Strong Jump Traceability I

Rod Downey
Victoria University
Wellington
New Zealand

Background References

- Calibrating Randomness, (Downey, Hirschfeldt, Nies, Terwijn) Bulletin Symbolic Logic, 2006
- Algorithmic Randomness and Complexity, (Downey and Hirschfeldt) Springer-Verlag
- Randomness and Computability (Nies) OUP

References

- Strong Jump-Traceability I : the Computably Enumerable Case, (Cholak, Downey, Greenberg) Advances in Math.
- Strong jump-traceability II: (Downey and Greenberg) Israel J Math.
- Lowness properties and approximations of the jump. (Figueira, Nies, Stephan), APAL
- K-trivial degrees and the jump-traceability hierarchy, Barmpalias, Downey and Greenberg. Proc AMS
- Beyond strong jump traceability, (Ng Keng Meng) Proc LMS

- Characterising the strongly jump-traceable sets via randomness, Greenberg, Hirschfeldt and Nies, Adv Math.
- Strong jump-traceability and Demuth randomness, Greenberg and Turetsky. to appear Proc. LMS
- Benign cost functions and lowness notions, Greenberg and Nies. JSL.
- Pseudo-jump operators and SJTHard sets, (Downey and Greenberg) to appear Advances in Math.
- Inherent enumerability of strong jump traceability, (Diamondstone, Greenberg, Turetsky)

Notation

- Real is a member of Cantor space 2^ω with topology with basic clopen sets $[\sigma] = \{\sigma\alpha : \alpha \in 2^\omega\}$ whose measure is $\mu([\sigma]) = 2^{-|\sigma|}$.
- $\alpha \upharpoonright n$ is the first n bits of α.
- Strings = members of $2^{<\omega} = \{0, 1\}^*$.

KOLMOGOROV COMPLEXITY

- Capture the incompressibility paradigm. Random means hard to describe, incompressible: e.g. 1010101010.... (10000 times) would have a short program.
- A string σ is random iff the only way to describe it is by hardwiring it. (Formalizing the Berry paradox)
- Want the bits of τ to describe σ if $U(\tau) = \sigma$ for a device (Turing machine) U. Write $C(\sigma) = |\tau|$ for the shortest such τ, and can use a universal machine.
- Plain complexity like this has τ providing itself and its length so this is circumvented by using prefix-free complexity (telephone numbers) giving K for prefix-free machines.
- Using this, Levin, Chaitin, Schnorr proved that there are reals with $K(\alpha \upharpoonright n) \geq^{+} n$ for all n, called 1-random, and coinciding with earlier ones avoiding all effective null sets,

Martin-Löf randomness

- Generalized effective statistical test: an effectively shrinking computable collection of open sets: $\{U_n \mid n \in \mathbb{N}\}$ with $\mu(U_n) \leq 2^{-n}$.
- e.g. every second bit is 0: $U_1 = \{[10], [00]\}$, etc.
- For all such Martin-Löf tests $A \notin \cap_n U_n$.
- Other notions: Schnorr randomness ($\mu(U_n) = 2^{-n}$), Demuth Randomness (ω-effective approximations and other acceptances criterion), etc.

- As with life, relationships here are complex (Solovay)

$$K(x) = C(x) + C^{(2)}(x) + \mathcal{O}(C^{(3)}(x)).$$

and

$$C(x) = K(x) - K^{(2)}(x) + \mathcal{O}(K^{(3)}(x)).$$

- These 3's are sharp (Solovay) That is, for example, $K = C + C^2 + C^3 + O(C^4)$ is NOT true.

LOWNESS

- I would like to discuss the remarkable story of lowness generating K-triviality and then sjt.
- First sjt was an apparent artifact
- Then proved to be intimately related with randomness
- Later (also) giving insight into computability itself.
- I will try to explain the little boxes method, which is new and poorly understood.
- Theme: to what extent do computational lowness (the extent to which sets resemble computable ones) and being far from random align themselves?

Key facts

- **Theorem (Chaitin)**
  There is a constant d such that for all c, and n,

$$|\{\nu : |\nu| = n \wedge C(\nu) \leq C(n) + c\}| \leq d2^c.$$

Theorem (Levin, Chaitin)
There is a constant d such that for all c and n,

$$|\{\nu : |\nu| = n \wedge C(\nu) \leq C(n) + c\}| \leq d2^c.$$

INFORMATION CHARACTERIZATION OF COMPUTABILITY

- Chaitin proved that a real A is computable iff for all n, $C(A \upharpoonright n) \leq^+ \log n$, iff $C(A \upharpoonright n) \leq^+ C(n)$.
- This is proven using the fact that a Π^0_1 class with a finite number of paths has only computable paths, combined with the Counting Theorem $\{\sigma : C(\sigma) \leq C(n) + c \wedge |\sigma| = n\} \leq d2^c$. (Using the Meyer-Loveland Technique below)
- Meyer(-Loveland) had earlier shown A is computable iff $C(A \upharpoonright n|n) \leq c$ for some c and all n.

