Some new natural definable degrees

Rod Downey

Victoria University

Wellington

New Zealand

References

- Joint work with Noam Greenberg
- Totally ω-computably enumerable degrees and bounding critical triples (with R. Weber) submitted.
- Totally $< \omega^\omega$-computably enumerable degrees and m-topped degrees, this proceedings.
- Some new natural definable degree classes.
- Hierarchies of definable degrees. (tentative title)

Motivation

- Understanding the *dynamic* nature of constructions, and *definability* in the natural structures of computability theory such as the computably enumerable sets and degree classes.

- Beautiful examples: (i) definable solution to Post's problem of Harrington and Soare

  (ii) definability of the double jump classes for c.e. sets of Cholak and Harrington

(iii) (Nies, Shore, Slaman) Any relation on the c.e. degrees invariant under the double jump is definable in the c.e. degrees iff it is definable in first order arithmetic.

- The proof of (i) and (ii) come from analysing the way the automorphism machinery fails. (ii) only gives $L_{\omega_1,\omega}$ definitions.

Natural definability

- This work is devoted to trying to find "natural" definitions.
- For instance, the NSS Theorem involves coding a standard model of arithmetic into the c.e. degrees, using parameters, and then dividing out by a suitable equivalence relation to get the (absolute) definability result.
- As atriculated by Shore, we seek *natural* definable classes as per the following.

- A c.e. degree $\boldsymbol{a}$ is promptly simple iff it is not cappable. (Ambos-Spies, Jockusch, Shore, and Soare)

- (Downey and Lempp) A c.e. degree $\boldsymbol{a}$ is contiguous iff it is locally distributive, meaning that

$$\forall \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{b}(\boldsymbol{a}_1 \cup \boldsymbol{a}_2 = \boldsymbol{a} \wedge \boldsymbol{b} \leq \boldsymbol{a} \rightarrow \\ \exists \boldsymbol{b}_1, \boldsymbol{b}_2(\boldsymbol{b}_1 \cup \boldsymbol{b}_2 = \boldsymbol{b} \\ \wedge \boldsymbol{b}_1 \leq \boldsymbol{a}_1 \wedge \boldsymbol{b}_2 \leq \boldsymbol{a}_2)),$$

  holds in the c.e. degrees.

- (Ambos-Spies and Fejer) A c.e. degree $\boldsymbol{a}$ is contiguous iff it is not the top of the non-modular 5 element lattice in the c.e. degrees.

- (Downey and Shore) A c.e. truth table degree is low_2 iff it has no minimal cover in the c.e. truth table degrees.
- (Ismukhametov) A c.e. degree is array computable iff it has a strong minimal cover in the degrees.

Second motivation: Unification

- It is quite rare in computability theory to find a single class of degrees which capture precisely the underlying dynamics of a wide class of apparently similar constructions.
- Example: promptly simple degrees again.
- Martin identified the high c.e. degrees as the ones arizing from dense simple, maximal, hh-simple and other similar kinds of c.e. sets constructions.
- K-trivials and lots of people, especially Nies and Hirschfeldt.

- Our inspiration was the the *array computable degrees.*
- These degrees were introduced by Downey, Jockusch and Stob
- This class was introduced by those authors to explain a number of natural "multiple permitting" arguments in computability theory.
- Definition: A degree $\boldsymbol{a}$ is called array noncomputable iff for all functions $f \leq_{wtt} \emptyset'$ there is a a function g computable from $\boldsymbol{a}$ such that

$$\exists^{\infty} x(g(x) > f(x)).$$

- Looks like "non-low$_2$."

- Indeed many nonlow$_2$ constructions can be run with only the above. For example, every anc degree bounds a generic.

- c.e. anc degree are those that:
- (Kummer) Contain c.e. sets of infinitely often maximal Kolmogorov complexity
- (Downey, Jockusch, and Stob) bound disjoint c.e. sets A and B such that every separating set for A and B computes the halting problem
- (Cholak, Coles, Downey, Herrmann) The array noncomputable c.e. degrees form an invariant class for the lattice of Π^0_1 classes via the thin perfect classes

The first class

- (Downey, Greenberg, Weber) We say that a c.e. degree $\boldsymbol{a}$ is *totally* ω*-c.e.* iff for all functions $g \leq_T \boldsymbol{a}$, g is ω-c.e.. That is, there is a computable approximation $g(x) = \lim_s g(x, s)$, and a computable function h, such that for all x,

$$|\{s : g(x, s) \neq g(x, s+1)\}| < h(x).$$

- array computability is a uniform version of this notion where h can be chosen independent of g.

