Bases and confinement

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Let E and F be normed spaces, and let U be an open set in E and $f : U \longrightarrow F$. I shall understand differentiability of f in the sense of Fréchet. A function $f : E \longrightarrow F$ is "of bounded support" if the set $\{x \in E : f(x) \neq 0\}$ is bounded.

It is unusual for a Banach space to admit a nonzero real-valued C^∞ function of bounded support, or "smooth bump function". Any Banach space that does so contains a copy of either c_0 or ℓ^{2q} for some natural number q (for a full discussion, see the book of Deville, Godefroy, and Zizler, V.4). Hence, for most Banach spaces E, C^∞ partitions of unity are out of the question, and many familiar constructions are unavailable for C^∞ manifolds modelled on E. In particular: suppose M is a paracompact C^∞ manifold modelled on a separable Banach space E. Is there always a nonconstant real-valued C^∞ function on M? One would really like to have many such functions — but the question whether, in general, there exist any at all was raised by Elworthy in 1972 and, so far as I know, remained unanswered for more than 20 years. I believe that the non-separable case still remains open.

The crucial step is to substitute for the question of "bump functions" an extension question. Let $B(R)$ denote the ball of radius R about the origin in the separable normed space E. If one can prove that

> [A] for any C^∞ function $f : B(R) \longrightarrow F$ into another normed space F, there are a C^∞ function $g : E \longrightarrow F$ and a radius r, $0 < r < R$, such that $g|B(r) = f|B(r)$,

then one can deduce, by a complicated inductive procedure of piecing-together, that

> [B] if M is a paracompact C^∞ manifold modelled on E, and N a closed C^∞ submanifold, and if $f : U \longrightarrow F$ is a C^∞ function on a neighbourhood U of N, then there are a neighbourhood V of N, with $V \subseteq U$, and a C^∞ function $g : M \longrightarrow F$, such that $g|V = f|V$.

(The neighbourhood V depends only on the geometry of M and V, not on the function f.) In particular, N might be a discrete set in M, and then one discovers that there is a C^∞ function $M \longrightarrow F$ with values prescribed near N.

The extension property [A] is satisfied if E has the following property (the name "confinement" is due to Lempert):

Definition. The normed space E is confined if there is a C^∞ mapping $\phi : E \longrightarrow E$ (a "shrinking map") such that, for some numbers r and R with $0 < r < R < \infty$,

$$\phi(E) \subseteq B(R) \quad \text{and} \quad \phi|B(r) = 1|B(r)\,.$$

(Like [A], this definition does not require E to be separable, and does not depend on the choice of norm in E. But [A] $\Rightarrow$ [B] does require separability.)

The "extension" g sought in [A] can then be $f \circ \phi$.

Confinement is merely a sufficient condition for [A], but it has other applications that make it of considerable independent interest. Thus it is pleasant to know:

Theorem. *If the Banach space E admits a Schauder basis, it is confined.*

Indeed, the "shrinking map" ϕ can be chosen to satisfy some extra conditions, as I shall explain in a moment. Let the basis be $\{e_1, e_2, e_3, \dots\}$, with coefficient functionals $\{\lambda_1, \lambda_2, \dots\}$; the projection P_n on the span E_n of $\{e_1, \dots, e_n\}$ is given by

$$P_n x := \sum\nolimits_{k=1}^{n} \lambda_k(x) e_k \, .$$

Then $\{P_n : n \in \mathbb{N}\}$ is uniformly bounded, and we can define a new norm on E,

$$|||x||| := \sup\{\|P_n x\|, \|x - P_n x\| : n \in \mathbb{N}\} \, ,$$

with is "bimonotone" with respect to the basis; that is, all the projections P_n and $P_n^{\perp} := I - P_n$ are norm-non-increasing. Let $E_n^{\perp} := P_n^{\perp}(E)$.

I claim, then, that, with respect to any bimonotone norm on E, there is for any choice of r and R such that $0 < r < R$ a C$^\infty$ mapping $\phi : E \longrightarrow E$ which is a C$^\infty$ diffeomorphism of E with an open subset of $B(R)$ and such that $\phi | B(r) = \mathbf{1}_{B(r)}$. This statement, notably stronger than the Theorem, has further useful consequences.

