# The Tutte polynomial

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## Abstract

Every graph has an associated polynomial in two variables called the Tutte polynomial. The Tutte polynomial encodes a considerable amount of information about the graph, including the number of spanning trees, the chromatic polynomial, the flow polynomial, the all-terminal reliability and the Jones polynomial of the associated alternating knot. We present proofs that these polynomials are specialisations of the Tutte polynomial. We also review the history of the Tutte polynomial. We discuss the computational complexity of the polynomial, and the fact that evaluating the Tutte polynomial of a graph at a point is #$P$-hard, apart from at nine special points and along a curve, for which it is computationally easy. Finally, we outline related research areas of recent interest and give potential future work.

# Contents

# Chapter 1

# Introduction

Every graph has an associated polynomial in two variables called the Tutte polynomial. This polynomial can give us a considerable amount of information about the graph—the number of ways it can be coloured, the number of ways we can make flows out of the edges of a graph, and the number of ways we can give orientations to edges such that there are no oriented cycles, to name just a few.

The information encoded in the Tutte polynomial has a number of applications, and is useful in a wide variety of domains. One such piece of information is the number of spanning trees of a graph, which is important in the theory of electrical networks. Another is the number of colourings of a graph. A well-known application of this information is finding whether a map (such as a map of the world) can be coloured using four colours with each adjacent region (or country) a different colour. However, many other applications exist. A graph might represent a scheduling problem where the edges correspond to items that cannot be scheduled at the same time. For example, consider a graph where the vertices correspond to exams, and there is an edge between vertices if there is at least one student taking both exams. Then, a vertex colouring, where each colour corresponds to a different day for an exam, gives a schedule where no student has to sit two exams in the same day.

Alternatively, a graph might correspond to a network of nodes, where something travels between the nodes. An obvious example is a computer network. The all-terminal reliability, which can be directly calculated given the Tutte polynomial, gives the probability that the whole network is still operational given the probability that any connection between nodes goes down. A flow is a graph with values assigned to the edges, which could be used to model traffic in a road system, fluid in pipes, or anything where something travels through the network of nodes. From the Tutte polynomial of a graph, we can find if a flow is possible for the graph.

The Jones polynomial, which follows easily when given the Tutte polynomial of the graph associated with the knot, can be used to identify unique knots, which has applications in biology and chemistry. For example, it can be used to distinguish between knotted molecules, where the same compound behaves in different ways depending on how the chain of molecules is knotted. Chemists are also interested in identifying when a compound has stereoisomers—molecules that differ only in their three-dimensional orientation, but may have different properties. Knowing which knots are *chiral* (when its mirror image is distinct from the original) can help do this. Given the Tutte polynomial (or more specifically, the Jones polynomial), it is straightforward to find if a knot is chiral.

A final application of the Tutte polynomial that we will mention is relevant to a branch of physics called statistical mechanics. This branch looks at the aggregate behaviour of large systems of particles by applying tools of probability theory. An important model in this field is the Ising model, which can model a system of particles as it changes state; from a liquid to a gas, for example. The Potts model is a generalisation which can also be used to model magnetism. Most of the aggregate quantities in a model of such a system can be described in terms of what is known as the *partition function*. Although we will not discuss it further, the partition function of the Ising or Potts model is another piece of information hiding in the Tutte polynomial.

Given this plethora of applications, an important question is how we calculate the Tutte polynomial, and the complexity of such a computation. In what follows, we start by giving the relevant graph theory preliminaries in Section 1.1, before defining the polynomial and how to find it for a graph in Section 1.2, and giving some examples of what this polynomial tells us about the graph in Section 1.3. We then look at how the polynomial came about and give a short history of its development in Chapter 2. In Chapter 3 we present proofs that three graph polynomials and a knot polynomial can be obtained from the Tutte polynomial. In Chapter 4 we give a visual interpretation of the Tutte polynomial in the plane, then we look at the complexity of calculating the polynomial in Chapter 5. Finally, in Chapter 6 we look at the recent developments in the area and potential future work.

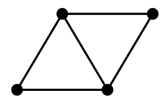


Figure 1.1: A planar graph on four vertices that we call $G_A$.

## 1.1 Graph theory preliminaries

A *graph* $G$ is a set of *vertices*, denoted $V(G)$, together with a set of *edges*, denoted $E(G)$. Each edge is *incident* to two vertices that need not be distinct; if the vertices are equal, then the edge is a *loop*. Two edges that are incident to the same vertex are *adjacent*. If two edges are incident to the same pair of distinct vertices, the edges are *parallel*. A graph with loops or parallel edges is sometimes called a *multigraph*—we allow graphs to have loops or parallel edges.

A graph can be drawn using a point or small circle for a vertex and drawing a line between two points if there is an edge incident to the corresponding pair of vertices. An example is given in Figure 1.1. There are different ways a graph can be drawn in the plane; each drawing is called an *embedding* of the graph. A *planar* graph can be drawn (that is, it has an embedding) with no edges crossing, as is the case in the figure.

A path between two vertices $v_0$ and $v_n$ is a sequence of edges $e_1, e_2, \ldots, e_n$ where each edge $e_i$ is incident to $v_{i-1}$ and $v_i$, and all of the $v_j$'s (for $0 \leq j \leq n$) are distinct. A graph is *connected* if it has a path between every pair of vertices, otherwise it is *disconnected*. A *cycle* is a path (of at least one edge) beginning and ending at the same vertex. A *forest* is a graph with no cycles—if the graph is also connected, it is a *tree*. A *spanning tree* of a connected graph $G$ is a tree containing all the vertices of $G$ and a subset (possibly all) of the edges. An *isthmus* is an edge that is not in any cycle; every edge in a forest is an isthmus.

If a graph is not connected, the vertex set $V(G)$ can be partitioned into disjoint subsets $V_1, V_2, \ldots, V_k$ where each vertex $v_i \in V_i$ has a path to each other vertex in $V_i$, but no path to any other vertex $v_j \in V_j$ where $j \neq i$. Each subset $V_i$, together with the edges incident to vertices in this subset, is called a *component* of $G$ and $G$ is said to have $k$ components.

A *subgraph* of $G$ is a graph $H$ where $E(H) \subseteq E(G)$ and $V(H) \subseteq V(G)$. One common operation on a graph $G$ that results in a subgraph is *edge-deletion*: the subgraph (denoted $G \backslash e$) is, as the name suggests, $G$ with a single edge $e$ removed, so $V(G \backslash e) = V(G)$ and $E(G \backslash e) = E(G) - \{e\}$. Another common operation on a graph $G$ and an edge $e$ of $G$ is *edge-contraction*, denoted $G/e$ (though $G/e$ is not

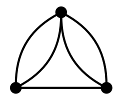


Figure 1.2: The plane dual of the planar graph in Figure 1.1.

a subgraph of $G$ unless $e$ is a loop). This graph is obtained from $G$ by replacing the edge $e$ and the vertices incident to it with a single vertex.

A *directed graph* assigns a direction to each edge, so for the two vertices incident to an edge one is the *head* and one is the *tail*, and the direction of the edge is towards the head vertex. A *signed* graph associates a sign ($+$ or $-$) with each edge.

Two graphs $G$ and $H$ are *isomorphic* if there exist bijections $\phi$ and $\psi$ between the edges in $G$ and the edges in $H$, and between the vertices in $G$ and the vertices in $H$, respectively, such that an edge $e$ in $G$ is adjacent to the vertices $v_1$ and $v_2$ if and only if the edge $\phi(e)$ in $H$ is adjacent to the vertices $\psi(v_1)$ and $\psi(v_2)$. Informally, two graphs can be thought of as isomorphic when we can go from one graph to the other by relabelling the edges and vertices.

Given an embedding of a planar graph, drawn with no edges crossing, we can obtain its *plane dual*. In an embedding of a graph, we call a minimal region enclosed by a cycle of edges a *face*. Each embedding also has an "outside" region—this is known as the *infinite face*. We obtain the plane dual $G^*$ of an embedding of a graph $G$ by making each face in the embedding of $G$ a vertex in $G^*$ (including the infinite face), with an edge between vertices in $G^*$ if the corresponding faces in the embedding of $G$ share an edge. As an example, a plane dual of $G_A$, using the embedding in Figure 1.1, is given in Figure 1.2.

Bondy and Murty give a thorough introduction to these and other concepts pertinent to graph theory in [10].

## 1.2 The Tutte polynomial

Every graph $G$ has an associated polynomial in two variables called the *Tutte polynomial* and denoted $T(G;x,y)$. In this section we define the polynomial in two equivalent ways: a recursive definition on the edges of the graph, and a decomposition on all the subgraphs of a graph with the same set of vertices and a subset of the edges. We also give examples of calculating the Tutte polynomial for graphs by either approach.

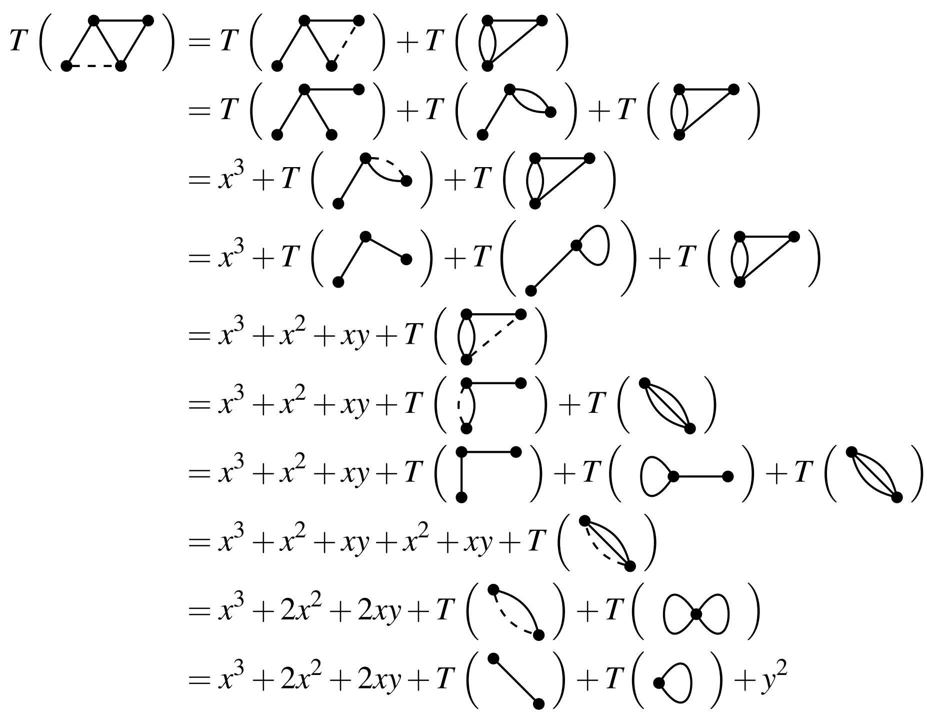
The Tutte polynomial for $G$ is given by the recursive definition:

$$T(G;x,y) = \begin{cases} 1 & \text{if } G \text{ has no edges,} \\ xT(G\backslash e;x,y) & \text{if } e \text{ is an isthmus,} \\ yT(G/e;x,y) & \text{if } e \text{ is a loop,} \\ T(G\backslash e;x,y) + T(G/e;x,y) & \text{otherwise,} \end{cases} \tag{1.1}$$

where $e$ is an edge of $G$.

