# NATURAL LARGE DEGREE SPECTRA

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Abstract. We show that the collection of array non-recursive degrees, the collection of non-jump-traceable degrees, and the collection of degrees which compute a function not dominated by any $\omega$-computably approximable function, are all degree spectra of countable structures.

## 1. Introduction

A central concern of computable model theory is the restriction that algebraic structure imposes on the information content of an object of study. One asks about a countable object, what information is coded intrinsically into this object, which cannot be avoided by passing to an isomorphic copy of the object? Given a countable structure $\mathcal{M}$, we define the degree spectrum of $\mathcal{M}$ to be

$$\operatorname{Spec}(\mathcal{M}) = \{X \in 2^{\omega} : \exists \mathcal{N} \cong \mathcal{M} \ \ (\mathcal{N} \leqslant_{\mathrm{T}} X)\},$$

where we identify $\mathcal{N}$ with its atomic diagram. In the language of mass problems, $\operatorname{Spec}(\mathcal{M})$ is the problem of computing a copy of $\mathcal{M}$. Since $\operatorname{Spec}(\mathcal{M})$ is degree-invariant, we often replace $\operatorname{Spec}(\mathcal{M})$ by the collection of Turing degrees of elements of $\operatorname{Spec}(\mathcal{M})$. One of the major aims of computable model theory is understanding which collections of Turing degrees can be the spectra of some countable structures. Intuitively, the isomorphism type of a structure $\mathcal{M}$ captures the computability-theoretic properties of $\operatorname{Spec}(\mathcal{M})$. In this way, classes of degrees which cannot be captured by any single countable set (as they may not have least elements), are nonetheless captured by a single countable structure. For example, Slaman [Sla98] and Wehner [Weh98] showed that the collection of nonzero Turing degrees is a degree spectrum, and so there is a structure which captures the property of being non-computable. Recently [GMS], the collection of non-hyperarithmetic degrees has also been shown to be a degree spectrum.

Of particular relevance to this paper are results by Csima and Kalimullin [CK10], who showed that the collection of hyperimmune degrees form a degree spectrum. Their interest in this class is part of effort to understand large degree spectra, for example co-null spectra. Certainly co-countable spectra (such as the non-computable and non-hyperarithmetic degrees) are large; Csima and Kalimullin were interested in large spectra whose complements are nonetheless uncountable. An underlying question was whether degree spectra can differentiate between measure and category; this was answered in the affirmative (and in both directions) in [GMS].

In this paper we show that a number of classes closely investigated by computability theorists – namely the class of array non-recursive degrees, of non jump-traceable degrees, and the degrees which compute a function not dominated by any

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$\omega$-computably-approximable function – all form degree spectra; indeed, they are degree spectra of countable families of sets. The first is a natural class which is null and co-meagre, again showing a separation between measure and category among degree spectra.

Our techniques owe much to those of Csima's and Kalimullin's. The underlying theme, though, is that domination properties, rather than sparsity properties of sets, are particularly amenable to being captured by degree spectra. The situation vis-a-vis traceability notions is less clear. In particular we ask, is the collection of non-c.e.traceable degrees a degree spectrum?

## 2. Enumerating families of functions

Knight (see [AK00]) and Khoussainov [Kho86] have shown how to code families of sets into graphs, converting the problem of enumerating or computing the family of sets to the problem of computing a copy of the graph. Thus, collections of degrees which can enumerate a particular family of sets are particularly nice degree spectra, and have been studied, for example, by Kalimullin in [Kal08], who tries to find conditions on a family of countable families of sets which are related to having the collection of degrees enumerating the family co-null.

Recall that a Turing degree $\mathbf{a}$ can *enumerate* a countable family $\mathcal{F}$ of sets of natural numbers if there is a uniformly $\mathbf{a}$-c.e. array $\langle A_n \rangle$ such that $\{A_n : n \in \omega\} = \mathcal{F}$; repetitions are allowed, and so we may assume that each set appears infinitely often in an enumeration. To code a particular set $A \subseteq \omega$, we let $H(A)$ be the "flower graph" starting with a central vertex $v$ and adding a loop from $v$ to itself of length $n+3$ for each $n \in A$. The "bouquet graph" $H(\mathcal{F})$ of a family of sets $\mathcal{F}$ consists of infinitely many disjoint copies of the flower graph $H(A)$ for each $A \in \mathcal{F}$. Then it is easy to see that a Turing degree $\mathbf{a}$ can enumerate a family $\mathcal{F}$ if and only if it computes a copy of $H(\mathcal{F})$.

