a}_1 \cup \mathbf{b}$ and there is a c.e. degree $\mathbf{c} \leq \mathbf{a}_0, \mathbf{a}_1$ with $\mathbf{a}_0 \leq \mathbf{b} \cup \mathbf{c}$.
- $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ form a *critical triple* in a lattice L, if $\mathbf{a} \cup \mathbf{b}_0 = \mathbf{a} \cup \mathbf{b}_1$, $\mathbf{b}_0 \not\leq \mathbf{a}$ and for $\mathbf{d}$, if $\mathbf{d} \leq \mathbf{b}_0, \mathbf{b}_1$ then $\mathbf{d} \leq \mathbf{a}$.
- A lattice L has a weak critical triple iff it has a critical triple.
- Critical triples attempt to capture the "continuous tracing" needed in an embedding of the lattice M_5 below, first embedded by Lachlan.

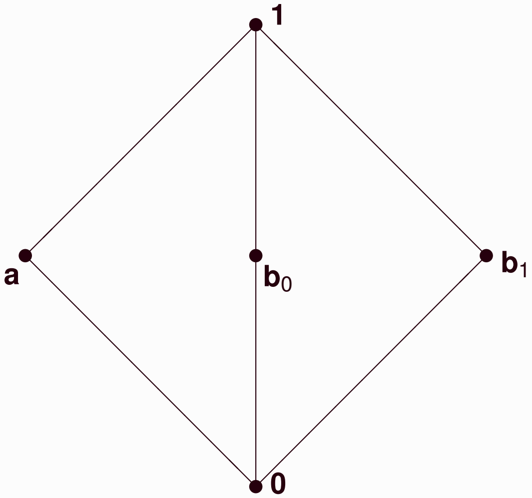


FIGURE: The lattice M_5

Theorem (Downey, Weinstein)

There are initial segments of the c.e. degrees where no lattice with a (weak) critical triple can be embedded.

Theorem (Downey and Shore)

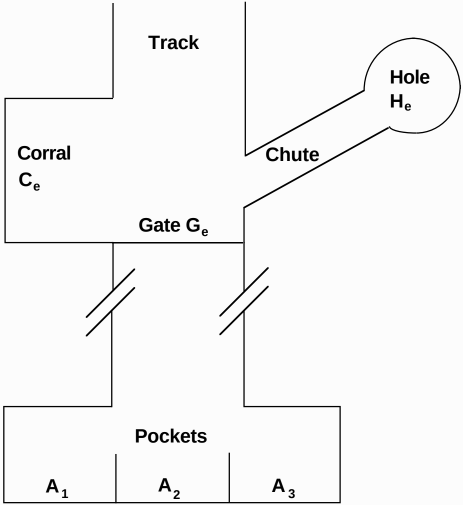
If $\mathbf{a}$ *is non-low*$_2$ *then* $\mathbf{a}$ *bounds a copy of* M_5.

Theorem (Walk)

Constructed a array noncomputable c.e. degree bounding no weak critical triple,

- and hence it was already known that array non-computability was not enough for such embeddings.

Analysing the construction



- $P_{e,i} : \Phi_e^A \neq B_i \, (i \in \{0, 1, 2\}, e \in \omega)$.
- $N_{e,i,j} : \Phi_e(B_i) = \Phi_e(B_j) = f$ total implies f computable in A, $(i, j \in \{0, 1, 2\}, i \neq j, e \in \omega.)$
- Associate $H_{\langle e,i \rangle}$ with $P_{e,i}$ and gate $G_{<e,i,j>}$ with $N_{e,i,j}$.

