COMPUTABILITY-THEORETIC CATEGORICITY AND SCOTT FAMILIES

EKATERINA FOKINA, VALENTINA HARIZANOV, AND DANIEL TURETSKY

Abstract. Computability-theoretic investigation of complexity of isomorphisms between countable structures is a key topic in computable model theory since Fröhlich and Shepherdson, Mal'cev, and Metakides and Nerode. A computable structure $\mathcal{A}$ is called Δ_n^0-categorical, for $n \geq 1$, if for every computable isomorphic $\mathcal{B}$ there is a Δ_n^0 isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. More generally, $\mathcal{A}$ is relatively Δ_n^0-categorical if for every isomorphic $\mathcal{B}$ there is an isomorphism that is Δ_n^0 relative to the atomic diagram of $\mathcal{B}$. Equivalently, $\mathcal{A}$ is relatively Δ_n^0-categorical if and only if $\mathcal{A}$ has a computably enumerable Scott family of computable (infinitary) Σ_n formulas. Relative Δ_n^0-categoricity implies Δ_n^0-categoricity, but not *vice versa*.

In this paper, we present an example of a computable Fraïssé limit that is computably categorical (that is, Δ_1^0-categorical) but not relatively computably categorical. We also present examples of Δ_2^0-categorical but not relatively Δ_2^0-categorical structures in natural classes such as trees of finite and infinite heights, abelian p-groups, and homogenous completely decomposable abelian groups. It is known that for structures from these classes computable categoricity and relative computable categoricity coincide.

By relativizing the notion of a computable categoricity to a Turing degree $\mathbf{d}$, we obtain a notion of $\mathbf{d}$-computable categoricity. The categoricity spectrum of a computable structure $\mathcal{M}$ is the set of all Turing degrees $\mathbf{d}$ such that $\mathcal{M}$ is $\mathbf{d}$-computably categorical. The degree of categoricity of $\mathcal{M}$ is the least degree in the categoricity spectrum of $\mathcal{M}$, if such a degree exists. Here we compute degrees of categoricity for relatively Δ_3^0-categorical Boolean algebras and relatively Δ_2^0-categorical abelian p-groups.

1. Introduction and preliminaries

In computable model theory we use the tools and techniques of computability theory to investigate algorithmic content of notions and constructions in classical mathematics. We consider only countable structures for computable languages, which are often finite. Such an infinite structure $\mathcal{A}$ is *computable* if its universe can be identified with the set ω of natural numbers in such a way that the relations and operations of $\mathcal{A}$ are uniformly computable. A finite structure is always computable. A structure $\mathcal{A}$ is called *n-decidable*, for $n \geq 1$, if the Σ_n-diagram of $\mathcal{A}$ is decidable. Computable categoricity is one of the main topics in computable model theory. It dates back to Fröhlich and Shepherdson [22] who produced examples of computable fields that are not computably isomorphic. A computable structure $\mathcal{A}$ is called *computably categorical* if for every computable structure $\mathcal{B}$ isomorphic to

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$\mathcal{A}$, there exists a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. For example, Ershov [20] established that a computable algebraically closed field is computably categorical if and only if it has a finite transcendence degree over its prime subfield. Miller and Schoutens [47] recently constructed a computably categorical field of infinite transcendence degree over the field of rational numbers.

The notion of computable categoricity can be extended to higher level of hyperarithmetic hierarchy. Let α be a computable ordinal. A computable structure $\mathcal{A}$ is Δ^0_α-*categorical* if for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists a Δ^0_α isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. More generally, a computable structure $\mathcal{A}$ is *relatively* Δ^0_α-*categorical* if for every $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, which is Δ^0_α relative to the atomic diagram of $\mathcal{B}$. Clearly, a relatively Δ^0_α-categorical structure is Δ^0_α-categorical. The converse is not always true.

Relative Δ^0_α-categoricity has a syntactic characterization that involves the existence of certain Scott families of computable formulas. Roughly speaking, computable formulas are infinitary formulas with disjunctions and conjunctions over computable enumerable (c.e.) sets. A *Scott family* for a structure $\mathcal{A}$ is a countable family Φ of $L_{\omega_1\omega}$-formulas with finitely many fixed parameters from A such that:

(i) Each finite tuple in $\mathcal{A}$ satisfies some $\psi \in \Phi$;

(ii) If $\overline{a}$, $\overline{b}$ are tuples in $\mathcal{A}$, of the same length, satisfying the same formulas in Φ, then there is an automorphism of $\mathcal{A}$, which maps $\overline{a}$ to $\overline{b}$.

Ash [3] defined computable Σ_α and Π_α formulas of $L_{\omega_1\omega}$, where α is a computable ordinal, recursively and simultaneously and together with their Gödel numbers. The computable Σ_0 and Π_0 formulas are the finitary quantifier-free formulas. The computable $\Sigma_{\alpha+1}$ formulas are of the form

$$\bigvee_{n\in W_e} \exists \overline{y}_n \psi_n(\overline{x}, \overline{y}_n),$$

where for $n \in W_e$, ψ_n is a Π_α formula indexed by its Gödel number n, and $\exists\overline{y}_n$ is a finite block of existential quantifiers. Similarly, $\Pi_{\alpha+1}$ formulas are c.e. conjunctions of $\forall\Sigma_\alpha$ formulas. If α is a limit ordinal, then Σ_α (Π_α, respectively) formulas are of the form $\bigvee\limits_{n\in W_e} \psi_n$ ($\bigwedge\limits_{n\in W_e} \psi_n$, respectively), such that there is a sequence $(\alpha_n)_{n\in W_e}$ of ordinals less than α, given by the ordinal notation for α, and every ψ_n is a Σ_{α_n} (Π_{α_n}, respectively) formula. For a more precise definition see [3].

A *formally* Σ^0_α *Scott family* is a Σ^0_α Scott family consisting of computable Σ_α formulas. It follows that a formally c.e. Scott family is also a c.e. Scott family of finitary existential formulas.

The following equivalence (i)–(ii)–(iii) for a computable structure $\mathcal{A}$ was established by Goncharov [26] for $\alpha = 1$, and by Ash, Knight, Manasse, and Slaman [4] and independently by Chisholm [11] for any computable ordinal α:

(i) The structure $\mathcal{A}$ is relatively Δ^0_α-categorical.

(ii) The structure $\mathcal{A}$ has a formally Σ^0_α Scott family.

(iii) The structure $\mathcal{A}$ has a c.e. Scott family consisting of computable Σ_α formulas.

Infinitary language is essential for Scott families. Cholak, Shore, and Solomon [14] proved the existence of a computably categorical rigid graph that does not have a Scott family of finitary formulas. It follows that this structure is not relatively computably categorical.

Goncharov [25] was the first to show that computable categoricity of a computable structure does not imply relative computable categoricity. The result of Goncharov was lifted to higher levels in the hyperarithmetic hierarchy by Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon for successor ordinals [28], and by Chisholm, Fokina, Goncharov, Harizanov, Knight, and Quinn for limit ordinals [12]. Hence, for every computable ordinal α, there is a Δ^0_α-categorical but not relatively Δ^0_α-categorical structure. If follows from results by Hirschfeldt, Khoussainov, Shore, and Slinko in [33] that there are (computable) computably categorical but not relatively computably categorical structures in the following classes: partial orders, lattices, 2-step nilpotent groups, commutative semigroups, and integral domains of arbitrary characteristic. Recently, Hirschfeldt, Kramer, R. Miller, and Shlapentokh [31] showed that there is a computably categorical algebraic field, which is not relatively computably categorical.

Cholak, Goncharov, Khoussainov, and Shore [13] showed that there is a computable structure, which is computably categorical, but ceases to be after naming any element of the structure. Clearly, this structure is not relatively computably categorical. Khoussainov and Shore [37] proved that there is a computably categorical structure $\mathcal{A}$, which is not relatively computably categorical, but the expansion of $\mathcal{A}$ by any finite number of constants is computably categorical. Previously, T. Millar [43] showed that if a computably categorical structure $\mathcal{A}$ is 1-decidable, then any expansion of $\mathcal{A}$ by finitely many constants remains computably categorical.

Goncharov's graph in [25], which is computably categorical but not relatively computably categorical, is rigid, and hence computably stable but not relatively computably stable. A structure $\mathcal{A}$ is Δ^0_α*-stable* if for every computable copy $\mathcal{B}$ of $\mathcal{A}$, all isomorphisms from $\mathcal{A}$ onto $\mathcal{B}$ are Δ^0_α. Similarly, we define relatively Δ^0_α-stable structures. A *defining family* for a structure $\mathcal{A}$ is a set Φ of $\mathcal{L}_{\omega_1\omega}$ formulas with one free variable and a fixed finite tuple of parameters from $\mathcal{A}$ such that:

(i) Every element of $\mathcal{A}$ satisfies some formula $\psi \in \Phi$;

(ii) No formula of Φ is satisfied by more than one element of $\mathcal{A}$.

The existence of a defining family is equivalent to rigidity relative to a finite set of parameters. A countable structure is rigid if and only if it has a defining family with no parameters. A computable structure $\mathcal{A}$ is relatively Δ^0_α-stable if and only if it has a formally Σ^0_α defining family.

Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky [16] proved that for every computable ordinal α, there is a computably categorical structure, which is not relatively Δ^0_α-categorical. In fact, it follows from their construction that the structure is rigid. Thus, they answered positively the following question from [28, 12]: For a computable ordinal $\alpha > 1$, is there a computable structure $\mathcal{A}$ that is Δ^0_α-stable but not relatively Δ^0_α-stable? On the other hand, a natural open question arising from [16] is whether there is a computably categorical structure that is not relatively hyperarithmetically categorical.

Ash [2] proved that a computable structure $\mathcal{A}$ is Δ^1_1-categorical if and only if $\mathcal{A}$ is Δ^0_α-categorical for some computable ordinal α. It is not known whether every computable Δ^1_1-categorical structure is relatively Δ^1_1-categorical. A similar question has been resolved for relations on structures – intrinsically Δ^1_1 and relatively intrinsically Δ^1_1 relations are the same (see [29]). Namely, it follows from a result by Soskov [51] that for a computable structure $\mathcal{A}$ and a relation R on $\mathcal{A}$, if R is invariant under automorphisms of $\mathcal{A}$, and Δ^1_1, then R is definable in $\mathcal{A}$ by a computable infinitary formula with no parameters. This is used to establish that if R is intrinsically Δ^1_1 on $\mathcal{A}$, then R is relatively intrinsically Δ^1_1 on $\mathcal{A}$.

The authors of [16] also proved that the index set of computable categorical structures is Π^1_1-complete. Hence computable categoricity has no simple syntactic characterization. On the other hand, the index set of relatively computably categorical structures is Σ^0_3-complete (see [16]).

An injection structure is a structure (A, f) where $f : A \to A$ is a $1-1$ function. For a linear order [27, 48], a Boolean algebra [27, 49], a tree of finite height [40], an abelian p-group [24, 50, 7], an equivalence structure [9], an injection structure [10], and an algebraic field with a splitting algorithm [46], computable categoricity coincides with relative computable categoricity.

