MATH 452

General Topology

Assignment 2

1. Prove that the Zariski topology on $\mathbb{C}$ is just the cofinite topology.

2. Prove Lemma 3.3.

3. Prove that the mapping of power classes $f^{-1} : \mathcal{P}(\Psi) \longrightarrow \mathcal{P}(\Omega)$ induced by a mapping $\Omega \longrightarrow \Psi$ is surjective if and only if f is injective, and is injective if and only if f is surjective. [This has nothing to do with topology!]

4. Prove Lemma 3.14.

5. Either: give an example of topological spaces Ω and Ψ and a mapping $f : \Omega \longrightarrow \Psi$ such that, for any $A \in \mathcal{P}(\Omega)$, $f(\mathrm{cl}_\Omega(A)) \subseteq \mathrm{cl}_\Psi(f(A))$, and yet f is not continuous; or, prove the converse of 3.14, viz.that, if $f : \Omega \longrightarrow \Psi$ is such that, for every $A \in \mathcal{P}(\Omega)$, $f(\mathrm{cl}_\Omega(A)) \subseteq \mathrm{cl}_\Psi(f(A))$, then f must be continuous.

6. In the set $\Omega := \mathbb{N} \cup \{*\}$, where $*$ is some point not belonging to $\mathbb{N}$ (think of it as the "point at infinity"), define $\mathcal{G}$ to be the subclass of $\mathcal{P}(\Omega)$ consisting of all those subsets of Ω that are *either* subsets of $\mathbb{N}$ *or* are subsets of Ω containing $*$ and with finite complements. Show that $\mathcal{G}$ is a topology in Ω.

7. Let Ω be a topological space. It is described as a T_1-space (we shall meet this terminology later) if every singleton in Ω is closed. Show that Ω is a T_1-space if and only if every singleton in Ω is the intersection of all the open sets that include it.

8. In $\mathbb{R}$, let $\mathcal{B}$ be the set of half-open intervals $[a, b)$ (open on the right, closed on the left). This is the base of a topology $\mathcal{T}$. Show that in this topology $\mathbb{R}$ is first countable but not second countable.

9. Let A and B be subsets of a topological space Ω. We say that A is *dense* in B if $\mathrm{cl}_\Omega(A) \supseteq B$. Ω is said to be *separable* if there is a countable subset of Ω that is dense in Ω.

   Now let Ω be a metric space. If Ω is separable, show that any subspace of Ω is separable. Prove also that, if Ω is separable, it must be second countable.

10. Prove that, in the topology $\mathcal{T}$ of exercise 8, $\mathbb{R}$ is separable, and any subspace of $\mathbb{R}$ is separable. [This, with question 9, shows that $\mathcal{T}$ cannot be defined by a metric.]