We also note that a countable list $\langle \mathcal{F}_n \rangle_{n\in\omega}$ of families can be coded by passing to the family $\{\{\{n\} \oplus A : A \in \mathcal{F}_n\} : n \in \omega\}$.[1] This shows that the collection of Turing degrees $\mathbf{a}$ which, uniformly in $n$, can enumerate $\mathcal{F}_n$, is a degree spectrum. Note that here it is important that we enumerate the families in order; this is not the same as the problem of enumerating a countable *family* of families of sets.

The technique of coding the enumeration of sets into graphs can be used to code partial functions. If $\mathcal{P}$ is a countable family of partial functions from $\omega$ to $\omega$, then we say that a Turing degree $\mathbf{a}$ can *effectively list* the family $\mathcal{P}$ if there is a uniformly $\mathbf{a}$-partial computable list of functions $\langle \psi_n \rangle$ such that $\{\psi_n : n \in \omega\} = \mathcal{P}$. Again, repetitions are allowed. Now it is easy to see that effectively listing a family of functions $\mathcal{P}$ is equivalent to enumerating the family of their graphs. Thus:

**Proposition 2.1.** *The collection of degrees which can effectively list a countable family of partial functions $\mathcal{P}$ is a degree spectrum; so is the collection of degrees which, uniformly in $n$, can effectively list a countable family $\mathcal{P}_n$ of partial functions.*

We will also need a property stronger than merely listing partial functions. If $\mathcal{P}$ is again a family of partial functions, then we say that a Turing degree $\mathbf{a}$ can *compute* $\mathcal{P}$ if there is a uniformly $\mathbf{a}$-computable list $\langle \psi_n \rangle$ which forms a listing

[1] Here $\oplus$ denotes the usual join operation between two sets of natural numbers;

$$B \oplus A = \{2n : n \in B\} \cup \{2n+1 : n \in A\}.$$

of $\mathcal{P}$, such that the sequence of sets $\langle \operatorname{dom} \psi_n \rangle_{n\in\omega}$ is uniformly $\mathbf{a}$-computable. By passing from a partial function $\psi$ to a total extension mapping each $n \notin \operatorname{dom}\psi$ to some designated marker (say $-1$), we see that we can reduce the problem of computing a family of partial functions to the problem of enumerating a family of partial functions. Hence:

**Proposition 2.2.** *The collection of degrees which can compute a countable family of partial functions $\mathcal{P}$ is a degree spectrum; so is the collection of degrees which, uniformly in $n$, can compute a countable family $\mathcal{P}_n$ of partial functions.*

## 3. Array non-recursive degrees

Recall [DJS96] that a Turing degree $\mathbf{a}$ is *array non-recursive* if for every function $f \leqslant_{\mathrm{wtt}} \varnothing'$ there is some function $g \in \mathbf{a}$ which is not dominated by $f$. The infinite collection of functions (those weak truth-table reducible to $\varnothing'$) can be replaced by a single function: let $\mu = m_{\varnothing'}$ be the modulus function for $\varnothing'$. Then a degree is array non-recursive if and only if it contains some function which is not dominated by $\mu$. As mentioned above, the array non-recursive degrees have measure 0 but are co-meagre.

**Theorem 3.1.** *The array non-recursive degrees form a degree spectrum.*

*Proof.* Let

$$\mathcal{P} = \{\varphi \mid \varphi \text{ is a finite partial function and } (\exists m \in \operatorname{dom}\varphi)[\varphi(m) > \mu(m)]\}.$$

To prove Theorem 3.1, we show that a Turing degree is array non-recursive if and only if it can compute $\mathcal{P}$, and then we appeal to Proposition 2.2.