In words, we can easily calculate the Tutte polynomial of a graph containing only loops and isthmuses: for such a graph $G$ we have $T(G;x,y) = x^r y^s$ where $r$ is the number of isthmuses and $s$ the number of loops. If this is not the case, we can pick any edge $e$ that is not a loop or an isthmus and find the Tutte polynomials for $G\backslash e$ and $G/e$. Finding either of these may require recursively repeating this process, until we have a graph of just loops and isthmuses. The Tutte polynomial $T(G;x,y)$ is then the sum of these two polynomials.

For example, the Tutte polynomial of the graph $G_A$, given in Figure 1.1, is calculated below. $T(G;x,y)$ is written as $T(G)$ for readability, and the dashed edge indicates the edge that is picked when applying the recurrence equation.




$$= x^3 + 2x^2 + 2xy + x + y + y^2 \tag{1.2}$$

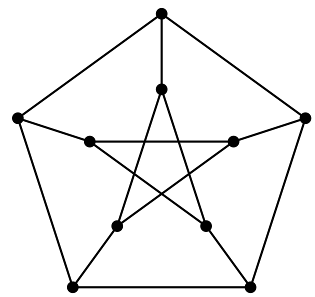


Figure 1.3: The Petersen graph, which we label $G_P$.

As a second example, consider the Petersen graph, given in Figure 1.3. The Petersen graph is a non-planar graph on 10 vertices. Its Tutte polynomial is given by

$$\begin{aligned} T(G_P;x,y) = {} & x^9 + 6x^8 + 21x^7 + 56x^6 + 114x^5 + 12x^5y + 170x^4 + 70x^4y \\ & + 180x^3 + 170x^3y + 30x^3y^2 + 120x^2 + 240x^2y + 105x^2y^2 \\ & + 15x^2y^3 + 36x + 168xy + 171xy^2 + 65xy^3 + 10xy^4 + 36y \\ & + 84y^2 + 75y^3 + 35y^4 + 9y^5 + y^6. \end{aligned} \tag{1.3}$$

This can be calculated in the same manner, but the process is rather lengthy. As an indication, after performing all the edge-deletions and edge-contractions that are required (1286 times an edge will be picked and the edge-deletion and edge-contraction performed on the graph), there are 2000 graphs containing just isthmuses and loops to consider.

An equivalent way to compute the Tutte polynomial is using the notion of rank. For a graph $G$ where $A \subseteq E(G)$, we denote by $k(A)$ the number of connected components of the graph with edges $A$ and all the vertices of $G$[1]. The *rank* of a subgraph of $G$ with vertices $V(G)$ and edges $A$ is then defined as

$$r(A) = |V(G)| - k(A). \tag{1.4}$$

The Tutte polynomial is then given by

$$T(G;x,y) = \sum_{A \subseteq E(G)} (x-1)^{r(E(G))-r(A)}(y-1)^{|A|-r(A)}. \tag{1.5}$$

This is equivalent to the recursive definition given in (1.1) [34].

[1]Note each isolated vertex is a connected component, so adds 1 to $k(G)$.

| | number of edges | | | | | |
|---|---|---|---|---|---|---|
| rank | 0 | 1 | 2 | 3 | 4 | 5 |
| 3 | | | | 8 | 5 | 1 |
| 2 | | | 10 | 2 | | |
| 1 | | 5 | | | | |
| 0 | 1 | | | | | |

Table 1.1: The number of subgraphs of $G_A$ with given rank and number of edges.

We can see from this definition that the Tutte polynomial is well-defined. In particular, the polynomial is the same regardless of the order in which edges are selected when applying (1.1).

Consider again the graph in Figure 1.1. This graph has five edges; since each can be present or absent in a subgraph there are $2^5 = 32$ subgraphs on the same set of vertices. Clearly there is one subgraph with all five edges that, with four vertices and one component, has rank three by (1.4). Thus it contributes a $(y-1)^2$ term. There are five subgraphs with four edges, one for each edge that can be absent. Each, being connected so again of rank three, contributes a $(y-1)$ term. There are ten graphs with three edges: eight are spanning trees so contribute a 1, the other two have rank two and thus contribute $(x-1)(y-1)$. There are ten graphs with two edges: all ten have two components and contribute $(x-1)$ terms. There are five graphs with a single edge, each having three components, so they contribute an $(x-1)^2$ term each. Finally, there is one graph with no edges, contributing $(x-1)^3$. Table 1.1 summarises the number of subgraphs of a given rank with a given number of edges. The sum of these terms

$$(y-1)^2 + 5(y-1) + 8 + 2(x-1)(y-1) + 10(x-1) + 5(x-1)^2 + (x-1)^3$$

matches the value of the Tutte polynomial for the graph of Figure 1.1 found earlier.

We could find the Tutte polynomial of the Petersen graph (Figure 1.3) in the same way. However, with 15 edges, there are $2^{15} = 32768$ subgraphs to consider. We saw earlier that the Tutte polynomial is not "easy" to calculate using the recursive approach either. The difficulty of finding the Tutte polynomial for a given graph is looked at further in Chapter 5.

### 1.2.1 Internal and external activities

In the previous section we saw two equivalent definitions of the Tutte polynomial. In this section we describe a third; in fact, this definition was how the Tutte polynomial was originally defined [33]. Although the definitions given in the previous section are usually the most practical, we will make use of this definition, in particular to demonstrate the Jones polynomial is an evaluation of the Tutte polynomial in Section 3.4.3.

Tutte defined what he called the dichromate $\chi(G;x,y)$ of a connected graph $G$ as follows: if $G$ has no edges, then $\chi(G;x,y) = 1$. Otherwise, give the edges an arbitrary ordering. Let $T$ be a spanning tree of $G$. Removing any edge of $T$ results in a forest of exactly two components. Alternatively, for any edge $e \in E(G) - E(T)$, the subgraph of $G$ with edge set $E(T) \cup \{e\}$ has a single cycle containing $e$. An edge $e$ is *internally active* in $T$, where $e \in E(T)$, if it precedes all other edges in $G$ with ends in different components of $T \backslash e$. An edge $e$ is *externally active* in $T$, where $e \notin E(T)$, if it precedes all other edges in the single cycle contained in the subgraph of $G$ with edge set $E(T) \cup \{e\}$. The *internal activity* and *external activity* of $T$ are then the numbers of edges that are internally active and externally active for $T$, respectively. Then

$$\chi(G;x,y) = \sum_{T \subset G} x^r y^s, \tag{1.6}$$

where the sum is taken over each spanning tree $T$ of $G$, $r$ is the internal activity of $T$, and $s$ is the external activity of $T$.

Tutte showed that the number of spanning trees with internal activity $r$ and external activity $s$ is independent of the ordering given to the edges, so $\chi(G;x,y)$ is well-defined for any graph $G$.

The definition can be extended to a disconnected graph $G$ with components $G_1, \ldots, G_k$ as follows:

$$\chi(G;x,y) = \prod_{i=1}^{k} \chi(G_i;x,y) \tag{1.7}$$

Tutte also showed that this definition is equivalent to the recursive definition given in (1.1).

Since the sum in (1.6) is over each spanning tree of a connected graph, it is clear to see that setting $x$ and $y$ to 1 gives each term a value of 1, so the sum gives the number of spanning trees. However this is but one of many pieces of information about a graph encoded in this polynomial, as we will see in the next section.

## 1.3 Data encoded in the Tutte polynomial

In this section we give an overview of some of the information encoded in the Tutte polynomial.

As previously mentioned, the Tutte polynomial $T(G;x,y)$ of a graph $G$ evaluated at $(x,y) = (1,1)$ gives the number of spanning trees for a connected graph $G$. Given $T(G;x,y)$, suitable values for $x$ and $y$ give other information about the graph. We will see that the number of spanning subgraphs is one such piece of information.

We also look at some one-variable polynomials that can be obtained from the Tutte polynomial. These polynomials are referred to as *specialisations* of the Tutte polynomial, as they are given by first performing a suitable substitution for $x$ and $y$ in terms of a variable, say $\lambda$, and then normalising by multiplying by a positive or negative monomial of $\lambda$. In Chapter 3 we will demonstrate that these polynomials—in particular, the chromatic polynomial, flow polynomial, all-terminal reliability and Jones polynomial—are specialisations of the Tutte polynomial.

To illustrate this information, we will continue to make reference to $G_A$, the graph given in Figure 1.1 on page 3, and the Petersen graph $G_P$ (Figure 1.3, page 6).

As useful as the Tutte polynomial is, not all information about a graph is encoded in it. Two non-isomorphic graphs can have the same Tutte polynomial. For example, the two graphs in Figure 1.4 both contain three edges, all of which are isthmuses. Thus they both have Tutte polynomial $x^3$. However, we can see they are not isomorphic. More generally, we can see from (1.1) that the Tutte polynomial is not concerned with which vertices isthmuses (or loops) are connected to. Therefore we can delete any isthmus (or loop) from a graph $G$, and add an edge $e$ to the resulting graph such that $e$ is an isthmus (or loop), and the Tutte polynomial of this graph is the same as $T(G;x,y)$. The Tutte polynomial is also unaffected by isolated vertices.

However, the question remains as to whether two markedly different graphs, and not just differing in the ways described, can have the same Tutte polynomial. Are there some families of graphs such that each graph in the family has a unique Tutte polynomial? These are recent areas of research that will be discussed further in Chapter 6.

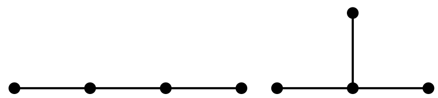


Figure 1.4: Two non-isomorphic graphs with the same Tutte polynomial.

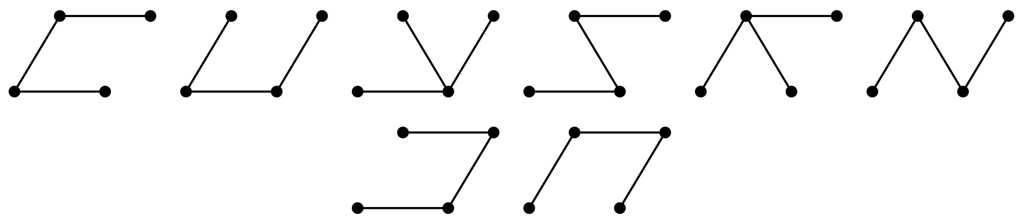


Figure 1.5: The eight spanning trees of the graph in Figure 1.1.

### 1.3.1 Counting spanning trees and spanning subgraphs

We first consider all possible spanning trees for the connected graph in Figure 1.1. We can see that there are eight, as given in Figure 1.5. The evaluation $T(G_A;1,1)$ of this graph's Tutte polynomial, see (1.2), also gives eight as we expect. Similarly, evaluating the Tutte polynomial of the Petersen graph at $(1,1)$ tells us that it has 120 spanning trees.

Both of these examples are instances of connected graphs; the question remains as to what the evaluation at $(1,1)$ gives for graphs that are not connected. A *maximal spanning forest* of $G$ is a subgraph of $G$, containing all the vertices of $G$, that is a forest with the same number of components as $G$. In other words, a maximal spanning forest of $G$ is the union of spanning trees for each component of $G$. For a graph $G$ that is not connected, $T(G;1,1)$ gives the number of maximal spanning forests.

A spanning subgraph of $G$ is like a spanning tree in that it is a graph with all the vertices of $G$ and for every two vertices with a path between them in $G$, there is a path between them in the subgraph. However a spanning subgraph relaxes the requirement that this subgraph's edge set is minimal (that is, it does not have to be a tree). The spanning subgraphs not already in Figure 1.5 are given in Figure 1.6.

It turns out that the number of spanning subgraphs is given by $T(G;1,2)$. In the case of the graph in Figure 1.1, $T(G_A;1,2) = 14$. This is consistent with the 14 subgraphs in Figures 1.5 and 1.6.

