Post's Programme Revisited

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Supported by the Marsden Fund and Alexander von Humboldt Stiftung

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Asian Logic Conference, June 2019

History

- Post (1944) suggested "thinness" properties of complements of sets might solve his problem of finding a Turing incomplete c.e. set.
- We know that it in its original form this proposal fails since
  1. By e.g. Martin 1965, there are complete maximal c.e. sets (recall M is maximal if it is a co-atom in L^*.)
  2. Soare (1975) showed that all maximal sets are automorphic so no "extra" property will suffice to guarantee incompleteness.
  3. By Cholak, Downey, Stob (1992) no property of $\overline{M}$ alone can guarantee incompleteness.
- On the other hand, Harrington and Soare (1991) showed that there is a definable property Q such that if $Q(A)$ then A is incomplete.

More History

- On the other hand, there are fascinating interactions with strong reducibilities.
- Simple sets solve Post's problem for m-degrees. (Post, 1944)
- (η-) Maximal sets have minimal m-degrees. (Ershov 1971, Lachlan 1972).
- Simple sets are not btt-cuppable. (Downey, 2000)
- Dense simple sets are not tt-cuppable. (Kummer and Schaefer, 2007)
- Hypersimple sets (recall A is h-simple iff it meets all infinite strong arrays) are wtt-incomplete (Friedberg and Rogers, 1959) and indeed not wtt-cuppable (Downey and Jockusch, 1987).

Starting point

- Recall that A is totally ω-c.a. iff for all functions $f \leq_T A$, f is ω-c.a.. This means that for each f there is a computable approximation $f(x) = \lim_s f(x,s)$ and a computable h, $|\{s \mid f(x,s+1) \neq f(x,s)\}| < h(x)$.

Theorem (Barmpalias, Downey and Greenberg 2010)

A c.e. $\mathbf{a}$ *is is totally* ω*-c.a. iff every (c.e.) set in* $\mathbf{a}$ *is wtt-reducible to a h-simple c.e. set.*

- The totally ω-c.a. degrees have turned out to be really interesting, with many characterizations, and systematizing the combinatorics of a number of constructions. They are definable in the c.e. degrees, etc. So they have natural independent interest.

Maximal sets

- A preliminary result.
- recall that A is *superlow* if $A' \equiv_{tt} \emptyset'$.
- Equivalently, for c.e. A, there is a computable h such that $J^A(e)$ is h-c.a. Here J^A is the universal partial A-computable function.

Theorem (Ambos-Spies, D, Monath)

If A is c.e. and superlow, then there is a maximal set M with $A \leq_{wtt} M$. Indeed $A \leq_{ibT} M$.

Proof

- Requirements: $A \leq_{wtt} M$.
  $R_e : W_e \cap \overline{M}$ infinite implies $W_e \supseteq^* \overline{M}$.
  $N_e : \lim_s m_{e,s} = m_e$ exists where $m_{0,s} < m_{1,s} \ldots$ lists $\overline{M_s}$.
- Standard maximal set construction maximizes e-states. The e-state of $z \in \overline{M_s}$ is $\{j \leq e \mid z \in W_{j,s}\}$, a string.
- Standard maximal set construction tries to put almost all of $\overline{M}$ into the same e-state.

Modification

- If $\Gamma^M = A$ is the wtt-reduction, then if some $x \in A_{s+1} - A_s$, we need to change $M \upharpoonright \gamma(x)$.
- This can only be done if there is some element in $\overline{M_s}$ which is below $\gamma(x)$ which can be put into $M - M_s$. We refer to this as *coding*.
- The e-state machinery puts lots of elements from $\overline{M_t}$ into M_{t+1}, so we must be careful to leave enough elements to cope with this coding. For example, even for the 0-state, i.e. for a single requirement, all of the elements in $W_{0,s} \cap \overline{M_s}$ might be bigger than $\gamma(0)$ so we could not code 0 entering A.
- We only do this e-state action when we can use the jump computation to tell us that we are safe, and few elements will enter A.

One requirement R_0

- We have some part of the jump we control using the recursion theorem say $J^A(\langle g(0), j\rangle \mid j \in \mathbb{N}\}$. Jump computations on this have an approximation (known in advance) $J^A(\langle g(0), j\rangle)[s]$ changing at most $h(\langle g(0), j\rangle)$ many times.
- Anticipating things somewhat we write that as $f_0^A(j)[s]$ and the mind change number $n(0, j)$.
- For a single requirement, set aside a block of elements B_1 with at least $n(0,0)+1$ many elements which we don't raise the 0-state of.
- When we see at least $n(0,1)+1$ many elements ($> \max B_1$) in the high state, the plan is to use these for B_2 and we define $f_0^A(0)[s]\downarrow$, with huge use s, and wait for the approximation to be confirmed. The interval $I_1 = [\max B_1, s]$. (We no longer code below $\max B_1$.)
- Now we declare that B_1 will code I_1, and dump all elements not in $B_1 \sqcup B_2$ below $s_0 = s$ into $M_{s+1} - M_s$.
- Each time some element enters A between $\max B_1$ and $\max I_1$ we redefine the jump on argument 0 with use s_0 and on recovery we code all such this using an element in B_1.