($\Longleftarrow$) Suppose that $\langle \psi_n \rangle_{n\in\omega}$ is a listing of $\mathcal{P}$ which witnesses that a degree $\mathbf{a}$ computes $\mathcal{P}$; so $\langle \operatorname{dom}\psi_n \rangle$ are uniformly computable from $\mathbf{a}$. We construct a function $f$ computable from $\mathbf{a}$ which is not dominated by the modulus function $\mu$.

The idea is to string together functions $g_0, g_1, \dots$ from $\mathcal{P}$ whose domains are disjoint; certainly this produces a function which is not dominated by $\mu$. For example, we can let $g_0 = \psi_0$ and inductively let $g_{i+1}$ to be $\psi_n$ for some $n$ such that $\min\operatorname{dom}\psi_n > \max\operatorname{dom} g_i$. Since $\mathbf{a}$ does not have access to a canonical finite index for the functions $\psi_n$, such a sequence cannot be computed from $\mathbf{a}$; but it can be approximated, in a process of finite injury.

*The sequence $\langle g_j \rangle$.* For any stage $s$, we enumerate a finite sequence $g_{0,s}, \dots, g_{s,s}$, each $g_{j,s}$ designated by an index $n = n_{j,s}$ such that $g_{j,s} = \psi_n$. At stage $s$, the "observable universe" consists of the natural numbers up to $s$. Restricted to this observable universe, we require that $\max\operatorname{dom} g_{j,s} < \min\operatorname{dom} g_{j+1,s}$ for all $j \leqslant s$. Formally, this means that we require that for all $j = 1, \dots, s$,

$$\otimes_{j,s}: \qquad \max\left(\operatorname{dom} g_{j-1,s} \cap \{0, \dots, s\}\right) < \min\operatorname{dom} g_{j,s};$$

Note that if $\min\operatorname{dom} g_{j,s} > s$ then the condition $\otimes_{j,s}$ holds.

At stage 0, we start by letting $g_{0,0} = \psi_0$. At stage $s > 0$ we are given $g_{0,s-1}, \dots, g_{s-1,s-1}$. We let $j_s$ be the least $j \in \{1, \dots, s-1\}$ such that $\otimes_{j,s}$ would fail if we didn't change the functions $g_j$ and $g_{j-1}$. Equivalently, it is the least $j$ such that $s \in \operatorname{dom} g_{j-1,s-1}$. If there is no such $j$, we let $j_s = s$. We then initialise at $j_s$; this means that for $j < j_s$ we let $g_{j,s} = g_{j,s-1}$, but for all $j \in \{j_s, \dots, s\}$ we let $g_{j,s}$ be some function $\psi_n$ such that $\min\operatorname{dom}\psi_n > s$.

This concludes the construction. To obtain the required sequence $\langle g_j \rangle$, we show that injury is finite: for all $i$, for almost all $s$, we have $j_s > i$; we let $s_i$ be the greatest stage $s \geqslant i$ such that $j_s \leqslant i$ (and then we let $g_i = g_{i,s_i} = g_{i,s}$ for all $s \geqslant s_i$ be the "limit function"). That $s_i$ exists for all $i$ is proved by induction on $i \geqslant 0$. We have $s_0 = 0$. Certainly for all $s$ we have $j_s > 0$. For $i \geqslant 0$, assuming that $s_i$ exists, and so $g_i$ is defined. At stage $s_i$ the permanent value of $g_i$ is chosen, with $\min \operatorname{dom} g_i > s_i$. For $s > s_i$, we have $j_s = i+1$ if and only if $s \in \operatorname{dom} g_i$. Hence $s_{i+1}$ exists and equals $\max \operatorname{dom} g_i$.

As we choose $\min \operatorname{dom} g_{i+1} > s_{i+1}$ at stage $s_{i+1}$, we see that indeed $\max \operatorname{dom} g_i < \min \operatorname{dom} g_{i+1}$ for all $i$.

*The function $f$.* We now implement the original plan. We show that there is an **a**-computable function $f$ such that for all $i$, for all $n \in \operatorname{dom} g_i$, we have $f(n) = g_i(n)$. Since the domain of each $g_i$ contains some $n$ such that $g_i(n) > \mu(n)$, we see that $f$ will not be dominated by $\mu$.