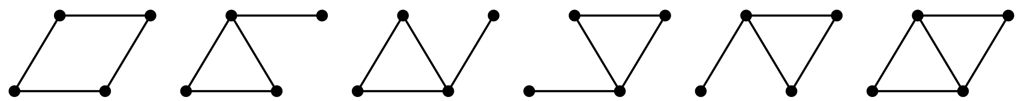


Figure 1.6: Six spanning subgraphs of the graph in Figure 1.1, on four or five edges.

### 1.3.2 The chromatic polynomial

A *vertex colouring*, also called a *proper colouring* or just a *colouring*, of a graph assigns a colour to each vertex so that no vertices connected by an edge share the same colour. The problem of finding such a graph colouring using $\lambda$ colours (known as a $\lambda$*-colouring*) has a long and prolific history—most notably, a proof to the long-standing conjecture that any loopless planar graph has a 4-colouring was long sought after. A controversial proof was found by Appel and Haken in 1976 [4], and following the same general idea Robertson et al. came up with a simpler proof in 1996 [29]. Along the way, efforts to solve the four colour problem led to a number of new developments in graph theory—one of which was the concept of the chromatic polynomial [8].

The chromatic polynomial $P(G,\lambda)$ gives the number of ways a graph $G$ can be coloured with $\lambda$ colours. For example, a graph of $v$ isolated vertices has $P(G,\lambda) = \lambda^v$ since each vertex can be coloured with any of the $\lambda$ colours. Similarly, a graph $G$ made up of $k$ components $G_1, G_2, \ldots, G_k$ has $P(G,\lambda) = \prod_{i=1}^{k} P(G_i,\lambda)$. A tree $G$ with $v$ vertices has chromatic polynomial $P(G,\lambda) = \lambda(\lambda-1)^{v-1}$ (we can start at any vertex and colour it any of the $\lambda$ colours, then each adjacent vertex can be coloured any of the other $\lambda-1$ colours, and we can repeat this process until the tree is completely coloured). Any graph with a loop has chromatic polynomial 0, as there is no way to colour the vertex at both ends of the loop with different colours.

The chromatic polynomial can be found by evaluating the Tutte polynomial $T(G;1-\lambda,0)$ and multiplying by a positive or negative monomial in $\lambda$ that depends on the number of vertices and components of the graph $G$ (more details are given in Section 3.1.1). For the graph in Figure 1.1, the chromatic polynomial is:

$$\begin{aligned} P(G_A,\lambda) &= (-1)^3 \lambda T(G_A; 1-\lambda, 0) \\ &= -\lambda\big((1-\lambda)^3 + 2(1-\lambda)^2 + (1-\lambda)\big) \\ &= \lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda \end{aligned}$$

Note that this polynomial evaluates to zero when $\lambda$ is one or two, but $P(G_A, 3)$

equals six. Thus the graph $G_A$ is 3-colourable, and can be coloured in six ways using three colours.

### 1.3.3 The flow polynomial

Another essential area of graph theory concerns finding flows for graphs [20]. A *flow* is an assignment of a value to each edge of a directed graph so that, for each vertex, the sum of the values of all incident edges where the vertex is the tail (that is, "outgoing" edges) is equal to the sum of the values of all incident edges where the vertex is the head ("incoming" edges).

A *nowhere-zero* flow also requires that each edge value be non-zero. If a graph has a flow assigning values of an abelian group[2] $H$, it is called an *$H$-flow*. A *$k$-flow* is a $\mathbb{Z}$-flow where edges are assigned values between 0 (or 1, if nowhere-zero) and $k-1$.

The *flow polynomial* $F(G,\lambda)$ gives the number of nowhere-zero $H$-flows for a graph $G$ and abelian group $H$ of order $\lambda$. We can calculate the flow polynomial of the graph in Figure 1.1 from the Tutte polynomial:

$$\begin{aligned} F(G_A,\lambda) &= (-1)^2 T(G_A;0,1-\lambda) \\ &= (1-\lambda)+(1-\lambda)^2 \\ &= \lambda^2-3\lambda+2 \end{aligned}$$

More detail is given on this evaluation of the Tutte polynomial in Section 3.2.1.

## 1.4 The recipe theorem

Before demonstrating that the data described in the previous section can be obtained from the Tutte polynomial, it pays to first introduce an important tool in this process. If a property of a graph can be shown to demonstrate a few simple rules, there is a general-purpose formula giving this property as an evaluation of the Tutte polynomial. This formula was discovered by Oxley and Welsh [28] and this result is now known as the "recipe theorem".

[2] An abelian group is a set of elements with an associative, commutative binary operation such that one element is the identity and every element has an inverse.

A generalised[3] *Tutte-Grothendieck invariant* (TG-invariant) is a map $f$ taking a graph as input, that has the same value for isomorphic graphs and satisfies

$$f(G) = af(G\backslash e) + bf(G/e) \tag{1.8}$$

where $e \in E(G)$ is not a loop or isthmus, and

$$f(G_1 \cup G_2) = f(G_1)f(G_2) \tag{1.9}$$

for any $G_1$ and $G_2$ where the edge sets are disjoint, and the union of a spanning tree of $G_1$ and a spanning tree of $G_2$ is a spanning tree of $G_1 \cup G_2$[4]. Note that $G_1 \cup G_2$ is defined as the graph $G$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$.

The recipe theorem states that any such $f$ is a specialisation of the Tutte polynomial that can be expressed as

$$f(G) = a^{|E|-r(E(G))} b^{r(E(G))} T(G; \frac{x_0}{b}, \frac{y_0}{a}) \tag{1.10}$$

where $x_0$ and $y_0$ are the values $f$ takes for a graph consisting of a single isthmus and single loop respectively.

This will be used in Chapter 3 to show some well-known graph polynomials are just specialisations of the Tutte polynomial. But first, we look at the history of the Tutte polynomial.

[3]A TG-invariant has $a = 1$, $b = 1$ in (1.8), whereas a generalised TG-invariant relaxes this condition. We refer to generalised TG-invariants simply as TG-invariants in the remainder of this report.

[4]These conditions on $G_1$ and $G_2$ may seem unusual, but follow more readily when working with matroids instead of graphs (in which case $f(M(G_1) \oplus M(G_2)) = f(M(G_1))f(M(G_2))$. Often when just concerned with graphs, authors simplify this condition to disjoint graphs $G_1$ and $G_2$ (where the vertex sets are also disjoint).

# Chapter 2

# History

The Tutte polynomial is closely related to another bivariate polynomial, known as the *Whitney rank-generating polynomial*, and no discussion of the history of the Tutte polynomial would be complete without mention of this polynomial. Due to their close connection, these polynomials are collectively referred to as the Whitney-Tutte (or Tutte-Whitney) polynomials. They have also been given a variety of other names over the years, and their nomenclature can be the source of some confusion. In this section we discuss how these polynomials came about, and give a history of the names they have been given.

Like many other areas of graph theory, the origins of the Tutte polynomial can be traced back to attempts to find a solution to the "4-colour problem"; that is, given any map, can the regions be coloured using four colours and no adjacent regions have the same colour? In 1913, Birkhoff approached the problem by looking at the number of ways a map can be coloured using $\lambda$ colours [7]. He came up with a formula for the number of such colourings that was a polynomial in $\lambda$, with coefficients (that he denoted $(p,s)$) depending on the number of subgraphs of $p$ edges and $s$ components, with $p$ and $s$ fixed for each term.

Whitney further developed Birkhoff's work, publishing two papers in 1932 related to finding the number of vertex colourings of a graph [37, 38]. The problem of finding a colouring for the regions of a map is equivalent to converting the map into a graph, treating the boundaries as edges and positioning a vertex at each point where boundaries meet, and then finding a vertex colouring for the plane dual of the graph. Whitney interpreted the coefficients $(p,s)$ as $m_{ij}$ (the number of sets of edges of rank $i$ and nullity $j$—where the nullity is the number of edges minus the rank). The $m_{ij}$ give the coefficients of the Whitney rank-generating function, which we will discuss shortly, with the $i$ index in reverse order (for example, $m_{ij}$

for the graph in Figure 1.1 is given by the value in the $i$th row and $j$th column of Table 1.1, starting at index 0). For this reason, these papers of Whitney's are often cited as the source of these polynomials, even though they are not explicitly defined in them.

Whitney's paper was also significant in being the first publication of the deletion-contraction relation that is now a cornerstone of the theory of Tutte-Whitney polynomials [37]. However, it is included almost as an afterthought, and gives credit to R. M. Forster for the result. It stated that for a non-loop edge $e$, the $m_{ij}$ satisfy the relation $m_{ij}(G) = m_{ij}(G\backslash e) + m_{i-1,j}(G/e)$.

Birkhoff and Lewis later authored a detailed account on the theory of the one-variable polynomials for map colourings, introducing the name "chromatic polynomials" [8]. It is also worth noting that the problem of finding a map colouring is closely related to the number of $H$-flows for the associated graph. In fact, as we will see in Section 3.2.1, the single-variable polynomial for the number of vertex colourings of a graph's plane dual is analogous, up to a scaling factor, to the polynomial known as the flow polynomial.

Tutte published a seminal work in 1947 that first explicitly defined an unnamed two-variable polynomial denoted $Q(G;x,y)$ that would come to be known as the Whitney rank-generating function (we will discuss this polynomial shortly) [31]. He arrived at this polynomial by looking at functions on linear graphs that take the same value on isomorphic graphs and satisfy a simple deletion-contraction relation (as in (1.8) with $a = 1$ and $b = 1$). This laid the groundwork for what we now know as a Tutte-Grothendieck invariant. The paper also looked at flows (under the name $\beta$-colourings) and gave the key result that the number of $H$-flows of a graph for an abelian group $H$ depends only on the order of $H$.

In 1954, Tutte introduced another two-variable polynomial $\chi(G;x,y)$ that generalised, for a given graph, the two single-variable polynomials: the chromatic polynomial and the flow polynomial[1] [33]. Tutte found this polynomial using the internal and external activities of a graph, as described in Section 1.2.1. He called this polynomial the *dichromate* of a graph—it would only later be known as the Tutte polynomial. He also identified that the chromatic and flow polynomials satisfy simple deletion-contraction relations (as we will see later, in equations (3.1) and (3.2.1)), and that the Tutte polynomial does as well, as in the final case of (1.1).

The Whitney rank-generating function $R(G;u,v)$, as we now know it, is given

[1] The flow polynomial was not given by this name at this point in time—it is simply referred to as the number of "colour cycles" over a non-empty finite set.

by

$$R(G;u,v) = \sum_{A \subseteq E(G)} u^{r(E(G))-r(A)} v^{|A|-r(A)}.$$

The coefficient of a term $u^i v^j$ counts the number of subgraphs of rank $i$ with $j$ edges. For example, recall Figure 1.1 that has subgraphs of rank and edge set size as given in Table 1.1. Its Whitney rank-generating function is

$$R(G;u,v) = u^3 + 5u^2 + 10u + 2uv + 8 + 5v + v^2.$$

In 1967, Tutte further developed the theory of the polynomial he denoted $Q(G;x,y)$, now calling it the *dichromatic polynomial* of a graph [34]. As mentioned earlier, this polynomial is in effect the Whitney rank-generating function, in terms of $x$ and $y$, but it is multiplied by a factor of $x^{k(G)}$. Compare this to the Tutte polynomial, given in terms of rank, in (1.5). It is clear that these polynomials are equivalent: we can go from the Tutte polynomial to the Whitney rank-generating function, for example, by substituting $x = u+1$ and $y = v+1$. This connection was also identified by Tutte [34] and Crapo [14].