- Every array computable degree (and hence every contiguous degree) is totally ω-c.e..

Critical triples and lattice embeddings

- Lattice embedding into the c.e. degrees. (Lerman, Lachlan, Lempp, Solomon etc.)
- One central notion:
- (Downey, Weinstein) Three incomparable c.e. degrees $\boldsymbol{a}_0, \boldsymbol{b}, \boldsymbol{a}_1$ form a weak critical triple iff $\boldsymbol{a}_0 \cup \boldsymbol{b} = \boldsymbol{a}_1 \cup \boldsymbol{b}$ and there is a c.e. degree $\boldsymbol{c} \leq \boldsymbol{a}_0, \boldsymbol{a}_1$ with $\boldsymbol{a}_0 \leq \boldsymbol{b} \cup \boldsymbol{c}$.
- $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ form a *critical triple* in a lattice L, if $\mathbf{a} \cup \mathbf{b}_0 = \mathbf{a} \cup \mathbf{b}_1$, $\mathbf{b}_0 \not\leq \mathbf{a}$ and for $\mathbf{d}$, if $\mathbf{d} \leq \mathbf{b}_0, \mathbf{b}_1$ then $\mathbf{d} \leq \mathbf{a}$.

- A lattice L has a weak critical triple iff it has a critical triple.

- Critical triples attempt to capture the "continuous tracing" needed in an embedding of the lattice M_5 below, first embedded by Lachlan.