For each $s$, we note that there is at most one $i \leqslant s$ such that $s \in \operatorname{dom} g_{i,s}$. If there is such $i$, we let $f(s) = g_{i,s}(s)$; otherwise, we let $f(s) = 0$. The function $f$ is certainly **a**-computable. Let $i \in \omega$ and $s \in \operatorname{dom} g_i$. We observed above that $s > s_i$, and so $g_{i,s} = g_i$, which shows that $f(s)$ is chosen to be $g_i(s)$ at stage $s$.

($\Longrightarrow$) We need to show that if $f$ is a function which is not dominated by $\mu$, then $f$ can compute $\mathcal{P}$. As is often the case, we show that there is an $f$-computable procedure which takes a finite partial function $\psi$ (given by a canonical finite index) and produces an $f$-computation of a function $\psi^* \in \mathcal{P}$. That is, we produce an $f$-partial computable index for $\psi^*$ and an $f$-computable index for $\operatorname{dom} \psi^*$. We ensure that if $\psi \in \mathcal{P}$ then $\psi^* = \psi$. Thus the sequence $\langle \psi^* \rangle$, as $\psi$ ranges over all finite partial functions, witnesses that $f$ computes the family $\mathcal{P}$.

So we are given a finite partial function $\psi$. The idea is to extend $\psi$ to $\psi^*$ gradually; whenever we discover that $\psi^* \restriction_s$, the function we have so far, is dominated by $\mu$, we add a new element to the domain of $\psi^*(s)$ by copying $f$. The problem is that $f(s) > \mu(s)$ only for infinitely many $s$, not for all of them, and so copying only infinitely many values of $f$ does not by itself guarantee escaping domination from $\mu$. On the other hand, because we need to compute $\operatorname{dom} \psi^*$, we cannot define $\psi^*(s)$ much later than stage $s$, while it may take a long time to discover that $\psi^* \restriction_s$ is dominated by $\mu$. The solution is to use the fact that $f$ escapes domination by $\mu$ another time, this time when giving a bound for our search for domination by $\mu$.

Let $\langle \mu_s \rangle$ be the standard computable approximation for $\mu$: $\mu_s$ is the modulus function for $\varnothing'_s$. The property of $\langle \mu_s \rangle$ that we use (apart from the fact that it is increasing with time) is that for all $n$, for all $s \geqslant \mu(n)$, we have $\mu_s \restriction_{n+1} = \mu \restriction_{n+1}$.

We define $\psi^*$ by first letting $\psi^*$ agree with $\psi$ on all inputs $s \leqslant \max \operatorname{dom} \psi$. At stage $s > \max \operatorname{dom} \psi$, we have determined $\psi^* \restriction_s$, and we need to define $\psi^*(s)$. If there is some $n \in \operatorname{dom} \psi^* \restriction_s$ with $\psi^*(n) > \mu_{f(s)}(n)$, we decide that $s \notin \operatorname{dom} \psi^*$. Otherwise, we let $\psi^*(s) = f(s)$.

This defines $\psi^*$; we need to show that $\psi^* \in \mathcal{P}$ and that $\psi^* = \psi$ if $\psi \in \mathcal{P}$. The latter is immediate; if $\psi(n) > \mu(n)$ for some $n$, then for all $s > \max \operatorname{dom} \psi$ we have $\psi^*(n) > \mu_{f(s)}(n)$ and so we decide that $s \notin \operatorname{dom} \psi^*$ for all $s \notin \operatorname{dom} \psi$. For the former, let $t$ be the least $s > \max \operatorname{dom} \psi$ such that $f(t) > \mu(t)$. At stage $t$ we act so that there is some $n \in \operatorname{dom} \psi^* \restriction_{t+1}$ such that $\psi^*(n) > \mu_{f(t)}(n)$. Since

$f(t) > \mu(t)$, and since we may assume that $f$ is increasing, for all $s > t$ we have $\psi^*(n) > \mu_{f(s)}(n) = \mu(n)$. Thus, $\operatorname{dom}\psi^* \subseteq \{0, \ldots, t\}$ and $\psi^*(n) > \mu(n)$, so $\psi^* \in \mathcal{P}$. □

## 4. Escaping domination by a countable collection of functions

During the proof of Theorem 3.1 we isolated the properties of the modulus function $\mu$ which allowed the construction to succeed.