Although this project focuses on the Tutte polynomial as it relates to graphs, the definition can be extended to matroids (abstract structures that capture the notion of dependence), as was done by Crapo [14]. His 1969 paper was also significant in introducing the names "rank-generating function" and "Tutte polynomial". Brylawski extended the theory of Tutte invariants, as he called them, to matroids [13] (his paper also identified the relevancy of the Grothendieck ring; the "TG-invariant" naming convention came later). Both Crapo's and Brylawski's extensions of the Tutte polynomial to matroids were in fact in Tutte's PhD thesis, completed at Cambridge University in 1948, but these results were not published [19].

In 1979, Oxley and Welsh generalised the set of conditions for a TG-invariant, as given in (1.8), and introduced what would come to be known as the recipe theorem [28], as was described in Section 1.4.

A considerable amount of literature around this time and since has been identifying connections to the Tutte polynomial, over a broad spectrum of areas. Of particular note is the connection with the Jones polynomial [23]. Vaughan Jones discovered this polynomial while working on the previously unrelated branch of von Neumann algebras when he came across a set of relations bearing a striking resemblance to the braid group. Apart from being a breakthrough in terms classifying knots (initiating a flurry of research on the previously dormant area of knot polynomials), this tied together previously unconnected areas of mathematics and he was awarded the Fields medal for this work in 1990. Thistlethwaite identified the connection between the Jones polynomial and the Tutte polynomial in 1987 [30] which we will discuss in Section 3.4.3.

Since the Tutte polynomial was discovered, a large number of specialisations have been found for graphs, matroids, and even other structures, such as linear codes. We have touched on a few of these already, and in the following chapter we concentrate on four of them.

# Chapter 3

# Specialisations of the Tutte Polynomial

In this chapter we look in more detail at three graph polynomials and a knot polynomial that are specialisations of the Tutte polynomial. We can obtain these one-variable polynomials (in $\lambda$, say) by performing a substitution for $x$ and $y$ in terms of $\lambda$, and "scaling" by an easily-calculated factor—a monomial in $\lambda$. In this chapter rather than just stating it is so, we aim to show *why* each specialisation can be obtained from the Tutte polynomial. We present proofs that these polynomials are specialisations, and give the scaling factors for each case.

This chapter is by no means a complete account of all Tutte-polynomial specialisations. Specialisations also include evaluations such as the number of spanning trees (as mentioned in Section 1.2.1) and the number of spanning subgraphs (as mentioned in Section 1.3.1). In particular, it is clear that $(1,1)$ gives the number of spanning trees from the definition of the Tutte polynomial in terms of internal and external activities (Section 1.2.1). In Chapter 4 we see a visual representation of how these specialisations relate.

Specialisations are not even limited to graphs polynomials, knot polynomials and enumerative properties of graphs. For more specialisations of the Tutte polynomial, refer to [12, 17, 36].

## 3.1 The chromatic polynomial

Recall that the chromatic polynomial $P(G,\lambda)$ counts the number of ways a graph $G$ can be coloured in $\lambda$ colours. The chromatic polynomial can be given by the recursive definition:

$$P(G,\lambda) = P(G\backslash e,\lambda) - P(G/e,\lambda) \tag{3.1}$$

To see why this is so, first consider $P(G\backslash e,\lambda)$. There are at least as many colourings of the graph $G\backslash e$ as there are for the graph $G$: $P(G\backslash e,\lambda)$ includes all the proper colourings of $G$ but also colourings where the vertices at either end of $e$ are the same colour. Since $G/e$ treats this edge and its endpoints as a single vertex, $P(G/e,\lambda)$ enumerates exactly those "improper" colourings of $G\backslash e$. Thus the difference gives the number of proper colourings of $G$.

### 3.1.1 As a specialisation of the Tutte polynomial

We now present a proof that the chromatic polynomial is a specialisation of the Tutte polynomial. This is not a new result; Tutte outlined an inductive proof by comparing, firstly, the deletion-contraction relations of the Tutte polynomial and chromatic polynomial, and secondly, the formulas for graphs containing only loops and edges of the two polynomials [33]. More recently, Brylawski and Oxley gave a proof using the recipe theorem [12]. In both, they first prove that a related polynomial $\theta(G,\lambda)$ (following Tutte's notation) is a specialisation of the Tutte polynomial. The polynomial $\theta(G,\lambda)$ is related to the chromatic polynomial by $\theta(G,\lambda) = \lambda^{-k(G)}P(G,\lambda)$. This approach is necessary when using the recipe theorem, as we will discuss shortly.

We take a different approach, bypassing this intermediate polynomial. We give an inductive proof using simple, known properties of the chromatic polynomial that were described in Section 1.3.2.

**Proposition 3.1.** *The chromatic polynomial is a specialisation of the Tutte polynomial given by:*

$$P(G,\lambda) = (-1)^{r(E(G))}\lambda^{k(G)}T(G;1-\lambda,0) \tag{3.2}$$

*Proof.* We will prove this by induction on the number of edges of $G$. Let $G_0$ be a graph with 0 edges. We know the chromatic polynomial for a graph with $|V|$ isolated vertices is given by $\lambda^{|V|}$. Additionally, since $T(G_0,1-\lambda,0) = 1$ by (1.1), $k(G_0) = |V|$ and $r(E(G_0)) = |V|-|V| = 0$, we have

$$P(G_0,\lambda) = (-1)^0\lambda^{|V|}(1) = \lambda^{|V|}.$$

So (3.2) holds for graphs with zero edges.

Now suppose (3.2) holds for any graph with $n-1$ edges; for such a graph $G_{n-1}$,

$$P(G_{n-1},\lambda) = (-1)^{r(E(G))}\lambda^{k(G_{n-1})}T(G_{n-1};1-\lambda,0). \tag{3.3}$$

This is our induction assumption.

Consider $G_n$. If $G_n$ is not a forest, it contains at least one edge that is not an isthmus. First consider the case where this edge $e$ is also not a loop. We can delete or contract such an edge to get $G_n\backslash e$ or $G_n/e$ respectively. Since the edge is not an isthmus, the number of connected components will not change for either case. Since the edge is not a loop, contracting an edge will decrease the number of vertices by one. Thus, the rank of these graphs is given by

$$\begin{aligned} r(E(G_n\backslash e)) &= |V(G_n)| - k(G_n) \\ &= r(E(G_n)) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} r(E(G_n/e)) &= |V(G_n)| - 1 - k(G_n) \\ &= r(E(G_n)) - 1. \end{aligned} \tag{3.5}$$

Now, from the recursive definition of the chromatic polynomial in (3.1), the chromatic polynomial of $G_n$ satisfies

$$\begin{aligned} P(G_n,\lambda) &= P(G_n\backslash e,\lambda) - P(G_n/e,\lambda) \\ &= (-1)^{r(E(G_n\backslash e))}\lambda^{k(G_n\backslash e)}T(G_n\backslash e;1-\lambda,0) \\ &\quad - (-1)^{r(E(G_n/e))}\lambda^{k(G_n/e)}T(G_n/e;1-\lambda,0) \end{aligned}$$

by the induction assumption in (3.3); so

$$\begin{aligned} P(G_n,\lambda) &= (-1)^{r(E(G_n))}\lambda^{k(G_n)}T(G_n\backslash e;1-\lambda,0) \\ &\quad - (-1)^{r(E(G_n))-1}\lambda^{k(G_n)}T(G_n/e;1-\lambda,0) \end{aligned}$$

from (3.4) and (3.5) and the fact that the number of connected components does not change when a non-isthmus edge is deleted or contracted. Thus

$$\begin{aligned} P(G_n,\lambda) &= (-1)^{r(E(G_n))}\lambda^{k(G_n)}(T(G_n\backslash e;1-\lambda,0) + T(G_n/e,1-\lambda,0)) \\ &= (-1)^{r(E(G_n))}\lambda^{k(G_n)}T(G_n;1-\lambda,0) \end{aligned}$$

from the Tutte-polynomial recurrence equation (1.1), satisfying (3.2) for this case.

Secondly, we consider the case where the graph contains a loop. We know for such a graph the chromatic polynomial is identically zero. By (1.1) the Tutte polynomial is $T(G_n, 1-\lambda, 0) = 0 \cdot T(G_n/e, 1-\lambda, 0) = 0$, so (3.2) holds in this case as well.

Finally, consider the case where all edges that are isthmuses; $G_n$ is a forest. As discussed in Section 1.3.2 the chromatic polynomial for a forest $G_F$ with components $G_{T_1}, G_{T_2}, \ldots, G_{T_k}$ is known to be

$$\begin{aligned} P(G_F, \lambda) &= \prod_{i=1}^{k} \lambda(\lambda-1)^{|V(G_{Ti})|-1} \\ &= \lambda^k (\lambda-1)^{|V(G_F)|-k}. \end{aligned}$$

This directly follows from the fact that the chromatic polynomial of a disconnected graph is the product of each of its components, and the chromatic polynomial of a tree.

We can easily evaluate the Tutte polynomial for a forest $G_F$ using the recurrence equation (1.1), since every edge is an isthmus:

$$\begin{aligned} T(G_F; x, y) &= xT(G_F \backslash e; x, y) \\ &= x^{|E(G_F)|} \\ &= x^{|V(G_F)|-k} \end{aligned}$$

using the fact that each component of $G_F$ is a tree with one less edge than its number of vertices.

Thus

$$\begin{aligned} (-1)^{r(G_F)} \lambda^k T(G_F; 1-\lambda, 0) &= (-1)^{|V(G_F)|-k} \lambda^k (1-\lambda)^{|V(G_F)|-k} \\ &= \lambda^k (\lambda-1)^{|V(G_F)|-k} \\ &= P(G_F, \lambda). \end{aligned}$$

So (3.2) holds for any graph of $n$ edges, thus by induction the chromatic polynomial is a specialisation of the Tutte polynomial satisfying (3.2). □

As stated earlier, another approach to prove this result is to use the recipe theorem, as done by Brylawski and Oxley [12]. One might look at (1.10) and think it would be as simple as plugging in $a = 1$, $b = -1$, $x_0 = \lambda(\lambda-1)$ and

$y_0 = 0$. However, the chromatic polynomial in its normal form is not, strictly speaking, a TG-invariant.

Consider the connected graph $I_2$ of two isthmuses. We can decompose this graph into the union of two graphs, $I_1$ and $I_1'$, each with a single isthmus, sharing a vertex. The chromatic polynomial for these graphs is $P(I_1,\lambda) = P(I_1',\lambda) = \lambda(\lambda-1)$. But the chromatic polynomial for the union of these graphs is $P(I_2,\lambda) = \lambda(\lambda-1)^2$, not $\lambda^2(\lambda-1)^2$, violating (1.9).

A workaround for this problem is to instead use $\theta(G,\lambda) = \lambda^{-k(G)}P(G,\lambda)$, which is a TG-invariant, when invoking the recipe theorem. For the aforementioned example, we see that $\theta(I_1,\lambda) = \theta(I_1',\lambda) = \lambda - 1$ and $\theta(I_2,\lambda) = (\lambda-1)^2$, so (1.9) holds in this case. In fact, (1.9) always holds [12]. To then find the specialisation for the chromatic polynomial, we just multiply the specialisation of $\theta(G,\lambda)$ by $\lambda^{k(G)}$.

## 3.2 The flow polynomial

Recall that the *flow polynomial* $F(G,\lambda)$ gives the number of nowhere-zero $H$-flows for a graph $G$ and abelian group $H$ of order $\lambda$. The flow polynomial for a graph $G$ is given by

$$F(G,\lambda) = \sum_{A\subseteq E(G)} (-1)^{|E(G)-A|}\lambda^{|A|-r(A)}. \tag{3.6}$$

This is a result of the fact that for a connected graph $G$ with a spanning tree $T$, if we consider the $H$-flows where the edge-values for each edge in $G$ but not $T$ are fixed (that is, they are given some value in $H$), there is exactly one $H$-flow [20].

