MATH 452

General Topology

Assignment 1

1. Let Ω be any set. Prove that the cofinite topology is indeed a topology on Ω.

2. Repeat Question 1 for the cocountable topology.

3. Show that $\mathbb{R}$, with its usual topology, is second countable.

4. Suppose that the topological space Ω is second countable. Show that, in that case, it has the following property.

[**P**]: for any family $\mathcal{U}$ of open sets in Ω, there is a *countable* subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that

$$\bigcup_{U \in \mathcal{V}} U = \bigcup_{U \in \mathcal{U}} U.$$

[If $\mathcal{U}$ is uncountable, most of its members are "redundant" in the union.]

5. Define a class $\mathcal{G}$ of sets in $\mathbb{R}$ by the rule that $G \in \mathcal{G}$ if and only if there is a set U, open in the Euclidean topology on $\mathbb{R}$, such that $G \subseteq U$ and $U \setminus G$ is finite or countable. Show that $\mathcal{G}$ is a topology in $\mathbb{R}$, and that it is the coarsest topology that is finer than both the standard topology on $\mathbb{R}$ and the cocountable topology on $\mathbb{R}$.

6. Prove that the topology $\mathcal{G}$ just defined in $\mathbb{R}$ is not first countable (*a fortiori* it is not second countable).

7. Prove that the topology $\mathcal{G}$ still has the property [**P**] of question 4. (Thus [**P**] is not equivalent to second countability.)

8. Suppose that A, B are subsets of a topological space Ω such that

$$\mathrm{cl}(A) \cap B = A \cap \mathrm{cl}(B) = \emptyset$$

(one sometimes says that A and B are *separated* in such a case). Prove that then

$$\mathrm{Fr}(A \cup B) = \mathrm{Fr}(A) \cup \mathrm{Fr}(B)\,.$$

Here $\mathrm{Fr}(A)$, the *frontier* of A, is defined to be $\mathrm{cl}(A) \setminus \mathrm{int}(A)$. [It is sometimes also called the *boundary* of A. Both names are a little unsatisfactory, since they suggest to the mind a "geometrical" interpretation that is quite inappropriate.]

9. Suppose A is a subset of the topological space Ω. Define $A^{(1)} := A'$, the derived set of A; inductively, let $A^{(n+1)} := (A^{(n)})'$, the $(n+1)$th derived set of A.

Give examples of subsets X of $\mathbb{R}$ (where $\mathbb{R}$ has the Euclidean topology) such that

(a) $X^{(1)} = \emptyset$; *(b)* $X^{(2)} = \emptyset \neq X^{(1)}$;
(c) $X^{(3)} = \emptyset$, but $X^{(2)}, X^{(1)}, X$ are all different;
(d) all of the sets $X, X^{(1)}, X^{(2)}, \ldots$ are different.

over ⟶

10. Give an example of a first countable topological space that is not second countable.

MATH 452

General Topology

Assignment 1

1. Let Ω be any set. Prove that the cofinite topology is indeed a topology on Ω.

2. Repeat Question 1 for the cocountable topology.

3. Show that $\mathbb{R}$, with its usual topology, is second countable.

4. Suppose that the topological space Ω is second countable. Show that, in that case, it has the following property.

[**P**]: for any family $\mathcal{U}$ of open sets in Ω, there is a *countable* subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that

$$\bigcup_{U \in \mathcal{V}} U = \bigcup_{U \in \mathcal{U}} U.$$

[If $\mathcal{U}$ is uncountable, most of its members are "redundant" in the union.]

5. Define a class $\mathcal{G}$ of sets in $\mathbb{R}$ by the rule that $G \in \mathcal{G}$ if and only if there is a set U, open in the Euclidean topology on $\mathbb{R}$, such that $G \subseteq U$ and $U \setminus G$ is finite or countable. Show that $\mathcal{G}$ is a topology in $\mathbb{R}$, and that it is the coarsest topology that is finer than both the standard topology on $\mathbb{R}$ and the cocountable topology on $\mathbb{R}$.

6. Prove that the topology $\mathcal{G}$ just defined in $\mathbb{R}$ is not first countable (*a fortiori* it is not second countable).

7. Prove that the topology $\mathcal{G}$ still has the property [**P**] of question 4. (Thus [**P**] is not equivalent to second countability.)