**Definition 4.1.** We let $\mathfrak{S}$ denote the collection of $\Delta^0_2$ functions $h$ which have a computable approximation $\langle h_s \rangle$ with the following properties:

- For all $n$ and $s$, $h_s(n) \leqslant h_{s+1}(n)$;
- If $h_s(n) \neq h_{s-1}(n)$ then $h_s(n) \geqslant s$, indeed, for all $m \geqslant n$ we have $h_s(m) \geqslant s$.

The collection $\mathfrak{S}$ is really the collection of modulus functions of convergent computable approximations. The second condition implies that for all $n$ and all $t \geqslant h(n)$ we have $h_t \restriction_{n+1} = h \restriction_{n+1}$.

For any function $h$, let $\mathcal{P}_h$ be the collection of finite partial functions $\psi$ for which there is some $n \in \operatorname{dom}\psi$ such that $\psi(n) > h(n)$. The proof of Theorem 3.1 actually shows:

**Proposition 4.2.** *If $h \in \mathfrak{S}$ then the following are equivalent for a Turing degree* $\mathbf{a}$*:*

(1) $\mathbf{a}$ *contains a function which is not dominated by $h$;*
(2) $\mathbf{a}$ *computes $\mathcal{P}_h$.*

And so the class of degrees which compute a function not dominated by $h$ is a degree spectrum.

Proposition 4.2 can be extended to uniform subfamilies of $\mathfrak{S}$. We say that a sequence of functions $\langle h_n \rangle$ is *uniformly in $\mathfrak{S}$* if there are uniformly computable approximations $\langle h_{n,s} \rangle_{s\in\omega}$, each approximation $\langle h_{n,s} \rangle$ witnessing that $h_n \in \mathfrak{S}$.

**Proposition 4.3.** *Suppose that $\langle h_n \rangle$ is a sequence of functions, uniformly in $\mathfrak{S}$. The following are equivalent for a Turing degree* $\mathbf{a}$*:*

(1) $\mathbf{a}$ *contains a function which is not dominated by any $h_n$;*
(2) *Uniformly in $n$,* $\mathbf{a}$ *computes $\mathcal{P}_{h_n}$.*

And so, Proposition 2.2 guarantees that the collection of degrees computing a function not dominated by any $h_n$ forms a degree spectrum.

*Proof.* (1)$\Longrightarrow$(2): This is just the observation that the second part of the proof of Theorem 3.1 shows that uniformly from an approximation $\langle h_s \rangle$ which witnesses that $h \in \mathfrak{S}$ and from a function $f$ not dominated by $h$ we can compute $\mathcal{P}_h$.

(2)$\Longrightarrow$(1): This is a simple adaptation of the first part of the proof of Theorem 3.1, to working with a list of functions. The only change is that for all $s$ and $j \leqslant s$, if $j$ codes a pair $(k, m)$ then we choose $g_{j,s} \in \mathcal{P}_{h_k}$. □

The restriction to functions in $\mathfrak{S}$ is not essential. This is because we can always pass from a $\Delta^0_2$ function to its modulus.

**Lemma 4.4.** *There is a computable procedure $\theta$ which, given (a computable index for) a computable approximation $\langle h_s \rangle$ for a $\Delta^0_2$ function $h$, outputs a computable approximation for a function in $\mathfrak{S}$ which dominates $h$.*

Slightly abusively, we write $\theta(h)$ for the function in $\mathfrak{S}$ whose approximation is given by $\theta$.

*Proof.* The function $\theta(h)$ is the maximum of $h$ with its modulus. We construct an approximation $\langle g_s \rangle$ for $\theta(h)$ obeying the conditions for $\theta(h) \in \mathfrak{S}$; at stage $s$, if $h_s(n) \neq h_{s-1}(n)$ (with $n$ least such) then for all $m \geqslant n$ we set $h_s(m) = \max(\{s\} \cup \{h_t(m) : t \leqslant s\})$; for $m < n$ we set $g_s(m) = g_{s-1}(m)$. □