In turn, this implies that the number of $H$-flows for an abelian group $H$ depends only on the order of the group. As a further consequence to this, Tutte showed that a directed graph has a nowhere-zero $k$-flow if and only if it has a nowhere-zero $H$-flow for an abelian group $H$ of order $k$ [32, 33]. Thus (3.6) also gives the number of $k$-flows for a graph $G$.

### 3.2.1 The flow polynomial and the recipe theorem

As with the chromatic polynomial, Tutte first gave an outline of a proof that the flow polynomial was a specialisation of the Tutte polynomial [33]. It used the same approach: comparing the deletion-contraction relations for edges that are not loops or isthmuses, and comparing formulas for graphs of just loops or isthmuses.

Brylawski and Oxley give a proof using the recipe theorem [12]—our proof is similar.

**Proposition 3.2.** *The flow polynomial is a specialisation of the Tutte polynomial, given by*

$$F(G,\lambda) = (-1)^{|E(G)|-|V(G)|+k(G)}T(G;0,1-\lambda). \tag{3.7}$$

*Proof.* To show this, we use the recipe theorem (1.10). In order to apply this theorem the flow polynomial must be a TG-invariant, so must satisfy the recursion (1.8) for any edge $e$ that is not an isthmus or loop. Consider the flow polynomial as given in (3.6). We can pick any edge $e$ that is not an isthmus or a loop, and split the terms of the sum into two groups: one for subsets $A$ containing $e$, and one for subsets $A$ that do not. The resulting equation is

$$F(G,\lambda) = \sum_{\substack{e\in A\\ A\subseteq E(G)}} (-1)^{|E(G)-A|}\lambda^{|A|-r(A)} + \sum_{\substack{e\notin A\\ A\subseteq E(G)}} (-1)^{|E(G)-A|}\lambda^{|A|-r(A)}. \tag{3.8}$$

First, examine the left sum. Since $e \in A$, for each such $A$ there is a corresponding $A-\{e\}$ (we label this $B$ in subsequent summands to avoid confusion) such that $|E(G)-A| = |E(G/e)-(A-\{e\})|$, because both $E(G/e)$ and $A-\{e\}$ will have one less edge. Since we are just concerned with when $e$ is not a loop, to satisfy (1.8), $r(A/e) = r(A)-1$ (there is one less vertex and the same number of connected components) so $|A|-r(A) = |A-\{e\}|-r(A/e)$. Moreover, the subsets of $E(G)$ that contain the edge $e$ are precisely the subsets of $E(G/e)$. Thus we have

$$\begin{aligned}
\sum_{\substack{e\in A\\ A\subseteq E(G)}} (-1)^{|E(G)-A|}\lambda^{|A|-r(A)} &= \sum_{\substack{e\in A\\ A\subseteq E(G)}} (-1)^{|E(G/e)-(A-\{e\})|}\lambda^{|A-\{e\}|-r(A/e)}\\
&= \sum_{B\subseteq E(G/e)} (-1)^{|E(G/e)-B|}\lambda^{|B|-r(B)}\\
&= F(G/e,\lambda).
\end{aligned}$$

In the right-hand side sum of (3.8), since $e \notin A$, the subsets of $E(G)$ not containing $e$ are exactly the subsets of $E(G\backslash e)$. Furthermore, $|E(G)-A| = |E(G\backslash e)-A|+1$ since $E(G)$ contains $e$ and $A$ does not.

$$\begin{aligned}
\sum_{\substack{e\notin A\\ A\subseteq E(G)}} (-1)^{|E(G)-A|}\lambda^{|A|-r(A)} &= \sum_{A\subseteq E(G\backslash e)} (-1)^{|E(G\backslash e)-A|+1}\lambda^{|A|-r(A)}\\
&= -F(G\backslash e,\lambda)
\end{aligned}$$

Therefore

$$F(G,\lambda) = F(G/e,\lambda) - F(G\backslash e,\lambda)$$

satisfying the first condition for a TG-invariant (1.8).

The second condition (1.9) is also satisfied; if we have two edge-disjoint graphs $G_1, G_2$ such that the union of a spanning tree of $G_1$ and a spanning tree of $G_2$ is a spanning tree of $G$, each flow of $G_1$ can be combined with every flow of $G_2$ to get a flow for $G_1 \cup G_2$ (the flows of the edge-disjoint graphs are independent) so $F(G_1 \cup G_2, \lambda) = F(G_1, \lambda)F(G_2, \lambda)$. Thus, applying the recipe theorem, as in (1.10), gives (3.7). □

This specialisation bears a resemblance to that for the chromatic polynomial in (3.2)—the flow polynomial is a specialisation at $(x,y) = (0, 1-\lambda)$ while the chromatic polynomial is a specialisation at $(x,y) = (1-\lambda, 0)$. The Tutte polynomial of a planar graph $G$ has the following relationship with the Tutte polynomial of a plane dual $G^*$:

$$T(G;x,y) = T(G^*;y,x).$$

This can be proven from (1.5) [12]. Thus, we have a relationship between the chromatic polynomial of a graph $G$ and the flow polynomial of a plane dual of $G$. In particular, the chromatic polynomial of $G$ can be obtained by scaling the flow polynomial of $G^*$ by a positive or negative power of $\lambda$. For example, since it is known that every loopless planar graph has a 4-colouring, every planar graph without an isthmus has a nowhere-zero 4-flow.

## 3.3 All-terminal reliability

The *all-terminal reliability* of a connected graph $G$ gives the probability that a corresponding graph $G'$ is also connected, where each edge of $G$ is, independently of other edges, retained in $G'$ with a probability $p$ and removed with a probability $q = 1-p$. If the graph were to represent a network where each link is operational (or *open*) with probability $p$, the reliability would give the probability that there would still be a path between every pair of nodes. The reliability $R(G;p)$ of a graph $G$ with probability $p$ that each edge is open is given by

$$R(G;p) = \sum_{A \in S} p^{|A|} q^{|E(G)-A|}, \tag{3.9}$$

where $S$ is the set of edge sets for each spanning subgraph of $G$, and $q = 1-p$.

### 3.3.1 All-terminal reliability and the recipe theorem

The following result was first given by Oxley and Welsh [28], but under the name the "percolation probability". Our proof follows the same approach as theirs, but we provide more detail as to why the all-terminal reliability is a TG-invariant.

**Proposition 3.3.** *The all-terminal reliability is given by:*

$$R(G;p) = q^{|E|-|V|+1} p^{|V|-1} T(G;1,\frac{1}{q}).$$

*Proof.* As with the flow polynomial, we can rearrange the all-terminal reliability as given in (3.9) to show it is a TG-invariant. If we pick any edge $e$ that is not an isthmus or a loop and partition $S$ into the sets containing $e$ and those not containing $e$, the reliability is given by

$$R(G;p) = \sum_{\substack{e\notin A\\ A\in S}} p^{|A|} q^{|E(G)-A|} + \sum_{\substack{e\in A\\ A\in S}} p^{|A|} q^{|E(G)-A|}.$$

The first sum evaluates to

$$\begin{aligned}\sum_{e\notin A} p^{|A|} q^{|E(G)-A|} &= \sum_{e\notin A} p^{|A|} q^{|E(G)-(\{e\}\cup A)|+1}\\ &= q\sum_{e\notin A} p^{|A|} q^{|E(G\backslash e)-(A)|}\\ &= qR(G\backslash e;p).\end{aligned}$$

Now consider the second sum. Every $A$ contains $e$ and is a subgraph of $G$ spanning $V(G)$. If we were to contract $e$ for each $A$, we would have all the subgraphs of $G/e$ spanning $V(G/e)$. Therefore, the right sum is

$$\begin{aligned}\sum_{e\in A} p^{|A|} q^{|E(G)-A|} &= \sum_{e\in A} p^{|A-\{e\}|+1} q^{|E(G)-((A-\{e\})\cup\{e\})|}\\ &= p\sum_{e\in A} p^{|A-\{e\}|} q^{|E(G/e)-(A-\{e\})|}\\ &= pR(G/e;p).\end{aligned}$$

So the all-terminal reliability satisfies the first requirement (1.8) of a TG-invariant:

$$R(G;p) = qR(G\backslash e;p) + pR(G/e;p)$$

with $a = q$ and $b = p$.

We now show that the all-terminal reliability satisfies the second requirement of a TG-invariant. For a graph $G$ that is the edge-disjoint union of $G_1$ and $G_2$ and the union of the spanning trees (or equivalently, spanning subgraphs) of $G_1$ and $G_2$ are spanning trees (or spanning subgraphs) of $G$, we can consider $G_1$ and $G_2$ separately: the probability that $G$ is still connected is the probability $G_1$ and $G_2$ are still connected. So

$$R(G_1 \cup G_2; p) = R(G_1; p)R(G_2; p),$$

and the second requirement (1.9) is also satisfied.

We can now apply the recipe theorem (1.10). We have found $a = q$ and $b = p$. Looking at the values of $R(G; p)$ for a graph of a single edge that is an isthmus or loop respectively:

$$\begin{aligned} x_0 &= R(G_{isthmus}; p) = p^1(1-p)^0 = p \\ y_0 &= R(G_{loop}; p) = p^1(1-p)^0 + p^0(1-p)^1 = 1 \end{aligned}$$

So the all-terminal reliability is another specialisation of the Tutte polynomial, as follows:

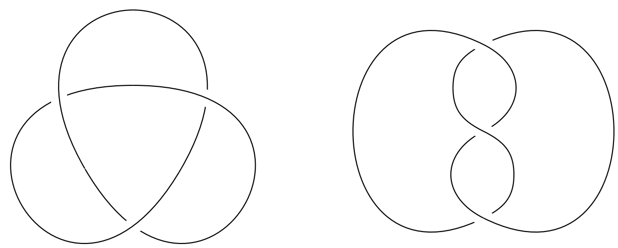
$$\begin{aligned} R(G; p) &= a^{|E|-r(E(G))} b^{r(E(G))} T(G; \frac{x_0}{b}, \frac{y_0}{a}) \\ &= q^{|E|-|V|+1} p^{|V|-1} T(G; 1, \frac{1}{q}) \end{aligned}$$

Note $r(E(G)) = |V| - 1$ since $G$ must be connected.

□

## 3.4 The Jones polynomial

A key problem in knot theory, a branch of topology that looks at the theory of mathematical knots, is identifying whether two drawings of a knot are really the same knot. The Jones polynomial gives a way of associating a polynomial to a knot drawing such that if two drawings have different polynomials, the knots are different. The Jones polynomial is a specialisation of the Tutte polynomial, which we will demonstrate in Section 3.4.3. First we introduce some preliminaries in Section 3.4.1 for readers unfamiliar with mathematical knots, and then describe the connection between knots and graphs in Section 3.4.2.



(a) A knot diagram (b) Another knot diagram of the same knot

Figure 3.1: Ambient isotopic diagrams of the trefoil knot.

### 3.4.1 Knot theory preliminaries

A *knot* is a closed curve sitting in three-space that does not intersect with itself. It is convenient to consider a knot as projected onto a plane, but at each *crossing*, where two *strands* (parts of the curve) are projected to the same position on the plane, one strand is considered above (an *over-crossing*) and the other strand is below (an *under-crossing*). A knot *diagram* is a drawing of a knot; an over-crossing is demonstrated using a solid line and an under-crossing by a broken line. Such a knot diagram is given in Figure 3.1(a). A knot diagram partitions two-space into a finite number of *region*s. The outermost region is called the *infinite region*. We say that regions are *adjacent* if they share a strand.

