Math 442

Exercise set 5

1. Let E be a *complex* inner product space. Prove the so-called *polarization identity*:

$$(\forall x, y \in E) \quad 4\langle x, y\rangle = (\|x+y\|^2 - \|x-y\|^2) + i(\|x+iy\|^2 - \|x-iy\|^2)\,.$$

2. Deduce from the previous question that, if E is a *real* normed space for which the "Apollonian identity"

$$(\forall x, y \in E) \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

holds, then the norm may be derived from a real inner product in E.

[It is easy to guess what the inner product ought to be, but showing that it is *additive* in the first variable requires ingenuity. It is not at all obvious how to do it.]

3. Deduce further from the polarization identity that if E is a *complex* normed space for which the Apollonian identity holds, then the norm may be derived from a (complex, i.e. Hermitian) inner product in E.

[Exploit — and assume, if necessary — the previous question. You have already defined a *real* inner product on E; has it any properties with respect to multiplication by i in E?]

4. Suppose that E is an inner product space. A linear mapping $S \in L(E;E)$ is described as *skew-adjoint* if, for all $x, y \in E$, $\langle Sx, y\rangle = -\langle x, Sy\rangle$. Show that, when E is a real inner product space, the real-linear mapping S will be skew-adjoint if and only if, for all $x \in E$, $\langle Sx, x\rangle = 0$.

If, on the other hand, E is a complex inner product space and $S : E \longrightarrow E$ is a *complex* linear mapping, prove that it will be skew-adjoint if and only if $\langle Sx, x\rangle$ is purely imaginary for all $x \in E$.

5. A complex-valued function is described as C^∞ if both its real and its imaginary parts are C^∞ as real-valued functions. Let E be the complex vector space of functions $\mathbb{R} \longrightarrow \mathbb{C}$ that are C^∞ in this sense and periodic with period 2π (the linear operations are to be defined pointwise). Define an inner product on E by

$$\langle f, g\rangle := \int_0^{2\pi} f(t)\overline{g(t)}\,dt\,,$$

and let $D : E \longrightarrow E$ be the linear operator given by

$$(\forall t \in \mathbb{R}) \quad (Df)(t) := f'(t)\,.$$

Show that the differentiation operator D is skew-adjoint; and find its image and its kernel.

6. Prove that the inner-product space E of the previous exercise is *not* a Hilbert space. [The easiest proof is probably *not* one that shows directly that it is metrically incomplete! See the previous question.]

7. A set Q in any (real or complex) vector space is *convex* if, whenever $a, b \in Q$ and $t \in [0,1]$, then $ta + (1-t)b \in Q$ too. (What this means geometrically is that, together with any two of its points, Q must contain the whole straight-line segment between them.) An *extreme point* of a convex set Q is a point $x \in Q$ that cannot be an interior point of any line segment in Q; formally, this means

$$(\forall a, b \in Q)(\forall t \in (0,1)) \quad x = ta + (1-t)b \Longrightarrow x = a = b\,.$$

Now suppose that H is a Hilbert space. The *closed unit ball* of H is $K := \{x \in H : \|x\| \le 1\}$. Show that K is convex, and that a point $x \in K$ is an extreme point of K if and only if $\|x\| = 1$.

8. Let C_1, C_2 be two closed convex subsets of a Hilbert space H, and suppose that $x \notin C_1 \cup C_2$. Prove that there are unique points $y_1 \in C_1$, $y_2 \in C_2$ such that

$$\|x - y_1\|^2 + \|x - y_2\|^2 = \inf\{\|x - z_1\|^2 + \|x - z_2\|^2 : z_1 \in C_1 \ \& \ z_2 \in C_2\}\,.$$

9. *(a)* Let A be any subset of the Hilbert space H. Prove that $(A^\perp)^\perp$ is the closure of the linear span of A [which was, for convenience, sometimes denoted $\Lambda(A)$ in the notes.]

(b) Prove that a finite-dimensional vector subspace of a Hilbert space is necessarily closed. [The same is true in any Hausdorff topological vector space, but the proof is long.]

10. Let E be any Banach space. Recall that an indexed subset $\{x_\alpha : \alpha \in A\}$ of E is said to have the *unordered sum* $x \in E$ if, for any $\epsilon > 0$, there is a finite subset D of A such that, whenever C is a finite subset of A and $D \subseteq C$, $\left\|x - \sum_{\alpha \in C} x_\alpha\right\| < \epsilon$. [The point of this definition is twofold. Firstly, only *finite* sums in E make algebraic sense. Secondly, if A is not a subset of $\mathbb{N}$, there is no obvious way of taking the limit of a sequence of sums.]

Prove that $\{x_\alpha : \alpha \in A\}$ has an unordered sum if and only if it satisfies the “Cauchy condition” that, for every $\epsilon > 0$, there is a finite subset D of A such that, for any finite subset B of A disjoint from D, $\left\|\sum_{\alpha \in B} x_\alpha\right\| < \epsilon$. Show also that, in that case, the set $Q := \{\alpha \in A : x_\alpha \ne 0\}$ is (finite or) countable, and that, if it is indexed in any way as a (finite or infinite) sequence $\{q(1), q(2), \dots\}$, the corresponding series $\sum_{n=1}^\infty x_{q(n)}$ converges to x. [“Only if” is fairly easy. For “if”, take a specific arrangement of Q as a sequence and show that the Cauchy condition forces the series to converge to a sum that can be taken for “x”. The moral is that “unordered sums” are, except for zeros, sums of series that have the property of remaining convergent, and with the same sum, no matter how they are rearranged: the so-called “unconditionally convergent” series.]