For an injection structure $\mathcal{A} = (A, f)$ and $a \in A$, we define the orbit of a:

$$\mathcal{O}_f(a) = \{b \in A : (\exists n \in \omega)[f^n(a) = b \vee f^n(b) = a]\}.$$

Cenzer, Harizanov, and Remmel [10] established that a computable injection structure is Δ^0_2-categorical if and only if it has finitely many orbits of type ω or finitely many orbits of type $\mathbb{Z}$. They showed that every Δ^0_2-categorical injection structure is relatively Δ^0_2-categorical. It is not hard to see that every computable injection structure is relatively Δ^0_3-categorical.

Calvert, Cenzer, Harizanov, and Morozov [9] proved that a computable equivalence structure is relatively Δ^0_2-categorical if and only if it either has finitely many infinite equivalence classes, or there is an upper bound on the size of its finite equivalence classes. They also have partial results towards characterizing Δ^0_2-categoricity. First we need some definitions. A function $f : \omega^2 \to \omega$ is a Khisamiev *s-function* if for every i and s, $f(i, s) \leq f(i, s + 1)$, and the limit $m_i = lim_t f(i, t)$ exists. If, in addition, $m_i < m_{i+1}$ for every i, then we say that f is a Khisamiev s_1*-function.* If an equivalence structure $\mathcal{A}$ has no upper bound on the size of the finite equivalence classes, then Khisamiev s_1-function for $\mathcal{A}$ is such that $\mathcal{A}$ contains an equivalence class of size m_i for every i. If an equivalence structure $\mathcal{A}$ has infinitely many infinite equivalence classes, no upper bound on the size of its finite equivalence classes, and has a computable Khisamiev s_1-function, then $\mathcal{A}$ is not Δ^0_2-categorical (see [9]). Kach and Turetsky [35] showed that there exists a Δ^0_2-categorical equivalence structure $\mathcal{M}$, which is not relatively Δ^0_2-categorical. Their equivalence structure $\mathcal{M}$ has infinitely many infinite equivalence classes and unbounded character, but has no computable Khisamiev's s_1-function, and has only finitely many equivalence classes of size k for any finite k. Every computable equivalence structure is relatively Δ^0_3-categorical.

Goncharov and Dzgoev [27], and independently Remmel [48] proved that a computable linear order is computably categorical (also, relatively computably categorical) if and only if it has only finitely many adjacencies (successor pairs). In [41], McCoy characterized relatively Δ^0_2-categorical linear orders as follows. By ω^* we denote the reverse order of ω, and by η the order type of rationals. A computable linear order is relatively Δ^0_2-categorical if and only if it is a sum of finitely many intervals, each of type m, ω, ω^*, $\mathbb{Z}$ or $n \cdot \eta$, so that each interval of type $n \cdot \eta$ has a supremum and an infimum. McCoy [41] also characterized, after adding certain extra predicates, Δ^0_2-categorical linear orders. However, it still remains open whether there is a Δ^0_2-categorical linear order, which is not relatively Δ^0_2-categorical. In [42], McCoy proved that there are $2^{\aleph_0}$ relatively Δ^0_3-categorical linear orders.

Goncharov and Dzgoev [27], and independently Remmel [49] established that a computable Boolean algebra is computably categorical (also, relatively computably categorical) if and only if it has finitely many atoms (see also LaRoche [39]). In [41], McCoy characterized computable relatively Δ^0_2-categorical Boolean algebras as those that can be expressed as finite direct sums of subalgebras $\mathcal{C}_0 \oplus \cdots \oplus \mathcal{C}_k$ where each $\mathcal{C}_k$ is either atomless, an atom, or a 1-atom. Using McCoy's characterization, Bazhenov [8] showed that for Boolean algebras the notions of Δ^0_2-categoricity and relative Δ^0_2-categoricity coincide. Harris gave another proof in [30]. In [42], McCoy gave a complete description of relatively Δ^0_3-categorical Boolean algebras.

Fokina, Kalimullin, and R. Miller [21] introduced the following notions trying to capture the set of all Turing degrees capable of computing isomorphisms between computable structures. Let $\mathcal{A}$ be a computable structure. The *categoricity spectrum* of $\mathcal{A}$ is the following set of Turing degrees:

$$CatSpec(\mathcal{A}) = \{\mathbf{x} : \mathcal{A} \text{ is } \mathbf{x}\text{-computably categorical}\}.$$

The *degree of categoricity* of $\mathcal{A}$, if it exists, is the least Turing degree in CatSpec($\mathcal{A}$). If $\mathbf{d}$ is a non-hyperarithmetic degree, then $\mathbf{d}$ cannot be the degree of categoricity of a computable structure. A Turing degree $\mathbf{d}$ is called *categorically definable* if it is the degree of categoricity of some computable structure. Fokina, Kalimullin, and R. Miller [21] investigated which arithmetic degrees are categorically definable. Csima, Franklin, and Shore [15] extended their results to hyperarithmetic degrees. For sets X and Y, we say that Y is *c.e. in and above* (c.e.a. in) X if Y is c.e. relative to X, and $X \leq_T Y$. Csima, Franklin, and Shore [15] proved that for every computable ordinal α, $\mathbf{0}^{(\alpha)}$ is categorically definable. They also established that for a computable successor ordinal α, every degree $\mathbf{d}$ that is c.e.a. in $\mathbf{0}^{(\alpha)}$ is categorically definable. There were also negative results in [21, 15]. Anderson and Csima [1] showed that there exists a Σ^0_2 set the degree of which is not categorically definable. They also showed that no noncomputable hyperimmune-free degree is categorically definable. It is an open question whether all Δ^0_2 degrees are categorically definable.

Not every computable structure has the degree of categoricity. The first negative example was built by R. Miller [44]. Further interesting examples of structures without the degrees of categoricity were built by Fokina, Frolov, and Kalimullin [19]. It is an open question whether there is a computable structure the categoricity spectrum of which is the set of all noncomputable Turing degrees.

In this paper, we present some new examples of structures in natural classes, which are computably categorical but not relatively computably categorical, as well as Δ^0_2-categorical but not relatively Δ^0_2-categorical. In Section 2, we present 1-decidable structure that is a Fraïssé limit, which is computably categorical but not relatively computably categorical. In Section 3, we build computable Δ^0_2-categorical but not relatively Δ^0_2-categorical trees of finite and infinite heights. Here, a tree can be viewed both as a partial order and as a directed graph. In Section 4, we present an abelian p-group that is Δ^0_2-categorical but not relatively Δ^0_2-categorical. In Section 5, we prove that there is a homogenous completely decomposable abelian group, which is Δ^0_2-categorical but not relatively Δ^0_2-categorical. In Section 6, we compute the degrees of categoricity for relatively Δ^0_2-categorical abelian p-groups. This parallels Frolov's work in [23] where he computed degrees of categoricity for relatively Δ^0_2-categorical linear orders. We further compute the degrees of categoricity for relatively Δ^0_3-categorical Boolean algebras. This extends Bazhenov's work in [8] where he computed the degrees of categoricity for relatively Δ^0_2-categorical Boolean algebras.

2. Computably categorical but not relatively computably categorical Fraïssé limits

For a computable ordinal α, the notions of Δ^0_α-categoricity and relative Δ^0_α-categoricity of a computable structure $\mathcal{A}$ coincide if $\mathcal{A}$ satisfies certain extra decidability conditions (see Goncharov [26] and Ash [2]). Goncharov [26] proved that if $\mathcal{A}$ is 2-decidable, then computable categoricity and relative computable categoricity of $\mathcal{A}$ coincide. Kudinov [38] showed that the assumption of 2-decidability cannot be weakened to 1-decidability, by giving an example of 1-decidable and computably categorical structure, which is not relatively computably categorical. On the other hand, Downey, Kach, Lempp, and Turetsky [17] showed that any 1-decidable computably categorical structure is relatively Δ^0_2-categorical.

The proofs by Goncharov and by Downey, Kach, Lempp, and Turetsky use the decidability of the structure to determine if certain finitely generated substructures can be extended to various larger finitely generated substructures. Because of the special properties of a Fraïssé limit, one might expect that all such questions would be trivial to determine, and so the decidability condition could be weakened or dropped entirely for such structures. However, this is not the case. Here, we give an example of 1-decidable and computably categorical Fraïssé limit which is not relatively computably categorical.

Let us recall the definition of a Fraïssé limit (see [34, Chapter 6]). The *age* of a structure $\mathcal{M}$ is the class of all finitely generated structures that can be embedded in $\mathcal{M}$. Fraïssé showed that a (nonempty) finite or countable class $\mathbb{K}$ of finitely generated structures is the age of a finite or a countable structure if and only if $\mathbb{K}$ has the hereditary property and the joint embedding property. A class $\mathbb{K}$ has the *hereditary property* if whenever $\mathcal{C} \in \mathbb{K}$ and $\mathcal{S}$ is a finitely generated substructure of $\mathcal{C}$, then $\mathcal{S}$ is isomorphic to some structure in $\mathbb{K}$. A class $\mathbb{K}$ has the *joint embedding property* if for every $\mathcal{B}, \mathcal{C} \in \mathbb{K}$ there is $\mathcal{D} \in \mathbb{K}$ such that $\mathcal{B}$ and $\mathcal{C}$ embed into $\mathcal{D}$. A structure $\mathcal{U}$ is *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\mathcal{U}$ extends to an automorphism of $\mathcal{U}$.

Definition 1. (see [34, Chapter 6]) A structure $\mathcal{A}$ is a *Fraïssé limit* of a class of finitely generated structures $\mathbb{K}$ if $\mathcal{A}$ is countable, ultrahomogeneous, and has age $\mathbb{K}$.

Fraïssé proved that the Fraïssé limit of a class of finitely generated structures is unique up to isomorphism. We say that a structure $\mathcal{A}$ is a Fraïssé limit if for some class $\mathbb{K}$, $\mathcal{A}$ is the Fraïssé limit of $\mathbb{K}$. First we show that every Fraïssé limit is relatively Δ^0_2-categorical.

Theorem 1. *Let $\mathcal{A}$ be a computable structure which is a Fraïssé limit. Then $\mathcal{A}$ is relatively Δ^0_2-categorical.*

Proof. Because of ultrahomogeneity, we can construct isomorphisms between $\mathcal{A}$ and an isomorphic structure $\mathcal{B}$ using a back-and-forth argument, as long as we can determine, for every $\overline{a} \in \mathcal{A}$ and $\overline{b} \in \mathcal{B}$, whether there is an isomorphism from the structure generated by $\overline{a}$ to the structure generated by $\overline{b}$ that maps $\overline{a}$ to $\overline{b}$ in order. This can be determined by $(\mathcal{B})'$, since there is such an isomorphism precisely if there is no atomic formula ϕ with $\mathcal{A} \models \phi(\overline{a})$ and $\mathcal{B} \not\models \phi(\overline{b})$. This is a Π^0_1 condition relative to $\mathcal{A} \oplus \mathcal{B} \equiv_T \mathcal{B}$.

Therefore, we can use $(\mathcal{B})'$ as an oracle to perform the back-and-forth construction of an isomorphism, and so there is a $\Delta^0_2(\mathcal{B})$ isomorphism. □

Note that if the language of $\mathcal{A}$ is finite and relational, then there are only finitely many atomic formulas ϕ to consider, and the set of such formulas can be effectively determined. Hence, if the language is finite and relational, then a Fraïssé limit is necessarily relatively computably categorical.