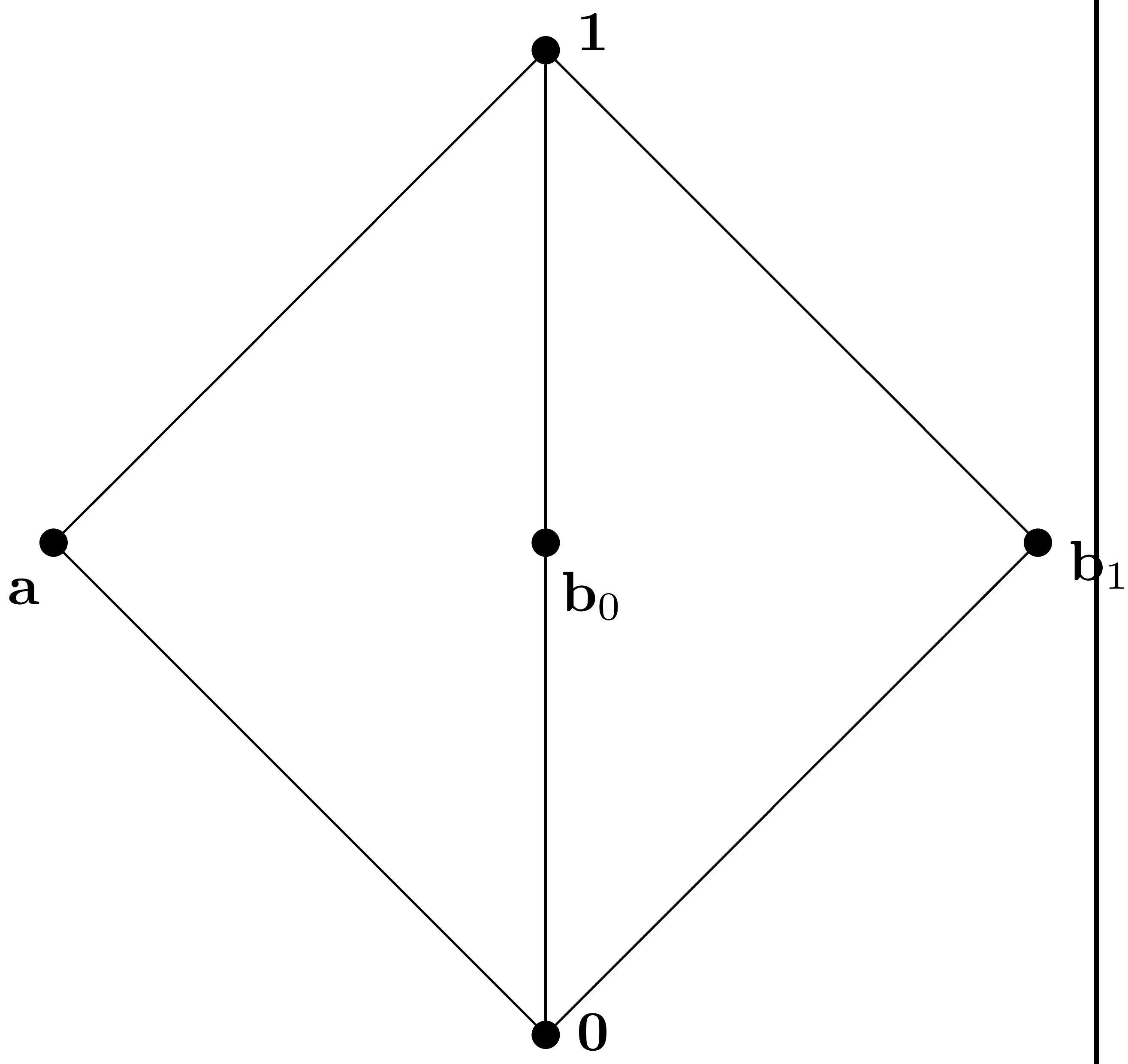

Figure 1: The lattice M_5

- Theorem (Downey, Weinstein) There are initial segments of the c.e. degrees where no lattice with a (weak) critical triple can be embedded.
- (Downey and Shore) If $\boldsymbol{a}$ is non-low$_2$ then $\boldsymbol{a}$ bounds a copy of M_5.
- (Walk) Constructed a array noncomputable c.e. degree bounding no weak critical triple, and hence it was already known that array non-computability was not enough for such embeddings.