A knot can be *deformed*, whereby any part can undergo a continuous stretching or shrinking in any direction in three-space such that the curve remains connected, closed, and avoids intersection. The result of such a continuous deformation, called an *ambient isotopy*, is still considered the same knot—we are concerned only with how the curve is intertwined with itself. As a result, a knot has a number of diagrams. For example, two diagrams for a knot (known as the left-handed *trefoil* knot) are given in Figure 3.1; the two diagrams are ambient isotopic.

A *link* is a collection of knots sitting in the same three-space, that may be intertwined. Each knot is a *component* of the link. A link is *splittable* if there is a diagram such that each component has no crossings with any other component. An *invariant* of a link is a property that does not change by ambient isotopy, for example the number of components. If a link has a diagram such that, when we

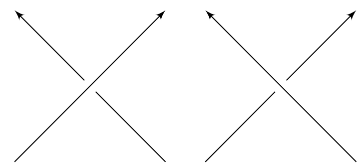

(a) $+1$ (b) $-1$

Figure 3.2: The values associated with crossings when calculating the writhe.

trace a path along any component of the link, the crossings alternate between an over-crossing and an under-crossing (or vice versa, depending on the starting position), the link is *alternating*. The trefoil knot in Figure 3.1 is alternating.

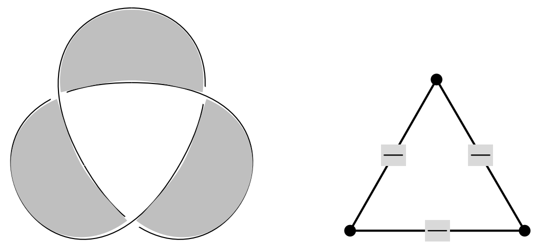
An *oriented* link assigns a direction to each component of the link. The *writhe* of an oriented link is a measure of how "twisted" the link is. We find the writhe by assigning a value to each crossing; the writhe is the sum of each these values. If the crossing is viewed so the two strands point towards the northeast and northwest, taking into account their orientation, then the two possible crossings and the values they are given are as in Figure 3.2. The writhe of the left-handed trefoil knot in Figure 3.1 is $-3$, as each of the three crossings contributes a $-1$. The writhe of a knot does not change if the orientation is changed (this will change the direction of both strands in each crossing) but the writhe of a link can be affected by changing the orientation of some but not all of its components.

A key problem in knot theory is identifying whether two knot diagrams correspond to the same knot. One approach that can assist in this process is associating polynomials to knots. If we can do this in such a way that the polynomial is an invariant (that is, every ambient isotopy has the same polynomial) then it is called a *knot polynomial*. If two knots have a different knot polynomial, the knots are distinct up to ambient isotopy.

The *Jones polynomial* is a knot polynomial that is an invariant for oriented links containing powers of the term $t^{\frac{1}{2}}$ with integer coefficients [23]. For example, the Jones polynomial of the left-handed trefoil knot, whether calculated for the diagram in Figure 3.1(a) or Figure 3.1(b), is $V(t) = t^{-1} + t^{-3} - t^{-4}$. Another relevant polynomial for unoriented links is Kauffman's *bracket polynomial* [24]. It is closely related to the Jones polynomial as given by

$$V_L(A^{-4}) = (-A)^{-3w(L)}\langle D\rangle, \tag{3.10}$$

where $L$ is the oriented link corresponding to the unoriented link $D$, $w(L)$ is the writhe of $L$, $V_L(t)$ is the Jones polynomial of $L$ and $\langle D\rangle$ is the bracket polynomial




(a) Shading of the trefoil knot

(b) The corresponding graph

Figure 3.3: The associated signed planar graph of the trefoil knot.

of the link $D$ in terms of the variable $A$. The Jones polynomial for a link can be found by calculating the bracket polynomial as described by Kauffman [24], calculating the writhe, and then using (3.10).

A thorough introduction to knot theory is given by Adams [1].

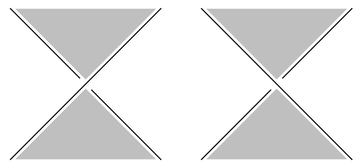
### 3.4.2 Links and signed graphs

There is a one-to-one correspondence between links and signed planar graphs: every knot or link diagram has an associated signed planar graph, and conversely every signed planar graph has a corresponding link diagram.

To find the graph corresponding to a link, we first need to find a *shading* (also commonly known as a *black-white* or *Tait colouring*) of a link diagram. For any link diagram, we can shade some of the regions, such that for any two adjacent regions exactly one is shaded. There are precisely two shadings of any link diagram satisfying these criteria. By convention, we are interested in the shading where the infinite region is not shaded[1]. The shading of the trefoil knot is given in Figure 3.3(a).

Once we have a shading of a link, we obtain the associated signed planar graph by creating a vertex for each shaded region and an edge for each crossing, joining the vertices of the two shaded regions that share a crossing. The sign of the edge depends on the type of crossing, as given in Figure 3.4. For an alternating knot, the signs on all the edges of the graph are the same. The graph for the left-handed trefoil knot is in Figure 3.3(b).

[1]The other shading would give the resulting graph's planar dual.



(a) + (b) −

Figure 3.4: The sign given to the edge corresponding to a crossing of a link.

### 3.4.3 As a specialisation of the Tutte polynomial

As discussed in Section 3.4.1, the Jones polynomial is a knot polynomial that is an invariant for oriented links. It is also another specialisation of the Tutte polynomial [30].

**Proposition 3.4.** *For an alternating link L with associated connected, planar graph G, the Jones polynomial of L is given by the Tutte polynomial of G evaluated at $(-t, -t^{-1})$ multiplied by a weighting; that is,*

$$V_L(t) = f_L(t)T(G; -t, -t^{-1}).$$

*The weighting is given by*

$$f_L(t) = (-1)^{w(L)} t^{\frac{1}{4}(|E|-2(|V|-1)+3w(L))},$$

*where $w(L)$ is the writhe of L, and $|E|$ and $|V|$ are the number of edges and vertices of G respectively.*

We will now demonstrate this connection and show why the weighting is as given above.

As described in Section 1.2.1, Tutte constructed the Tutte polynomial for a graph by decomposing the graph into spanning trees, each corresponding to one term of the polynomial. Each internally active edge of the spanning tree contributes an $x$ to the term, and each externally active edge contributes a $y$, so a spanning tree with internal activity $r$ and external activity $s$ will have the following term:

$$x^r y^s \tag{3.11}$$

Thistlethwaite was the first to observe the connection between the Jones polynomial and the Tutte polynomial [30]. He builds a polynomial of one variable for the graph associated with a link in a similar way, where each internally active edge contributes $-A^{-3}$ and each externally active edge contributes $-A^{3}$. But whereas edges of each spanning tree that are not internally active or not externally active (called *internally inactive* and *externally inactive* edges respectively) have no effect on the corresponding term in the Tutte polynomial, each internally and externally inactive edge also contributes an $A$ and an $A^{-1}$ respectively to Thistlethwaite's polynomial. Thus a spanning tree with internal activity $r$, external activity $s$, internal inactivity $u$ and external inactivity $w$ will have the following term:

$$(-A^{-3})^{r}A^{u}(-A^{3})^{s}(A^{-1})^{w} \tag{3.12}$$

This is referred to as the *weight* of each spanning tree. Note that the contributions for internally active (or inactive) edges and externally active (or inactive) edges are inverses of each other (remembering our substitutions for $x$ and $y$ in the Jones polynomial specialisation of the Tutte polynomial, that is $-t$ and $-t^{-1}$, will be also). Thistlethwaite demonstrated that this polynomial is in fact the bracket polynomial.

In order to understand the relationship between the Tutte polynomial and Jones polynomial, we need to look at the difference between the bracket polynomial and Jones polynomial, as well as the effect of the different weights.

Recall that the Jones polynomial is given by $V_L(A^{-4}) = (-A)^{-3w(L)}\langle D\rangle$ where $\langle D\rangle$ is the bracket polynomial in terms of the variable $A$. In other words, the Jones polynomial in terms of $t$ is found by performing a variable substitution of $A = t^{-\frac{1}{4}}$ and then multiplying by a power of $-t^{\frac{3}{4}}$, the power depending on the writhe of the link.

Now we look at the effect of the different weights. Consider a graph of $|V|$ vertices and $|E|$ edges. Let each spanning tree of the graph have $f$ edges—we know that $f = |V| - 1$. Furthermore, let $g = |E| - f$: the number of edges in the graph that are not in the spanning tree. Consider a spanning tree with internal activity $r$ and external activity $s$. Each term in the Tutte polynomial is as given in (3.11), whereas for the bracket polynomial we have

$$(-A^{-3})^{r}A^{f-r}(-A^{3})^{s}(A^{-1})^{g-s} = (-1)^{r+s}A^{f-g}A^{-4r+4s}.$$

Since the Jones polynomial substitutes each $A$ with a $t^{-\frac{1}{4}}$ (ignoring the extra factor depending on the writhe for now), and $r$ and $s$ are non-negative integers (so

$(-1)^{r+s} = (-1)^{r-s}$) we see that for each term we have

$$\begin{aligned}(-A^{-3})^r A^{f-r}(-A^3)^s (A^{-1})^{g-s} &= (-1)^{r-s} t^{\frac{1}{4}(g-f)} t^{r-s} \\ &= t^{\frac{1}{4}(g-f)}(-t)^{r-s} \\ &= t^{\frac{1}{4}(g-f)}(-t)^r(-t^{-1})^s. \qquad (3.13)\end{aligned}$$

Comparing (3.13) to each term of the Tutte polynomial, see (3.11), we can conclude that the Jones polynomial is an evaluation of the Tutte polynomial multiplied by a factor, with $(x,y) = (-t,-t^{-1})$.

The factor, still excluding the effect of writhe, is $t^{\frac{1}{4}(g-f)}$. Remembering $g = |E| - f$ and $f = |V| - 1$ (the graph is connected), we get $t^{\frac{1}{4}(|E|-2(|V|-1))}$. Finally, we obtain the Jones polynomial by multiplying the bracket polynomial, in terms of $t$, by $(-t^{\frac{3}{4}})^{w(L)}$. Thus we can compute the factor:

$$\begin{aligned}f_L(t) &= t^{\frac{1}{4}(|E|-2(|V|-1))} \cdot (-1)^{w(L)}(t^{\frac{3}{4}w(L)}) \\ &= (-1)^{w(L)} t^{\frac{1}{4}(|E|-2(|V|-1)+3w(L))}\end{aligned}$$

Since the writhe is the sum of either a $+1$ or $-1$ associated with each edge, the writhe must be odd when the number of edges is odd, and even when the number of edges is even. Thus $3w(L) + |E|$ is even, and hence the factor is plus or minus an integer power of $t^{\frac{1}{2}}$.