We say that a collection $\mathcal{S}$ of $\Delta^0_2$ function is *effectively listable* if there is a uniformly computable sequence of computable approximations $\langle h_{n,s} \rangle_{s<\omega}$ such that denoting the limit of $\langle h_{n,s} \rangle$ by $h_n$, we have $\{h_n : n \in \omega\} = \mathcal{S}$. Lemma 4.4 and Proposition 4.3 together imply:

**Proposition 4.5.** *Suppose that $\mathcal{S}$ is an effectively listable collection of $\Delta^0_2$ functions which is closed under applying the operation $\theta$. Then the collection of degrees* **a** *which compute a function not dominated by any function in $\mathcal{S}$ is a degree spectrum.*

The family of examples we have in mind are the $\alpha$-computably approximable functions for sufficiently closed ordinals $\alpha$. If $\alpha$ is a (notation for a) computable ordinal, then an *$\alpha$-computable approximation* is a computable approximation $\langle f_s \rangle$ for a function $f$ which is coupled with a witness $\langle o_s \rangle$ for the approximation settling down: the sequence $\langle o_s \rangle$ is a uniformly computable sequence of functions from $\omega$ to $\alpha$ such that for all $n$ and $s$,

- $o_{s+1}(n) \leqslant o_s(n)$; and
- if $f_{s+1}(n) \neq f_s(n)$ then $o_{s+1}(n) < o_s(n)$.

A $\Delta^0_2$ function is *$\alpha$-computably approximable* (abbreviated *$\alpha$-c.a.*) if it has a computable approximation which is an $\alpha$-computable approximation. Most prominent is the case $\alpha = \omega$; a function $f$ is $\omega$-c.a. if and only if $f \leqslant_{\mathrm{wtt}} \varnothing'$.[2]

It is not difficult to see that for any ordinal $\alpha$, the collection of all $\alpha$-c.a. functions is effectively listable; see [DG, Prop.I.2.7]. The second observation is counting the changes in the approximation $\langle g_s \rangle$ for $\theta(h)$ compared with the approximation $\langle h_s \rangle$ for $h$. We note that if $g_{s+1}(m) \neq g_s(m)$ then $h_{s+1}(n) \neq h_s(n)$ for some $n \leqslant m$. This shows that if $h$ is $\alpha$-c.a. and $\alpha$ is closed under addition, then the class of $\alpha$-c.a. functions is closed under the operation $\theta$. This uses the operation of commutative addition [DG, Lem.I.2.42], and assumes that $\alpha$ is given by a strong notation [DG, Sec.I.2]. The ordinals which are closed under addition are the nonzero powers of $\omega$. Hence:

**Theorem 4.6.** *For any nonzero ordinal $\alpha \leqslant \epsilon_0$, the class of degrees which compute a function not dominated by any $\omega^\alpha$-c.a. function is a degree spectrum.*

Call a degree which computes a function not dominated by any $\alpha$-c.a. function *$\alpha$-c.a.-non-dominated*. The interest in the notion of $\omega^\alpha$-c.a.-domination stems from the fact that it seems to be the correct generalisation of the notion, restricted to the c.e. degrees, of being totally $\omega^\alpha$-c.a. This notion is investigated in [DG, DGW07, BDG10], where it is shown, for example, that a c.e. degree **a** is totally $\omega$-c.a. if and only if it does not bounds a critical triple, if and only if every set in **a** is wtt-reducible to a proper initial segment of a scattered computable linear ordering.

[2]This can be iterated; a function is $\omega^\alpha$-c.a. if and only if it is weak truth-table reducible to the $\alpha^{\text{th}}$ iteration of the function wtt-jump. See [DG, Thm.I.2.37].

**Lemma 4.7.** *Let $\alpha \leqslant \epsilon_0$ be nonzero. A c.e. degree* $\mathbf{a}$ *is $\omega^\alpha$-c.a.-dominated if and only if it is totally $\omega^\alpha$-c.a.*

*Proof.* The point is that a c.e. degree can compute a modulus for any function it computes. So if $\langle h_s \rangle$ is an approximation of $h$ which is given by a computation from some c.e. set $A$ ($h_s = \Gamma(A)[s]$) then $A$ can compute $\theta(h)$.