Theorem 2. *There is a* 1*-decidable structure* $\mathcal{F}$ *that is a Fraïssé limit and computably categorical, but not relatively computably categorical. Moreover, the language for such* $\mathcal{F}$ *can be finite or relational.*

Proof. The proof is a modification of the first construction in Theorem 3.3 by Downey, Kach, Lempp, and Turetsky [17]; the only new ingredient we add is to make the resulting structure a Fraïssé limit. Instead of repeating the entire construction here, we only explain the modifications we must make.

The original construction was an undirected graph. We assured that the structure is made not relatively computably categorical by creating infinitely many connected components that were all accumulation points in the Σ_1 type-space; this is similar to the technique used in Kudinov's construction in [38]. Then for any potential Scott family of Σ_1 formulas, there must be some accumulation point in a component disjoint from the finitely many parameters of the family. Any Σ_1 formula from the Scott family, which holds of the accumulation point would also need to hold of any other point that is "sufficiently close" in the type space, contradicting the definition of a Scott family.

The original construction created these accumulation points as vertices with loops of various sizes coming out of them. For each accumulation point, there would be a pair of computable sequences $\{n_k\}_{k\in\omega}$ and $\{m_k\}_{k\in\omega}$. For every k, there would be a vertex v_k with attached loops of sizes $n_0, \dots, n_k$ and a loop of size m_k. There would also be a vertex v_∞ with attached loops $n_0, n_1, \dots$. Each v_k and v_∞ would also have infinitely many *rays* – non-branching infinite paths originating from the vertex. The Σ_1 type of v_∞ was then the limit of the Σ_1 types of the v_k.

The original construction took place on a tree of strategies, where each accumulation point was created by an individual strategy. Because a strategy might be visited only finitely many times in the construction, not all strategies would create the full set of vertices described above. Each time a strategy was visited, it performed one of the following steps, in alternation:

- Increment k, choose n_{k+1} and attach a loop of size n_{k+1} to v_∞.

- Choose m_k. Create the full v_k component.

Thus, if a strategy was only visited finitely many times, the v_∞-component would have loops of sizes $n_0, \ldots, n_{k+1}$, and the components $v_0, \ldots, v_{k-1}$ would have all been created, and possibly v_k as well. Numbers n_k and m_k are always chosen larger than the current stage, and two distinct strategies choose completely distinct numbers n_k and m_k. That is, any number is chosen by at most one strategy.

We describe now two ways of modifying this construction so that the structure is a Fraïssé limit. The first uses a finite language with function symbols, while the second uses an infinite relational language. Let $\mathcal{L}_1 = \{E, f, g, h\}$, where E is a binary relation symbol and f, g and h are unary function symbols. Let

$$\mathcal{L}_\infty = \{E\} \cup \{U_{i,j} : j < i \wedge i, j \in \omega\} \cup \{V_{i,j} : j \leq i \wedge i, j \in \omega\} \cup \{R_i : i \in \omega\} \cup \{S_i : i \in \omega\},$$

where E is a binary relation symbol and each $U_{i,j}$, $V_{i,j}$, R_i and S_i is a unary relation symbol.

The intention is that E is the edge relation of the graph from the original construction. That is, in both cases, the reduct of the structures we make to the language $\{E\}$ will be the original structure in [17]. We will now describe the new functions and relations on the structure.

Suppose that v is one of the v_k or v_∞, and $a_0, \ldots, a_{n_k-1}$ are vertices with vEa_0, a_iEa_{i+1} for all $i < n_k - 1$, and $a_{n_k-1}Ev$; that is, $v, a_0, \ldots, a_{n_k-1}$ is the loop of size n_k attached to v. Suppose also that a_0 has lower Gödel number than a_{n_k-1}, so that we have chosen a particular orientation of the loop. Then we define $f(a_i) = a_{i+1}$, and $f(a_{n_k-1}) = v$. We also define $g(a_{i+1}) = a_i$ and $g(a_0) = v$. So f "walks" along the loop in one direction, and g "walks" along it in the other direction. We also define $U_{n_k,i}(a_i)$ to hold for every $i < n_k$, while $U_{n_k,i}(x)$ fails to hold for any other x.

For v_k, suppose that $a_0, \ldots, a_{m_k-1}$ are vertices as above, so that $v_k, a_0, \ldots, a_{m_k-1}$ is the loop of size m_k attached to v_k, again with a chosen orientation. Then we define $f(a_i) = a_{i+1}$, $f(a_{m_k-1}) = v_k$ and $f(v_k) = a_0$. We also define $g(a_{i+1}) = a_i$, $g(a_0) = v_k$ and $g(v_k) = a_{m_k-1}$. So again f and g walk along the loop in the opposite directions, but the walks continue through v_k. We also define $V_{m_k,i}(a_i)$ to hold, and $V_{m_k,i}(x)$ fails to hold for any other x, for every $i < m_k$. Finally, we define $V_{m_k,m_k}(z)$ to hold for every vertex z in the same component as v_k.

Suppose that v is one of the v_k's or v_∞, and consider a ray of the form $a_0, a_1, \ldots$ with vEa_0 and a_iEa_{i+1} for all $i \in \omega$. For infinitely many of these rays, we define $f(a_i) = a_{i+1}$, $g(a_{i+1}) = a_i$ and $g(a_0) = v$, and for infinitely many rays we define $g(a_i) = a_{i+1}$, $f(a_{i+1}) = a_i$ and $f(a_0) = v$. So for infinitely many rays, f walks away from v, while g walks towards v, and for infinitely many rays the reverse holds. For every ray, we define $R_i(a_i)$ to hold.

For v_∞, we choose some a_0 from some ray with $g(a_0) = v_\infty$ and define $f(v_\infty) = a_0$. We choose some b_0 from some ray with $f(b_0) = v_\infty$ and define $g(v_\infty) = b_0$.

Suppose that v is one of the v_k's or v_∞, and a is part of the loop of size n_0 with $g(a) = v$. Then we define $h(v) = a$. For every other x, we define $h(x) = f(x)$.

For every vertex x in every component created by strategy i from the priority tree, we define $S_i(x)$ to hold.

Claim 1. *In both $\mathcal{L}_1$ and $\mathcal{L}_\infty$, if $\overline{x}$ and $\overline{y}$ generate substructures that are isomorphic* via *an isomorphism mapping $\overline{x}$ to $\overline{y}$, then there is an automorphism of the full structure $\mathcal{F}$ mapping $\overline{x}$ to $\overline{y}$.*

Proof. We prove the result for singletons x and y. The general case proceeds similarly. The point is that if $x \neq y$, then they must both be vertices from rays within the same component, and they must be the same length along those rays. Then, for any two rays, there is an automorphism switching those rays and fixing the remainder of the structure. The argument is slightly longer for $\mathcal{L}_\infty$, because rays come in two sorts, and there are two distinguished rays in the component of v_∞.

In $\mathcal{L}_1$, through f or g, the substructure generated by x contains some vertex v_k or v_∞. The same is true for y. Through h, the substructure also contains the entire loop of size n_0. Since n_0 is unique to some strategy from the priority tree, x and y are both placed by the same strategy.

In $\mathcal{L}_\infty$, there is some i such that $S_i(x)$ and $S_i(y)$ hold. So x and y must again both be placed by the same strategy.

In $\mathcal{L}_1$, if the substructure generated by x contains v_k, then through $f(v_k)$ it also contains the loop of size m_k. If the substructure contains v_∞, then through $f(v_\infty)$ it also contains an infinite ray with $f(v_\infty) = a_0$. The same holds for y. This loop or ray uniquely characterizes the component, so x and y must be part of the same component.

In $\mathcal{L}_\infty$, if the component of x contains v_k, then $V_{m_k,m_k}(x)$ holds. If instead it contains v_∞, then no $V_{m_k,m_k}(x)$ holds for any k. The same is true for y. So x and y must be part of the same component.

In $\mathcal{L}_1$, there are four possibilities: $f^i(x) = v$ and $g^j(x) = v$ for some i and j; $f^i(x) = v$ for some i but $g^j(x) \neq v$ for all j; $g^j(x) = v$ for some j but $f^i(x) \neq v$ for all i; or $x = v$. Note that v is uniquely characterized by having degree greater than 2, even in the substructures generated by x or y. In the first case, x must be a_{j-1} from the loop of size $i+j$. In the second case, x must be a_{i-1} from one of the rays in which f walks towards v. In the third case, x must be a_{j-1} from one of the rays in which g walks towards v. The same holds for y. The first case is unique in the component, so in this case we know that $x = y$. If $v \neq v_\infty$, there is a single orbit containing every instance of the second case, and another containing every instance of the third case, so there must be an automorphism mapping x to y. If $v = v_\infty$, then the second case breaks into two subcases: $g(v) = f^{i-1}(x)$, and $g(v) \neq f^{i-1}(x)$. The first subcase is unique in the component, so $x = y$, while the second subcase again comprises a single orbit. We reason similarly in the third case. The fourth case is again unique in the component.

In $\mathcal{L}_\infty$, if x is part of some loop, then there is some $U_{i,j}$ or $V_{i,j}$ that holds of x and no other point. So $x = y$. If x is part of some ray, then there is some R_i that holds of x and only of the points on rays, which are distance i from v. So y is also a point on a ray, which is distance i from v. So there is an automorphism of the structure switching those two rays, and in particular sending x to y.

In $\mathcal{L}_\infty$, v_k is uniquely characterized by $V_{m_k,m_k}(v_k)$ holding, some $S_i(v_k)$ holding, and no other unary relation holding. So if $x = v_k$, then $y = v_k$. Also, v_∞ is uniquely characterized by some $S_i(v_\infty)$ holding and no other unary relation holding. So if $x = v_\infty$, then $y = v_\infty$. □

It follows that the structures we have described are Fraïssé limits. Observe that they are defined in a computable fashion. Finally, our expanded language does not provide an obstacle to 1-decidability, since n_k and m_k are always chosen larger than the current stage. Thus any statement about $f^s(x)$, $g^s(x)$, $h^s(x)$, $U_{s,j}(x)$, $V_{s,j}(x)$, $R_s(x)$ or $S_s(x)$ can be decided by considering the construction up through stage s. □

3. Δ_2^0-categorical but not relatively Δ_2^0-categorical trees

We consider trees as partial orders. R. Miller [45] established that no computable tree of infinite height is computably categorical. Lempp, McCoy, R. Miller, and Solomon [40] characterized computably categorical trees of finite height, and showed that for these structures, computable categoricity coincides with relative computable categoricity. There is no known characterization of Δ_2^0-categoricity or higher level categoricity for trees of finite height. Lempp, McCoy, R. Miller, and Solomon [40] proved that for every $n \geq 1$, there is a computable tree of finite height, which is Δ_{n+1}^0-categorical but not Δ_n^0-categorical. We will establish the following result, which also holds when a tree is presented as a directed graph.

Theorem 3. *There is a computable Δ_2^0-categorical tree of finite height, which is not relatively Δ_2^0-categorical.*

Proof. While building a computable tree $\mathcal{T}$ (with domain ω), we diagonalize against all potential c.e. Scott families of computable Σ_2 formulas with finitely many parameters. Thus, we consider all pairs $(\mathcal{X}, \overline{p})$, where $\mathcal{X}$ is a c.e. family of computable Σ_2 formulas and $\overline{p}$ is a finite tuple of elements from the domain of $\mathcal{T}$, and we must ensure that for each pair $(\mathcal{X}, \overline{p})$, $\mathcal{X}$ with parameters $\overline{p}$ is not a Scott family for $\mathcal{T}$. At the same time, we have to assure that every isomorphic computable tree is $\mathbf{0}'$-isomorphic to $\mathcal{T}$. The construction will be an infinite injury construction where strategies are arranged on a priority tree with the true path defined as usual.

