MATH 452

General Topology

Test, 2008

1. *(a)* Prove that, for any set Ω, there is a weakest T_1 topology $\mathcal{G}$ on Ω.
(b) Show that, if Ω is any T_1 space and $A \in \mathcal{P}(\Omega)$, then the derived set of A is closed.

2. If the topological space Ω is second countable, show that *any* base for its topology includes a countable base.

3. Suppose that $\{\Psi_n : n \in \mathbb{N}\}$ is a countable family of topological spaces, Ω a set, and that, for each n, $f_n : \Omega \longrightarrow \Psi_n$ is a mapping. Let $\mathcal{G}$ be the topology on Ω induced by the mappings f_n (the "weak topology" defined by these mappings). Prove that, if each Ψ_n is first countable, then so is $(\Omega, \mathcal{G})$; if each Ψ_n is second countable, then so is $(\Omega, \mathcal{G})$; and, if each Ψ_n is pseudometrizable (i.e. its topology may be defined by a pseudometric), then so is $(\Omega, \mathcal{G})$.

4. Suppose that $f : \Omega \longrightarrow \Psi$ is a mapping between topological spaces, that $\Omega = A \cup B$, and that $f|A : A \longrightarrow \Psi$ and $f|B : B \longrightarrow \Psi$ are continuous with respect to the subspace topologies on A and on B. Prove that then $f : \Omega \longrightarrow \Psi$ is continuous with respect to the topology on Ω in each of the following cases:
(a) if A and B are both open,
(b) if A and B are both closed.

5. Recall that subsets X, Y of a topological space Ω may be described as *separated* if $\mathrm{cl}(X) \cap Y = \emptyset = X \cap \mathrm{cl}(Y)$. Prove that, in the last question, $f : \Omega \longrightarrow \Psi$ is continuous if $A \setminus B$ and $B \setminus A$ are separated.
Prove also this condition subsumes both the previous conditions 3*(a)* and 3*(b)*. [Evidently the most economical approach would be to prove this result first, as it includes the cases of qu. 4, but *(a)* and *(b)* are the conditions usually quoted and are very much easier to handle. This question is quite difficult; it isn't hard to guess what ought to work, but the set-theoretic manipulations are awkward.]

6. Prove that the one-point compactification of $\mathbb{R}^n$ for $n \geq 1$ (with its standard topology) is homeomorphic to S^n. [It is possible to write down a "formula", but I should be satisfied with a clear description how to construct a homeomorphism. To get an idea how to proceed, you may find it illuminating to do the bottom dimensions $n = 1$ and $n = 2$ first; in these low dimensions, one can use one's geometrical imagination. But there is nothing else special about them.]

7. A set E in the topological space Ω is *sequentially compact* if every sequence in E has a subsequence that converges to a point of E. Show that, if Ω is first countable, any compact subset of Ω is also sequentially compact.

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Let E be a sequentially compact subset of Ω. Prove that any *countable* open covering of E has a finite subcovering. [Hint: this is very like an argument in 312.]

8. Let $f : \Omega \longrightarrow \Psi$ be a sequentially continuous mapping (see 6.4) between topological spaces. Prove that, if E is a sequentially compact subset of Ω, then $f(E)$ is a sequentially compact subset of Ψ.

Show that the union of two sequentially compact subsets of Ω is also sequentially compact.

9. Prove that any subspace of a normal space is Tikhonov. [We shall soon see that, conversely, any Tikhonov space is homeomorphic to a subspace of a suitably chosen normal space.]

10. Show that the product of countably many second countable topological spaces is second countable. Is the same statement true for the coproduct of countably many second countable spaces?

NOTE. Beware of possible errors in the questions.