- Balls may be follower balls (which are emitted from holes), or trace balls.
- $x = x^i_{e,n}$. that x is a follower is targeted for A_i for the sake of requirement $P_{e,i}$ and is our n^{th} attempt at satisfying $P_{e,i}$.
- otherwise it is a trace ball: $t^j_{e,i,m}(x)$ which indicates it is targeted for B_j and is the m^{th} trace:
- at any stage s things look like:
  $x^i_{e,n}, t^{j_1}_{e,i,1}, t^{j_2}_{e,i,2}, \ldots, t^{j_m}_{e,i,m}$
- The key observation of Lachlan was that a requirement $N_{e,i,j}$ is only concerned with entry of elements into both B_i and B_j between expansionary stages

- When a ball is sitting at a hole it either gets released or it gets a new trace.
- When released a 1-2 sequence, say, moves down together and then stops at the first unoccupied 1-2 gate. All but the last one are put in the corral. The last one is the lead trace.
- Here things need to go thru one ball at a time and we retarget the lead trace as a 1-3 sequence or a 2-3 sequence. The current last trace is targeted for 1 it is a 1-3 sequence, else a 2-3 sequence.

One Gate

- The notion of a critical triple is reflected in the behaviour of one gate. This can be made precise with a tree argument.
- We have a 1-2 sequence with all but the last in the corral.
- the last needs to get thru. It's traces while waiting will be a 2-3 sequence, say. (or in the case of a critical triple, a sequence with a trace and a trace for the middle set A).
- Once it enters its target set, the the next comes out of corral and so forth.
- Now suppose that we want to do this below a degree **a**. We would have a lower gate where thru drop waiting for some permission by the relevant set D.
- We know that if **d** is not totally ω-c.a. then we have a function $g \leq_T D$, $\Gamma^D = g$ which is not ω-c.a. for any witness f.
- We force this enumeration to be given in a stage by stage manner $\Gamma^D = g[s]$.
- We ignore gratuitous changes by the opponent.

- Now we try to build a ω approximation to g to force D to give many permisssions.
- thus, when the ball and its A-trace drop to the lower gate, then we enumerate an attempt at a ω-c.a. approximation to $\Gamma^D(n)[s]$.
- This is repeated each time the ball needs some permission.

A characterization

Theorem (Downey, Greenberg, Weber)

(i) *Suppose that* **a** *is totally* ω*-c.a.. Then* **a** *bounds no weak critical triple.*

(ii) *Suppose that* **a** *is not totally* ω*-c.a.. Then* **a** *bounds a weak critical triple.*

(iii) *Hence, being totally* ω*-c.a. is naturally definable in the c.e. degrees.*

- The proof of (i) involves simulating the Downey-Weinstein construction enough and guessing nonuniformly at the ω-c.a. witness.
- The other direction is a tree argument simutaling the “one gate” scenario, as outlined.

A corollary

- Recall, a set B is called superlow if $B' \equiv_{tt} \emptyset'$.

Theorem (Downey, Greenberg, Weber)

The low degrees and the superlow degrees are not elementarily equivalent. (Nies question)

- Proof: There are low copies of M_5.
- Also: Cor. (DGW) There are c.e. degrees that are totally ω-c.a. and not array computable.

Other Similar Results

Theorem (Downey, Greenberg, Weber)

A c.e. degree $\mathbf{a}$ *is totally* ω*-c.a. iff there are c.e. sets* A*,* B *and* C *of degree* $\leq_T \mathbf{a}$*, such that*

(i) $A \equiv_T B$

(ii) $A \not\leq_T C$

(iii) *For all* $D \leq_{wtt} A, B$*,* $D \leq_{wtt} C$*.*

- (Downey and Greenberg) Actually D can be made as the infimum.

Presenting reals

- A real A is called left-c.e. if it is a limit of a computable non-decreasing sequence of rationals.
- (eg) $\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$, the halting probability.
- A c.e. prefix-free set of strings $A \in 2^{<\omega}$ presents left c.e. real α if $\alpha = \sum_{\sigma \in A} 2^{-|\sigma|} = \lambda(A)$.

Theorem (Downey and LaForte)

There exist noncomputable left c.e. reals α whose only presentations are computable.

Theorem (Downey and Terwijn)

The wtt degrees of presentations forms a Σ_3^0 ideal. Any Σ_3^0 ideal can be realized.

Theorem (Downey and Greenberg)

The following are equivalent.