If a function $f$ dominates $\theta(h)$, then $h \leqslant_{\mathrm{wtt}} f$. By [DG, Thm.I.2.37], if $f$ is $\omega^\alpha$-c.a. then so is $h$. □

We get the following:

**Theorem 4.8.** *Let $\alpha \leqslant \epsilon_0$ be nonzero. There is a countable structure $\mathcal{M}$ such that a c.e. degree* $\mathbf{a}$ *computes a copy of $\mathcal{M}$ if and only if it is not totally $\omega^\alpha$-c.a.*

## 5. Jump traceable degrees

Among the by-now-familiar notions of traceability, such as c.e. traceability and computable traceability, jump traceability is particularly interesting due to its interaction with classes such as the $K$-trivial degrees [BDG09, HKM09]. What enables a fine look at this class is the fact that unlike other notions of traceability, the growth-rate of the order function bounding the traces matters a great deal.

Recall that an order function is a non-decreasing and unbounded computable function $h\colon \omega \to \omega\backslash\{0\}$, and that if $h$ is an order function, then an $h$-trace is a uniformly c.e. sequence $\langle T_x \rangle$ such that for all $x$, $|T_x| \leqslant h(x)$. Also recall that for a *partial* function $\varphi$, we say that a trace $\langle T_x \rangle$ is a trace for $\varphi$ if for all $x \in \operatorname{dom}\varphi$ we have $\varphi(x) \in T_x$.

To deal with an arbitrary degree which is not jump traceable, we need to work uniformly relative to an order function, and this is why we need to consider partial order functions.

**Definition 5.1.** A *partial order function* is a partial computable function $h\colon \omega \to \omega\backslash\{0\}$ whose domain is an initial segment of $\omega$, which is non-decreasing on its domain, and is unbounded if it is total.

If $h$ is a partial order function, then an $h$-trace is a uniformly c.e. sequence $\langle T_x \rangle$ such that for all $x \in \operatorname{dom} h$, $|T_x| \leqslant h(n)$, and for all $x \notin \operatorname{dom} h$, $T_x$ is empty.

It is not difficult to construct a computable list $\langle h_n \rangle$ of partial order functions which contains all order functions. To see this, start with an effective list $\langle g_n \rangle$ of all partial computable functions, and note that the collection of $n$ for which $g_n$ is a total order function is $\Pi^0_2$. Let $R$ be a computable predicate such that $g_n$ is a total order function if and only if $\forall y \exists z\, R(n,y,z)$. Define $h_n$ by letting, at stage $s$, $h_n(x) = g_n(x)$, if $h_n(x-1)$ has already been defined (in the case that $x > 0$) and further, for all $y \leqslant x$ there is some $z \leqslant s$ such that $R(n,y,z)$ holds.

Uniformly in $n$ we can list all $\sqrt{h_n}$-traces $\langle T^{0,n}_x \rangle, \langle T^{1,n}_x \rangle, \ldots$. Letting $V^n_x = \bigcup_{e \leqslant \sqrt{h_n(x)}} T^{e,n}_x$ we obtain, uniformly in $n$, a "universal" $h_n$-trace $\langle V^n_x \rangle_{x \in \omega}$. It is universal in the sense that if $h_n$ is total and a partial function $\psi$ has a c.e. $\sqrt{h_n}$-trace then for almost all $z \in \operatorname{dom}\psi$ we have $\psi(z) \in V^n_z$. Also, note that if $h$ is an order function then so is $h^2$, and so the collection of total functions among the functions $\sqrt{h_n}$ coincides with the collection of all order functions.

Recall that an oracle $A$ is *jump traceable* if there is some order function $h$ such that every $A$-partial computable $\varphi$ has an $h$-trace. Instead of looking at all $A$-partial computable functions, it suffices to require a computably-bounded trace for $J^A$, the universal $A$-partial computable function.

Our listing and universality of the traces $\langle V^n_x \rangle$ give the following:

- A set $A$ is jump traceable if there is some $n$ such that for every $A$-partial-computable function $\varphi$, for all but finitely many elements $x$ of $\operatorname{dom}\varphi$, $x \in V^n_x$ (for such $n$, $h_n$ of course must be total, and so an order function).
- A set $A$ is jump traceable if and only if there is some $n$ such that for all but finitely many elements $x$ of $\operatorname{dom} J^A$, $J^A(x) \in V^n_x$.

