Math 442

Exercise set 3

1. Let $g \in C([0,1];\mathbb{K})$. Show that the mapping

$$C([0,1];\mathbb{K}) \longrightarrow \mathbb{K} : f \mapsto \int_{[0,1]} g(t)f(t)\,dt$$

is a continuous linear functional on $C([0,1];\mathbb{K})$.

Give (with proof) an example of a continuous linear functional on $C([0,1];\mathbb{K})$ that cannot be described by an integral of this kind.

2. Show that $d(f,g) := \displaystyle\int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}\,dx$ defines a metric on $C([0,1];\mathbb{K})$, and that the linear operations are continuous with respect to this metric.

3. Similarly, show that $d'(f,g) := \displaystyle\int_0^1 \min(|f(x)-g(x)|,1)\,dx$ defines a metric on $C([0,1];\mathbb{K})$, and that the linear operations are continuous with respect to this metric.

4*. Show that the metrics of questions 2 and 3 define the same topology. [It will suffice to show that any open d-ball about the origin includes an open d'-ball and vice versa.]

5. Is the *topology* defined by the metric d' of question 3 definable by a norm? [This is not trivial. One way of doing it is to find a sequence (x_n) such that $x_n \to 0$ with respect to d, but the arithmetic means $\dfrac{x_{n+1}+x_{n+2}+\cdots+x_{2n}}{n}$ do not tend to 0. If the topology came from a norm, they would have to — why?]

6. Find an *algebraic* basis for the space c_{00} over the field $\mathbb{K}$ (*(iv)* of the list in 8.6).

7. A topological space Ω is *Hausdorff* if, for any two points x, y of Ω (with $x \neq y$, there are open sets U, V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Let E be a topological vector space. Show that, if the set $\{0\}$ is a closed subset of E, then E is Hausdorff. [Use 9.3*(b)*.]

8. Let E be a topological vector space. Show that the intersection of all the open sets in E that contain the origin is a closed vector subspace K of E, and that E is Hausdorff if and only if $K = \{0\}$. [Again, 9.3 is helpful.]

9. Show that the only possible topologies under which $\mathbb{K}$ is a topological vector space over itself are the "indiscrete" topology (in which the only open sets are $\emptyset$ and $\mathbb{K}$ itself) and the usual topology of $\mathbb{K}$. [HINT: consider *balanced* open sets containing the origin; show they must be "balls" in the usual sense in $\mathbb{K}$. This is not as difficult as it may look at first.]