(I) **a** *is not totally* ω*-c.a.*

(II) **a** *bounds a left c.e. real* α *and a c.e. set* $B <_T \alpha$ *such that if* A *presents* α*, then* $A \leq_T B$*.*

- For example, this generalizes work of Stephan and Wu who proved part of this for K-trivials, which of course are array computable.
- Notice that if **a** is array computable, it means that we can always present it via prefic free set of the same degree.

A HIERARCHY

- Lets re-analyse the 1-3-1 example.
- With more than one gate then when it drops down, it needs to have the same consitions met.
- That is, for each of the $f(i)$ many values j at the first gate there is some value $f(j, s)$ at the second.
- This suggest ordinal notations.
- (Strong Notation) Notations in Kleene's sense, except that we ask that the notation for an ordinal is given by an effective Cantor Normal Form.
- There is no problem for the for ordinals below ϵ_0, and such notations are computably unique.

- Now we can define for a notation for an ordinal $\mathcal{O}$, a function to be $\mathcal{O}$-c.a. in an analogous was as we did for ω-c.a..
- e.g. g is $2\omega + 3$ c.e., if it had a computable approximation $g(x, s)$, which initially would allow at most 3 mind changes.
- Perhaps at some stage s_0, this might change to $\omega + j$ for some j, and hence then we would be allowed j mind changes, and finally there could be a final change to some j' many mind changes.
- All low$_2$.

ω^ω

- Analysing the 1-3-1 case, you realize that that construction needs at least ω^ω.

Theorem (Downey and Greenberg)

a *is not totally* $< \omega^\omega$*-c.a. iff* **a** *bounds a copy of* M_5*.*

- The proof in one way uses direct simulation of the pinball machine plus "not $< \omega^\omega$" permissions, building functions at the gates. At gate n build at level ω^n for each P_e of higher peiority.
- In the reverse direction, we use level ω-nonuniform arguments where the inductive strategies are based on the failure of the previous level. Kind of like a level ω version of Lachlan non-diamond, using the Downey-Weinstein construction as a base.
- Corollary There are c.e. degrees that bound lattices with critical triples, yet do not bound copies of M_5.

ADMISSIBLE RECURSION

THEOREM (GREENBERG, THESIS)

Let $\alpha > \omega$ be admissible. Let **a** *be an incomplete α-ce degree. TFAE.*

(1) **a** *computes a counting of α*

(2) **a** *bounds a 1-3-1*

(3) **a** *bounds a critical triple.*

- Uses a theorem of Shore that if **a** computes a cofinal sequence iff it computes a counting. Then the weak critical triple machinery can actually have a limit. (Plus Maass-Freidman)

THEOREM (DOWNEY AND GREENBERG)

*Let ψ be the sentence "***a** *bounds a critical triple but not a 1-3-1" and let α be admissable. Then α satisfies ψ iff $\alpha = \omega$.*

- This is the first natural difference between R_ω and $R_{\omega_1^{CK}}$.
- Differences in Greenberg's thesis are all about coding.

m-TOPPED DEGREES

- Recall that a c.e. degrees $\mathbf{a}$ is called m-topped if it contains a c.e. set A such that for all c.e. $W \leq_T A$, $W \leq_m A$.

THEOREM (DOWNEY AND JOCKUSCH)

Incomplete ones exist, and are all low$_2$. None are low.

THEOREM (DOWNEY AND SHORE)

If $\mathbf{a}$ is a c.e. low$_2$ degree then there is an m-topped incomplete degree $\mathbf{b} > \mathbf{a}$.

Theorem (Downey and Greenberg)

Suppose that **b** *is totally* $< \omega^\omega$*-c.a. Then* **a** *bounds no m-topped degree.*

- The point is that making an m-top is kind of like making $\emptyset'$ on a tree: $Phi_e^A = W_e$ implies $W_e \leq_m A$, with $\Phi_e^A \neq B$.
- (Downey and Greenberg) There is, however, a totally ω^ω degree that is an m-top (and hence the full power of nonlow$_2$ permitting is not needed), and arbitarily complex degrees that are not.

EXPLORING THE HIERARCHY

- Theorem (Downey and Greenberg) If $n \neq m$ then the classes of totally ω^n-c.a. and totally ω^m-degrees are distinct. Also there is a c.e. degree **a** which is not totally $< \omega^\omega$-c.a. yet is totally ω^ω-c.a..
- Also totally $< \omega^\omega$ not ω^n for any n.
- This is also true at limit levels higher up.