The Meyer-Loveland Technique

- If $C(\alpha \upharpoonright n|n) \leq c$ then there are only c programmes possibly computing initial segments of α.
- This computes a tree of *strings* of maximal width c.
- Therefore only at most c paths. Say $\hat{c}$.
- Imagine the situation that there is only *one* path in a tree of maximal width 2.
- Enumerate until only one remains.

K-TRIVIALITY

- What is a consequence of $K(A \upharpoonright n) \leq^+ K(n)$ for all n? We call such reals K-trivial. Does A K-trivial imply A computable?
- Write $A \in KT(d)$ iff for all n, $K(A \upharpoonright n) \leq K(n) + d$.

The argument fails

- It is still true that $\{\sigma : K(\sigma) \leq K(|\sigma|) + d\}$ is $O(2^d)$, so it would appear that we could run the Π^0_1 class argument used for C. But no...
- The problem is that we don't know $K(n)$ in any computable interval, therefore the tree of K-trivials we would construct would be a Π^0_1 class relative to $\emptyset'$.

THEOREM (CHAITIN, ZAMBELLA)

There are only $O(2^d)$ members of $KT(d)$. They are all Δ_2^0.

THEOREM (SOLOVAY)

There are noncomputable K-trivial reals.

THEOREM (ZAMBELLA)

Such reals can be c.e. sets.

A remarkable class

- K-trivials form a remarkable class as we will see.
- First they solve Post's problem.
- Theorem: (DHNS) If A is K-trivial then $A <_T \emptyset'$.

Similar methods allow for us to show the following

THEOREM (NIES)

All K-trivials are superlow $A' \equiv_{tt} \emptyset'$, and are tt-bounded by c.e. K-trivials. In fact they are Jump Traceable as we see below.

Thus triviality is essentially an "enumerable" phenomenom.

There are other antirandomness notions.

Definition (Kučera and Terwijn)

We say A is low for randomness iff the reals Martin-Löf random relative to A are exactly the Martin-Löf random reals.

Definition (Hirschfeldt, Nies, Stephan)

A is a a base for randomness iff $A \leq_T B$ with B A-random.

THEOREM

The following are equivalent to being K-trivial.

(I) *(Nies) A is low for randomness.*

(II) *(Hirschfeldt and Nies) A is K-low in that $K^A =^+ K$.*

(III) *(Hirschfeldt, Nies, Stephan) A is a base for randomness.*

(IV) *(Downey, Nies, Weber, Yu+Nies, Miller) A is low for weak-2-randomness.*

(V) *+ 15 others!*

Questions and a proper subclass

It is open if this is the same as a number of other "cost function" classes such as the reals which are Martin-Löf coverable.
It is known there is a proper subclass defined by cost function.

Definition

- (Zambella, Terwijn, later Nies) Let h be an order. We say that A is jump traceable for the order h iff there is a computable collection of c.e. sets $W_{g(e)}$ with $|W_{g(e)}| < h(e)$ and $J^A(e) \in W_{g(e)}$, for every partial A-computable function J^A.
- (Figueira, Nies, Stephan) A is strongly jump traceable iff it is jump traceable for every computable order.

Think about classical set theory notions of capturing a function by specifying possibilities. (Raisonnier, Shelah, ..., Zambella).

Combinatorial ideas

Inspired by the result

Theorem (Terwijn-Zambella, then Bedregal, Kjos-Hanssen, Nies, Stephan)

A is low for Schnorr randomness iff A is computably traceable. That is, for every function $f \leq_T A$ there is a canonical collection of finite sets $\{D_{g(n)} \mid n \in \mathbb{N}\}$ such that $|D_{g(n)}| \leq n + 1$ and for all n, $f(n) \in D_{g(n)}$.

This generalized old traceing notions such as highness (Martin) lowness (Soare), hyperhperimmunity (Miller-Martin).

THEOREM (NIES)

A is K-triv implies that there is an order h ($n \log n$) relative to which A is jump traceable.

Improved to $M \log n$ for some M by Hölzl, Kräling, Merkle.

THEOREM (FIGUEIRA, NIES, STEPHAN)

Noncomputable sjt c.e. sets exist.

THEOREM (FIGUEIRA, NIES, STEPHAN)

If A is strongly superlow the A is sjt.

strongly superlow means that A' has very tame approximations

THEOREM (FIGUEIRA, NIES, STEPHAN)

Sjt is equivalent to $C(n) \leq^+ C^A(n) + h(C^A(n))$ for all orders h.

ie A is lowly for C

THEOREM (CHOLAK, DOWNEY, GREENBERG)

The c.e. sjt's are a proper subclass of the K-trivials. They form an ideal.

Turetsky has recently shown that there is a K-trivial which is not $o(\log n)$-jump traceable.

THEOREM (DOWNEY, GREENBERG)

If A is sjt then A is Δ_2^0

THEOREM (DIAMONDSTONE, GREENBERG, TURETSKY)

If A is sjt then $A \leq_T B$ with B sjt and c.e.