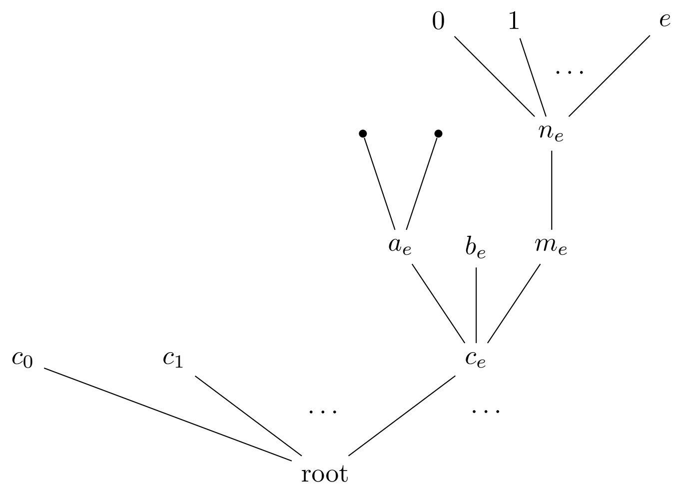
The root of $\mathcal{T}$ will have infinitely many "children," which we label $c_0, c_1, c_2, \ldots$. Each c_e will have 3 children, a_e, b_e and m_e. The purpose of m_e is to uniquely identify c_e. The node m_e will have a child n_e, and n_e will have $e+1$ many children. See the diagram.

At stage 0, a_e will have 2 children and b_e will have no children. Through the action of some strategy, more children may be added to a_e and b_e at later stages.

Let $(\mathcal{X}_i, \overline{p}_i)_i$ be an enumeration of pairs, where $\mathcal{X}_i$ is a c.e. family of computable Σ_2 formulas, and $\overline{p}_i$ is a tuple drawn from ω, the domain of $\mathcal{T}$. We must meet the following categoricity and isomorphism requirements. Let $M_0, M_1, \ldots$ be an effective enumeration of all computable structures.

R_i : $\mathcal{X}_i$ with parameters $\overline{p}_i$ is not a Scott family for $\mathcal{T}$.

Q_j : If $M_j \cong \mathcal{T}$, then there is a $\mathbf{0}'$-computable isomorphism between M_j and $\mathcal{T}$.



Strategy for R_i

Our strategy will appear on a priority tree. When the strategy is visited, s is always the current stage, and $t < s$ is the last stage at which the strategy took outcome ∞ (or $t = 0$ if the strategy has never before taken outcome ∞). The first time the strategy is visited, we choose a large e to work with. In particular, a_e and b_e must not occur in $\overline{p}_i$, and $e > s$.

We will take advantage of the fact that if $\phi(\overline{x})$ is a computable Σ_2 formula and $\overline{a} \in \mathcal{T}$, then we have a computable approximation $(\mathcal{T}_s)_s$ to $\mathcal{T}$ such that $\mathcal{T} \models \phi(\overline{a})$ if and only if $\mathcal{T}_s \models \phi(\overline{a})$ for co-finitely many stages s. By modifying our assumption, we may assume that $\mathcal{T}_s \not\models \phi(\overline{a})$ for any $\overline{a}$ if $\phi(\overline{x})$ is not one of the first s elements of $\mathcal{X}_i$.

We proceed as follows.

(1) Among the first s elements of $\mathcal{X}_i$, locate the $\phi(\overline{x})$ that minimizes the u such that $\mathcal{T}_r \models \phi(a_e, \overline{p}_i) \wedge \phi(b_e, \overline{p}_i)$ for every $r \in (u, s]$. Note that $u = s$ always works. Decide ties by favoring earlier elements of $\mathcal{X}_i$.
(2) Wait until there is an $r \in (t, s]$ with $\mathcal{T}_r \not\models \phi(a_e, \overline{p}_i) \wedge \phi(b_e, \overline{p}_i)$.
(3) Add a child to both a_e and b_e, ensuring that these children are not elements of $\overline{p}_i$.
(4) Return to Step (1).

We perform at most one step at every stage at which the strategy is visited. In particular, we never add more than 1 child to a_e at a single stage. This will be important for interactions with higher priority categoricity requirements. Note also that at every stage, a_e has exactly 2 more children than b_e.

The strategy has infinitely many outcomes: ∞ and $\mathtt{fin}_k$ for $k \in \omega$. Every time we reach Step (4), we take outcome ∞ for a single stage. At all other stages, we take outcome $\mathtt{fin}_k$, where k is the number of previous stages at which we have taken outcome ∞.

Strategy for Q_j

Suppose σ is a strategy for Q_j. This strategy will also appear on the priority tree. When σ is visited, s is always the current stage and $t < s$ is the last stage at which the strategy took outcome ∞ (or $t = 0$ if σ has never before taken outcome ∞).

We construct the isomorphism on c_e and its descendants independently of the isomorphism for all the other $c_{e'}$'s. We begin by searching for a tuple $(r, c, m, n) \in M_j$ with

$$r \lhd_{M_j} c \lhd_{M_j} m \lhd_{M_j} n,$$

and n having $e+1$ many children. When we find such a tuple, we map c_e to c; m_e to m; n_e to n; and the children of n_e to the children of n. Of course, we may later see that the $(e+2)$nd child of n_e appear, in which case we have made a mistake. If this happens, we will discard our mapping and begin again. If $M_j \cong \mathcal{T}$, eventually the tuple in M_j that respects the isomorphism is the Gödel least satisfying the above, and so we will define the correct mapping. The oracle $\mathbf{0}'$ will be able to predict our mistakes, and so can ignore all mappings before the correct one.

Under the assumption that we have correctly mapped c_e, we must map a_e and b_e. This part will not rely on the oracle. We wait until σ is visited and $s > e$. If e has not been chosen by an R_i-strategy by this point, we know by construction that it will be never chosen. In this case, we search for an $a \rhd_{M_j} c$ such that a has two children and map a_e to a. We then search for any child $b \rhd_{M_j} c$ other than m or a, and map b_e to b.

If e has been chosen by an R_i-strategy, and that strategy is incomparable with σ on the tree, then, under the assumption that σ is along the true path, the strategy that chose e will never be visited again. So let p^e be the number of children of a_e. We search for an $a \rhd_{M_j} c$ such that a has p^e children, and map a_e to a. We then search for any $b \rhd_{M_j} c$ other m or a, and map b_e to b.

If e has been chosen by an R_i-strategy τ with $\tau^\frown\infty \subseteq \sigma$, then, under the assumption that σ is along the true path, a_e and b_e are automorphic. So we search for any $a, b \rhd_{M_j} c$ other than m, and map a_e to a and b_e to b.

If e has been chosen by an R_i-strategy τ with $\tau^\frown\mathtt{fin}_k \subseteq \sigma$, then, under the assumption that σ is along the true path, a_e and b_e will never gain any more children. So let p^e be the number of children on a_e. We search for an $a \rhd_{M_j} c$ such that a has p^e children, and map a_e to a. We then search for any $b \rhd_{M_j} c$ other than m or a, and map b_e to b.

If e has been chosen by an R_i-strategy τ with $\sigma^\frown\mathtt{fin}_k \subseteq \tau$, then we wait until a stage t when σ is accessible and $t > e$. At this stage, we know that τ will never again be accessible (since τ was visited before t, σ had taken outcome ∞ at least k times strictly before t, so at least $k+1$ times by any stage after t, so any future outcomes of σ must be ∞ or $\mathtt{fin}_{k'}$ for $k' > k$). So let p^e be the number of children on a_e. We search for an $a \rhd_{M_j} c$ such that a has p^e children, and map a_e to a. We then search for any $b \rhd_{M_j} c$ other than m or a, and map b_e to b.

If e has been chosen by an R_i-strategy τ with $\sigma^\frown\infty \subseteq \tau$, then let p^e_s be the number of children on a_e at the beginning of stage s. We search for an $a \rhd_{M_j} c$ such that a has p^e_s children, and map a_e to a. We then search for any $b \rhd_{M_j} c$ other than m or a, and map b_e to b. Note that, unlike in the other cases, p^e_s may change, which is why we have subscripted it with the stage number.

The strategy has infinitely many outcomes: ∞ and $\mathtt{fin}_k$ for $k \in \omega$. At stage s, if the isomorphism is defined on a_e for every $e < s$, which has been chosen by a τ extending $\sigma^\frown\infty$, and further the image of a_e in M_j has p^e_s many children for every such e, then we take outcome ∞. Otherwise, we take outcome $\mathtt{fin}_k$ where k is the number of previous stages at which we have taken outcome ∞.

Construction

Arrange the strategies on a tree in some effective fashion, and at every stage allow strategies to be visited according to the outcome of previous strategies at that stage in the usual fashion.

Verification

Define the true path in the usual fashion for a $\mathbf{0}''$-construction.

Lemma 1. *Suppose that τ is an R_i-strategy along the true path. Then τ ensures R_i is satisfied.*

Proof. Since τ is along the true path, it is visited infinitely often. Suppose there is some $\phi(\overline{x}) \in \mathcal{X}_i$ such that $\mathcal{T} \models \phi(a_e, \overline{p}_i) \wedge \phi(b_e, \overline{p}_i)$. Let u be such that $\mathcal{T}_r \models \phi(a_e, \overline{p}_i) \wedge \phi(b_e, \overline{p}_i)$ for every $r \in (u, \infty]$. Then for any $\psi(\overline{x}) \in \mathcal{X}_i$, which is not one of the first $u + 1$ elements of $\mathcal{X}_i$, we know that τ will never choose $\psi(\overline{x})$ because it will always prefer $\phi(\overline{x})$.

So if τ were to take outcome ∞ infinitely many times, by the pigeon hole principle, it would choose one of the first $u + 1$ elements of $\mathcal{X}_i$ infinitely many times. But if there are infinitely many r with $\mathcal{T}_r \not\models \psi(a_e, \overline{p}_i) \wedge \phi(b_e, \overline{p})$, then eventually τ will prefer ϕ over ψ, and so will stop choosing ψ. Thus, τ cannot choose an element of $\mathcal{X}_i$ infinitely many times. So τ has true outcome $\mathtt{fin}_k$ for some k, and a_e and b_e have different finite numbers of children. This means that a_e and b_e are not automorphic, so ϕ witnesses the failure of $(\mathcal{X}_i, \overline{p}_i)$ as a Scott family.

Suppose instead that there is no such ϕ. Then for any ϕ, there are infinitely many r with $\mathcal{T}_r \not\models \phi(a_e, \overline{p}_i) \wedge \phi(b_e, \overline{p}_i)$. So with any chosen ϕ we eventually reach Step (3), so a_e and b_e have infinitely many children. So a_e and b_e will be automorphic, and in particular there will be an automorphism permuting a_e and b_e and pointwise fixing $\overline{p}_i$. So for any ϕ with $\mathcal{T} \models \phi(a_e, \overline{p}_i)$, we know that $\mathcal{T} \models \phi(b_e, \overline{p}_i)$. Hence there can be no such $\phi \in \mathcal{X}_i$, and thus $\mathcal{X}_i$ fails to be a Scott family. ☐

Lemma 2. *Suppose that σ is a Q_j-strategy along the true path, that $M_j \cong \mathcal{T}$, and e is chosen by some $\tau \supseteq \sigma^\frown\infty$. Then σ eventually correctly maps a_e and b_e.*

Proof. Certainly, σ eventually correctly maps c_e and m_e, and defines some map for a_e and b_e. If τ has true outcome ∞, then a_e and b_e are automorphic, so this is a correct map.