Analysing the construction

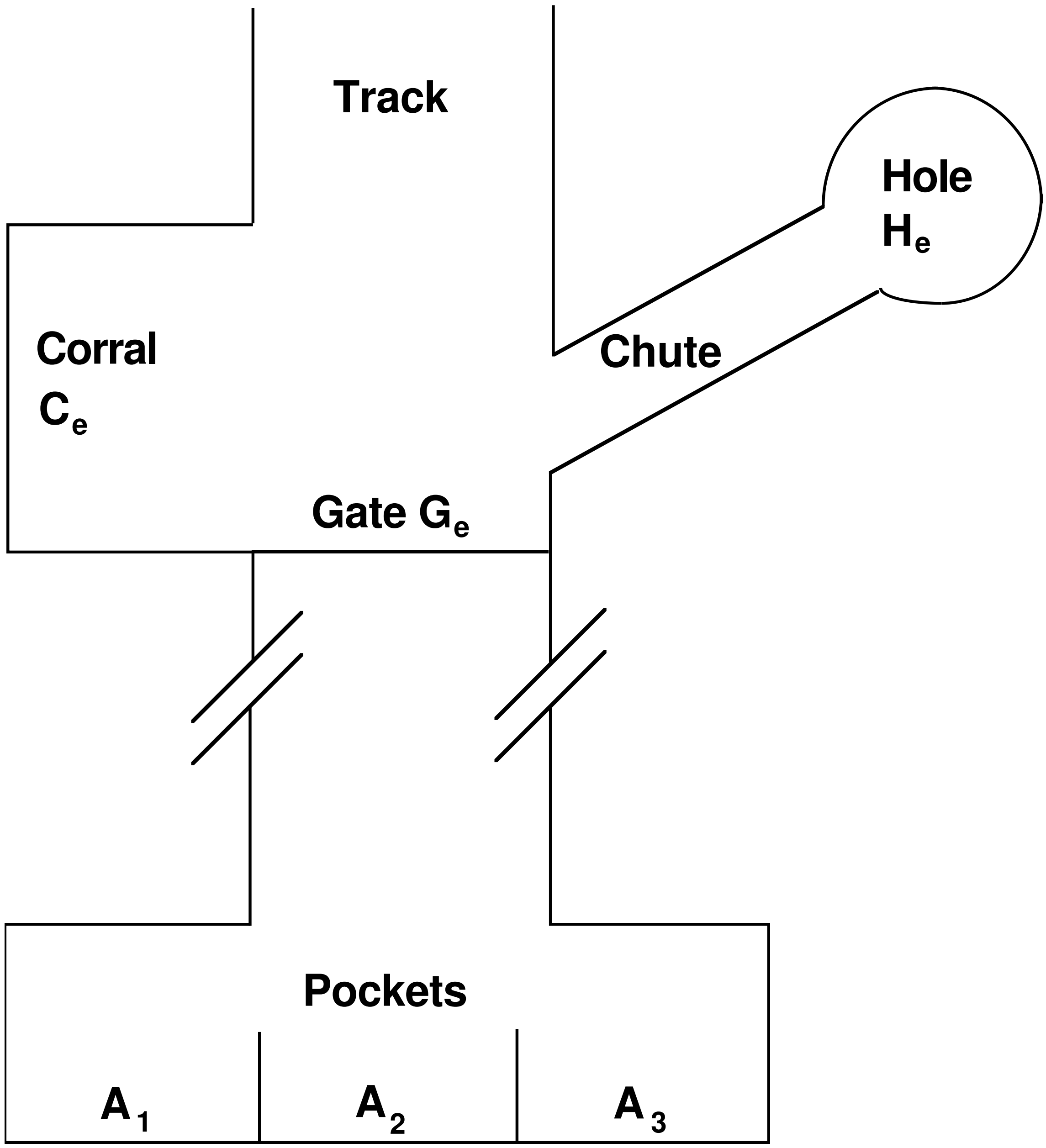

Figure 2: A pinball machine

- $P_{e,i} : \Phi_e^A \neq B_i \, (i \in \{0,1,2\}, e \in \omega)$.
- $N_{e,i,j} : \Phi_e(B_i) = \Phi_e(B_j) =$
  f total implies f computable in A,
  $(i, j \in \{0,1,2\}, i \neq j, e \in \omega.)$
- Associate $H_{\langle e,i \rangle}$ with $P_{e,i}$ and gate $G_{<e,i,j>}$ with $N_{e,i,j}$.

- Balls may be follower balls (which are emitted from holes), or trace balls.
- $x = x^i_{e,n}$. that x is a follower is targeted for A_i for the sake of requirement $P_{e,i}$ and is our n^{th} attempt at satisfying $P_{e,i}$.
- otherwise it is a trace ball: $t^j_{e,i,m}(x)$ which indicates it is targeted for B_j and is the m^{th} trace:
- at any stage s things look like: $x^i_{e,n}, t^{j_1}_{e,i,1}, t^{j_2}_{e,i,2}, ..., t^{j_m}_{e,i,m}$
- The key observation of Lachlan was that a requirement $N_{e,i,j}$ is only concerned with entry of elements into both B_i and B_j between expansionary stages

- When a ball is sitting at a hole it either gets released or it gets a new trace.
- When released a 1-2 sequence, say, moves down together and then stops at the first unoccupied 1-2 gate. All but the last one are put in the corral. The last one is the *lead trace.*
- Here things need to go thru one ball at a time and we retarget the lead trace as a 1-3 sequence or a 2-3 sequence. The current last trace is targeted for 1 it is a 1-3 sequence, else a 2-3 sequence.

One Gate

- The notion of a critical triple is reflected in the behaviour of *one gate*. This can be made precise with a tree argument.
- We have a 1-2 sequence with all but the last in the corral.
- the last needs to get thru. It's traces while waiting will be a 2-3 sequence, say. (or in the case of a critical triple, a sequence with a trace and a trace for the middle set A).
- once it enters its target set, the the next comes out of corral and so forth.

- Now suppose that we want to do this below a degree $\boldsymbol{a}$. We would have a *lower gate* where thru drop waiting for some permission by the relevant set D.

- *We know that if* $\mathbf{d}$ *is* not *totally* ω*-c.e. then we have a function* $g \leq_T D$, $\Gamma^D = g$ *which is not* ω*-c.e. for any witness* f.

- We force this enumeration to be given in a stage by stage manner $\Gamma^D = g[s]$.

