Real, randomness and reducibilities

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- Joint work with Goeff Laforte, and some with Richard Coles

Basic Definition(s)

- Setting : reals in $(0,1)$. How to specify it?
- a has an infinite (e.g.) decimal expansion, $a = .a_1a_2....$
- $a = \Sigma_{n \in A} 2^{-n}$, where $A \subset \mathbb{N}$.
- $a = \Sigma_{n \in A} 2^{-|n|}$, where $A \subset \Sigma^*$.
- a is the limit of an increasing sequence of rationals. That is, $a = \lim_s \{q_s : q_s \in C\}$ and $q_{s+1} \geq q_s$, the Dedekind cut definition. We say that the sequence specified by C *represents* a.

Computably Enumerable Reals

- We say that a real a is *computably enumerable* if it is the limit of a c.e. sequence (iff $L(a) =_{def} \{q \in \mathbb{Q} : q \leq a\}$ is c.e.) Such reals are very natural and arize in the study of e.g. randomness. Chaitin's Ω numbers are c.e. reals. The can be characterized as special kinds of Δ^0_2 sets.

Theorem 1 (Calude Khoussainov Hertling Wang) *The following are equivalent.*

(i) a *is computably enumerable.*

(ii) *The set* $L(a) = \{q \in \mathbb{Q} : q < a\}$ *is computably enumerable.*

(iii) *There is an infinite computably enumerable prefix free set* A *with* $a = \Sigma_{n \in A} 2^{-|n|}$.

(iv) *There is a computable prefix free set* A *such that* $a = \Sigma_{n \in A} 2^{-|n|}$.

(iv) *There is a computable function* $f(x, y)$ *of two variables, such that*

(iva) *If, for some* k, s *we have* $f(k, s+1) = 0$ *yet* $f(k, s) = 1$ *then*

there is some $k' < k$ *such that*
$f(k', s) = 0$ *and* $f(k', s+1) = 1$.

(ivb) $a = \cdot a_1 a_2 ...$ *is a dyadic expansion of* a *with* $a_i = \lim_s f(i, s)$.

(v) There is a computable increasing sequence of rationals with limit a

- Calude, Coles, Hertling Khoussainov investigated the relationship between *representations* of a real a and $L(a) =_{\text{def}} \{q \in \mathbb{Q} : q \leq a\}$.

- Note that Soare proved that there are c.e. reals a such that for no c.e. set A is $a = .\chi_A$. We call such reals *strongly c.e.*.

  The following is easy

  Lemma 2 (Soare, Calude et. al.) *Let $L(a)$ denote the lower cut of a real a. That is, $L(a)$ is the collection of rationals less than a. Then if A is a representation of a, $A \leq_T L(a)$.*

Lemma 3 (Calude et. al.) *Suppose that A represents a. Then A is an infinite half of a splitting of $L(a)$.*

Proof. Clearly, if A represents a then A must be an infinite c.e. subset of $L(a)$. The thing to note is that $L(a) - A$ is also c.e.. Given rational q, if q occurs in $L(a)$, we need only wait till either q occurs in A or some rational bigger than q does. $\square$

Lemma 4 (Calude et. al.) *Let A be a representation of a. For subsets B of A, the following are equivalent.*

- *B represents a.*
- *B is half of a splitting of A.*

- It would be nice to be able to reverse the above. That is, prove that every splitting of $L(a)$ is a representation of a. Clearly this is false since some splittings won't even be representations.

- Calude et al did prove a has a representation of degree $\deg(L(a))$, and (ii) every representation can be extended to one of degree $\deg(L(a))$.

Theorem 5 (Downey) *The following are equivalent*

- $\boldsymbol{b}$ *is the weak truth table degree of a splitting of* $L(a)$*.*

- $\boldsymbol{b}$ *is the weak truth table degree of a representation of* a*.*

Presentations

- We say that a c.e. set A *presents* a real a if $a = \Sigma_{n \in A} 2^{-|n|}$.
- By the earlier theorem every c.e real can be presented by a comutable set A. What else can be said?

  Theorem 6 (Downey and Laforte) *There is a c.e. real α which is not computable, but such that if W presents α, then W is computable.*

- In fact the real can be chosen to be of high Turing degree.

- Seems related to the "speed" of enumeration. For example:

Theorem 7 *Suppose α has promptly simple degree. Then there is a presentation, A, of α that is noncomputable.*

- And related to strong reducibilities:

Theorem 8 *Let α be a computably enumerable real, with $\alpha = .\chi_A$ for some set A. Suppose that B is any presentation of α. Then $B \leq_{wtt} A$ with use function the identity.*

Theorem 9 *If A is a presentation of a c.e. real α and $C \leq_{wtt} A$ is computably* *enumerable, then there is a presentation B of α with $B \equiv_{wtt} C$.*