MATH 452

General Topology

Assignment 3

1. Prove Lemma 3.6.

2. I commented that my statements of the Axiom of Choice and of the Multiplicative Axiom differed. What is the difference? Show that the two statements are in fact equivalent (if the standard assumptions of set theory are accepted; it is not necessary to discuss the question in terms of a rigorous formalization of set theory).

3. If Ω is a topological space and $\Psi \subseteq \Omega$, show that the subspace topology as defined at 5.1 is indeed a topology on Ψ (Lemma 5.3); and that, for any $A \subseteq \Psi$, the closure of A in the subspace topology on Ψ is just $\Psi \cap \mathrm{cl}_\Omega(A)$ (Lemma 5.5).

4. If E is a vector space over a field $\mathbb{F}$, a *linear functional* on E is a linear mapping (over $\mathbb{F}$) from E to $\mathbb{F}$ itself (regarded as a vector space of dimension 1 over $\mathbb{F}$). Take $\mathbb{F} := \mathbb{R}$, $E := \mathbb{R}^n$, and let $\mathcal{F}$ be the family of all linear functionals $E \longrightarrow \mathbb{R}$. The standard topology on $\mathbb{R}$ and the family $\mathcal{F}$ then define a topology $\mathcal{G}$ on E, as at 5.12. Show that this "weak topology on $\mathbb{R}^n$" is just the standard topology.

5. Prove 5.10.

6. Let (Ω_n, d_n) be a metric space for each $n \in \mathbb{N}$.

(a) On $\Omega_1 \times \Omega_2$, define a metric d by

$$d((x_1, x_2), (y_1, y_2)) := d(x_1, y_1) + d(x_2, y_2)\,.$$

Show that the topology defined by d is the product topology on $\Omega_1 \times \Omega_2$.

(b) On $\Omega := \prod_{n=1}^{\infty} \Omega_n$, define

$$d_\infty((x_n), (y_n)) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}\,. \tag{1}$$

Prove that this is a metric on Ω, and that it defines the product topology. [Thus the *countable* product of metric spaces is metrizable, i.e. its topology can be defined by a metric.]

(c) There are many other formulæ that will define a metric on Ω consistent with (i.e. having the same open sets as) the product topology. Present two (that will, in typical cases, yield metrics genuinely different from d_∞ and from each other).

[This is an example of the advantages of considering topological spaces rather than metrics; the formula (1) is clearly rather artificial and has many alternatives, but the product topology is natural.]

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7. If $\mathcal{B}_i$ is a base for the topology $\mathcal{G}_i$ in Ω_i, where $i = 1, 2$, show that

$$\{B_1 \times B_2 : B_i \in \mathcal{B}_i\}$$

is a base for the product topology in $\Omega_1 \times \Omega_2$.

8. Extend the result of the last question to a product of any indexed class of topological spaces.

9. Prove that the coordinate projections of any product of topological spaces are open.

10. Prove lemma 5.21.