# Math 442

## Exercise set 4

1. Let $c_0$ be the vector space of $\mathbb{K}$-valued sequences $(\xi_n)$ such that $\xi_n \to 0$ as $n \to \infty$, with termwise operations. Show that it is a Banach space under the norm

$$\|(\xi_n)\| := \sup\{|\xi_n| : n \in \mathbb{N}\}\,. \tag{1}$$

2. Given $(\eta_n) \in l^1$, define $\phi : c_0 \longrightarrow \mathbb{K}$ by

$$(\forall (\xi_n) \in c_0) \quad \phi((\xi_n)) = \sum\nolimits_{n=1}^{\infty} \xi_n \eta_n\,. \tag{2}$$

Prove that this definition makes sense, that $\phi \in c_0'$, and that the mapping $(\eta_n) \mapsto \phi$ is an isometric isomorphism between $l^1$ (with its usual norm) and $c_0'$ (with the norm dual to (1)). [One customarily says that the dual of $c_0$ *is*, or at least *is naturally identified with,* $l^1$.]

3. Use the identification of $c_0'$ with $l^1$ to present an example of a continuous linear functional $\phi$ on $c_0$ such that $|\phi(x)| < \|\phi\|\|x\|$ for any non-zero $x \in c_0$. (Compare 12.2.)

4. Show that the dual of $l^1$ may be isometrically identified with $l^\infty$ by the formula (2). Deduce that $c_0$ is not reflexive.

5. Let $F$ be a closed vector subspace of the normed space $E$. Use the Hahn-Banach theorem to show that $F'$ is naturally identified (isometrically) with a quotient space of $E'$. Similarly, show that $(E/F)'$ is naturally identified (isometrically) with a closed subspace of $E'$. Deduce that any closed vector subspace of a reflexive Banach space is itself reflexive.

6. A closed vector subspace $F$ of a Banach space $E$ is described as *complemented* if there is a second closed vector subspace $F_1$ of $E$ such that $F \cap F_1 = \{0\}$ and $F + F_1 = E$. Prove that in this case the mapping $F \oplus F_1 \longrightarrow E : (f, f_1) \mapsto f + f_1$ is a continuous linear isomorphism with continuous inverse.

7. Show that the vector subspace $F$ of the Banach space $E$ is closed and complemented if and only if there is a bounded linear map $T : E \longrightarrow E$ such that $T^2 = T$ ($T$ is then described as *idempotent* or a *projection*) and $T(E) = F$.

8. Is $l^1$ reflexive? Is $l^\infty$ reflexive? (Give reasons.)

9*. Let $E$ be a Banach space, and let $J : E' \longrightarrow E'''$ be the bidual map of $E'$. Show that $J(E')$ is a complemented subspace of $E'''$. (HINT: it is sufficient — why? — to construct $T \in L(E''', E')$ such that $TJ$ is the identity of $E'$. This is a lovely piece of "abstract nonsense" that is far less subtle than it threatens.)

10. Let $E$ be a Banach space. Define $c_0(E)$ to be the normed space of $E$-valued sequences tending to $0_E$, with the norm $\|(\xi_n)\| := \sup\{\|\xi_n\|_E : n \in \mathbb{N}\}$. Show that the dual of $c_0(E)$ may be naturally isometrically identified with a sequence space $l^1(E')$ which is defined in a similar fashion.