The idea is as follows. We take R only slightly larger than r, and write $B_n(r)$ for the r-ball about the origin in E_n. As the norm is bimonotone, $B_n(r) = P_n(B(r))$ etc. For each $n > 0$, construct in E_{n+1} the following two neighbourhoods of E_n:

$$C_n := B_{n+1}(r) \cup (B_{n+1}(\epsilon) + E_n)\,,$$
$$D_n := B_{n+1}(r_n) \cup (B_{n+1}(\epsilon_n) + E_n)\,,$$

where $r_n \downarrow r$ and $\epsilon_n \downarrow \epsilon$ and $r_0 + 2\epsilon_0 < R$ (say). E_{n+1} is finite-dimensional, so there is a C$^\infty$ function $\chi_n : E_{n+1} \longrightarrow (0, 1]$ which is constantly 1 on C_n such that

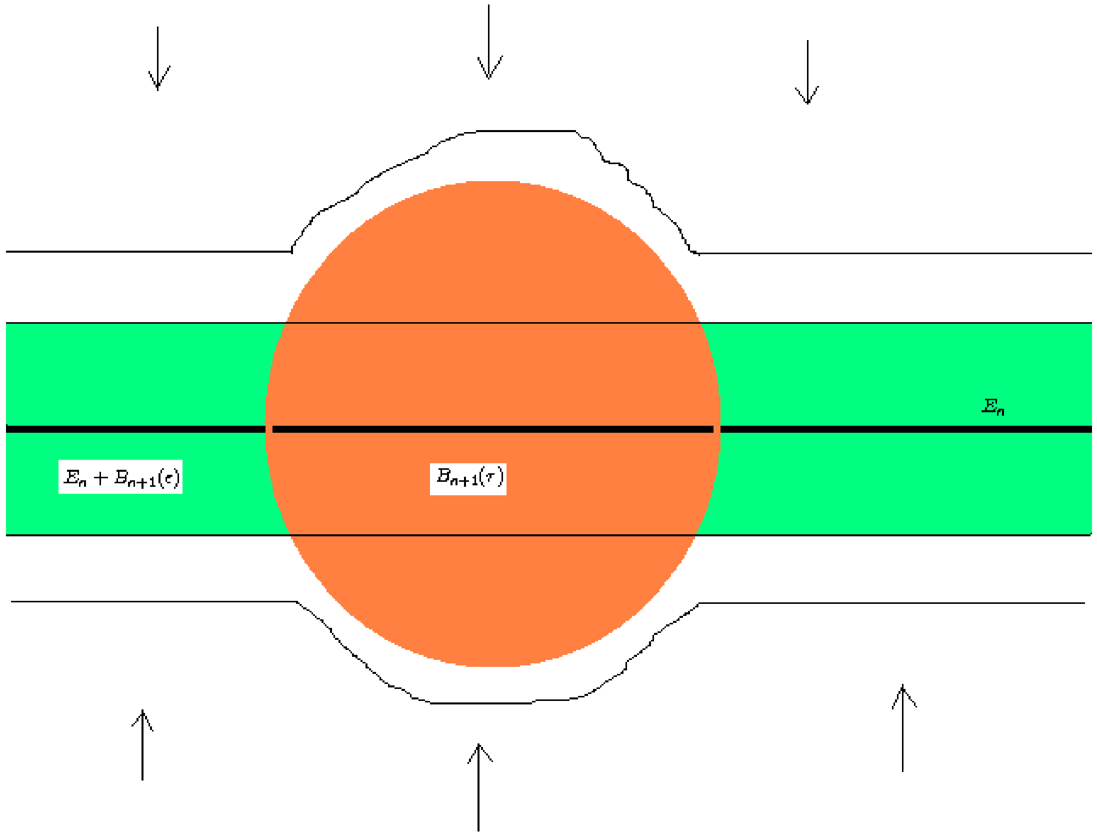
$$\Phi_n(y) := P_n y + \chi_n(y) P_n^\perp(y)$$

defines a C$^\infty$ diffeomorphism of E_{n+1} with an open subset of D_n that is the identity on C_n. Now take a C$^\infty$ function $\psi_n : E_{n+1} \longrightarrow (0, 1]$ which is 1 on C_n, does not exceed χ_n anywhere, and is very small off D_n. The mapping

$$\Psi_n : E \longrightarrow E : x \mapsto \Phi_n(P_{n+1}x) + \psi_n(P_{n+1}x) P_{n+1}^\perp(x)$$

is a C$^\infty$ diffeomorphism of E with an open subset.

For any $x \in E$, there is some N such that $x \in B(\epsilon) + E_N = B_{N+1}(\epsilon) + E_N$. This implies that $\Psi_n(x') = x'$ for all x' sufficiently close to x and all $n \geq N$, so that the "backwards infinite product" $\Psi_0 \circ \Psi_1 \circ \Psi_2 \circ \cdots$ is in effect locally finite. This product is the "shrinking map".



E_n
$E_n + B_{n+1}(\epsilon)$
$B_{n+1}(\tau)$