8. Suppose that A, B are subsets of a topological space Ω such that

$$\mathrm{cl}(A) \cap B = A \cap \mathrm{cl}(B) = \emptyset$$

(one sometimes says that A and B are *separated* in such a case). Prove that then

$$\mathrm{Fr}(A \cup B) = \mathrm{Fr}(A) \cup \mathrm{Fr}(B)\,.$$

Here $\mathrm{Fr}(A)$, the *frontier* of A, is defined to be $\mathrm{cl}(A) \setminus \mathrm{int}(A)$. [It is sometimes also called the *boundary* of A. Both names are a little unsatisfactory, since they suggest to the mind a "geometrical" interpretation that is quite inappropriate.]

9. Suppose A is a subset of the topological space Ω. Define $A^{(1)} := A'$, the derived set of A; inductively, let $A^{(n+1)} := (A^{(n)})'$, the $(n+1)$th derived set of A.

Give examples of subsets X of $\mathbb{R}$ (where $\mathbb{R}$ has the Euclidean topology) such that

(a) $X^{(1)} = \emptyset$; *(b)* $X^{(2)} = \emptyset \neq X^{(1)}$;

(c) $X^{(3)} = \emptyset$, but $X^{(2)}, X^{(1)}, X$ are all different;

(d) all of the sets $X, X^{(1)}, X^{(2)}, \ldots$ are different.

over $\longrightarrow$

10. Give an example of a first countable topological space that is not second countable.

MATH 452

General Topology

Assignment 1

1. Let Ω be any set. Prove that the cofinite topology is indeed a topology on Ω.

2. Repeat Question 1 for the cocountable topology.

3. Show that $\mathbb{R}$, with its usual topology, is second countable.

4. Suppose that the topological space Ω is second countable. Show that, in that case, it has the following property.

[**P**]: for any family $\mathcal{U}$ of open sets in Ω, there is a *countable* subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that

$$\bigcup_{U\in\mathcal{V}} U = \bigcup_{U\in\mathcal{U}} U.$$

[If $\mathcal{U}$ is uncountable, most of its members are "redundant" in the union.]

5. Define a class $\mathcal{G}$ of sets in $\mathbb{R}$ by the rule that $G \in \mathcal{G}$ if and only if there is a set U, open in the Euclidean topology on $\mathbb{R}$, such that $G \subseteq U$ and $U \setminus G$ is finite or countable. Show that $\mathcal{G}$ is a topology in $\mathbb{R}$, and that it is the coarsest topology that is finer than both the standard topology on $\mathbb{R}$ and the cocountable topology on $\mathbb{R}$.

6. Prove that the topology $\mathcal{G}$ just defined in $\mathbb{R}$ is not first countable (*a fortiori* it is not second countable).

7. Prove that the topology $\mathcal{G}$ still has the property [**P**] of question 4. (Thus [**P**] is not equivalent to second countability.)

8. Suppose that A, B are subsets of a topological space Ω such that

$$\mathrm{cl}(A) \cap B = A \cap \mathrm{cl}(B) = \emptyset$$

(one sometimes says that A and B are *separated* in such a case). Prove that then

$$\mathrm{Fr}(A \cup B) = \mathrm{Fr}(A) \cup \mathrm{Fr}(B)\,.$$

Here $\mathrm{Fr}(A)$, the *frontier* of A, is defined to be $\mathrm{cl}(A) \setminus \mathrm{int}(A)$. [It is sometimes also called the *boundary* of A. Both names are a little unsatisfactory, since they suggest to the mind a "geometrical" interpretation that is quite inappropriate.]

9. Suppose A is a subset of the topological space Ω. Define $A^{(1)} := A'$, the derived set of A; inductively, let $A^{(n+1)} := (A^{(n)})'$, the $(n+1)$th derived set of A.

Give examples of subsets X of $\mathbb{R}$ (where $\mathbb{R}$ has the Euclidean topology) such that

(a) $X^{(1)} = \emptyset$; *(b)* $X^{(2)} = \emptyset \neq X^{(1)}$;

(c) $X^{(3)} = \emptyset$, but $X^{(2)}, X^{(1)}, X$ are all different;

(d) all of the sets $X, X^{(1)}, X^{(2)}, \ldots$ are different.

over ⟶

10. Give an example of a first countable topological space that is not second countable.