**Theorem 5.2.** *The collection of Turing degrees which are not jump traceable forms a degree spectrum.*

*Proof.* For $n \in \omega$, let $\mathcal{P}_n$ be the collection of finite partial functions $\psi$ such that for some $x \in \operatorname{dom}\psi$, $\psi(x) \notin V^n_x$; and let $\mathcal{P}_{n,e}$ be the collection of functions $\psi \in \mathcal{P}_n$ such that $\operatorname{dom}\psi \subset \omega^{[e]}$ (the $e^{\text{th}}$ column of $\omega$). Note that the fact that each $V^n_x$ is finite implies that for all $e$, $\mathcal{P}_{e,n}$ is infinite.

We show that a degree $\mathbf{a}$ is not jump traceable if and only if uniformly in $n$ and $e$, $\mathbf{a}$ can effectively *list* $\mathcal{P}_{n,e}$. The theorem then follows from Proposition 2.1.

($\Longleftarrow$) Suppose that uniformly in $n$ and $e$, $\langle \psi^{n,e}_k \rangle_{k\in\omega}$ is a $\mathbf{a}$-effective listing of $\mathcal{P}_{n,e}$. For each $n$ such that $h_n$ is an order function, we wish to enumerate a partial function $f$ such that for infinitely many $x \in \operatorname{dom} f$, $f(x) \notin V^n_x$. This would show that no order function can witness that $\mathbf{a}$ is jump traceable.

The construction is easy. For each $e$, let $g_e = \psi^{n,e}_0$ be the first function enumerated in $\mathcal{P}_{n,e}$, and let $f = \bigcup_e g_e$, which is well-defined because the functions $g_e$ have pairwise disjoint domains. The function $f$ is as required.

($\Longrightarrow$) Suppose that a set $A$ is not jump traceable. Uniformly in $n$ and $e$ we obtain an $A$-effective list of $\mathcal{P}_{n,e}$ by again, using the oracle $A$ to extend any given finite partial function $\psi$ whose domain is contained in $\omega^{[e]}$ to a function $\psi^* \in \mathcal{P}_{n,e}$, making sure that if $\psi \in \mathcal{P}_{n,e}$ then $\psi^* = \psi$. This time around, we only need to enumerate the graph of $\psi^*$, and we do not need to compute its domain.

The point is that uniformly in $e$, $A$ can enumerate the graph of a partial function $f_e\colon \omega^{[e]} \to \omega$ such that for all $n$, there are infinitely many $x \in \operatorname{dom} f_e$ such that $f_e(x) \notin V^n_x$. We use the universal function $J^A$ and copy it: we simply let $f_e(e,x) = J^A(x)$. If for all but finitely many $x \in \operatorname{dom} f_e$ we have $f_e(e,x) \in V^n_{(e,x)}$ then we can convert $\langle V^n_x \rangle$ to give a c.e. trace for $J^A$ bounded by the order function $x \mapsto h_n(e,x)$, contradicting the assumption that $A$ is not jump traceable.

Fix $n$, $e$ and a finite partial function $\psi$ (given by a canonical finite index), whose domain is contained in $\omega^{[e]}$. We define the function $\psi^*$ by enumerating its graph; at stage $s$ we have enumerated the graph of a function $\psi^*_s$. We start with $\psi^*_0 = \psi$. At stage $s$, if we see that for all $x \in \operatorname{dom}\psi^*_s$ we have $\psi^*(x) \in V^n_x$, and we see an input $x \in \operatorname{dom} f_e \backslash \operatorname{dom}\psi^*_s$, then we set $\psi^*_{s+1}(x) = f_e(x)$ for the least such $x$. Otherwise, we let $\psi^*_{s+1} = \psi^*_s$. It is now easy to verify that if $\psi \in \mathcal{P}_{n,e}$ then $\psi^* = \psi$, and that for any $\psi$, $\psi^* \in \mathcal{P}_{n,e}$, as required. ∎

We end with a question.

**Question 5.3.** *Does the collection of degrees which are not c.e. traceable form a degree spectrum? What about the degrees which are not strongly jump traceable?*

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