Suppose instead that τ has true outcome $\mathtt{fin}_k$ (thus a_e has $k+2$ children, and b_e has k children). Let s_0 be the stage at which σ correctly maps c_e, and let t_0 be the final stage at which τ takes outcome ∞. Suppose that $s_0 > t_0$. Then at stage s_0, σ searches for an $a \rhd_{M_j} c$ with $p^e_{s_0} = k+2$ children, and maps a_e to a. By assumption, a_e never gains any more children, so, since $M_j \cong \mathcal{T}$, the correct image of a_e is the only such child of c. The element b_e is correctly mapped by elimination.

If instead $s_0 \leq t_0$, then let a be the element to which σ has mapped a_e at stage t_0. (Such an element necessarily exists because σ must have taken outcome ∞ at stage t_0.) Since a_e can gain at most one child during stage t_0, and will gain no children after stage t_0, it has at least $k+1$ children at the start of stage t_0. Since σ has outcome ∞ at stage t_0, a has at least $p^e_{t_0} = k+1$ children. Since $M_j \cong \mathcal{T}$, the correct image of a_e is the only child of c with at least $k+1$ children, so a_e is correctly mapped. The element b_e is correctly mapped by elimination. □

Lemma 3. *Suppose that σ is a Q_j-strategy along the true path, and that $M_j \cong \mathcal{T}$. Then σ has true outcome ∞.*

Proof. Suppose otherwise. Let t_0 be the final stage at which σ takes outcome ∞. Then there are only finitely many e that are chosen by strategies extending $\sigma^\frown\infty$, and, by Lemma 2, σ eventually correctly maps a_e for each of these e's. Since $M_j \cong \mathcal{T}$, σ eventually sees $p^e_{t_0}$ many children below the target of a_e for each e, and so σ will take outcome ∞ at some stage after t_0, contrary to our assumption. □

Lemma 4. *If $M_j \cong \mathcal{T}$, then there is a Δ^0_2 isomorphism between M_j and $\mathcal{T}$.*

Proof. Non-uniformly fix σ that is the Q_j-strategy along the true path. As argued before, σ eventually correctly maps every c_e and m_e, and $\mathbf{0}'$ can determine when this occurs. By Lemma 2, or by the description of σ's action, σ correctly maps a_e and b_e once c_e has been correctly mapped. The only new ingredient is the observation that since σ has true outcome ∞, there is eventually a stage s with $t > e$, thus treating those e's chosen by strategies extending $\sigma^\frown\mathtt{fin}_k$.

Once a_e and b_e are mapped, their children can be mapped by a simple back-and-forth argument. Thus $\mathbf{0}'$ can build an isomorphism. □

This completes the proof. Note that every step we have described above can be performed equally well for partial orders and directed graphs. □

We can modify the construction in the proof of the previous theorem to make the tree have infinite height by extending every child of a_e, b_e and n_e to an infinite non-branching path. Once a_e, b_e and n_e are correctly mapped, we then need to use the $\mathbf{0}'$-oracle to correctly map their descendants. Hence we have the following result, which is interesting, in particular, since there is no computably categorical tree of infinite height.

Theorem 4. *There is a computable Δ^0_2-categorical tree of infinite height, which is not relatively Δ^0_2-categorical.*

4. Δ^0_2-CATEGORICAL BUT NOT RELATIVELY Δ^0_2-CATEGORICAL ABELIAN p-GROUPS

In this section, we will focus on Δ^0_2-categorical abelian p-groups for a prime number p. A group G is called a *p-group* if for all $g \in G$, the order of g is a power of p. By $\mathbb{Z}(p^n)$ we denote the cyclic group of order p^n. By $\mathbb{Z}(p^\infty)$ we denote the quasicyclic (Prüfer) abelian p-group, the direct limit of the sequence $\mathbb{Z}(p^n)$, and also the set of rationals in $[0, 1)$ of the form $\frac{i}{p^n}$ with addition modulo 1. The *length* of an abelian p-group G, $\lambda(G)$, is the least ordinal α such that $p^{\alpha+1}G = p^\alpha G$. Here, $p^0G = G$, $p^{\alpha+1}G = p(p^\alpha G)$, and $p^\lambda G = \bigcap_{\alpha<\lambda} p^\alpha G$ for limit λ. The divisible part of G is $Div(G) = p^{\lambda(G)}G$ and it is a direct summand of G. The group G is said to be *reduced* if $Div(G) = \{0\}$. For an element $g \in G$, the *height* of g, $ht(g)$, is ∞ if $g \in Div(G)$, and is otherwise the least α such that $g \notin p^{\alpha+1}G$. For a computable group G, $ht(g)$ can be an arbitrary computable ordinal. The height of G is the supremum of $\{ht(g) : g \in G\}$. Let $o_G(g)$ be the order of g in G. The *period* of G is $max\{o(g) : g \in G\}$ if this quantity is finite, and it is ∞ otherwise.

Barker [6] proved that for every computable ordinal α, there is a $\Delta^0_{2\alpha+2}$-categorical but not $\Delta^0_{2\alpha+1}$-categorical abelian p-group. Goncharov [24] and Smith [50] independently characterized computably categorical abelian p-groups as those that can be written in one of the following forms:

(i) $\bigoplus_l \mathbb{Z}(p^\infty) \oplus F$ for $l \leq \omega$ and F is a finite group; or

(ii) $\bigoplus_n \mathbb{Z}(p^\infty) \oplus H \oplus \bigoplus_\omega \mathbb{Z}(p^k)$, where $n, k \in \omega$ and H is a finite group.

For these groups, computable categoricity and relative computable categoricity coincide (for a proof see also [7]).

In [7], Calvert, Cenzer, Harizanov, and Morozov established that a computable abelian p-group G is relatively Δ^0_2-categorical if and only if:

(i) G is isomorphic to $\bigoplus_l \mathbb{Z}(p^\infty) \oplus H$, where $l \leq \omega$ and H has finite period; or

(ii) All elements in G are of finite height (equivalently, G is reduced with $\lambda(G) \leq \omega$).

They also have partial results towards characterizing Δ^0_2-categoricity. For example, if G is a computable group with reduced part H such that H has a computable copy and infinitely many elements of height $\geq \omega$, then G is not Δ^0_2-categorical. If G is a computable group isomorphic to $\bigoplus_l Z(p^\infty) \oplus H$, where all elements of H are of finite height, then G is relatively Δ^0_3-categorical (see [7]).

Theorem 5. *There is a computable Δ^0_2-categorical abelian p-group, which is not relatively Δ^0_2-categorical.*

Proof. Let $((\omega, +_n, e_n))_{n\in\omega}$ be an enumeration of all partial computable abelian groups with universe ω. Let $\langle\cdot,\cdot\rangle$ be a standard pairing function. By $p^m \cdot_n z$ we indicate $z +_n z +_n \cdots +_n z$ where there are p^m summands. Define the following set:

$$\begin{aligned} k \in A \Leftrightarrow_{def} \neg(\exists n < k)\,(\exists x < k)\,(\exists m < k)\,[(2\langle n, x\rangle < k) \wedge (p^m \cdot_n x \neq e_n) \\ \wedge\, (p^{m+1} \cdot_n x = e_n) \\ \wedge\, \exists z\,(p^{k-m-1} \cdot_n z = x) \\ \wedge\, \neg\exists w\,(p^{k-m} \cdot_n w = x)]. \end{aligned}$$

Ignoring bounded quantifiers, A is defined by the conjunction of a Σ^0_1 formula and a Π^0_1 formula, and is thus Δ^0_2. Furthermore, note that every $\langle n, x\rangle$ can be the witness to at most one $k \notin A$. That is, if $\langle n, x\rangle$ witnesses some $k \notin A$, and $k' < k$, then fix the m with $p^m \cdot_n x \neq e_n$ and $p^{m+1} \cdot_n = e_n$, and fix some z with $p^{k-m-1} \cdot_n z = x$. Then $w = p^{k-k'-1} \cdot z$ is such that $p^{k'-m} \cdot_n w = x$. Since $\langle n, x\rangle$ can only be the witness to $k \notin A$ if $k > 2\langle n, x\rangle$, it follows that A is infinite.

Define

$$G = \bigoplus_{\omega} \mathbb{Z}(p^\infty) \oplus \bigoplus_{k \in A} \mathbb{Z}(p^k).$$

Since A is Δ^0_2, it can be easily shown that G has a computable isomorphic copy. The form of G shows that it is not relatively Δ^0_2-categorical (see [7]). We claim that G is Δ^0_2-categorical.

Lemma 5. *Suppose that $(\omega, +_n, e_n) \cong G$. Then the divisible part of $(\omega, +_n, e_n)$ (the isomorphic image of $\bigoplus_{\omega} \mathbb{Z}(p^\infty)$) is computably enumerable.*

Proof. An element $x \neq e_n$ is in the divisible part of $(\omega, +_n, e_n)$ precisely if the following holds:

$$\exists m\, \exists k\, [(2\langle n, x\rangle < k) \wedge (p^m \cdot_n x \neq e_n) \wedge (p^{m+1} \cdot_n x = e_n) \wedge \exists z\,(p^{k-m-1} \cdot_n z = x)].$$

Clearly, if x is in the divisible part, then there are m, k and z as desired. Conversely, suppose that x is not in the divisible part. Fix m such that $p^m \cdot_n x \neq e_n$ and $p^{m+1} \cdot_n x = e_n$. Since x is not divisible, fix k such that $\exists z\,(p^{k-m-1} \cdot_n z = x)$ and $\neg\exists z\,(p^{k-m} \cdot_n z = x)$. Then since $(\omega, +_n, e_n) \cong G$, it must be that $\mathbb{Z}(p^k)$ is a summand in G, and thus $k \in A$. By definition, this requires that $k \leq 2\langle n, x\rangle$, and so x cannot satisfy the above formula. $\square$

Now suppose that $(\omega, +_n, e_n) \cong G$. We can construct a $\mathbf{0}'$-computable isomorphism as follows: since we can enumerate the divisible parts, we run a computable back-and-forth construction on those; meanwhile, for each $k \in A$, we use the $\mathbf{0}'$-oracle to locate an element x with $p^k \cdot_n x = e_n$ but $\neg\exists z\,(p \cdot_n z = x)$, and use this to map the image of $\mathbb{Z}(p^k)$. $\square$

5. Δ^0_2-categorical but not relatively Δ^0_2-categorical homogenous completely decomposable abelian groups

We will now consider certain torsion-free abelian groups. A *homogenous completely decomposable abelian group* is a group of the form $\bigoplus_{i \in \kappa} H$, where H is a subgroup of the additive group of the rationals, $(\mathbb{Q}, +)$. Note that we have only a single H in the sum – any two summands are isomorphic. It is well known that such a group is computably categorical if and only if κ is finite; the proof is similar to the analogous result that a computable vector space is computably categorical if and only if it has finite dimension. In the remainder of this section, we will restrict our attention to groups of infinite rank κ.