Theorem (Downey and Greenberg)

There are maximal (e.g.) totally ω-c.a. degrees. These are totally ω-c.a. and each degree above is not totally ω-c.a. degree.

- Thus they are another definable class.
- As are maximal totally $< \omega^\omega$-c.a. degrees.

Theorem (Downey and Greenberg)

a *is totally* ω^2*-c.a. implies there is some totally* ω*-c.a. degree* **b** *below* **a** *with no critical triple embeddable in* $[\mathbf{b}, \mathbf{a}]$*.*

- Question: Are totally ω^n-c.a. degrees are all definable.
- Other assorted results about contiguity higher up.

The prompt case

- What about zero bottom? It is posssible to get the infimum to be zero.
- (DG) For the classes $\mathcal{C}$ aoove, we can define a notion of being promptly $\mathcal{C}$ then show that if **a** is such for the ω case, then it bound a critical triple with infumum **0**.
- (DG) **a** bounds a pairs of separating clases the degrees of whose members form minimal pairs.
- etc.

NORMAL NOTATIONS?

THEOREM (DOWNEY AND GREENBERG)

Suppose that $\mathbf{a}$ *is low*$_2$. *Then there is a notation* $\mathcal{O}$ *relative to which* $\mathbf{a}$ *is totally* ω^2*-c.a.*

- Δ^0_3 nonuniform version of Epstein-Haass-Kramer/Ershov.

Finite randomness

- Replace tests by finite tests. Several variations.
- If no conditions then on Δ_2^0 reals MLR and finite random coincide.
- If the test $\{V_n : n \in \omega\}$ has $|V_n| < g(n)$ for computable g, we say it is computably finitele random. (Ie if it passes all such tests.)

Theorem (Brodhead, Downey, Ng)

The c.e. degrees $\mathbf{a}$ *containing no such real are the totally* ω*-c.a. degrees.*

- Compare with

Theorem (Downey and Greenberg)

The c.e. segrees containing sets of packing dimension 1 are exactly the anc degrees.

Working above such degrees

- With George Barmpalias, Noam and I began to look at the effect of being able to compute such a degree, but with strong reducibilities.

Theorem (Barmpalias, Downey, Greenberg)

Every set in (c.e.) **a** *is wtt reducible to a ranked one iff every set in* **a** *is wtt reducible to a hypersimple set iff* **a** *is totally* ω*-c.a.*

Theorem (Barmpalias, Downey, Greenberg)

A computably enumerable **a** *computes a pair of left c.e. reals with no upper bound in the cL degrees iff* **a** *computes a left c.e. real not cL reducible to a random left c.e. real iff* **a** *is anc.*

OTHER WORK

- A set I is called indifferent for A and class C if changing A on any position in I keeps A in C.
- For example I is indifferent ifor A for 1-genericity if anything I-equivalent to A is 1-generic.
- (Day) **a** can compute a 1-generic B which can compute and indifferent subset of itself if **a** is not totally $< \omega^\omega$-c.a.. Conversely if **a** can do this it must not be totally ω-c.a.
- Nice open question to sort this one out.

How unbounded is Sacks Splitting?

Theorem (Ambos-Spies, D, Monath)

If A is c.e. then there are totally ω^3-c.a. c.e. sets splitting A.

Theorem (D and Ng)

*There are c.e. degrees **a** which are not the sup of two totally ω-c.a. c.e. degrees.*

Generalizations

- In a recent paper, D, Ambos-Spies and Monath, extends this to *wtt*-reductions.
- They characterize those c.e. sets $\leq_{wtt}$ a maximal set.
- Yet more hierarchies to analyse.

CONCLUSIONS

- We have defined a new hierarchy of degree classes within low_2.
- This hierarchy unifies many constructions, and
- Provides new natural degree definable degree classes.
- Many questions remain. eg, is array computable definable in the degrees. Are these classes definable in the degrees?
- Can they be used higher up in relativized form, say?