- We repeat the process planning to use B_2 to code some interval $[\max I_1, s_1] = I_2$.
  Now the coding is in the high state
- That is, we's wait for at least $n(0,2)+1$ many elements in the high state (and these *must* be $> \max I_1$) for block B_3, etc.
- So block B_n looks after I_n.
- For more than one states, this is done inductively. First note that B_1 might never be used as maybe the 0-state is not well-resided in $\overline{M}$. So there would need to be a version of "B_1 for R_1 guessing R_0 is inactive" and working in the same way as above for R_0, and getting re-stated each time the 0 state acts up.
- There would be a version of R_1 guessing R_0 is infinitely often active. This demands that B_2 above would have a part of its block devoted to $B_1^{\infty\infty}$. It is only used when we see enough elements in state $\infty\infty$ and these are verified by a part of the jump we build for this guess $f^A_{\infty\infty}(1)$.

Observations

- All of the definitions above are wtt-jump computations, in that the use never changes once defined.
- That is, the proof only needs a new concept: We say that A is wtt-sl iff there is a uniformly computable approximation $g(x,s)$ of $\hat{J}^A(x)$ with $h(x)$ many mind changes where this is the partial wtt-jump. (i.e. (Φ_e, φ_e).) That is, the value of the the wtt-jump relative to A is $\leq_{wtt} \emptyset'$.

Theorem (ADM)

$A \leq_{ibT} M$ for M maximal if A is c.e. and wtt-sl.

Theorem (ADM)

There are wtt-sl Turing complete c.e. sets.

A Characterization

- A modified version works for the following. A is eventually uniformly *wtt*-array computable iff there are computable functions k, g and h, with $k(n,s) \leq k(n,s+1) \leq 1$, $\lim_s k(n,s)$ exists for all n such that
  1. $\lim_s g(x,s) = \hat{J}^A(x)$ for all x.
  2. $k(n,s) \leq k(n,s+1) \leq 1$, $\lim_s k(n,s)$ exists for all n
  3. If $k(x,s) = 1$ then $g(x,t)$ has at most $h(x)$ further mind changes for $t > s$ (hence wlog $k(x,t) = 1$ for all $t > s$).
  4. If $\hat{J}^A(\langle e,y\rangle)\downarrow$ for all y, then for almost all s, $\lim_s k(\langle e,y\rangle, s) = 1$.
- More or less the same proof gives one direction of:

Theorem (ADM)

For a c.e. A, $A \leq_{ibT} M$ iff $A \leq_{wtt} M$ for M maximal iff A is eventually uniformly wtt-array computable.

The Other Direction

- Suppose that $\Gamma^M = A$, and A not eventually uniformly wtt-ac. Choose $h(n) = 2^n$ for simplicity.
- Let $\ell(s) = \max\{z \mid \leq y\Gamma^M \upharpoonright z = A \upharpoonright z[s]\}$. Our assumptions about the enumerations of A, M and the jump are that once $\ell(s) > n$, if $A_{s+1}(n) \neq A_s(n)$, $M_{s+1} \upharpoonright \gamma(n) \neq M_s \upharpoonright \gamma(n)$.
- We know that for any approximation for the wtt-jump $g(\langle e, x\rangle, s)$ there will be total $\hat{\Phi}_e^A(x)$ changing more than $h(\langle e, x\rangle)$ many times on infinitely many x.
- Initially have $k(\langle e, x\rangle, s) = 0$ for all e, x and keep it like this unless told otherwise for $t > s$. The approximation $g(\langle e, x\rangle, s)$ is the natural one observing halting computations.
- For each e carry out the following construction. When we see $\hat{\Phi}_e^A(0)\downarrow[s]$ let $I_0^e = [0, \gamma(\phi_e(0)))$. Wlog the $\hat{\Phi}_e^A$ are monotone, and we can continue $I_1^e = [\gamma(\phi_e(0)), \gamma(\phi_e(1)))$, defining a sequence of disjoint e-intervals $\{I_n^e \mid n \in \mathbb{N}\}$. If ever we see $|\overline{M}_s \cap [0, \max I_n^e)| < 2^{\langle e, n\rangle}$, define $k(\langle e, n\rangle, s) = 1$. (Note that this ensures $k(\langle e, n\rangle, t) = 1$ for all $t > s$, also).

- The assumption is that $\Gamma^M = A$, and M is maximal.
- Notice that if we define $k(\langle e, n\rangle, s) = 1$, $A \upharpoonright \phi_e(n)$ can change only $< 2^{\langle e,n\rangle} = h(\langle e, n\rangle)$ many times, since each change induces a change in $M \upharpoonright \gamma(\phi_e(n))$ and hence $M \upharpoonright \max I_n^e$. There are not enough elements to enter $M - M_s$ for this to happen more than $2^{\langle e,n\rangle} - 1$ many times. So 3 holds.
- Now suppose that $\hat{\Phi}_e^A$ is total. Then for each n, we define I_n^e.
- Moreover, since M is maximal, we know that for almost all n, $|\overline{M} \cap I_e^n| \leq 1$.
- Thus, for almost all n, there is an s with $|\overline{M}_s \cap [0, \max I_n^e)| < 2^n \leq 2^{\langle e,n\rangle}$, and hence for all e with $\hat{\Phi_e}^A$ total, and for almost all n, s, $k(\langle e, n\rangle, s) = 1$.
- Therefore A is eventually uniformly wtt-ac, a contradiction.
- Same argument works for e.g. dense simple, hh-simple also.

- Further work involves exploring approximations to wtt-functionals.
- Thank you.