COROLLARY (DIAMONDSTONE, GREENBERG, TURETSKY)

A is sjt is equivalent to A is strongly super low.

Randomness

- It might seem that sjt's are an *artifact* of randomness and potential combinatorial characterizations of notions. However (Noam will be discussing these results):

Theorem (Greenberg, Hirschfeldt, Nies)

sjt=superlow$^\diamond$=superhigh$^\diamond$. Here $\mathcal{C}^\diamond$ is exactly the c.e. sets below all random members of $\mathcal{C}$.

Theorem (Greenberg, Nies)

A is sjt iff it is computable from all Δ_2 MRL which are not weakly Demuth random.

Theorem (Kucera-Nies, Greenberg-Turetsky)

*A c.e. degree **a** is sjt iff it is computable from some Demuth random real.*

- Roughly need orders $\sqrt{\log n}, o(\log n)$. Is there a combinatorial characterization?
- Conjecture : A is K-trivial iff A is jump traceable for all computable orders h with $\sum_{n\geq 1} 2^{-h(n)} < \infty$.

A hint of the proof techniques

- To show that if A and B are c.e. sjt, so is $A \oplus B$.
- Given h we can construct a slower order k such that if A and B are jump traceable via k then $A \oplus B$ is jump traceable via h
- Opponent gives: $W_{p(x)}$ jump tracing A and $W_{q(x)}$ jump tracing B, such that $|W_{p(x)}|, |W_{p(x)}| < k(x)$.
- We: V_z tracing $J^{A \oplus B}(z)$ with $|V_z| < h(z)$.

TWO OBSTACLES

- We see an apparent jump computation $J^{A \oplus B}(x) \downarrow [s]$.
- Should we believe? We only have $h(x)$ many slots in the trace V_x to put possible values.
- Opponent can change A or B after stage s on the use.
- We build parts of jump (recursion thm) testing A and B
- Basic idea: For some $a = a(x)$ and $b = b(x)$ we will define

$$J^B[s](b) = j_B(x, s) \text{ and } J^A[s](a) = j_A(x, s),$$

where $j_C(x, s)$ denotes the C-use of the $J^{A \oplus B}(x)[s]$ computation.

- Ignore *noncompletion*: that is the $A \oplus B$ computation changes before these procedures return.
- Simplest case: $W_{p(a)}$ and $W_{q(b)}$ were of size 1 (1-boxes)
- Then if return: $A \oplus B$ is correct
- Now 2-boxes. If the $A \oplus B$ computation is wrong, at least one of the A or B ones are too.

- If we are lucky and there is are false jump computations in both of the $W_{p(a)}$ and $W_{q(b)}$.
- The 2-boxes are now, in effect 1-boxes. (Very good)
- Can't allow to only point at one side. Use up all the 2-boxes.
- For example if always the A sides was the wrong part, and there were k 2-boxes then after k attacks, all the 2-boxes would be useless and the information in the B-side is correct, hence the box is used.

Multiple Boxes

- Idea: use multiple 2-boxes. E.g. at the beginning use two 2-boxes for the same computation.
- A side was wrong. Then now we have *two* promoted 1-boxes.
- Since the A-computation now must be correct, if the believed computation is wrong, it must be the B side which wrong the next time, now creating a new B-1-box. Finally the third time we test, we would have two 1-boxes.

NON-RETURN

- Now we face the ignored problem. We test and before the computation returns, the jump computation is changed by an A or B change, but possibly one of the A or B uses is correct. Now nothing is promoted. This seems very bad.
- Even with 1-boxes.
- Use descending sequences of boxes, and monster boxes.
- Complicated, combinatorial.

- The idea is that for a computation whose target is, say, 2-boxes, begin ever further out. Begin by testing at, say, s-boxes.
- Monster boxes called metaboxes.
- If both A and B return at the s-box, go to $s - 1$ etc. Only believe if you get back to the 2-boxes. The idea that a failure at k promotes $k + 1, \ldots, s$-boxes, at least on one side.
- A combinatorial argument if used to show that cannot favour one side forever.

PROPER SUBCLASS OF THE K-TRIVIALS

- How to make a properly ω-c.e. K-trivial?
- Use descending costs....
- If the trace grows slowly enough then can make K-trivial and not jt at that order. Much the same idea, the key point being the to change the trace and use a a box location, the use is very big, and the opponent needs more tailweight.

Ng Keng Meng's Theorems

- The c.e. sjt's are Π^0_4 complete.
- This solves a problem of Nies: there is no minimal order.

Beyond jump traceability

- Say A is C-sjt iff for all orders h^B, for $B \in C$, A is h^B-jt.
- (Ng) No real is C-sjt where $C = \Delta_2$.
- (Ng) There are c.e. reals sjt for all c.e. sets.
- (Ng) They cannot be promptly simple, the first such class.
- (Ng) No real is K^B-trivial for all B, or c.e. B.