- We ignore gratuituos changes by the opponent.

- Now we try to build a ω approximation to g to *force* D to give many permisssions.
- thus, when the ball and its A-trace drop to the lower gate, then we enumerate an attempt at a ω-c.e. approximation to $\Gamma^D(n)[s]$.
- This is repeated each time the ball needs some permission.

A characterization

- Theorem (Downey, Greenberg, Weber)

  (i) Suppose that $\boldsymbol{a}$ is totally ω-c.e.. Then $\boldsymbol{a}$ bounds no weak critical triple.

  (ii) Suppose that $\boldsymbol{a}$ is not totally ω-c.e.. Then $\boldsymbol{a}$ bounds a weak critical triple. ‘

  (iii) Hence, being totally ω-c.e. is naturally definable in the c.e. degrees.

- The proof of (i) involves simulating the Downey-Weinstein construction *enough* and guessing nonuniformly at the ω-c.e. witness.
- The other direction is a tree argument simutaling the "one gate" scenario, as outlined.

A corollary

- Recall, a set B is called *superlow* if $B' \equiv_{tt} \emptyset'$.
- Theorem(DGW) The low degrees and the superlow degrees are not elementarily equivalent. (Nies question)
- Proof: There are low copies of M_5.
- Also: Cor. (DGW) There are c.e. degrees that are totally ω-c.e. and not array computable.

Other Similar Results

- Theorem (Downey, Greenberg, Weber) A c.e. degree $\boldsymbol{a}$ is totally ω-c.e. iff there are c.e. sets A, B and C of degree $\leq_T \boldsymbol{a}$, such that

  (i) $A \equiv_T B$

  (ii) $A \not\leq_T C$

  (iii) For all $D \leq_{wtt} A, B$, $D \leq_{wtt} C$.

- (Downey and Greenberg) Actually D can be made as the infimum.

Presenting reals

- A real A is called left-c.e. if it is a limit of a computable non-decreasing sequence of rationals.
- (eg) $\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$, the halting probability.
- A c.e. prefix-free set of strings $A \in 2^{<\omega}$ *presents* left c.e. real α if $\alpha = \sum_{\sigma \in A} 2^{-|\sigma|} = \mu(A)$.
- Theorem (Downey and LaForte) There exist noncomputable left c.e. reals α whose only presentations are computable.

- Theorem (Downey and Terwijn) The wtt degrees of presentations forms a Σ^0_3 ideal. Any Σ^0_3 ideal can be realized.

- Theorem (Downey and Greenberg) The following are equivalent.

  (i) $\boldsymbol{a}$ is not totally ω-c.e..

  (ii) $\boldsymbol{a}$ bounds a left c.e. real α and a c.e. set $B <_T \alpha$ such that if A presents α, then $A \leq_T B$.

A Hierarchy

- Lets re-analyse the 1-3-1 example.
- With more than one gate then when it drops down, it needs to have the same consitions met.
- That is, for each of the $f(i)$ many values j at the first gate there is some value $f(j, s)$ at the second.
- This suggest *ordinal notations*.
- (Strong Notation) Notations in Kleene's sense, except that we ask that the notation for an ordinal is given by an effective Cantor Normal Form.

- There is no problem for the for ordinals below ϵ_0, and such notations are computably unique.

- Now we can define for a notation for an ordinal $\mathcal{O}$, a function to be $\mathcal{O}$-c.e. in an analogous was as we did for ω-c.e..

- e.g. g is $2\omega + 3$ c.e., if it had a computable approximation $g(x, s)$, which initially would allow at most 3 mind changes.

- Perhaps at some stage s_0, this might change to $\omega + j$ for some j, and hence then we would be allowed j mind changes, and finally there could be a final change to some j' many mind changes.

- All low$_2$.

ω^ω

- Analysing the 1-3-1 case, you realize that *that* construction needs at least ω^ω.
- Theorem (Downey and Greenberg) $\boldsymbol{a}$ is not totally $< \omega^\omega$-c.e. iff $\boldsymbol{a}$ bounds a copy of M_5.