For P a set of primes, define $Q^{(P)}$ to be the subgroup of $(\mathbb{Q},+)$ generated by $\{\frac{1}{p^k} : p \in P \wedge k \in \omega\}$. Downey and Melnikov [18] showed that a computable homogenous completely decomposable abelian group of infinite rank is Δ^0_2-categorical if and only if it is isomorphic to $\bigoplus_\omega Q^{(P)}$, where P is c.e. and the set (Primes $-$ P) is semi-low. Recall that a set $S \subseteq \omega$ is *semi-low* if the set $H_S = \{e : W_e \cap S \neq \emptyset\}$ is computable from $\emptyset'$. Here, we will first fully characterize the computable relatively Δ^0_2-categorical homogenous completely decomposable abelian groups of infinite rank.

Theorem 6. *A computable homogenous completely decomposable abelian group of infinite rank is relatively Δ^0_2-categorical if and only if it is isomorphic to $\bigoplus_\omega Q^{(P)}$, where P is a computable set of primes.*

Proof. Suppose that G is relatively Δ^0_2-categorical. Since this implies that G is Δ^0_2-categorical, by the above mentioned result of Downey and Melnikov, we know that $G \cong \bigoplus_\omega Q^{(P)}$ for P a c.e. set of primes. We will show that P is also co-c.e.

Fix $\mathcal{X}$, a c.e. Scott family of computable Σ_2 formulas for G, with parameters $\overline{a} \in G^{<\omega}$. Fix an element $b \in G$, which is independent of $\overline{a}$. Then $b \mapsto p \cdot b$ can be extended to an automorphism of G fixing $\overline{a}$ if and only if $p \in P$. Fix some formula $\exists \overline{x}\, \theta(\overline{z}, \overline{x}, y) \in \mathcal{X}$, where θ is a computable Π_1 formula and $G \models \exists \overline{x}\, \theta(\overline{a}, \overline{x}, b)$. Fix some tuple $\overline{c} \in G$ such that $G \models \theta(\overline{a}, \overline{c}, b)$.

Now, decompose the elements of $\overline{c}$ as $c_i = d_i + e_i$, where d_i is a rational multiple of b, and b is independent of $\{\overline{a}, \overline{e}\}$. One way to achieve this is to fix an isomorphism $f : \bigoplus_\omega Q^{(P)} \to G$ such that $b = f((q, 0, 0, 0, \dots))$ for some $q \in Q^{(P)}$, and then define v_i to be the projection of $f^{-1}(c_i)$ onto the first coordinate, $d_i = f(v_i)$, and $e_i = c_i - d_i$. Observe that the map $b \mapsto p \cdot b$ can be extended to an automorphism of G fixing $\overline{a}$ and $\overline{e}$ if and only if $p \in P$, and any such isomorphism would need to map $d_i \mapsto p \cdot d_i$.

Define $\overline{c}^p$ by $c^p_i = p \cdot d_i + e_i$. Note that an isomorphism sending $b \mapsto p \cdot b$ and fixing $\overline{a}$ and $\overline{e}$ would necessarily map $\overline{c} \mapsto \overline{c}^p$. So, if there is such an isomorphism, then $G \models \theta(\overline{a}, \overline{c}^p, p \cdot b)$. Conversely, if $G \models \theta(\overline{a}, \overline{c}^p, p \cdot b)$ then $G \models \exists \overline{x}\, \theta(\overline{a}, \overline{x}, p \cdot b)$, and, by the definition of Scott family, there must be an isomorphism fixing $\overline{a}$ and mapping $b \mapsto p \cdot b$. Thus,

$$p \in P \Leftrightarrow G \models \theta(\overline{a}, \overline{c}^p, p \cdot b).$$

Since θ is a computable $\Pi_1$formula, and $\overline{c}^p$ can be obtained effectively from p, it follows that P is co-c.e. $\square$

Since there exist co-c.e. sets that are semi-low and noncomputable, we obtain the following categoricity result.

Corollary 1. *There is a computable homogenous completely decomposable abelian group, which is Δ^0_2-categorical but not relatively Δ^0_2-categorical.*

6. Degrees of categoricity of certain Boolean algebras and abelian p-groups

Cenzer, Harizanov, and Remmel established in [10] that the degrees of categorictiy of computable injections structures can only be $\mathbf{0}$, $\mathbf{0}'$ and $\mathbf{0}''$. Frolov [23] showed that the degrees of categoricity of relatively Δ^0_2-categorical linear orders can only be $\mathbf{0}$ and $\mathbf{0}'$. Using the characterization of relatively Δ^0_2-categorical Boolean algebras by McCoy in [41], Bazhenov [8] established that the degrees of categoricity of relatively Δ^0_2-categorical (equivalently, Δ^0_2-categorical) Boolean algebras can only be $\mathbf{0}$ and $\mathbf{0}'$. In this section, we will extend Bazhenov's result to relatively Δ^0_3-categorical Boolean algebras.

A Boolean algebra $\mathcal{B}$ is *atomic* if for every $a \in \mathcal{B}$ there is an atom $b \leq a$. An equivalence relation $\sim$ on a Boolean algebra $\mathcal{A}$ is defined by:

$$a \sim b \text{ iff each of } a \cap \overline{b} \text{ and } b \cap \overline{a} \text{ is } \emptyset \text{ or a union of finitely many atoms of } \mathcal{A}.$$

A Boolean algebra $\mathcal{A}$ is a 1-*atom* if $\mathcal{A}/\sim$ is a two-element algebra. A Boolean algebra $\mathcal{A}$ is *rank* 1 if $\mathcal{A}/\sim$ is a nontrivial atomless Boolean algebra. McCoy [42] proved that a countable rank 1 atomic Boolean algebra is isomorphic to $I(2 \cdot \eta)$.

In [41], McCoy established that a Boolean algebra is relatively Δ^0_2-categorical if and only if it is a finite direct sum of algebras that are atoms, atomless, or 1-atoms. Furthermore, in [42], McCoy characterized relatively Δ^0_3-categorical Boolean algebras as those computable Boolean algebras that can be expressed as finite direct sums of algebras that are atoms, atomless, 1-atoms, rank 1 atomic, or isomorphic to the interval algebra $I(\omega+\eta)$. In our next theorem, we will use this characterization and the following isomorphism result of Remmel [49] .

Lemma 6 (Remmel)**.** *If $\mathcal{A}$ is a Boolean algebra, $\mathcal{B} \subseteq \mathcal{A}$ is a subalgebra, $\mathcal{B}$ has infinitely many atoms, every atom in $\mathcal{B}$ is a finite join of atoms in $\mathcal{A}$, and $\mathcal{A}$ is generated by $\mathcal{B}$ and the elements below the atoms of $\mathcal{B}$, then $\mathcal{B} \cong \mathcal{A}$.*

Theorem 7. *The degrees of categoricity of relatively Δ^0_3-categorical Boolean algebras can only be $\mathbf{0}$, $\mathbf{0}'$ and $\mathbf{0}''$.*

Proof. Fix a relatively Δ^0_3-categorical Boolean algebra $\mathcal{B}$. If $\mathcal{B}$ is a finite join of atoms, 1-atoms and atomless Boolean algebras, then $\mathcal{B}$ is relatively Δ^0_2-categorical, and so its degree of categoricity is either $\mathbf{0}$ or $\mathbf{0}'$. Otherwise, $\mathcal{B}$ has a summand which is either rank 1 atomic or isomorphic to the interval algebra $I(\omega+\eta)$.

All of the potential summands in the characterization of relatively Δ^0_3-categorical Boolean algebras have computable isomorphic copies in which the set of finite elements (that is, the elements a with $a \sim 0$) is computable. We will show that both the rank 1 atomic algebra and $I(\omega+\eta)$ have computable isomorphic copies where the set of finite elements is Σ^0_2-complete. It will follow that $\mathcal{B}$ has a computable isomorphic copy in which the set of finite elements is computable, and another computable isomorphic copy in which it is Σ_2^0-complete, and so any isomorphism between these two copies will compute $\emptyset''$.

We begin with the rank 1 atomic algebra. Let $\mathcal{C}$ be a computable copy of this algebra in which the set of atoms is computable. Let $\{a_i : i \in \omega\}$ be the atoms of $\mathcal{C}$. We will create an algebra $\mathcal{A}$ by extending $\mathcal{C}$. Let $\phi(i, x)$ be a computable formula such that

$$i \in \emptyset'' \Leftrightarrow \exists^{<\infty} x\, \phi(i, x).$$

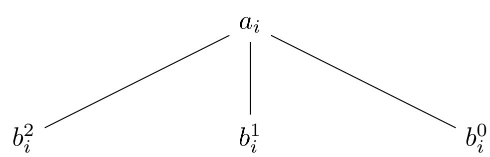
At every step s, we will consider whether $\phi(i, s)$ holds. The first time $\phi(i, s)$ holds, we choose three large elements b_i^0, b_i^1 and b_i^2 and use them to partition a_i into three pieces. That is,

$$b_i^0 \wedge b_i^1 = b_i^1 \wedge b_i^2 = b_i^2 \wedge b_i^0 = 0$$

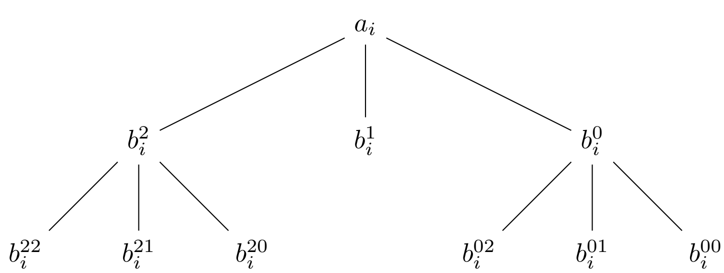
and

$$b_i^0 \vee b_i^1 \vee b_i^2 = a_i.$$

At the second stage at which we see $\phi(i, s)$ hold, we repeat the process on b_i^0 and b_i^2. See the following diagrams.



Working with rank 1 atomic, the first time we see $\phi(i, s)$ hold.

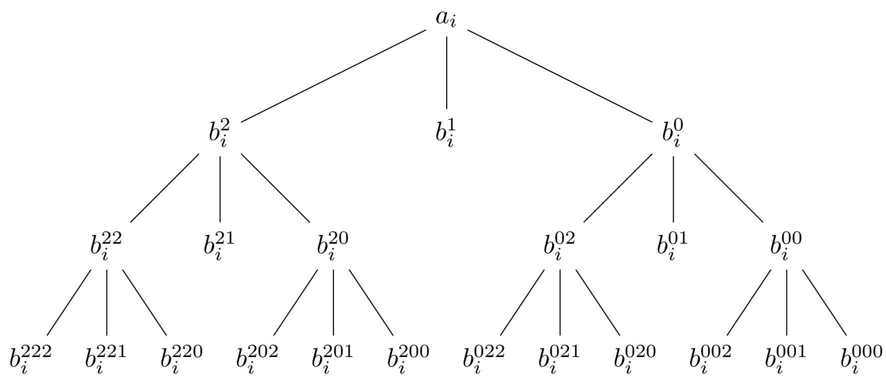


Working with rank 1 atomic, the second time we see $\phi(i, s)$ hold.

We then let $\mathcal{A}$ be the Boolean algebra generated by $\mathcal{C}$ along with these new elements we have added. Note that every element of $\mathcal{A}$ is the join of an element from $\mathcal{C}$ and some of these new elements (among b_i^σ's). That is, for all $d \in \mathcal{A}$, $d = c \vee b_{i_0}^{\sigma_0} \vee b_{i_1}^{\sigma_1} \vee \cdots \vee b_{i_k}^{\sigma_k}$ for some $c \in \mathcal{C}$ and some $b_{i_0}^{\sigma_0}, \ldots, b_{i_k}^{\sigma_k}$.