We can generalise this result to links with corresponding planar graphs that are not connected. Such a link is splittable as each component of the graph corresponds to a component of the link that has no crossings in common with any other component. A property of the Jones polynomial (and bracket polynomial) is that if $L$ is composed of two splittable links $L_1$ and $L_2$ that each share no crossings with the other, the Jones polynomial is given by $V_L(t) = (-t^{-\frac{1}{2}} - t^{\frac{1}{2}})V_{L_1}(t)V_{L_2}(t)$ [24]. Since the Tutte polynomial of a disconnected graph $G$ is the product of the Tutte polynomial for each component as given in (1.7), we need to include a power of $-t^{-\frac{1}{2}} - t^{\frac{1}{2}}$ for each disconnected component to the weighting. Let $L$ be a link as described of $k$ such components with a total of $|V|$ vertices and $|E|$ edges in the corresponding graph, and let $|V_i|$ and $|E_i|$ be the number of vertices and edges, respectively for the $i$th component. The weighting is given by

$$\begin{aligned}f_L(t) &= (-t^{-\frac{1}{2}} - t^{\frac{1}{2}})^k \prod_{i=1}^{k} (-1)^{w(L_i)} t^{\frac{1}{4}(|E_i|-2(|V_i|-1)+3w(L_i))} \\ &= (-1)^{w(L)} t^{\frac{1}{4}(|E|-2(|V|-k)+3w(L))}(-t^{-\frac{1}{2}} - t^{\frac{1}{2}})^k.\end{aligned}$$

# Chapter 4

# The Tutte Plane

We can visualise the various specialisations and evaluations of the Tutte polynomial by looking at the *Tutte plane* as given in Figure 4.1.

In this figure we plot each piece of information about a graph that we have discussed in Section 1.3 and Chapter 3 (see [17, 35] for a visualisation of the Tutte, or Tutte-Whitney, plane containing more evaluations that we have not mentioned). For example, recall that the chromatic polynomial is given in (3.2) on page 19, so evaluations of the chromatic polynomial $P(G;\lambda)$ are (after scaling) evaluations of the Tutte polynomial at $(x,y) = (1-\lambda, 0)$. Therefore, the number of colourings for a graph are found along the line $y = 0$, varying the value for $x$. Similarly, the flow polynomial is given along the line $x = 0$ and the all-terminal reliability is along $x = 1$ (but only values $y \geq 1$ are possible since $0 \leq p \leq 1$). The Jones polynomial, being a specialisation at $(x,y) = (-t, -t^{-1})$ is given by the hyperbola $xy = 1$. We also plot the points discussed in Section 1.3.1: the number of maximal spanning forests at $(1,1)$ and spanning subgraphs at $(1,2)$.

The Tutte plane allows us to envision the different specialisations and evaluations for the Tutte polynomial in general. For a specific Tutte polynomial (the Tutte polynomial for a graph $G$, say), the bivariate polynomial gives us a 3-dimensional surface that we can plot on the Cartesian coordinate system.

For example, recall the graph we saw in Figure 1.1 (on page 3) with Tutte polynomial as given in (1.2). A plot of this surface is in Figure 4.2. The intersection of this surface with the plane $y = 0$ gives a curve, along which a scaled version of the chromatic polynomial is defined. Similarly, the curve sitting in the surface given by restricting the surface to values where $x = 0$ gives the flow polynomial.

Recall also the Petersen graph we saw in Figure 1.3 (on page 6) with Tutte polynomial as given in (1.3). A plot of this surface is in Figure 4.3.

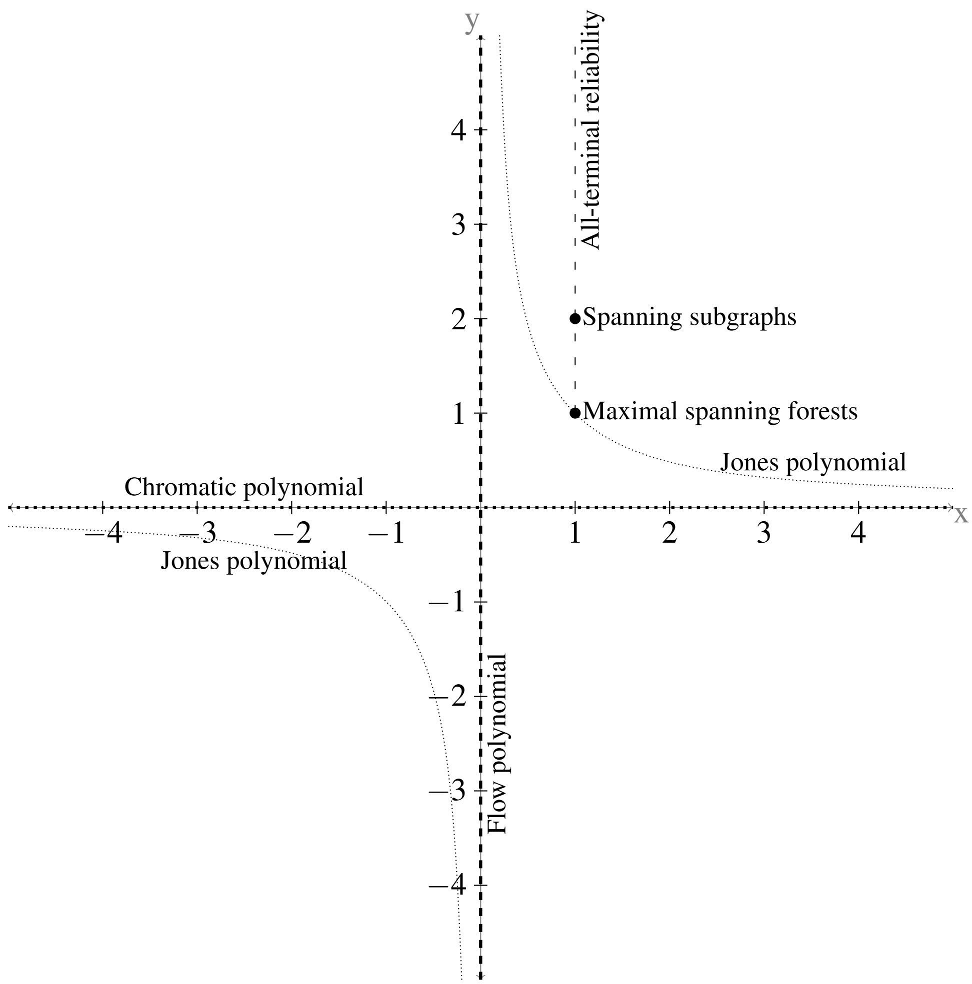



Figure 4.1: The Tutte plane.

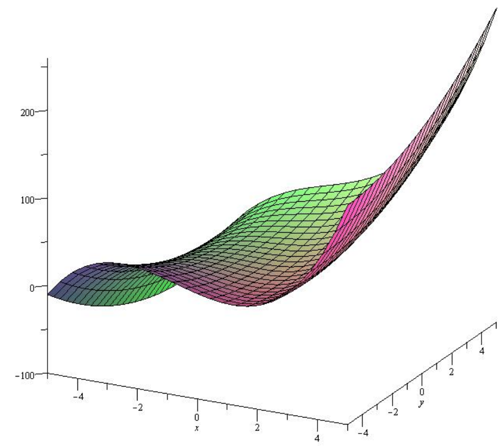



Figure 4.2: Plot of the Tutte polynomial for the graph in Figure 1.1.

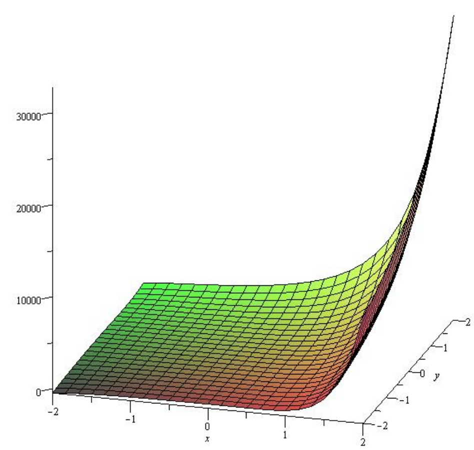



Figure 4.3: Plot of the Tutte polynomial for the Petersen graph.

# Chapter 5

# Complexity of the Tutte Polynomial

Given the usefulness of the Tutte polynomial, an important question is: can we compute the Tutte polynomial for an arbitrary graph "quickly"? If we can, then we can easily find the specialisations we have discussed earlier, like whether a graph is $\lambda$-colourable, or if it has an $H$-flow. In this section we answer this question, but first we need to introduce the relevant concepts of computational complexity, which we do in Section 5.1. We then detail the complexity of the Tutte polynomial in Section 5.2.

## 5.1 Preliminaries

In order to identify when an *algorithm* (a well-defined process to solve a problem) is a viable approach, even when the problem size is large, we need a definition of the computational complexity of an algorithm.

Although we are ultimately interested in the length of time an algorithm will take to run, this depends on the speed at which the computations are performed. This variability makes such a time measurement meaningless; it is more valuable to consider the number of steps in an algorithm. Since this value is still proportional to the time taken, it is still loosely referred to as such.

This "time" also varies wildly between different instances of a problem, so it is most beneficial to look at this value relative to the size of the problem. Suppose the size of the problem is $n$ and the time taken by the algorithm in a worst-case scenario can be expressed as a polynomial in $n$—this is called a *polynomial-time* computable algorithm.

A *decision problem* is a problem that takes a simple "yes" or "no" as an answer. For example, one decision problem is: does a graph $G$ have a proper vertex-colouring using four colours? The class of all decision problems that can be computed in polynomial time is denoted $P$. These problems can be solved using algorithms that are considered “fast” or “tractable”, as the time taken does not grow too rapidly as the size of the problem increases.

The aforementioned decision problem of finding if a graph has a 4-colouring is easy to check—that is, given a proposed 4-colouring for a graph, we can find in polynomial time if it is a proper colouring. Such decision problems, where we can verify a solution in polynomial time, are called *non-deterministic polynomial-time* computable and belong to the complexity class $NP$. $P$ is a subclass of $NP$, but it is an open problem (and in fact a key unsolved problem in mathematics) as to whether $P = NP$.

A problem $A$ is *polynomial-time reducible* to $B$, written $A \propto B$, if it is possible to solve $A$ with subroutine calls to $B$ in polynomial time when treating each call to $B$ as a single step. An $NP$-problem is also in the class *NP-complete* if any other $NP$ problem can be polynomial-time reduced to it. As a consequence, if a $NP$-complete problem can be solved in polynomial time, then so can all other problems in $NP$, and $P = NP$. $NP$-complete problems can be thought of as the hardest of the $NP$-problems. A problem (that need not be in $NP$) is $NP$-hard if some $NP$-complete problem is polynomial-time reducible to it; informally, the $NP$-hard problems are those that are at least as difficult as the $NP$-complete problems.

Whereas a decision problem finds if there is a solution to some problem, an enumeration problem finds the number of solutions. We have seen numerous examples of enumeration problems that we can solve by a simple substitution once we have the Tutte polynomial: the number of spanning trees of a graph, the number of $\lambda$-colourings of a graph, and the number of $H$-flows of a graph for an abelian group of order $|H|$.

The complexity class $\#P$ is the analogue of $NP$ for enumeration problems; namely, the structures being counted can be recognised in polynomial time. As with the $NP$-complete class for decision problems, the *#P-complete* class contains a problem $A$ in $\#P$ if for any other problem $B$ in $\#P$, $B \propto A$. Just as a problem is $NP$-hard if some $NP$-complete problem is polynomial-time reducible to it, a problem is $\#P$-hard if some $\#P$-complete problem is polynomial-time reducible to it.

For an introduction to computational complexity, in the context of problems relating to graphs, see [2]. For a more detailed account of these concepts, refer to [35].

## 5.2 Computational complexity of the Tutte polynomial

Jaeger, Vertigan and Welsh showed that evaluating the Tutte polynomial of a graph (or more generally, a matroid) is #*P*-hard, apart from for nine special points and along a curve $(x-1)(y-1)=1$, for which it is easy [21].