- The proof in one way uses direct simulation of the pinball machine plus "not $< \omega^\omega$" permissions, building functions at the gates. At gate n build at level ω^n for *each* P_e of higher peiority.
- In the reverse direction, we use level ω-nonuniform arguments where the inductive strategies are based on the failure of the previous level. Kind of like a level ω version of Lachlan non-diamond, using the Downey-Weinstein construction as a base.
- Corollary There are c.e. degrees that bound lattices with critical triples, yet do not bound copies of M_5.

Admissible Recursion

- Theorem (Greenberg, Thesis) Let $\alpha > \omega$ be admissible. Let $\boldsymbol{a}$ be an incomplete α-ce degree. TFAE.

  (1) $\boldsymbol{a}$ computes a counting of α

  (2) $\boldsymbol{a}$ bounds a 1-3-1

  (3) $\boldsymbol{a}$ bounds a critical triple.

- Uses a theorem of Shore that if $\boldsymbol{a}$ computes a cofinal sequence iff it computes a counting. Then the weak critical triple machinery can actually have a limit. (Plus Maass-Freidman)

- Theorem (Downey and Greenberg) Let ψ be the sentence "$\boldsymbol{a}$ bounds a critical triple but not a 1-3-1" and let α be admissable. Then α satisfies ψ iff $\alpha = \omega$.

- This is the first natural difference between R_ω and $R_{\omega_1^{CK}}$.
- Differences in Greenberg's thesis are all about coding.

m-topped degrees

- Recall that a c.e. degrees $\boldsymbol{a}$ is called m-topped if it contains a c.e. set A such that for all c.e. $W \leq_T A$, $W \leq_m A$.
- Theorem (Downey and Jockusch) Incomplete ones exist, and are all low$_2$. None are low.
- Theorem (Downey and Shore) If $\boldsymbol{a}$ is a c.e. low$_2$ degree then there is an m-topped incomplete degree $\boldsymbol{b} > \boldsymbol{a}$.

- (Downey and Greenberg) Suppose that $\boldsymbol{b}$ is totally $< \omega^\omega$-c.e. Then $\boldsymbol{a}$ bounds no m-topped degree.
- (Downey and Greenberg) There is, however, a totally ω^ω degree that is an m-top, and arbitarily complex degrees that are not.

Exploring the hierarchy

- Theorem (Downey and Greenberg) If $n \neq m$ then the classes of totally ω^n-c.e. and totally ω^m-degrees are distinct. Also there is a c.e. degree $\boldsymbol{a}$ which is not totally $< \omega^\omega$-c.e. yet is totally ω^ω-c.e..
- Also totally $< \omega^\omega$ not ω^n for any n.
- This is also true at limit levels higher up.

- Theorem (Downey and Greenberg) There are *maximal* (e.g.) totally ω-c.e. degrees. These are totally ω-c.e. and each degree above is *not* totally ω-c.e. degree.
- Thus they are another definable class. item As are maximal totally $< \omega^{\omega}$-c.e. degrees.

- Theorem (Downey and Greenberg) $\boldsymbol{a}$ is totally ω^2-c.e. implies there is some totally ω-c.e. degree $\boldsymbol{b}$ below $\boldsymbol{a}$ with no critical triple embeddable in $[\boldsymbol{b}, \boldsymbol{a}]$.
- Question: Are totally ω^n-c.e. degrees are all definable.
- Other assorted results about contiguity higher up.

The prompt case

- What about zero bottom? It is posssible to get the infimum to be zero.
- (DG) For the classes $\mathcal{C}$ aoove, we can define a notion of being *promptly* $\mathcal{C}$ then show that if $\boldsymbol{a}$ is such for the ω case, then it bound a critical triple with infumum $\mathbf{o}$.
- etc.

What about normal notations?

- Theorem (DG) Suppose that $\boldsymbol{a}$ is low$_2$. Then there is a notation $\mathcal{O}$ relative to which $\boldsymbol{a}$ is totally ω^2-c.e.
- Δ^0_3 nonuniform version of Epstein-Haass-Kramer/Ershov.

Conclusions

- We have defined a new hierarchy of degree classes within low_2.
- This hierarchy *unifies* many constructions, and
- Provides *new* natural degree definable degree classes.
- Many questions remain. eg, is array computable definable in the degrees. Are these classes definable in the degrees?
- Can they be used higher up in relativized form, say?