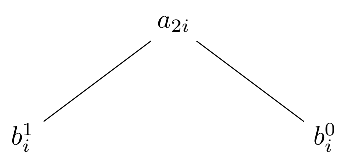
Observe that a_i is infinite in $\mathcal{A}$ if and only if $\phi(i, x)$ holds for infinitely many x, which is if and only if $i \notin \emptyset''$. Also, a_i necessarily bounds an atom in $\mathcal{A}$, e.g., b_i^1. Finally, if a_i is infinite, then it can be partitioned into two infinite elements, e.g., b_i^0 and $b_i^1 \vee b_i^2$. Since every element of $\mathcal{C}$ bounds an atom, and every infinite element of $\mathcal{C}$ can be partitioned into two infinite elements, it follows that the same holds for every element of $\mathcal{A}$. This characterizes the rank 1 atomic algebra. Thus $\mathcal{A} \cong \mathcal{C}$, and $\mathcal{A}$ is as desired.

Next, consider $I(\omega + \eta)$. Again, let $\mathcal{C}$ be a computable copy of $I(\omega + \eta)$ in which the set of atoms is computable. Let $\{a_i : i \in \omega\}$ be the atoms of $\mathcal{C}$. We again create

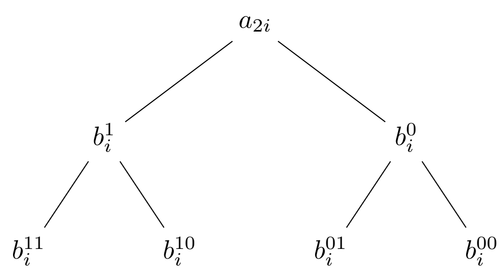


Working with rank 1 atomic, the third time we see $\phi(i,s)$ hold.

$\mathcal{A}$ extending $\mathcal{C}$. Let $\phi(i,x)$ be as before. At every step s, if $\phi(i,s)$ holds, we add new elements below a_{2i}. The first time $\phi(i,s)$ holds, we partition $a_{2i} = b_i^0 \vee b_i^1$. The second time it holds, we partition b_i^0 and b_i^1. See the diagrams.



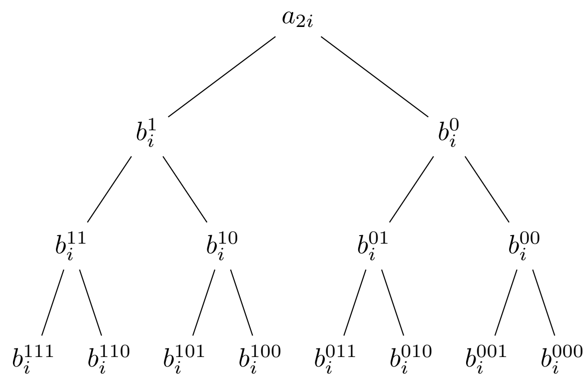
Working with $I(\omega+\eta)$, the first time we see $\phi(i,s)$ hold.



Working with $I(\omega+\eta)$, the second time we see $\phi(i,s)$ hold.

We again let $\mathcal{A}$ be the Boolean algebra generated by $\mathcal{C}$ along with these new elements. The isomorphism type of $I(\omega+\eta)$ is characterized by three properties: there are infinitely many atoms; any element which bounds infinitely many atoms also bounds an atomless element; and no two disjoint elements both bound infinitely many atoms. Since every atom of $\mathcal{A}$ is bounded by an atom of $\mathcal{C}$, every atomless element of $\mathcal{C}$ is still atomless in $\mathcal{A}$, and every atom of $\mathcal{C}$ is either atomless or finite in $\mathcal{A}$, the second and the third properties are inherited from $\mathcal{C}$ to $\mathcal{A}$. Meanwhile, the first property is ensured by the fact that each a_{2i+1} is still an atom of $\mathcal{A}$. Thus $\mathcal{A} \cong \mathcal{C}$. Also, a_{2i} is finite if and only if $i \in \emptyset''$, so $\mathcal{A}$ is as desired.

This completes the proof. $\square$



Working with $I(\omega + \eta)$, the third time we see $\phi(i, s)$ hold.

It follows from proofs in [9] that the degrees of categoricity of computable relatively Δ^0_2-categorical equivalence structures can only be $\mathbf{0}$ and $\mathbf{0}'$. Using the characterization of relatively Δ^0_2-categorical abelian p-groups in [7] we can show the following.

Proposition 1. *The categoricity degrees of computable relatively Δ^0_2-categorical abelian p-groups can only be $\mathbf{0}$ and $\mathbf{0}'$.*

Proof. Suppose that G is a computable abelian p-group, which is relatively Δ^0_2-categorical but not computably categorical. We will show that G has degree of categoricity $\mathbf{0}'$. From the earlier described classifications of categoricity, it follows that G is of one of the following two forms:

(1) $\bigoplus_\omega \mathbb{Z}(p^k) \oplus \bigoplus_\omega \mathbb{Z}(p^m) \oplus H$, where $0 < k < m \leq \omega$; or
(2) Every element of G has finite height, but G contains elements of arbitrarily large finite heights.

We will handle the two cases separately.

First Case

Consider elements $x \in G$ with $x \neq 0$, $p \cdot x = 0$ and $ht(x) = k - 1$. Note that $\mathbb{Z}(p^k)$ contains such an element (indeed, $p - 1$ such elements). By the observation that $G \cong \bigoplus_\omega \mathbb{Z}(p^k) \oplus G$, we may assume that we have an effective enumeration $\{a_n : n \in \omega\}$ of elements of this sort.

Fix μ the modulus function of $\emptyset'$. We will build a second computable copy A such that the first $\mu(n)$ elements of A contain at most n elements of the desired sort. Then given any isomorphism $f : G \cong A$, the function $n \mapsto f(a_n)$ would necessarily dominate μ. Thus, any isomorphism from G to A would compute $\emptyset'$.

The construction is now straightforward. By $dom(F)$ we denote the domain and by $ran(F)$ the range of a function F. We will build a Δ^0_2 homomorphism $F : G \cong A$ and arrange that $A = ran(F) \oplus \bigoplus_\omega \mathbb{Z}(p^m)$. We begin with $F_0 = \emptyset$.

At stage $s + 1$, for every $n \leq s$, we consider every $x \in G$ with $n \leq x \leq s$, $x \neq 0$, $p \cdot x = 0$ and $[ht(x)]^{G_s} < k$. For each such element, if $F_s(x) \leq \mu_s(n)$, we define

$F_{s+1}(x)$ as some new large element. This requires that we also define $F_{s+1}(y)$ for every y dividing such an x, to be some new large element. We let $F_{s+1}(x) = F_s(x)$ for every other x. We then extend the domain of F_{s+1} to the next element of G. We let F_{s+1} induce the group operation on its range *via* pull-back.

Let $D_{s+1} = ran(F_s) - ran(F_{s+1})$. Note that every elements of D_s has height less than k. We add new elements to extend D_{s+1} to a copy of $\bigoplus_l \mathbb{Z}(p^m)$ for some $l < \omega$. Also, for every $a \in A_{s+1} - ran(F_{s+1})$ and every $b \in ran(F_{s+1})$, if A does not yet have an element corresponding to $a + b$, we add an appropriate element now. This completes stage $s + 1$.

Now we argue that F is a total Δ^0_2 function. Fix $x \in G$ with $x \neq 0$ and $p \cdot x = 0$. If $F_{s+1}(x) \neq F_s(x)$, then either our construction was deliberately redefining $F(x)$, or it was required to redefine $F(x)$ because it deliberately redefined $F(z)$ for some z that x divides. The only such z's are of the form $i \cdot x$ for $1 \leq i < p$. Let s_0 be such that $\mu_{s_0}(i \cdot x) = \mu(i \cdot x)$ for $1 \leq i < p$. Then at any stage $s > s_0$ with $F_{s+1}(x) \neq F_s(x)$, necessarily $F_{s+1}(i \cdot x) > \mu_s(i \cdot x) = \mu(i \cdot x)$, since $F_{s+1}(i \cdot x)$ is chosen to be large. Then at any stage $t > s$, $F_t(i \cdot x) > \mu(i \cdot x) = \mu_t(i \cdot x)$, and so we will have $F_{t+1}(x) = F_t(x)$, and thus $F(x)$ will reach a limit.

Now, consider $y \in G$ with $p^{\alpha+1} \cdot y = 0$. Then $p \cdot (p^\alpha \cdot y) = 0$, and $F_{s+1}(y) \neq F_s(y)$ only when $F_{s+1}(p^\alpha \cdot y) \neq F_s(p^\alpha \cdot y)$. Since we have just argued that $F(p^\alpha \cdot y)$ reaches a limit, it follows that $F(y)$ reaches a limit.

Note that $A = ran(F) \oplus \bigoplus_\omega \mathbb{Z}(p^m)$ by construction. It follows that $A \cong G$. It also follows that every $x \in A - ran(F)$ with $p \cdot x = 0$ has height at least $m - 1 \geq k$. Finally, our construction ensured that there are at most n elements $x \in G$ with $p \cdot x = 0$, $ht(x) < k$ and $F(x) < \mu(n)$. Thus, there are at most n elements $x \in A$ with $p \cdot x = 0$, $ht(x) < k$ and $x < \mu(n)$, as desired.

Second Case

By a result of Khisamiev [36] and independently of Ash, Knight and Oates [5], we know that

$$G \cong \mathbb{Z}(p^{k_0}) \oplus \mathbb{Z}(p^{k_1}) \oplus \cdots ,$$

where the sequence $(k_i)_{i \in \omega}$ is uniformly computable from below. That is, there is a computable function $g : \omega \times \omega \to \omega$ such that for all i and s, $g(i, s) \leq g(i, s + 1)$, and for all i, $k_i = \lim_s g(i, s)$. Fix such a function g. By our assumptions on G, we know that the k_i's are unbounded.

We will construct a computable function h and a Δ^0_2 function ι such that:

(1) For all i and s, $h(i, s) \leq h(i, s + 1)$;
(2) $\iota : \omega \to \omega$ is a bijection;
(3) For all i, $\lim_s h(i, s) = \lim_s g(\iota(i), s)$; and
(4) For all n and all $x \in G$ with $x < \mu(n)$ and $x \neq 0$, $ht(x) + 1 < \lim_s h(2n, s)$.

We will then let $A = \mathbb{Z}(p^{\lim_s h(0,s)}) \oplus \mathbb{Z}(p^{\lim_s h(0,s)}) \oplus \cdots$. By the first property above, this is a computable structure. By the second and the third properties, $A \cong G$. By the fourth property, given an isomorphism $f : A \cong G$, for any element x of the $(2n)$th summand of A with $x \neq 0$ and $p \cdot x = 0$, it must be that $f(x) \geq \mu(n)$. Thus, f computes $\emptyset'$.

It remains to construct h and ι. We begin with $\iota_0 = \emptyset$ and $h(i, 0) = 0$ for all i.