They identify three problems: finding the Tutte polynomial of a graph, $\pi_1[G]$; evaluating the Tutte polynomial of a graph at a point $(a,b)$, $\pi_2[G,a,b]$; and finding the Tutte polynomial along a curve $L$ in the $xy$-plane, $\pi_3[G,L]$[1]. It is straightforward that for a point $(a,b)$ on the curve $L$, $\pi_2[G,a,b] \propto \pi_3[G,L]$, since evaluating at a point is easy when we have the polynomial along a curve containing the point; and $\pi_3[G,L] \propto \pi_1[G]$, since evaluating along a curve is easy when we have the polynomial. However, they also show that the converse polynomial time reductions also hold in some cases.

They define a *special* curve as one where $(x-1)(y-1)=\alpha$ for some constant $\alpha$, and prove the following important result[2]:

**Theorem 5.1.** *Finding the Tutte polynomial along a special curve is polynomial time reducible to evaluating it for any point on the curve, other than nine special points.*

In other words, finding the Tutte polynomial at any non-special point is at least as difficult as finding it along a special curve containing the point.

For a special curve where $\alpha \geq 3$, the curve contains the point $(1-\alpha,0)$, for which evaluating the Tutte polynomial is known to be #*P*-hard since this corresponds to the problem of finding an $\alpha$-colouring of a graph. Therefore, finding the Tutte polynomial along any such special curve is #*P*-hard. In fact, for any special curve such that $\alpha \neq 1$, there is at least one point known to be #*P*-hard by a similar approach, so finding the Tutte polynomial along any such curve is #*P*-hard. Thus by Theorem 5.1, finding it at any non-special point, other than those sitting on $(x-1)(y-1)=1$, must also be #*P*-hard.

Evaluating the Tutte polynomial at any of the nine special points is known to be easy. One such point, $(0,0)$, is trivial. Another, $(1,1)$, which counts the number of maximal spanning forests, can be found in polynomial time by what is known

[1] In each case they state the input of the problem as a "succinct" matroid rather than a graph, but, in line with the rest of this report, we just consider graphs (which, essentially, fall into the class of succinct matroids).

[2] This proof, in their words, is "rather technical"—for details refer to the source material or [27].

as Kirchoff's matrix-tree theorem (see [6] for details). Of the six special points that have real values, the others are $(-1,-1)$, $(-1,0)$ and $(0,-1)$.

Along the curve $(x-1)(y-1)=1$, we can show evaluating the Tutte polynomial to be easy by giving a formula for the polynomial in this case, using only the rank and cardinality of $E(G)$. This result is often stated as trivial following from (1.5), but we give an explicit proof.

**Lemma 5.2.** *Evaluating the Tutte polynomial is easy for any* $(x,y)$ *on the curve* $(x-1)(y-1)=1$.

*Proof.* From (1.5) and using $(x-1)=(y-1)^{-1}$ we get

$$\begin{aligned} T(G;x,y) &= (y-1)^{-r(E(G))} \sum_{A \subseteq E(G)} (y-1)^{|A|} \\ &= (y-1)^{-r(E(G))} \sum_{i=0}^{|E(G)|} \left\{ \binom{|E(G)|}{i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} y^{i-j} \right\}, \end{aligned}$$

since the number of $i$-element subsets of $E(G)$ is given by $\binom{|E(G)|}{i}$ and from the binomial expansion of $(y-1)^{|A|}$. We can change the order of the summations and rearrange to get

$$T(G;x,y) = (y-1)^{-r(E(G))} \sum_{i=0}^{|E(G)|} \left\{ y^i \sum_{j=0}^{|E(G)|-i} (-1)^j \binom{|E(G)|}{j+i} \binom{j+i}{j} \right\},$$

and then use the fact that

$$\begin{aligned} \binom{|E(G)|}{j+i} \binom{j+i}{j} &= \frac{|E(G)|!}{(|E(G)|-j-i)!i!j!} \\ &= \binom{|E(G)|}{i} \binom{|E(G)|-i}{j} \end{aligned}$$

to get

$$T(G;x,y) = (y-1)^{-r(E(G))} \sum_{i=0}^{|E(G)|} \left\{ y^i \binom{|E(G)|}{i} \sum_{j=0}^{|E(G)|-i} (-1)^j \binom{|E(G)|-i}{j} \right\}. \tag{5.1}$$

It is a well-known property of binomial coefficients that

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = 0$$

when $n > 0$, therefore the terms of the rightmost sum in (5.1) are only non-zero when $i = |E(G)|$, giving

$$T(G;x,y) = (y-1)^{-r(E(G))} y^{|E(G)|}.$$

□

Collectively these give a main result of [21]:

**Theorem 5.3.** *Evaluating the Tutte polynomial of a graph is #P-hard, apart from for nine special points and along the curve $(x-1)(y-1) = 1$, for which it is easy.*

This is an important result which has a number of implications. For a given graph, although finding the number of spanning trees is easy by Kirchoff's matrix-tree theorem, finding the number of spanning subgraphs is #*P*-hard since $(1,2)$ is not a special point. Likewise, counting the number of nowhere-zero *k*-flows for $k \geq 3$ is #*P*-hard, as is finding the all-terminal reliability of a graph.

One further result of this paper worth mentioning is that determining the Tutte polynomial on the plane is polynomial time reducible to finding it along a non-special curve. Since this means that the Tutte polynomial for a non-special curve is at least as difficult as finding the polynomial for the whole plane, finding the Jones polynomial of an alternating link (a non-special curve on the Tutte plane) is also #*P*-hard. We discussed in Section 3.4.1 the importance of the Jones polynomial in finding if two knot diagrams correspond to the same knot—this result shows that finding the Jones polynomial in the interest of distinguishing knots is not straightforward.

Finally, the result of Theorem 5.3 has been strengthened when looking at a some specific classes of graph, such as planar graphs, or bipartite graphs. In particular, although polynomial-time algorithms for evaluating the Tutte polynomial can be all but ruled out, polynomial-time algorithms have been found we looking at the class of graphs with bounded tree-width. Noble gives a survey of these findings, and other results regarding the complexity of polynomials for graphs in [27].

# Chapter 6

# Recent Developments and Open Problems

The Tutte polynomial is still an active area of research. Kung (2008) gives a valuable overview of areas of recent interest in [26]. He identifies two key areas: investigating to what extent the Tutte polynomial determines a graph, and finding generalisations of the Tutte polynomial. Another area of recent interest that we will discuss briefly is finding approximate values for evaluations of the Tutte polynomial.

In Section 1.3 we saw that the Tutte polynomial is not a graph invariant—that is, there are graphs with the same Tutte polynomial that are not isomorphic. However the question remains, can this be considered the general case, or are such graphs in the minority? A graph $G$ for which any other graph that has the same Tutte polynomial is isomorphic to $G$ is called *$T$-unique*. Which graphs are $T$-unique? Recent work has answered these questions to some degree, but it is still very much an open area.

A graph $G$ for which the only graphs that have the same chromatic polynomial as $G$ are isomorphic to $G$ is called *$\chi$-unique*. A number of families of graph are known to be $\chi$-unique, and since the chromatic polynomial is a specialisation of the Tutte polynomial, it follows that they are also $T$-unique. These include *complete graphs* (a graph that contains a single edge between every pair of vertices), *cycle graphs* (a graph containing a single cycle on $n$ vertices that visits every vertex, denoted $C_n$) and *complete bipartite graphs* $K_{p,q}$ (graphs where the vertices can be put into two groups of size $p$ and $q$ such that any vertex is adjacent to every vertex of the other group) where $p,q \geq 2$ [25].

More recently, a number of families of graph have been found that are $T$-

unique but not necessarily $\chi$-unique. For example, a graph on $n+1$ vertices is called a *wheel* if it can be obtained from the cycle graph $C_n$ by adding a single vertex adjacent to every other. De Mier and Noy (2005) [15] showed that wheels (for $n \geq 3$) are $T$-unique. They also proved that *complete multipartite graphs* (graphs where the vertices can be partitioned into a number of *parts* such that a vertex is adjacent to every vertex not in the same part) and hypercubes (projections of a n-dimensional cube onto the plane, with the obvious vertices and edges) are also $T$-unique, among others. Duan et al. (2009) proved that another family known as *twisted wheels* are also $T$-unique [16]. Obvious future work is to find all such families, but, as stated by De Mier and Noy, a difficulty is that proving a family of graphs is $T$-unique seems to require a different approach each time.

As seen earlier, some graphs that are not $T$-unique are easy to find, such as those containing an isthmus[1]. However, the problem becomes more difficult as the *connectivity* (the minimum number of vertices that need to be removed to disconnect a graph) increases. A $k$-connected graph that is not $T$-unique has been shown to exist for any $k$ [9, 11]. However, work still remains to see if there are large families of $T$-unique graphs with high connectivity. It has also been conjectured that almost every graph is $T$-unique; more precisely, that the probability of a graph on $n$ vertices being $T$-unique tends to one as $n$ tends to infinity [9].

A considerable amount of research exists, and is still being developed, on generalisations of the Tutte polynomial. Farr gives a survey of generalisations in the literature as of 2006 in [17]. These generalisations take various forms that Farr neatly categorises into four types. The first type is generalisations to mathematical objects that have less structure than graphs. An obvious example, that we mentioned previously, is the generalisation of the Tutte polynomial to matroids. The Tutte polynomial has also been generalised to specific types of matroids (such as representable matroids) and structures more general than matroids (such as greedoids and semimatroids). The second type of generalisation is to objects that have more structure than graphs, for example signed graphs, or graphs with one or more weightings assigned to each edge. The third type is generalisations to objects analogous to graphs, and the fourth type is generalisations of the polynomials. We will touch on this final type some more; for details on these or the other types of generalisations refer to Farr [17].

A number of generalisations of the Tutte polynomial exist that contain as spe-

[1]Note that not every graph containing an isthmus is $T$-unique. The graphs containing an isthmus that might not have a non-isomorphic graph with the same Tutte polynomial are those such that any repositioning of the isthmus results in an isomorphic graph. A trivial example is the complete graph on two vertices.

cialisations other useful polynomials in graph theory, or even other related areas. Many of these introduce another variable (or in some cases, more than one). As recently as 2010, Averbouch, Godlin and Makowsky introduced a three-variable polynomial that generalises the Tutte polynomial [5]. A *matching* is a set of edges of a graph, none of which have any vertices in common. The matching polynomial, a generating function for the number of matchings of varying sizes for a graph, and in fact generalisations of this polynomial (such as the independent set polynomial and vertex-cover polynomial), are contained as specialisations. Their polynomial satisfies a recurrence relation in terms of not only edge deletions and contractions, but *edge extractions* (deletion of an edge, its incident vertices and adjacent edges). Furthermore, they show that their polynomial is the most general satisfying a linear recurrence relation with respect to the three edge-elimination operations.

We saw earlier that computing the Tutte polynomial for an arbitrary graph can be an insurmountable problem. However, recent research has investigated instead finding approximations to the evaluation for a graph at a given point. A *fully polynomial randomised approximation scheme* (FPRAS) is a polynomial-time approximation algorithm with a high probability of having arbitrarily small relative error. An FPRAS is known to exist for any point $(x,y)$ on the Tutte plane satisfying $(x-1)(y-1)=2$ and $y>1$ [22]. An FPRAS is also known to exist when placing certain restrictions on the input graph, in particular for *dense* graphs, where each vertex has at least $\alpha|V(G)|$ neighbours for some fixed $\alpha>0$ [3]. However, a number of points on the Tutte plane have been shown to have no FPRAS in the general case. Most recently, Goldberg and Jerrum (2008) substantially widened the area over which it is known no FPRAS exists [18]. The problem of finding all points where an FPRAS can and cannot exist is still open.

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