At stage $s+1$, if there is an n with $2n \in dom(\iota_s)$ and an $x \in G$ with $x < \mu(n)$, $x \neq 0$ and $[ht(x)]^{G_s} \geq h(2n,s)$, we search for a large pair (j,t) with $g(j,t) > h(2n,s)$, and define $\iota_{s+1}(2n) = j$ and $h(2n,s+1) = g(j,t)$. We then choose a large m and define $\iota_{s+1}(2m+1) = \iota_s(2n)$. We let $\iota_{s+1}(k) = \iota_s(k)$ for every other k.

We then choose the least $a \notin dom(\iota_{s+1})$ and the least $b \notin ran(\iota_{s+1})$, and define $\iota_{s+1}(a) = b$. Then, for every $i \in dom(\iota_{s+1})$ with $h(i,s+1)$ not yet defined, we define $h(i,s+1) = \max\{g(\iota_{s+1}(i), s+1), h(i,s)\}$. For every $i \notin dom(\iota_{s+1})$, we define $h(i,s+1) = 0$. This completes stage $s+1$.

First, note that, by construction, $h(i,s) \leq h(i,s+1)$ for every i and s.

Next, we argue that ι is a total Δ^0_2 function. Note that, by construction, for every i, there is eventually a stage s_0 with $\iota_s(i)$ defined for all $s \geq s_0$. If i is odd, then $\iota_s(i) = \iota_{s_0}(i)$ for all $s \geq s_0$. If instead $i = 2n$, then at every stage s with $\iota_s(i) \neq \iota_s(i+1)$, we have $h(i,s+1) \geq h(i,s)+1$. Let $u = \max\{ht(x) : x \in G \wedge x < \mu(n)\}$. So for sufficiently large s_1, $h(i,s_1) > u$, and then $h(i,s) = h(i,s_1)$ for all $s \geq s_1$.

Next, we argue that ι is surjective. If $b = \iota_{s_0}(a)$, then either $b = \iota_s(a)$ for all $s > s_0$, or there is a stage $s_1 > s_0$ with $b = \iota_{s_1}(c)$ for some odd c. By construction, ι never changes on odd inputs, so $b = \iota_s(c)$ for all $s \geq s_1$. By construction, every element is eventually added to the range of some ι_s, so every element is in $ran(\iota)$.

By induction on s, $h(i,s) \leq \lim_s g(\iota_s(i), s)$ for all i and s, and so in particular, $\lim_s h(i,s)$ exists and equals at most $\lim_s g(\iota(i), s)$. On the other hand, $h(i,s) \geq g(\iota_s(i), s)$ for all i and s by construction, and so $\lim_s h(i,s) = \lim_s g(\iota(i), s)$, as desired.

Finally, for all n and all $x \in G$ with $x < \mu(n)$ and $x \neq 0$, $ht(x)+1 < \lim_s h(2n,s)$, as we deliberately increase $h(2n,s)$ whenever this appears to be false. This completes the proof. $\square$

References

[1] B.A. Anderson and B.F. Csima, Degrees that are not degrees of categoricity, to appear in the *Notre Dame Journal of Formal Logic*.

[2] C.J. Ash, Categoricity in hyperarithmetical degrees, *Annals of Pure and Applied Logic* 34 (1987), pp. 1–14.

[3] C.J. Ash, Recursive labeling systems and stability of recursive structures in hyperarithmetical degrees, *Transactions of the American Mathematical Society* 298 (1986), pp. 497–514.

[4] C. Ash, J. Knight, M. Manasse, and T. Slaman, Generic copies of countable structures, *Annals of Pure and Applied Logic* 42 (1989), pp. 195–205.

[5] C. Ash, J.F. Knight, and S. Oates, Recursive abelian p-groups of small length, unpublished. An annotated manuscript: https://dl.dropbox.com/u/4752353/Homepage/AKO.pdf.

[6] E. Barker, Back and forth relations for reduced abelian p-groups, *Annals of Pure and Applied Logic* 75 (1995), pp. 223–249.

[7] W. Calvert, D. Cenzer, V. Harizanov, and A. Morozov, Effective categoricity of Abelian p-groups," *Annals of Pure and Applied Logic* 159 (2009), pp. 187–197.

[8] N.A. Bazhenov, Δ^0_2-categoricity of Boolean algebras, *Journal of Mathematical Sciences* 203 (2014), pp. 444–454.

[9] W. Calvert, D. Cenzer, V. Harizanov, and A. Morozov, Effective categoricity of equivalence structures, *Annals of Pure and Applied Logic* 141 (2006), pp. 61–78.

[10] D. Cenzer, V. Harizanov, and J. Remmel, Effective categoricity of injection structures, *Algebra and Logic* 53 (2014), pp. 39–69 (English translation).

[11] J. Chisholm, Effective model theory vs. recursive model theory, *Journal of Symbolic Logic* 55 (1990), pp. 1168–1191.

[12] J. Chisholm, E. Fokina, S. Goncharov, V. Harizanov, J. Knight, and S. Quinn, Intrinsic bounds on complexity and definability at limit levels, *Journal of Symbolic Logic* 74 (2009), pp. 1047–1060.
[13] P. Cholak, S. Goncharov, B. Khoussainov, and R.A. Shore, Computably categorical structures and expansions by constants, *Journal of Symbolic Logic* 64 (1999), pp. 13–37.
[14] P. Cholak, R.A. Shore, and R. Solomon, A computably stable structure with no Scott family of finitary formulas. *Archive for Mathematical Logic* 45 (2006), pp. 519–538.
[15] B.F. Csima, J.N.Y. Franklin, and R.A. Shore, Degrees of categoricity and the hyperarithmetic hierarchy, *Notre Dame Journal of Formal Logic* 54 (2013), pp. 215–231
[16] R. Downey, A. Kach, S. Lempp, A. Lewis, A. Montalbán, and D. Turetsky, The complexity of computable categoricity, to appear in *Advances in Mathematics*.
[17] R.G. Downey, A.M. Kach, S. Lempp, and D.D. Turetsky, Computable categoricity *versus* relative computable categoricity, *Fundamenta Mathematicae* 221 (2013), pp. 129–159.
[18] R.G. Downey, and A.G. Melnikov, Effectively categorical abelian groups. *J. Algebra* 373 (2013), 223–248.
[19] E. Fokina, A. Frolov, and I. Kalimullin, Spectra of categoricity for rigid structures, to appear in the *Notre Dame Journal of Formal Logic*.
[20] Yu.L. Ershov, Theorie der Numerierungen III, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 23 (1977), pp. 289–371.
[21] E.B. Fokina, I. Kalimullin, and R. Miller, Degrees of categoricity of computable structures, *Archive for Mathematical Logic* 49 (2010), pp. 51–67.
[22] A. Fröhlich and J. Shepherdson, Effective procedures in field theory, *Philosophical Transactions of the Royal Society*, ser. A, 248 (1956), pp. 407–432.
[23] A.N. Frolov, Categoricity degrees of computable linear orderings, in preparation.
[24] S.S. Goncharov, Autostability of models and abelian groups, *Algebra and Logic* 19 (1980), pp. 13–27 (English translation).
[25] S.S. Goncharov, The quantity of nonautoequivalent constructivizations, *Algebra and Logic* 16 (1977), pp. 169–185 (English translation).
[26] S.S. Goncharov, Selfstability and computable families of constructivizations, *Algebra and Logic* 14 (1975), pp. 647–680 (Russian).
[27] S.S. Goncharov and V.D. Dzgoev, Autostability of models, *Algebra and Logic* 19 (1980), pp. 28–37 (English translation).
[28] S. Goncharov, V. Harizanov, J. Knight, C. McCoy, R. Miller, and R. Solomon, Enumerations in computable structure theory, *Annals of Pure and Applied Logic* 136 (2005), pp. 219–246.
[29] S. Goncharov, V. Harizanov, J. Knight and R. Shore, Π^1_1 relations and paths through $\mathcal{O}$, *Journal of Symbolic Logic* 69 (2004), pp. 585–611.
[30] K. Harris, Δ^0_2-categorical Boolean algebras, preprint.
[31] D. Hirschfeldt, K. Kramer, R. Miller, and A. Shlapentokh, Categoricity properties for computable algebraic fields, *Transactions of the American Mathematical Society* (2015), pp. 3981–4017.
[32] D.R. Hirschfeldt, B. Khoussainov, and R.A. Shore, A computably categorical structure whose expansion by a constant has infinite computable dimension, *Journal of Symbolic Logic* 68 (2003), pp. 1199–1241.
[33] D.R. Hirschfeldt, B. Khoussainov, R.A. Shore, and A.M. Slinko, Degree spectra and computable dimensions in algebraic structures, *Annals of Pure and Applied Logic* 115 (2002), pp. 71–113.
[34] W. Hodges; *A Shorter Model Theory*, Cambridge University Press, 1997.
[35] A. Kach and D. Turetsky, Δ^0_2 categoricity of equivalence structures, *New Zealand Journal of Mathematics* 39 (2009), pp. 143–149.
[36] N.G. Khisamiev, Constructive Abelian p-groups, *Siberian Advances in Mathematics* 2 (1992), pp. 68–113 (English translation).
[37] B. Khoussainov and R.A. Shore, Computable isomorphisms, degree spectra of relations and Scott families, *Annals of Pure and Applied Logic* 93 (1998), pp. 153–193.
[38] O. Kudinov, An autostable 1-decidable model without a computable Scott family of $\exists$-formulas, *Algebra and Logic* 35 (1996), pp. 458–467.
[39] P. LaRoche, Recursively presented Boolean algebras, *Notices* AMS 24 (1977), A552–A553.
[40] S. Lempp, C. McCoy, R. Miller and R. Solomon, Computable categoricity of trees of finite height, *Journal of Symbolic Logic* 70 (2005), pp. 151–215.

[41] C.F.D. McCoy, Δ^0_2-categoricity in Boolean algebras and linear orderings, *Annals of Pure and Applied Logic* 119 (2003), pp. 85–120.
[42] C.F.D. McCoy, On Δ^0_3-categoricity for linear orders and Boolean algebras, *Algebra and Logic* 41 (2002), pp. 295–305 (English translation).
[43] T. Millar, Recursive categoricity and persistence, *Journal of Symbolic Logic* 51 (1986), pp. 430–434.
[44] R. Miller, **d**-computable categoricity for algebraic fields, *Journal of Symbolic Logic* 74 (2009), pp. 1325–1351.
[45] R. Miller, The computable dimension of trees of infinite height, *Journal of Symbolic Logic* 70 (2005), pp. 111–141.
[46] R. Miller and A. Shlapentokh, Computable categoricity for algebraic fields with splitting algorithms, to appear in the *Transactions of the American Mathematical Society.*
[47] R. Miller and H. Schoutens, Computably categorical fields *via* Fermat's Last Theorem, to appear in *Computability.*
[48] J.B. Remmel, Recursively categorical linear orderings, *Proceedings of the American Mathematical Society* 83 (1981), pp. 387–391.
[49] J.B. Remmel, Recursive isomorphism types of recursive Boolean algebras, *Journal of Symbolic Logic* 46 (1981), pp. 572–594.
[50] R.L. Smith, Two theorems on autostability in p-groups, in *Logic Year* 1979–80, Univ. Connecticut, Storrs, Lecture Notes in Mathematics 859, (Springer, Berlin, 1981), pp. 302–311.
[51] I.N. Soskov, Intrinsically hyperarithmetical sets, *Mathematical Logic Quarterly* 42 (1996), pp. 469–480.